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CASTELNUOVO'S INDEX OF REGULARITY AND REDUCTION NUMBERS

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For an ideal I of a local Noetherian ring Castelnuovo's index of regularity $\operatorname{reg} I$ is defined. It is shown that $\operatorname{reg} I$ yields upper bounds for the reduction numbers as well as the relation type of I. Under additional assumptions on I these estimates become equalities.

1. Introduction

Let (A, M) denote a local Noetherian ring with an infinite residue field A/M. If I is an ideal of A, recall that an ideal $J \subseteq I$ is called a *reduction* of I if $I^{n+1} = JI^n$ for some integer n. A reduction J is called a *minimal reduction* if it does not contain properly a reduction of I. These concepts were introduced and studied by D. G. Northcott and D. Rees in [8]. More recently, the reduction number $r_J(I)$, the smallest number n such that $I^{n+1} = JI^n$, attained much attention, see [10]. Another invariant related to I is the relation type $n(I)$, the largest degree of a minimal generating set of J, where J denotes the presentation ideal of the form ring

$$G(I) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}, \qquad G(I) \cong A/I\,[T_1, \ldots, T_r]/J,$$

r the minimal number of generators of I, see [5] and Section 4. For computational reasons it is important to estimate the relation type.

The purpose of this paper is to define another invariant $\operatorname{reg} I$ of I, Castelnuovo's index of regularity, determined in terms of the vanishing of local

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cohomology modules, see (2.1). On the other side, see (3.1) and (4.1), there are cohomological interpretations of $r_J(I)$ and $n(I)$ respectively. In the main result, see (2.2), there is an expression of $\operatorname{reg} I$ as the maximum over certain cohomologically defined numbers, among them $r_J(I)$ as well as $n(I)$. It turns out, see (4.2) and (3.2), that

$$n(I)-1 \leqslant \operatorname{reg} I \quad \text{and} \quad r_J(I) \leqslant \operatorname{reg} I$$

for any reduction J of I. There are results about equality in these bounds, see (3.3), (3.4), and (4.3). This is closely related to recent research by H. Cho [3], S. Huckaba [5], and N. V. Trung [12], where similar results are obtained.

Therefore Castelnuovo's index of regularity is an approriate invariant related to reduction numbers as well as the relation type. The idea to study $\operatorname{reg} I$ is inspired by D. Mumford's book [7], see also [1], [9], and [11] for an algebraic point of view. Because of the cohomological nature of $\operatorname{reg} I$ there are used local cohomology modules and Koszul cohomology as technical tools. In the sequel we work with graded rings. The above-mentioned results about ideals are obtained by passing to the corresponding form rings.

2. Castelnuovo's index of regularity

Throughout this section, let $R = \bigoplus_{n \geqslant 0} R_n$ denote a Noetherian commutative graded ring such that R_0 is a local ring and the R_0-algebra R is a generated by the elements of R_1. Put $R_+ = \bigoplus_{n>0} R_n$ the homogeneous ideal generated by all forms of positive degree. For a graded R-module E, denote by E_n, $n \in \mathbf{Z}$, the nth graded piece of E_n. Put

$$s(E) = \sup\{n \in \mathbf{Z}\colon\ E_n \neq 0\}.$$

In the case E is a finitely generated graded R-module $s(E) < \infty$ if there is an integer n such that $(R_+)^n E = 0$. In the following consider the local cohomology modules $H^i_{R_+}(E)$, $i \in \mathbf{Z}$, of E with respect to R_+. Note that they are graded R-modules with $H^i_{R_+}(E)_n = 0$ for all large n as follows by virtue of Serre's Theorem, see [4].

(2.1) Definition. Set

$$\operatorname{reg} E = \max\{s(H^i_{R_+}(E)) + i\colon\ i \in \mathbf{Z}\},$$

the *Castelnuovo index of regularity* of the R-module E.

This notion was introduced by D. Mumford [7] resp. A. Ooishi [9] in a geometric resp. algebraic context. For further information see also [1] and [11]. By a minimal reduction Q of R_+ we define a homogeneous ideal $Q \subseteqq R_+$ generated by forms of degree one such that $Q_n = R_n$ for all large n. In the following we use the notation of the Koszul cohomology $H^i(Q; E)$, where Q denotes a system of forms of R resp. for the simplicity of notation the ideal generated by them. The following result is the crucial point for that what follows.

(2.2) THEOREM. *For a finitely generated R-module E the following integers are equal*:

(a) $\operatorname{reg} E$.

(b) $\max\{s(H^i(Q; E))+i\colon i\in \mathbf{Z}\}$ *for any ideal Q generated by forms of degree one with* $\operatorname{rad} Q = \operatorname{rad} R_+$.

(c) $\max\{s(H^i(\boldsymbol{x}; E))+i\colon i\in \mathbf{Z}\}$ *for any minimal reduction* $\boldsymbol{x} = \{x_1, \dots, x_g\}$ *of R.*

In particular, the integers defined in (b) and (c) are independent of the choice of Q and $\boldsymbol{x}$ respectively.

Proof. Denote by a, b, c the integers defined in (a), (b), (c) respectively. For the proof of $a \geqslant b$ consider the following spectral sequence

$$E_2^{pq} = H^p(Q; H^q_{R_+}(E)) \Rightarrow E^{p+q} = H^{p+q}(Q; E).$$

By the definition of a, $H^i_{R_+}(E)_j = 0$ for all $i+j > a$. Therefore $H^{p+q}(Q; E)_n = 0$ whenever $p+q+n > a$ since it is a subquotient of $\bigoplus H^p(Q; H^q_{R_+}(E))_n$. Therefore $H^i(Q; E)_n = 0$ for all $i+n > a$, as required. Because of $\operatorname{rad} R_+ = \operatorname{rad} \boldsymbol{x}R$ for a minimal reduction $\boldsymbol{x}$ of R, $b \geqslant c$. In order to complete the proof let us show $c \geqslant a$. To this end use the above spectral sequence. For $i = 0$ it yields

$$H^0(\boldsymbol{x}; H^0_{R_+}(E)) \cong H^0(\boldsymbol{x}; E).$$

Assume that $s(H^0_{R_+}(E)) =: s > c$. Then

$$0 \neq H^0_{R_+}(E)_s \cong H^0(\boldsymbol{x}; H^0_{R_+}(E))_s \cong H^0(\boldsymbol{x}; E)_s,$$

a contradiction. Now let

$$H^i_{R_+}(E)_n = 0 \quad \text{for } i+n > c \text{ and all } i < j.$$

Assume that $s(H^j_{R_+}(E))+j =: s > c$. Then the spectral sequence degenerates partially to the following isomorphisms

$$0 \neq H^j_{R_+}(E)_{s-j} \cong H^0(\boldsymbol{x}; H^j_{R_+}(E))_{s-j} \cong H^j(\boldsymbol{x}; E)_{s-j},$$

a contradiction. This completes the proof.

(2.3) COROLLARY. $\operatorname{reg} E = \max\{s(H^i(R_+; E))+i\colon i\in \mathbf{Z}\}$.

In the case $\boldsymbol{x}$ forms an E-regular sequence one obtains the well-known fact

$$s(H^g_{R_+}(E))+g = s(E/\boldsymbol{x}E).$$

In particular, $s(E/xE)$ does not depend on x. For an ideal I of a local ring (A, M) put $\operatorname{reg} I = \operatorname{reg} G(I)$, where $G(I)$ denotes the form ring of A with respect to I. Call $\operatorname{reg} I$ Castelnuovo's index of regularity of I.

3. The reduction number

As above $R = \bigoplus_{n\geqslant 0} R_n$ denotes a Noetherian graded ring such that R_0 is a local ring and the R_0-algebra R is generated by R_1. For a reduction $Q = (f_1, \ldots, f_g)R$ of R_+ define $r(Q)$, the *reduction number* of Q, the least integer $n \in \mathbb{N}$ such that $Q_{n+1} = R_{n+1}$.

(3.1) Proposition. *$r(Q) = s(H^g(Q; R)) + g$, where g denotes the number of elements of Q.*

Proof. By properties of the Koszul cohomology it follows

$$H^g(Q; R) \cong (R/Q)(g).$$

Because Q is a reduction of R_+, $s(R/Q) = r(Q)$ as easily seen. Whence the result follows.

Next there are estimates of $r(Q)$ in terms of the local cohomology modules. This result was shown by a different argument in [12], (3.2).

(3.2) Proposition. *For any reduction Q of R_+*

$$s(H^g_{R_+}(R)) + g \leqslant r(Q) \leqslant \operatorname{reg} R,$$

where g denotes the number of generators of Q.

Proof. The upper bound follows by virtue of (3.1) and (2.2). For the lower bound recall that

$$H^g_{R_+}(R)_n \cong \varinjlim_k R_{kg+n}/x^{(k)} R_{k(g-1)+n},$$

see [4], where $Q = (x_1, \ldots, x_g)R$ and $x^{(k)} = \{x_1^k, \ldots, x_g^k\}$. Let $r = r(Q)$ and $n > r+g$. Then

$$R_{kg+n}/x^{(k)} R_{k(g-1)+n} = 0$$

by the same argument as given in [10], proof of Proposition 2.

In the following result we will show that under additional assumptions the bounds in (3.2) become equalities. This extends results of [2], [5], and [12]. To this end let I denote an ideal of a local ring (A, M) with A/M an infinite residue field and $G = G(I)$ its form ring. By $a(I)$ denote the analytic spread of I, see [8].

(3.3) Theorem. *Assume $a(I) = \operatorname{grade} I =: g$ and $\operatorname{grade} G_+ \geqslant g-1$. Then*

$$s(H^g_{G_+}(G)) + g = r_x(I) = \operatorname{reg} I$$

for any minimal reduction x of I.

Proof. Because of the assumptions and prime avoidance arguments one may assume for a given minimal reduction $\boldsymbol{x}$ of I the existence of a regular sequence $\{x_1, \ldots, x_g\}$ such that

$$\boldsymbol{x}A = (x_1, \ldots, x_g)A$$

and $\{x'_1, \ldots, x'_{g-1}\}$ forms a G-regular sequence, where x' denotes the initial form of an element $x \in A$. Thus without loss of generality assume $\boldsymbol{x}$ generated by elements of the required type. By (3.2) it is enough to show that

$$s(H^{g-1}_{G_+}(G)) + g - 1 < r_{\boldsymbol{x}}(I).$$

In order to prove this recall that $s(H^{g-1}_{G_+}(G)) = s(H^{g-1}(\boldsymbol{x}'; G))$ as follows by the above spectral sequence. Furthermore,

$$H^{g-1}(\boldsymbol{x}'; G) \cong H^0(x'_g; G(I/(x_1, \ldots, x_{g-1})))(g-1)$$

as follows because $\{x'_1, \ldots, x'_{g-1}\}$ is a G-regular sequence. So it is enough to show the case $g = 1$. Let $r = r_{\boldsymbol{x}}(I)$, $x = x_1$. Then

$$H^0(x'; G)_n \cong (I^{n+2} : x) \cap I^n/I^{n+1} = 0$$

for all $n \geqslant r - 1$ because

$$I^{n+2} : x = xI^{n+1} : x = I^{n+1} \quad \text{for } n \geqslant r - 1.$$

Recall that x is a nonzero divisor.

Another result about the invariance of the reduction number is related to the notion of a d-sequence, see C. Huneke [6] for the definition.

(3.4) THEOREM. *Let Q denote a reduction of R_+ generated by a d-sequence. Then $r(Q) = \operatorname{reg} R$.*

Proof. Let $Q = (x_1, \ldots, x_g)R$ with a d-sequence $\{x_1, \ldots, x_g\}$. Then it is a relative R-regular sequence with respect to Q in the sense of [3]. By [3] it follows that the induced maps

$$H^i(\boldsymbol{x}; Q) \to H^i(\boldsymbol{x}; R), \quad i \in \mathbf{Z},$$

are zero. Hence the short exact sequence

$$0 \to Q \to R \to R/Q \to 0$$

induces injections

$$0 \to H^i(\boldsymbol{x}; R) \to H^i(\boldsymbol{x}; R/Q), \quad i \in \mathbf{Z}.$$

Because of $H^i(\boldsymbol{x}; R/Q) \cong (R/Q)^{\binom{g}{i}}(i)$,

$$s(H^i(\boldsymbol{x}; R)) + i \leqslant r(Q) \quad \text{for all } i.$$

While $s(H^g(\boldsymbol{x}; R)) + g = r(Q)$ the claim follows by virtue of (2.2).

4. The relation type

Let $R = \bigoplus_{n \geqslant 0} R_n$ denote a Noetherian graded ring, where R_0 is a local ring and R_+ is minimally generated by forms of degree one, say $\{f_1, \ldots, f_e\}$. Then there is an epimorphism

$$R_0[T_1, \ldots, T_e] \to R, \quad T_i \mapsto f_i, \quad i = 1, \ldots, e,$$

where $T_1, \ldots, T_e$ are indeterminates over R_0. Let J denote the kernel, a homogeneous ideal of the polynomial ring $R_0[T_1, \ldots, T_e]$. Let J_n, $n \in \mathbf{N}$, denote the ideal generated by all homogeneous elements of J of degree at most n. Then $J_1 \subseteqq J_2 \subseteqq \ldots$ is an ascending chain of ideals. Define $n(R)$, the relation type of R, the smallest integer n such that $J = J_n$. Note that $n(R)$ denotes the highest degree of generators in a minimal generating set of J. Moreover, $n(R)$ does not depend on the choice of the minimal generating set of R_+.

(4.1) Proposition. $n(R) = s(H^{e-1}(R_+; R)) + e$, *where e denotes the minimal number of generators of R_+.*

Proof. Apply the Koszul cohomlogy $H^{\cdot}(\boldsymbol{T}; \square)$ to the short exact sequence of graded modules over $R_0[T_1, \ldots, T_e]$

$$0 \to J \to R_0[T_1, \ldots, T_e] \to R \to 0.$$

Then there are isomorphisms

$$H^{e-1}(R_+; R) \cong H^{e-1}(\boldsymbol{T}; R) \cong J/\boldsymbol{T}J(e).$$

Because of $n(R) = s(J/\boldsymbol{T}J)$ the claim follows.

Next let us relate the relation type $n(R)$ to Castelnuovo's index of regularity. Furthermore, there are some bounds on $n(R)$.

(4.2) Corollary. (a) $n(R) - 1 \leqslant \operatorname{reg} R$.
(b) *With the assumptions of* (3.3)

$$n(I) - 1 = r_{\boldsymbol{x}}(I)$$

for any minimal reduction $\boldsymbol{x}$ of I and $n(I) = n(G(I))$.

For a finitely generated graded R-module E one has $s(E/R_+ E) \leqslant \operatorname{reg} E$ since $(E/R_+ E)(e) \cong H^e(R_+; E)$. Moreover, $s(E/R_+ E)$ is equal to the maximal degree of forms in a minimal generating set of E. Thus, $\operatorname{reg} E$ gives a degree bound for a minimal generating set of E, see [1], [9], and [12] for further information.

We will continue with a result about the equality of $n(I) - 1$ and $\operatorname{reg} I$, which was shown by S. Huckaba in [5].

(4.3) Corollary. *Let I be as in* (3.3). *Assume that I requires at most $g+1$ generators. Then*

$$n(I)-1 = r_{\mathbf{x}}(I) = \operatorname{reg} I$$

for any minimal reduction $\mathbf{x}$ of I.

Proof. First assume that I requires $g+1$ generators. Then

$$s\big(H^{g+1}(G_+;\, G)\big)+g+1 = 0$$

and

$$\operatorname{reg} I = \max\big\{s\big(H^g(G_+;\, G)\big)+g,\ s\big(H^{g-1}(G_+;\, G)\big)+g-1\big\}$$

by (2.2). Now

$$s\big(H^{g-1}(G_+;\, G)\big) = s\big(H^{g-1}_{G_+}(G)\big) < \operatorname{reg} G-g+1$$

as shown in the proof of (3.3). Whence

$$\operatorname{reg} G = s\big(H^g(G_+;\, G)\big)+g = n(G)-1,$$

as required. The case of a g-generator ideal is more easy to handle by the same arguments.

Let us conclude with a bound of $\operatorname{reg} I$ in the situation of (3.3) where I is in addition an M-primary ideal.

(4.4) Theorem. *Let I denote an M-primary ideal of a d-dimensional Cohen–Macaulay ring (A, M) such that $G(I)$ is a Cohen–Macaulay ring. Then*

$$\operatorname{reg} I \leqslant e(I)-L(A/I),$$

where e resp. L denotes the multiplicity resp. the length of I.

Proof. Without loss of generality we may assume the existence of a minimal reduction $J = (x_1, \ldots, x_d)A$ of I such that $\{x'_1, \ldots, x'_d\}$ forms a $G(I)$-regular sequence. Because of

$$G/\mathbf{x}'G = G(IA/\mathbf{x}A), \qquad \operatorname{reg} G/\mathbf{x}'G = \operatorname{reg} G,$$

and

$$e(I) = e(\mathbf{x};\, A) = L(A/\mathbf{x}A)$$

we may assume $\dim A = 0$. Then

$$\operatorname{reg} G \leqslant L(G)-L(G_0)$$

which proves the claim.

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