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MAPPINGS OF REDUCIBLE 3-MANIFOLDS*

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This article is expository, although some of the material in section 3 is new. It is intended for the reader who has some familiarity with the theory of 3-manifolds. I would like to thank the Research Council of the University of Oklahoma, the Stefan Banach International Mathematical Center, and the Mathematical Sciences Research Institute for their support in its preparation. I would also like to thank Harrie Hendriks for reading an earlier version and suggesting several improvements.

Recall that a 3-manifold M is *prime* if whenever M is written as a connected sum $M_1 \# M_2$, then at least one of M_1 or M_2 is homeomorphic to the 3-sphere S^3. By the Kneser factorization theorem [K], [H8] every compact 3-manifold can be written as a connected sum of finitely many prime 3-manifolds. In the orientable case, the summands that are not S^3 are unique up to order [M2]. In the nonorientable case, the summands that are not S^3 are unique up to order and up to ambiguity arising from the fact that if N is nonorientable then $N \# (S^1 \times S^2) = N \# (S^1 \tilde{\times} S^2)$, where $S^1 \tilde{\times} S^2$ is the nonorientable S^2-bundle over S^1. A 3-manifold M is *irreducible* if every (tame) 2-sphere imbedded in M separates M and bounds a 3-ball in M. A prime 3-manifold must be homeomorphic to $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, or else be irreducible. The reader is referred to the book by Hempel [H8] for an excellent treatment of these facts.

At this point, most discussions of compact 3-manifolds specialize to the irreducible case. For classification of 3-manifolds, this is sufficient, because of the uniqueness of the factorization. But in the study of mappings between 3-manifolds, many phenomena arise in the reducible case that do not appear in the irreducible case. In this article, I will discuss some of these phenomena and the progress that has been made in understanding them.

We will need a few more standard definitions and facts, which may be

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found in [H8] or most other texts on 3-manifolds. A 3-manifold is $\boldsymbol{P}^2$-*irreducible* if it is irreducible and contains no 2-sided projective planes. A $\boldsymbol{P}^2$-irreducible 3-manifold M with infinite fundamental group must be aspherical. A surface F imbedded in a 3-manifold N is *properly* imbedded if $F \cap \partial M = \partial F$, and such a surface is called *incompressible* if it is 2-sided and the homomorphism $\pi_1(F) \to \pi_1(N)$ induced by inclusion is injective. A $\boldsymbol{P}^2$-irreducible 3-manifold M is *sufficiently large* if it contains an incompressible surface F with $F \neq S^2$. This implies that either M is a 3-ball or $\pi_1(M)$ is infinite. Note that any $\boldsymbol{P}^2$-irreducible 3-manifold with nonempty boundary is sufficiently large.

All homeomorphisms, imbeddings, and submanifolds will be PL, without explicit mention. The *mapping class group* $\mathscr{H}(M)$ is the group of path components of the homeomorphism group Homeo(M). The *group of homotopy equivalences* $\mathscr{E}(M)$ is the group of path components of the H-space of self-homotopy-equivalences Equiv(M). A *proper* map from M to N is a map which carries ∂M into ∂N, and a *proper homotopy equivalence* is a homotopy equivalence of pairs f: $(M, \partial M) \to (N, \partial N)$. The group of proper homotopy classes of proper self-homotopy-equivalences is denoted $\mathscr{E}(M, \partial M)$. If $A \subseteq M$ then $\mathscr{H}(M \text{ rel } A)$ is the group of path components of Homeo$(M \text{ rel } A) = \{h \in \text{Homeo}(M) \mid h|_A \text{ is the identity map } 1_A\}$.

If M is aspherical, the correspondence $\langle f \rangle \to f_\#$ induces an isomorphism from $\mathscr{E}(M)$ to Out$(\pi_1(M))$, the group of outer automorphisms of $\pi_1(M)$.

1. Splitting a homotopy equivalence along a 2-sphere

Suppose N is an n-dimensional manifold with $N = N_1 \# N_2$. We have $N = N_1' \cup N_2'$ where $N_1' \cap N_2'$ is a 2-sphere S. To *split* f along S means to find a homotopy equivalence g: $M \to N$, homotopic to f, with $g^{-1}(S)$ equal to an imbedded $(n-1)$-sphere Σ in M. When S separates N, then Σ must separate M into $M_1 \# M_2$ with $g^{-1}(N_i') = M_i'$. The maps $g|_{M_i'}$ can be extended to maps g_i: $M_i \to N_i$ by coning over the boundary components Σ and S, and the maps g_i will be homotopy equivalences. In this way the problem of understanding f can be reduced to an inductive procedure, beginning with the case of N prime.

In dimensions $n \geqslant 5$, the general problem of splitting a homotopy equivalence has been studied by many authors, notably by Cappell. In particular, he showed [C1] that for closed N of dimension $n = 2k+1 \geqslant 5$, if $\pi_1(N)$ contains no element of order 2 which is orientation-preserving (when k is even) or orientation-reversing (when k is odd) then any homotopy equivalence from a closed n-manifold to N is splittable; however if $\pi_1(N)$

does contain such an order 2 element, then there is a nonsplittable homotopy equivalence from some closed n-manifold to N. As we will see, these results hold for $n = 3$ as well.

Since $3 = 2k+1$ for $k = 1$, Cappell's condition in dimension 3 states that $\pi_1(N)$ contains an orientation-reversing element of order 2. In dimension 3, this has a simple geometric meaning:

PROPOSITION. *Let N be a compact 3-manifold. Then $\pi_2(N)$ contains an orientation-reversing element of order 2 if and only if N contains a 2-sided projective plane.*

Proof. The proposition is an immediate consequence of Theorem 8.2 of [E]. ∎

The positive splitting result in dimension 3 is due to Hendriks and Laudenbach:

SPLITTING THEOREM ([L1], [H–L1]). *Let N be a closed 3-manifold and S an imbedded 2-sphere in N. Suppose that N does not contain any 2-sided projective planes. Then any homotopy equivalence f: $M \to N$ can be split along S.*

The proof of the splitting theorem occurs in two steps. First, f is changed by homotopy so that $f^{-1}(S)$ consists of 2-spheres and tori. In [L1] this is accomplished by a delicate geometric argument, while in [H–L1] it is accomplished by using obstruction theory. Then, in both proofs, it is shown that f can be changed by homotopy to accomplish ambient surgery on the tori, changing them into 2-spheres, and it is easy to "pipe together" all the 2-spheres by a final homotopy to make $f^{-1}(S)$ into a single 2-sphere.

The Splitting Theorem has been extended to manifolds with boundary by Swarup [S2].

The negative splitting result in dimension 3 is due to Hendriks:

THEOREM ([H9]). *Let S be an imbedded 2-sphere in the closed 3-manifold N. If S separates N so that $N = N'_1 \cup_S N'_2$, suppose that neither N'_1 nor N'_2 is simply connected. Suppose N contains a 2-sided projective plane. Then for any 3-manifold M homotopy equivalent to N, there is a homotopy equivalence f: $M \to N$ which cannot be split along S.*

In order to give an example of such a nonsplittable f, we first describe a general method for producing homotopy equivalences. Let f: $(M, x_0) \to (N, f(x_0))$ be a homotopy equivalence between n-manifolds, and let B be an n-ball imbedded in M disjoint from x_0 and from the $(n-1)$-skeleton $M^{(n-1)}$. Assume $x_0 \in M^{(n-1)}$. By homotopy we may assume $f(B) = f(x_0)$. Let q: $M \to M \vee S^n$ be the quotient map that collapses ∂B to a point. If $\langle\tau\rangle \in \pi_n(N, f(x_0))$, then the map $(f \vee \tau) \circ q$: $M \to N$ is said to be obtained from f by a *modification within the ball B using τ*.

LEMMA. *Suppose M is an n-manifold, $n \geqslant 2$, whose universal cover is not* *closed. If f: $M \to N$ is a homotopy equivalence, and f' is obtained from f by a modification within a ball, then f' is also a homotopy equivalence.*

Proof. Since $n \geqslant 2$, f' induces an isomorphism on fundamental groups. Since the universal covering of M is not closed, it deformation retracts to a subcomplex of its $(n-1)$-skeleton, consequently any element of $\pi_q(M, x_0)$ can be represented by a map into the $(n-1)$-skeleton of M. But $f'|_{M^{(n-1)}} = f|_{M^{(n-1)}}$ so $(f')_{\#}$: $\pi_q(M, x_0) \to \pi_q(N, f(x_0))$ is an isomorphism for all $q \geqslant 2$. By Whitehead's criterion, f' is a homotopy equivalence. ∎

Remark. When $n = 3$, $M = N$, and $f = 1_M$, f' will be a *simple* homotopy equivalence [H9, section 3.2].

Return now to the situation of the previous theorem. Let P be a 2-sided projective plane in N. By a cutting and pasting argument we can find P disjoint from S. Let $\gamma \in \pi_2(N, p_0)$ be represented by the orientable double covering $(S^2, s_0) \to (P, p_0) \subseteq (N, p_0)$. Let α be a loop in N based at p_0 which intersects S in one point if S is nonseparating, or intersects S in two points and is not homotopic (rel. p_0) off of S, if S is separating. Let τ be the Whitehead product $[\gamma, \alpha \cdot \gamma] \in \pi_3(N, p_0)$. Now τ can be represented by a map t: $S^3 \to N$ so that $t^{-1}(S)$ consists of one torus (if S is nonseparating) or two tori (if S is separating). Let f: $M \to N$ be any split homotopy equivalence and let f' be obtained from f by a modification within a ball using t. The preimage $(f')^{-1}(S)$ consists of a 2-sphere and one or two tori, and for this f' the argument for the second step of the Splitting Theorem cannot be carried out. In [H9], it is proved that such an f' is not splittable, by constructing an invariant which gives an obstruction to eliminating the preimage tori by homotopy.

2. Deforming homotopy equivalences to homeomorphisms

In this section, we consider the question of when a homotopy equivalence f: $M \to N$ is homotopic to a homeomorphism. If M and N contain 2-sided projective planes, then as we have seen, f might not even be splittable. So we will assume M and N do not contain 2-sided projective planes.

Even in the irreducible case, a homotopy equivalence need not be homotopic to a homeomorphism. The standard example is:

EXAMPLE A. $M = L(7,1)$ and $N = L(7,2)$. Here M and N are 3-dimensional lens spaces. By (29.5) of [C3], there is a degree 1 homotopy equivalence from M to N. But by (30.1) of [C3], there is no simple homotopy equivalence, and hence no homeomorphism, from M to N.

We can rule out this example by assuming that there is *some* homeomorphism from M to N. But we still have:

EXAMPLE B. $M = N = L(12,1)$. Let γ be the standard generator of $\pi_1(M)$

$\cong \mathbf{Z}/12$. By (29.5) of [C3], there is a self-homotopy-equivalence of M taking γ to γ^5. But by (30.1) of [C3], there is no simple self-homotopy-equivalence, and hence no self-homeomorphism, of M taking γ to γ^5.

In the sufficiently large case, we have important positive results:

THEOREM ([W],[H7]). *Let N be a closed $\mathbf{P}^2$-irreducible sufficiently large 3-manifold. Then every homotopy equivalence from a closed 3-manifold (satisfying the Poincaré conjecture) to N is homotopic to a homeomorphism.*

THEOREM ([W], [L1]). *Let M be a closed $\mathbf{P}^2$-irreducible sufficently large 3-manifold. Let f and g be homeomorphisms from M to M. If f and g are homotopic, then they are isotopic.*

These two theorems can be combined into the statement that $\mathscr{H}(M) \to \mathscr{E}(M)$ is an isomorphism. In the bounded case, $\mathscr{H}(M) \to \mathscr{E}(M, \partial M)$ is an isomorphism.

These theorems have reducible versions due to Laudenbach:

THEOREM ([L1]). *Suppose the closed 3-manifold N is a connected sum $N_1 \# N_2 \# \ldots \# N_k$ where each N_i has the property that any homotopy equivalence from a closed 3-manifold (satisfying the Poincaré conjecture) to N_i is homotopic to a homeomorphism. Then every homotopy equivalence from a closed 3-manifold (satisfying the Poincaré conjecture) to N is homotopic to a homeomorphism.*

THEOREM ([L1]). *Suppose the closed 3-manifold N is a connected sum $N_1 \# N_2 \# \ldots \# N_k$ where each N_i is either an S^2-bundle over S^1 or a $\mathbf{P}^2$-irreducible sufficiently large 3-manifold. Suppose M satisfies the Poincaré conjecture. Then every homotopy equivalence from M to N is homotopic to a homeomorphism, unique up to isotopy.*

The first of these theorems is proved using the Splitting Theorem and induction on k. The uniqueness part of the second theorem uses Laudenbach's result that homotopic imbedded 2-spheres in M are isotopic, together with results of Hendriks.

In case $M = N$, it is *not* sufficient to assume that every self-homotopy-equivalence of each summand is homotopic to a homeomorphism:

EXAMPLE C. $M = L(7,1) \# L(7,2)$. Using [C3], every self-homotopy-equivalence of $L(7,1)$ or $L(7,2)$ is homotopic to a homeomorphism. From Example A, there is a degree 1 homotopy equivalence $g\colon L(7,1) \to L(7,2)$, but there is no homeomorphism. Let h be a homotopy inverse for g. We may assume that g carries a 3-ball $B_1 \subseteq L(7,1)$ homeomorphically to $B_2 \subseteq L(7,2)$, with $g^{-1}(B_2) = B_1$, and h carries B_2 homeomorphically to B_1, with $h^{-1}(B_1) = B_2$. We can use g and h to construct a self-homotopy-equivalence f of $M = (L(7,1)\text{-int}(B_1)) \cup_S (L(7,2)\text{-int}(B_2))$ that interchanges the sides of S. Suppose f were homotopic to a homeomorphism F. An easy argument using irreducibility of $L(7,1)$ and $L(7,2)$ shows that any essential imbedded 2-sphere in M is isotopic to S, so we may assume $F(S) = S$. Since $F_{\#}$ interchanges the factors of $\pi_1 M \cong \pi_1(L(7,1)) = \pi_1(L(7,2))$, F must interchange the sides of S. Splitting along S and filling in 3-balls would yield a homeomorphism from $L(7,1)$ to $L(7,2)$. Therefore, f cannot be homotopic to a homeomorphism.

There is, however, an important case for which a general result is available. In order to state it, we must first describe a certain kind of homeomorphism. Suppose S is a 2-sphere imbedded in the interior of M (or alternatively as a boundary component of M). Let $S^2 \times I$ be coordinates on a product neighborhood of S. Regard S^2 as the unit 2-sphere in $\boldsymbol{R}^3$, and let $SO(3)$ be the group of orthogonal linear transformations of $\boldsymbol{R}^3$ of determinant 1. It is known that $\pi_1(SO(3), 1_{\boldsymbol{R}^3}) \cong \boldsymbol{Z}/2$ (with generator carried by $SO(2) \subseteq SO(3)$). Let τ: $(I, 0, 1) \to (SO(3), 1_{\boldsymbol{R}^3}, 1_{\boldsymbol{R}^3})$ represent the generator, and define f_τ: $M \to M$ by $f_\tau(x, t) = (\tau(t)(x), t)$ for $(x, t) \in S^2 \times I$ and $f_\tau(y) = y$ for $y \notin S^2 \times I$. Observe that f_τ^2 is isotopic to f_{τ^2}, and a nullhomotopy of τ^2 gives an isotopy from f_{τ^2} to 1_M. Thus $\langle f_\tau \rangle$ is an element of $\mathscr{H}(M)$ of order $\leqslant 2$. Since f_τ moves points only in a simply connected subset of M, $(f_\tau)_{\#}$: $\pi_1(M, x_0) \to \pi_1(M, x_0)$ is the identity at any basepoint x_0 fixed by f. The isotopy class $\langle f_\tau \rangle$ does not depend on the choice of product neighborhood or the choice of τ. The homeomorphism f_τ is called a *rotation* about the 2-sphere S.

We can now state the result of Hendriks:

THEOREM ([H10], [H11]). *Let M be a closed 3-manifold which does not contain 2-sided projective planes, and let f: $(M, x_0) \to (M, x_0)$ be a homotopy equivalence having local degree 1 at x_0 and inducing the identity automorphism on $\pi_1(M, x_0)$. Then f is homotopic* (rel x_0) *to a rotation about a 2-sphere.*

There is a generalization of this result to bounded 3-manifolds in [K–M].

In many cases, the theorem of Laudenbach mentioned above implies that the rotation in the conclusion of Hendriks' theorem is unique up to isotopy. In fact, Laudenbach [L1, p. 133] conjectured that in general two products of rotations that are homotopic must be isotopic. A recent and surprising result of J. Friedman and D. Witt disproves this conjecture. Recall, for example from [O], that there is a free action of $D(2^k, m) \times C_n$ on S^3, where $D(2^k, m)$ is the extension of the cyclic group C_m by C_{2^k} which has presentation

$$\langle a, b \mid a^{2^k} = b^m = 1, aba^{-1} = b^{-1} \rangle$$

with $k \geqslant 2$, m odd and $m \geqslant 3$, and $(2^k m, n) = 1$. Let $N(k, m, n)$ be the quotient of S^3 by this action, and let $M(k, m, n)$ be the complement of an open ball in $N(k, m, n)$. Denote by R the rotation about the boundary 2-sphere of $M(k, m, n)$.

THEOREM ([F–W]). *For $k \geqslant 3$, R is not isotopic to the identity* (rel $\partial M(k, m, n)$).

This theorem is proved using results of Ivanov [I]. We have also a remarkable theorem of Hendriks:

THEOREM ([H10], [H11]). *A rotation about an imbedded 2-sphere S in M is homotopic to the identity* (rel ∂M) *if and only if S bounds a submanifold M_1 in M such that $M_1 \cup_S D^3$ is a connected sum of closed manifolds, each either with finite fundamental group whose 2-Sylow subgroup is cyclic, or homotopy equivalent to an S^2 or $\boldsymbol{P}^2$ bundle over S^1.*

Since the 2-Sylow subgroup of $\pi_1(N(k, m, n))$ is cyclic, the rotations in the theorem of Friedman and Witt are homotopic to the identity (rel $\partial M(k, m, n)$). To get a closed example, form $N = N(k, m, n) \# N(k', m', n')$ from summands as in their theorem. Using a result of Hatcher [H5], one shows that a rotation about the connected sum sphere of N is not isotopic to the identity, while Hendriks' theorem still applies.

3. The mapping class group of M

In Section 2, we discussed the homeomorphism called a rotation about a 2-sphere in M. More generally, when F is a properly-imbedded 2-sided 2-manifold in M and $\langle \tau \rangle$ is a nontrivial element of $\pi_1(\text{Homeo}(F), I_F)$ (see [H1], [H2], [H3] for calculation of $\pi_1(\text{Homeo}(F))$), we may define a similar homeomorphism f_τ with support in a product neighborhood of F. When $F = \boldsymbol{RP}^2$, we have $\pi_1(\text{Homeo}(\boldsymbol{RP}^2)) \cong \boldsymbol{Z}/2$ and f_τ is called a *rotation* about the projective plane F. When F is a 2-disc, $\pi_1(\text{Homeo}(D^2)) \cong \boldsymbol{Z}$ and f_τ is called a *twist* about the disc F. When F is a 2-sided annulus, torus, Möbius band, or Klein bottle, f_τ is called a *Dehn twist*. All of these homeomorphisms are said to be *of Dehn type* since they are 3-dimensional analogs of Dehn's generators for the orientation-preserving mapping class group of an orientable 2-manifold. The subgroup of $\mathscr{H}(M)$ generated by homeomorphism of Dehn type is called the *Johannson subgroup* and is denoted by $J(M)$.

Most results about the mapping class groups of 3-manifolds are of the following three kinds:

(1) comparison of $\mathscr{H}(M)$ with $\text{Out}(\pi_1(M))$ under the natural homomorphism that sends $\langle f \rangle$ to $f_\#$;

(2) statements that most mapping classes arise from a small number of geometrically-defined types of homeomorphisms;

(3) statements that various groups of mapping classes are finite, finitely presented, or finitely generated.

For example, when M is $\boldsymbol{P}^2$-irreducible and sufficiently large, and ∂M is incompressible, $\mathscr{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective except for the trivial exception of reflection in the fibers of I-bundles [W], [H7]. When the boundary is compressible, the kernel of $\mathscr{H}(M) \to \mathrm{Out}(\pi_1(M))$ is generated by twists about 2-discs for these manifolds, together with a reflection in the I-bundle case [L2], [M–M]. Johannson [J2] proved that when M is irreducible, orientable, sufficiently large, and has incompressible boundary, $J(M)$ has finite index in $\mathscr{H}(M)$, and $\mathscr{H}(M)$ is finitely generated. These results were extended to the boundary compressible case by McCullough and Miller [M–M]. My doctoral student P. Grasse has proved that these mapping class groups are finitely presented.

In the case of non-sufficiently-large aspherical 3-manifolds, all known examples have finitely presented groups of self homotopy equivalences. Consequently, if homotopic homeomorphisms are isotopic for these manifolds (this has recently been proved for most of the Seifert examples by P. Scott), then they will also have finitely presented mapping class groups.

For 3-manifolds with finite fundamental group, the mapping class group has been calculated in many cases [B], [B–R], [H–R], [R]. It seems likely that further progress can be made for the known 3-manifolds with finite fundamental group, and unlikely that there will be any surprising results. In particular, their mapping class groups will almost surely turn out to be finite.

We will now consider the reducible case. Let $M = M_1 \# M_2 \# \ldots \ldots \# M_n \# (\#_g S^1 \times S^2)$ be a compact orientable 3-manifold with irreducible oriented summands $M_1, M_2, \ldots, M_n$. We regard M as constructed in the following way. Take a 3-sphere and remove $n+2g$ open discs to obtain a punctured 3-cell W with boundary components $S_1, S_2, \ldots, S_n, S_{n+1,0}, S_{n+1,1}, S_{n+2,0}, \ldots, S_{n+g,1}$. In each M_i choose a 3-ball D_i and attach $M_i' = M_i\text{-int}(D_i)$ to S_i along ∂D_i for $1 \leqslant i \leqslant n$. For $n+1 \leqslant j \leqslant n+g$, let $S_j \times I$ be a copy of $S^2 \times I$, attached to W by identifying $S_j \times \{0\}$ with $S_{j,0}$ and $S_j \times \{1\}$ to $S_{j,1}$ to form an $S^1 \times S^2$ summand.

We will now describe four types of homeomorphisms of M. We remark that two orientation-preserving homeomorphisms of W are isotopic if and only if they induce the same permutation on the set boundary components of W.

1. *Homeomorphisms preserving summands.* These are the homemorphisms that restrict to the identity on W. They form a subgroup of $\mathrm{Homeo}(M)$ isomorphic to $\prod_{i=1}^{n} \mathrm{Homeo}(M_i \text{ rel } D_i) \times \prod_{j=1}^{g} \mathrm{Homeo}(S^2 \times I \text{ rel } S^2 \times \partial I)$. It is known that $\mathrm{Homeo}(S^2 \times I \text{ rel } S^2 \times \partial I)$ has two path components: that of the identity and that of a rotation about $S^2 \times \{\frac{1}{2}\}$.

2. *Interchanges of homeomorphic summands.* Suppose M_i and M_j are homeomorphic by an orientation-preserving homeomorphism. Then we can construct a homeomorphism of M fixing all other summands, leaving W

invariant, and interchanging M_i' and M_j'. Similarly we can interchange two $S^1 \times S^2$ summands, leaving W invariant.

3. *Spins of $S^1 \times S^2$ summands.* For each $n+1 \leqslant j \leqslant n+g$, we can construct a homeomorphism of M fixing all other summands, leaving W invariant, interchanging $S_{j,0}$ and $S_{j,1}$, and restricting to an orientation-preserving homeomorphism that interchanges the boundary components of $S_j \times I$.

4. *Slide homeomorphisms.* For $i \leqslant n$, let $\hat{M}$ be obtained from M by replacing M_i' with a 3-cell E. Let α be an arc in $\hat{M}$ meeting E only in its endpoints. Choose an isotopy J_t of $\hat{M}$, with $J_0 = 1_{\hat{M}}$ and $J_1|_E = 1_E$, so that J_t moves E around α. By a *slide homeomorphism* that *slides M_i around α* we mean a homeomorphism h defined by $h|_{M-M_i'} = J_1|_{\hat{M}-E}$ and $h|_{M_i'} = 1_{M_i'}$. A change in the choice of J_t changes h by isotopy and possibly by a rotation about S_i. Thus a choice of α might determine two isotopy classes of slide homeomorphism. This ambiguity will cause no difficulties. It is interesting to note that if T is the frontier of a regular neighborhood of $M_i' \cup \alpha$ in M, then T is a compressible torus and h is isotopic to a certain Dehn twist about T. Consequently, slide homeomorphisms lie in the Johannson subgroup of $\mathscr{H}(M)$.

By a similar construction, we can slide either end of $S_j \times I$ around an arc in $M - S_j \times (0,1)$, obtaining an element in the Johannson subgroup.

It can be shown that if α_1 and α_2 are two arcs meeting E only in their endpoints, and α is an arc representing the product of α_1 and α_2 in $\pi_1((M - M_i')^-, S_i)$, then a slide of M_i around α is isotopic to a composite of slides around α_1 and α_2. Similarly for sliding ends of $S_j \times I$'s. It follows that the subgroup of $\mathscr{H}(M)$ generated by slide homeomorphisms is finitely generated.

Remark. It is possible to treat spins of $S^1 \times S^2$ summands as homeomorphisms preserving summands. We prefer, however, that our homeomorphisms correspond to the generators of $\mathrm{Aut}(\pi_1(M, x_0))$ given by [F–R].

THEOREM. *Let M be a compact connected orientable 3-manifold. Then any orientation-preserving homeomorphism of M is isotopic to a composite of the four types of homeomorphisms described above.*

This implies:

COROLLARY. *Suppose each irreducible summand of M has finitely generated mapping class group. Then the mapping class group of M is finitely generated.*

The theorem appears without proof in the research announcement [C–R] (see the next section for discussion of [C–R]) and it appeared earlier in the thesis of César de Sá [C2], who gave an argument based on "partial slides." The proof we give here avoids the partial slide concept; it is based on an argument due to M. Scharlemann which appears in Appendix A of [B1].

Proof of the theorem. Let f be an orientation-preserving homeomorphism of M. We will apply isotopies, slide homeomorphisms, interchanges, and spins to change f to a homeomorphism preserving summands. The intermediate composites will again be called f in order to avoid an excess of notation.

Let $\Sigma = \bigcup_{i=1}^{n} S_i \cup \left(\bigcup_{j=n+1}^{n+g} (S_{j,0} \cup S_{j,1}) \right)$. We may assume $f(\Sigma)$ is transverse to Σ. We will modify f to reduce the number of components of $f(\Sigma) \cap \Sigma$. Let C be a circle of intersection that is innermost on $f(\Sigma)$, so that C bounds a disc $E_1 \subseteq f(\Sigma)$ with $\mathrm{int}(E_1)$ disjoint from Σ. If $E_1 \subseteq M_i'$ or $E_1 \subseteq S_j \times I$ then there is an isotopy pulling E_1 into W, eliminating C and possibly other circles of intersection as well. Suppose now that $\partial E_1 \subseteq S_i$ for some $1 \leqslant i \leqslant n$, and $E_1 \subseteq W$. If the other disc that ∂E_1 bounds in $f(\Sigma)$ has interior disjoint from Σ, then it must lie in M_i' and we can eliminate C by the preceding step. So assume this is not the case. Then we can choose an arc α_0 in $f(\Sigma) \cap M_i'$ with one endpoint in C and the other endpoint in S_i-C. Let E_2 be the disc in S_i which is the closure of the component of S_i-C that does not contain the other endpoint of α_0. The 2-sphere $E_1 \cup E_2$ bounds a punctured 3-cell $W_1 \subseteq W$. Suppose M_k' is attached to W_1 $(k \neq i)$. There is an arc α with endpoints in S_k that travels in W_1 to S_i, then through $M_i' - f(\Sigma)$ emerging in $W - W_1$ (i.e., follows along near α_0), and then through W to S_k. Such an arc α is shown in Fig. 1. Slide M_k around α (i.e., compose f with the slide

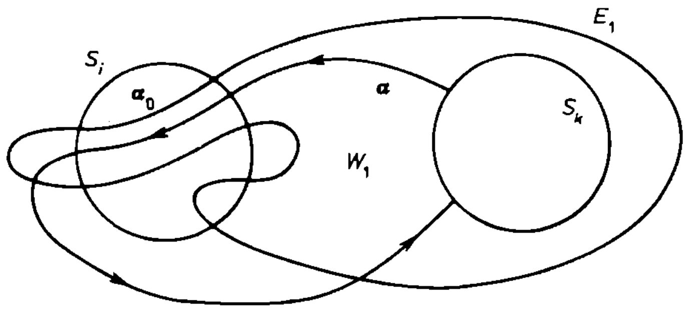

Fig. 1

homeomorphism that slides M_k around α). Repeating for each M_k attached to W_1, and each end of an $S_j \times I$ attached to W_1, we arrive at a situation where $E_1 \cup E_2$ bounds a 3-ball in W so that C can be eliminated by isotopy. Finally, suppose C lies on $S_{j,0}$ (the case of $S_{j,1}$ being similar). Try to find an arc α_0 in $f(\Sigma) \cap (S_j \times I)$ with one end in C and the other end in $S_{j,1}$. If this is not possible, then C can be eliminated by isotopy, so assume α_0 is found. This time, choose E_2 so that $S_{j,1}$ is not a boundary component of W_1. Then, proceed as before, sliding summands and ends of $S_k \times I$'s over $S_j \times I$ until C can be eliminated by isotopy.

Repeating this process as far as possible, we reach the situation $f(\Sigma) \cap \Sigma = \emptyset$.

Since no component of $f(\Sigma)$ can bound a 3-ball in M, it is not hard to show that $f(W)$ is isotopic to W, so we may assume $f(W) = W$. Applying interchanges of homeomorphic summands and spins of $S^1 \times S^2$ summands, we may assume f permutes the boundary components of W trivially. Since f is orientation-preserving, $f|_W$ is isotopic to 1_W, hence f is isotopic to a homeomorphism preserving summands. This completes the proof of the theorem. ∎

The previous proof yields another result:

THEOREM. *Suppose $J(M_i)$ has finite index in $\mathscr{H}(M_i)$ for each irreducible summand M_i of M. Then $J(M)$ has finite index in $\mathscr{H}(M)$.*

Proof. First, we note that $\mathscr{H}(M_i \text{ rel } D_i)$ and the orientation-preserving subgroup of $\mathscr{H}(M_i)$ differ only by homeomorphisms sliding D_i around loops in M_i, which are Dehn twists about tori, hence the hypothesis implies $J(M_i \text{ rel } D_i)$ has finite index in $\mathscr{H}(M_i \text{ rel } D_i)$.

There is a finite collection of products of interchanges of homeomorphic summands and spins of $S^1 \times S^2$ summands that suffices to carry out the argument of the previous theorem. Thus that argument shows that for any orientation-preserving homeomorphism f, there is a product k_1 of slide homeomorphisms, and one of these finitely many products k_2, so that $k_2 k_1 f$ is isotopic to a homeomorphism preserving summands. So we can write $\langle k_2 k_1 f\rangle = \langle j\rangle\langle k_3\rangle$ where $\langle j\rangle \in \text{image}\left(\prod_{i=1}^{n} J(M_i \text{ rel } D_i) \to \mathscr{H}(M)\right)$ and $\langle k_3\rangle$ is one of finitely many coset representatives of image $\left(\prod_{i=1}^{n} J(M_i \text{ rel } D_i) \to \mathscr{H}(M)\right)$ in

$$\text{image}\left(\prod_{i=1}^{n} \mathscr{H}(M_i \text{ rel } D_i) \times \prod_{j=1}^{g} \mathscr{H}(S^2 \times I \text{ rel } S^2 \times \partial I) \to \mathscr{H}(M)\right).$$

Therefore $\langle f\rangle = \langle k_1^{-1} k_2^{-1} j k_2\rangle\langle k_2^{-1} k_3\rangle$. Now $\langle k_1^{-1}\rangle\langle k_2^{-1} j k_2\rangle \in J(M)$ and there are only finitely many products of the form $k_2^{-1} k_3$. The result follows. ∎

A proof that the mapping class group of M is finitely presented if the same is true for its irreducible summands should be within range of current techniques. In fact, it is not unreasonable to conjecture that the mapping class group of any compact 3-manifold is finitely presented, and that the Johannson subgroup always has finite index.

4. The homotopy type of Homeo(M)

In Section 3 we considered the group of path components of the PL homeomorphism group Homeo(M). Since Homeo(M) is a topological group, its path components are homeomorphic, so to understand the homotopy type of these components we need only consider the connected component of the identity map 1_M. In the closed P^2-irreducible sufficiently large case, we have a very strong result due to Laudenbach and Hatcher:

THEOREM ([L1], [H4]). *For $q \geqslant 1$, $\pi_q(\text{Homeo}(M)) \to \pi_q(\text{Equiv}(M))$ is an isomorphism.*

Thus the Waldhausen–Heil isomorphism at the π_0 level extends to all higher homotopy groups. The group $\pi_q(\text{Equiv}(M))$ is easy to compute since M is aspherical; for $q = 1$, $\pi_1(\text{Equiv}(M)) \cong \text{center}(\pi_1(M))$ while for $q \geqslant 2$, $\pi_q(\text{Equiv}(M)) = 0$. For non-sufficiently-large aspherical 3-manifolds, we would hope for the same result.

When M is reducible, even with nice summands, the homotopy type of Homeo(M) is much more complicated. We have:

THEOREM ([M1]). *Let M be a connected sum of at least three closed aspherical 3-manifolds. Then $\pi_1(\text{Homeo}(M))$ is not finitely generated.*

In the proof of this theorem, a single geometric construction leads to many elements of $\pi_1(\text{Homeo}(M))$ (i.e., isotopies from 1_M to 1_M). The proof that they are not in any finitely generated subgroup is by very explicit obstruction-theoretic calculations that do not give direct insight into the structure of Homeo(M).

For the reducible case, the theorem of Laudenbach and Hatcher fails drastically. In fact, we have:

THEOREM ([J1]). *Let M_1 and M_2 be closed orientable irreducible sufficiently large 3-manifolds. Then $\pi_1(\text{Homeo}(M_1 \# M_2)) \to \pi_1(\text{Equiv}(M_1 \# M_2))$ is not an isomorphism.*

In [J1], Jahren proves much more, but this particular statement is immediate from later work: by [H5], $\pi_1(\text{Homeo}(M_1 \# M_2)) = \{1\}$ while from [M1], $\pi_1(\text{Equiv}(M_1 \# M_2))$ is not finitely generated.

Although these results might seem rather discouraging, an innovative new idea for studying Homeo(M) in the reducible case is leading to a greatly increased understanding of its homotopy type. In [C–R], César de Sá and Rourke gave a description of Homeo(M) making use of a "configuration space" C whose loop space is a direct factor of Homeo(M) up to weak homotopy. They were later unable to give complete proofs for the announced results, but Hendriks and Laudenbach [H–L2] overcame formidable technical obstacles and made the basic ideas of [C–R] go through in the orientable case. In order to give the result, a bit of notation will be needed.

We will work with the group of diffeomorphisms; analogous results hold for PL homeomorphisms. Suppose M is a connected compact orientable 3-manifold with a 2-sphere boundary component S_0. Then we may construct M as a connected sum of a 3-ball P_0, having boundary S_0, with n irreducible 3-manifolds $P_1, P_2, \ldots, P_n$ ($P_i \neq S^3$) and g copies of $S^1 \times S^2$. Let D_i be the connected sum 3-ball in P_i. There is a compact codimension-zero submanifold B of M, diffeomorphic to the complement of $n+2g$ disjointly imbedded 3-balls in P_0 so that

(a) $S_0 = \partial B \cap \partial M$;

(b) $M-B$ is the disjoint union of $P_i - D_i$, $1 \leqslant i \leqslant n$, and g copies of $(0,1) \times S^2$.

In [H–L2] is constructed a configuration space C_1 so that there are three H-space homomorphisms:

α: $(F_g)^n \to \mathrm{Diff}(M \text{ rel } \partial M)$,

β: $\Omega C_1 \to \mathrm{Diff}(M \text{ rel } \partial M)$,

γ: $\prod_{i=1}^{n} \mathrm{Diff}(P_i \text{ rel } \partial P_i \cup D_i) \times \prod_{j=1}^{g} \Omega O(3) \subseteq \mathrm{Diff}(M \text{ rel } \partial M)$.

Here, F_g the free group on g generators, corresponding to the subgroup of $\pi_1(M)$ coming from the $S^1 \times S^2$ summands. For $x = (x_1, x_2, \ldots, x_n) \in (F_g)^n$, $\alpha(x)$ corresponds to the composition of slide homeomorphisms of the summands P_i around loops representing x_i. The $\Omega O(3)$ factors correspond to $\mathrm{Diff}([0,1] \times S^2 \text{ rel } \{0,1\} \times S^2)$ by the Smale Conjecture [H6]. The main result of [H–L2] is:

THEOREM. *The map*

$$h\colon (F_g)^n \times \Omega C_1 \times \prod_{i=1}^{n} \mathrm{Diff}(P_i \text{ rel } \partial P_i \cup D_i) \times \prod_{j=1}^{g} \Omega O(3) \to \mathrm{Diff}(M \text{ rel } \partial M)$$

defined by $h(x, y, z) = \alpha(x)\beta(y)\gamma(z)$ *is a homotopy equivalence.*

The map h is not an H-space homomorphism.

Although the construction of the configuration space C_1 is quite difficult, C_1 can be identified with a much more concrete object. Let $\mathrm{Imb}(B, M \text{ rel } S_0)$ be the space of smooth imbeddings of B into M that restrict to the inclusion on S_0, and let $\mathrm{Imb}_0(B, M \text{ rel } S_0)$ be the subspace consisting of those imbeddings that extend to diffeomorphisms of M. Then we have the following results from [H–M], which actually follow rather easily from [H–L2]:

THEOREM. *The restriction map* p: $\mathrm{Diff}(M \text{ rel } \partial M) \to \mathrm{Imb}_0(B, M \text{ rel } S_0)$, *which is a principal fibre bundle with fibre*

$$\prod_{i=1}^{n} \mathrm{Diff}(P_i \text{ rel } \partial P_i \cup D_i) \times \prod_{j=1}^{g} \mathrm{Diff}([0,1] \times S^2 \text{ rel } \{0,1\} \times S^2),$$

is a product fibration.

THEOREM. *The composition* $p\circ(\alpha\beta)$: $(F_g)^n \times \Omega C_1 \to \mathrm{Diff}(M \text{ rel } \partial M) \to \mathrm{Imb}_0(B, M \text{ rel } S_0)$ *is a homotopy equivalence.*

As (nonimmediate) applications of the first of these two theorems, we have:

THEOREM. *Let* $\mathrm{Diff}(M, B \text{ rel } \partial M)$ *be the group of diffeomorphisms of* M, *fixed on* ∂M, *that take* B *to* B. *Suppose none of the irreducible summands* P_i *has universal cover homotopy equivalent to* S^3. *Then the inclusion map* $\mathrm{Diff}(M, B \text{ rel } \partial M) \to \mathrm{Diff}(M \text{ rel } \partial M)$ *induces injective homomorphisms on homotopy groups.*

THEOREM. *Let* Q *be the* 1*-point union of* $n+2g-1$ 2*-spheres. Suppose none of the irreducible summands* P_1 *has universal cover homotopy equivalent to* S^3. *Then for all* $q \geqslant 1$, *there is an injective homomorphism*

$$\pi_q(\Omega Q) \to \pi_q(\mathrm{Diff}(M \text{ rel } \partial M)).$$

This shows that $\pi_*(\mathrm{Diff}(M \text{ rel } \partial M))$ contains torsion of all orders, as long as $n+2g-1 \geqslant 1$. I hope that further investigation of $\mathrm{Imb}_0(B, M \text{ rel } S_0)$ will yield a much better understanding of $\mathrm{Diff}(M \text{ rel } \partial M)$.

5. Other topics

Here we briefly mention some other results. One natural question which has an interesting answer for reducible 3-manifolds is the following. Suppose $f\colon (M, \partial M) \to (N, \partial N)$ is a proper map inducing an isomorphism on fundamental groups. Is f a proper homotopy equivalence? It turns out that the presence of a "good" summand forces an affirmative answer:

THEOREM ([S1]). *Let* $f\colon M \to N$ *be a map between closed orientable* 3*-manifolds which induces an isomorphism on fundamental groups. If one of the free factors of* $\pi_1(M)$ *is infinite but not cyclic, then* f *is a homotopy equivalence.*

In the bounded case, we have:

THEOREM ([K–M]). *Let* $f\colon (M, \partial M) \to (N, \partial N)$ *be a proper map between compact* 3*-manifolds which induces an isomorphism on fundamental groups. Suppose* M *has no more* 2*-sphere boundary components than* N *has. If* M *has an irreducible aspherical summand which is not an* I*-bundle, or* N *has an irreducible aspherical summand which is not a product-with-handles, then* f *is a proper homotopy equivalence.*

A closely related problem is the determination of which isomorphisms on fundamental groups can be induced by homotopy equivalences. This has been studied by Swarup. He associates to a closed oriented n-manifold with basepoint an invariant $\tau(M, m) \in H_n(\pi_1(M, m); \mathbf{Z})$ and proves the following results:

THEOREM ([S1]). *Let* (M_1, m_1) *and* (M_2, m_2) *be two closed oriented* 3-

manifolds and let $\theta\colon \pi_1(M_1, m_1) \to \pi_1(M_2, m_2)$ *be an isomorphism. Then there is a continuous map* $f\colon(M_1, m_1) \to (M_2, m_2)$ *with* $f_\# = \theta$ *if and only if* $\theta_*(\tau(M_1, m_1)) = d\tau(M_2, m_2)$ *for some integer d. Moreover, when this condition is satisfied, we can realize an f of degree d.*

THEOREM ([S1]). *With hypotheses as in the previous theorem, there is an oriented homotopy equivalence* $f\colon (M_1, m_1) \to (M_2, m_2)$ *with* $f_\# = \theta$ *if and only if* $\theta_*(\tau(M_1, m_1)) = \tau(M_2, m_2)$.

These two theorems are generalized to not necessarily orientable Poincaré pairs in [H12].

Group actions on reducible 3-manifolds are quite well-understood due to the development of minimal surface techniques by Meeks and Yau. We refer the reader to [P] for the most general result to date.

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