

POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES
MATHEMATICAE
(ROZPRAWY MATEMATYCZNE)

KOMITET REDAKCYJNY

KAROL BORSUK redaktor

ANDRZEJ BIAŁYNICKI-BIRULA, BOGDAN BOJARSKI,
ZBIGNIEW CIESIELSKI, JERZY ŁOŚ, ZBIGNIEW SEMADENI,

WANDA SZMIELEW

CXLVIII

F. D. TALL

**Set-theoretic consistency results and topological theorems
concerning the normal Moore space conjecture
and related problems**

WARSZAWA 1977

PAŃSTWOWE WYDAWNICTWO NAUKOWE

ś. 7133



PRINTED IN POLAND

Copyright ©
by PWN — Polish Scientific Publishers,
Warszawa 1977

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

CONTENTS

Chapter 0

0. Introduction	5
1. Outline of thesis	7
2. Notation	8

Chapter I

0. Definitions	9
1. Relations among the covering axioms	10
2. Covering axioms and collectionwise normality	13
3. Covering axioms and countability conditions	18

Chapter II

0. Preliminaries	21
1. The consistency proof	24
2. Topological consequences	31
3. Bing's example	37
4. To be continued	39

Chapter III

0. Prehistory	42
1. History	43
2. Everything is equivalent	47

Bibliography	51
------------------------	----

CHAPTER 0*

0. Introduction. Back in the Cro-Magnon days of general topology — the 1920's and 30's — R. L. Moore introduced the notion of a development of a topological space. (We shall always refer to concepts by the currently accepted names rather than the original ones in case of conflict.)

0.1. DEFINITION. A *development* of a space X is a sequence $\{G_n\}_{n<\omega}$ of open covers of X , such that for each $p \in X$, and each open set U containing p , there is an n , such that every element of G_n which contains p is contained in U . A space is *developable* if there is a development of it.

Developability imposes a uniformity on a space that, although weaker than metrizable, still gives many of the theorems that hold in metric spaces, for example the equivalence of compactness and countable compactness.

0.2. DEFINITION. In honor of Moore, regular Hausdorff developable spaces are known as *Moore spaces*.

Moore [36] is the sourcebook on Moore spaces. It is readily verified that every metric space is a Moore space. Once it became known that not every Moore space was metric — we shall see a simple example of this in Chapter III — the natural question was, what property was equivalent to metrizable for Moore spaces. Jones [26] proposed normality, and proved the rather unlikely

0.3. THEOREM. $2^{\aleph_0} < 2^{\aleph_1}$ implies every separable normal Moore space is metrizable.

The question of whether every normal Moore space is metrizable occupied a number of topologists at various times in the next thirty years. See for example the papers of Traylor listed in the Bibliography, and the references contained therein. To those who worked on it, it naturally became “one of the outstanding questions of general topology”. Having been involved with it for two years, the writer reduces cognitive dissonance by agreeing. The only substantial results of these thirty years are contained in the two papers of Bing. In the first, he proves

* The author's doctoral dissertation — of which this is an updated version — was supported by NSF grants GP-5913 and GP-8501. The preparation of this paper was assisted by Grant A-7354 of the National Research Council of Canada.

0.4. THEOREM. *If there exists an uncountable set X of real numbers such that every subset of X is F_σ relative to X , then there is a separable, normal, non-metrizable Moore space.*

(Heath [22] proved the converse. See Chapter III.) He also introduces a strengthening of normality and shows it to be a necessary and sufficient condition for the metrizability of a Moore space.

0.5. DEFINITION. Let \mathcal{Y} be a collection of disjoint sets in a topological space X . \mathcal{Y} is said to be a *discrete* collection if the elements of \mathcal{Y} have mutually disjoint closures, and \mathcal{Y} is *closure-preserving*, i.e. if $\mathcal{Z} \subset \mathcal{Y}$, then $\overline{\bigcup \{Y : Y \in \mathcal{Z}\}} = \bigcup \{\overline{Y} : Y \in \mathcal{Z}\}$. X is said to be *collectionwise normal with respect to the discrete collection \mathcal{Y}* if there exist simultaneously disjoint open sets U_Y , $Y \in \mathcal{Y}$, $Y \subset U_Y$. X is *collectionwise normal* if it is collectionwise normal with respect to every discrete collection.

0.6. THEOREM. *A space is metrizable if and only if it is a collectionwise normal Moore space.*

Clearly, collectionwise normality implies normality. In this same paper, Bing provides an example of a normal, noncollectionwise normal space. We shall discuss his example in a later chapter.

In his second paper, Bing exhibits a set-theoretic translation of a restricted version of the normal Moore space problem. The set-theoretic consistency results of Chapter II, or rather the simplest of them, were originally obtained with the use of the translation. It has since become evident that stronger results may be obtained by working directly with the topology. However, the translation is still of interest, and it is discussed and generalized in Tall [46].

Through the investigations of the author, it has become clear that the normal Moore space problem has very little to do with Moore spaces. The real question is, when is a normal space collectionwise normal. Developability qua developability has only a peripheral bearing on this consequence. Moore spaces will therefore be discussed only occasionally in what follows.

In 1967, while trying to prove the consistency of the existence of a separable, normal, non-metrizable Moore space, the author mentioned Bing's 0.4 to Jack Silver. In the summer of that year, Silver noticed that in a model of set theory invented by Solovay, there existed such a set of real numbers. Thus

0.7. THEOREM. *If set theory (any "reasonable" system; for the purposes of this thesis, set theory will be Zermelo–Fraenkel, including the Axiom of Choice) is consistent, it remains consistent if the existence of a separable, normal, non-metrizable Moore space is assumed.*

Note that by Jones' 0.3, in conjunction with the fundamental work of Gödel,

0.8. THEOREM. *If set theory is consistent, it remains consistent if it is assumed that every separable, normal Moore space is metrizable.*

Thus the normal Moore space question is no ordinary topological problem. In particular, the axioms of set theory do not decide the question of whether every separable normal Moore space is metrizable. The separable case will be discussed in detail in Chapter III.

Whether it is consistent to assume that every normal Moore space is metrizable, is still an open question.

Bing's translation had provided a set-theoretic equivalent of the statement "Every normal Moore space is collectionwise normal with respect to any discrete collection of \aleph_1 points." In late 1967, the author conjectured that this statement held in the model obtained by adjoining \aleph_2 Cohen subsets of ω_1 to a model of set theory and the continuum hypothesis, but was unable to push through one crucial step in the proof. In the spring of 1968, Silver had also concluded that this was the appropriate model, and found a clever solution to the difficulty. Unfortunately, there was a gap in the proof, apparently unnoticed by Silver. Following a suggestion by David Booth, the author was able to fill the hole, so indeed

0.9. THEOREM. *If set theory is consistent, it remains consistent if it is assumed that every normal Moore space is collectionwise normal with respect to any discrete collection of \aleph_1 points.*

The author has since managed to greatly generalize this result, as will be seen in Chapter II.

The preceding pages constitute a historical introduction to the normal Moore space problem. In the next section, the structure of the thesis will be outlined.

1. Outline of thesis. Chapter I is a purely topological treatment of covering axioms, that is, conditions on a space of the form "Every open cover has a refinement which is...". The collectionwise normality of a space is closely related to the covering axioms the space satisfies, so Chapter I provides a topological background for the rest of the thesis. Given the inability of the author to walk the straight and narrow path, the chapter branches off in a number of natural directions. If the reader is a set-theorist, or is primarily interested in the consistency results of Chapters II and III, it is suggested that he skim Chapter I. For the topologist, Chapter I provides a comprehensive survey of the known relations among a large number of covering axioms, and their relations with collectionwise normality. It contains the first study of paralindelof spaces, a natural generalization of the class of paracompact spaces, that surprisingly has important applications in Chapter II. The final major topic discussed is the interplay between covering axioms and such count-

ability conditions as \aleph_1 -compactness, local separability, etc. The author originally planned to include a section of examples showing the limits of the theorems, but printing deadlines did not permit this.

In Chapter II, the necessary set-theoretic and topological machinery is developed, and then 0.9 and its generalizations are proved. For the benefit of the reader who is not conversant with both disciplines, an effort is made to keep the topology and set theory in separate sections. The chapter concludes with an indication of the author's current efforts toward improving the consistency results actually proved in the thesis. This is the only chapter in which a substantial amount of prior knowledge is assumed. In the set-theoretic sections, the reader is expected to be familiar with Cohen-style independence proofs. The topologist or uninitiated set-theorist might wish to consult the books of Cohen or Jech, or the article by Shoenfield.

Chapter III is devoted to the separable case of the normal Moore space problem. 0.7 and 0.8 are proved, modulo results of Solovay and Gödel respectively. A rather surprising range of set-theoretic and topological questions are shown to be equivalent to the existence of a separable, normal, non-metrizable Moore space.

The content of Chapters IV, V, VI of this thesis appears in expanded form in Tall [46], [47] and Charlesworth, Hodel, Tall [8], respectively.

2. Notation. $k.m.n$ will denote the n th statement worth numbering — be it theorem, definition, lemma, etc., — of the m th section of the k th chapter. When a cross-reference within a chapter is made, the initial digit and period will be omitted.

Items in the Bibliography are referred to by the author's last name, plus additional identification if necessary.

$\langle X, \mathcal{T} \rangle$ is the archetypal topological space, X for short, unless \mathcal{T} needs to be emphasized. Points are denoted by lower case Latin letters, spaces and sets by capital Latin letters, collections of sets by script Latin letters. i, j, k, m, n are used for natural numbers. Greek lower case letters from the beginning of the alphabet are generally used for ordinals, from the middle, cardinals. Models are denoted by German capitals. The first letter of the Hebrew alphabet is assigned its customary task.

\mathcal{Y}^* is often used instead of $\bigcup \{Y : Y \in \mathcal{Y}\}$. Additional special notation is introduced as needed.

CHAPTER I

0. Definitions

0.1. DEFINITION. A collection \mathcal{A} of sets in a topological space is *point-finite* (*point-countable*) if each point of the space is in at most finitely (countably) many members of \mathcal{A} . \mathcal{A} is *locally finite* (*locally countable*) if each point has an open neighborhood which intersects at most finitely (countably) many members of \mathcal{A} . \mathcal{A} is *star-finite* (*star-countable*) if each member of \mathcal{A} intersects at most finitely (countably) many members of \mathcal{A} . \mathcal{A} is *disjoint* if each member of \mathcal{A} intersects no other member of \mathcal{A} . Recall *discrete* was defined in 0.0.5.

0.2. DEFINITION. \mathcal{A} is σ -*point-finite* (σ -*locally countable*, etc.) if it is the union of countably many point-finite (locally countable, etc.) collections.

Covers and refinements are assumed to be open, unless stated otherwise. A refinement of a cover is assumed to be a cover. The following definitions, although not all in general usage, have the virtue of being systematic. Conflict with terminology in the literature is minimal.

0.3. DEFINITION. A space is *metacompact* (*paracompact*) (*hypocompact*) if every cover has a point-finite (locally finite) (star-finite) refinement. A space is *metalindelof* (*paralindelof*) (*hypolindelof*) if every cover has a point-countable (locally countable) (star-countable) refinement.

0.4. DEFINITION. A space is σ -*metacompact* if every cover has a σ -point-finite refinement. Similarly for the other concepts in the previous group.

0.5. DEFINITION. A space is *screenable* if every cover has a σ -disjoint refinement. A space is *strongly screenable* if every cover has a σ -discrete refinement.

0.6. DEFINITION. A space is F_σ -*screenable* if every cover has a σ -discrete *closed* refinement.

0.7. DEFINITION. A space is *countably metacompact* if every countable cover has a point-finite refinement. Similarly for the concepts in the previous groups.

There are other covering axioms, but our attention will mainly be focussed on those defined above.

1. Relations among the covering axioms

1.1. Remark. Note that hypo- is stronger than para- is stronger than meta-. But also note that σ -hypo does not necessarily imply σ -para.

Aside from the implications immediate from the definitions, the following theorems may be gleaned from the literature. For simplicity, assume all spaces are Hausdorff in this section.

1.2. THEOREM (Dieudonné [10], Bing [5]). *Every paracompact space is regular, normal, and collectionwise normal.*

1.3. THEOREM (Michael [32]). *For regular spaces, paracompactness is equivalent to strong screenability, indeed to σ -paracompactness.*

1.4. THEOREM (proved as part of the next theorem by Smirnov). *Every hypolindelof space is strongly screenable.*

1.5. THEOREM (Smirnov [43]. For a proof in English, see Engelking [15]). *Every hypolindelof regular space is hypocompact, hence paracompact.*

1.6. THEOREM (Nagami [38]). *Paracompactness is equivalent to screenability plus normality plus countable metacompactness.*

1.7. THEOREM (Michael [33], Nagami [38]). *For regular spaces, paracompactness is equivalent to metacompactness plus collectionwise normality.*

1.8. THEOREM (follows immediately from theorem in McAuley [30]). *Paracompactness is equivalent to F_σ -screenability plus collectionwise normality.*

1.9. THEOREM (Heath [22]). *Screenability is equivalent to σ -hypolindelofness.*

1.10. THEOREM (Heath [22]). *If every closed set in a screenable space is a G_δ , then the space is metacompact.*

1.11. THEOREM (Morita [37]). *For normal spaces, countable metacompactness, countable paracompactness, and countable hypocompactness are all equivalent.*

1.12. THEOREM (Mansfield [29]). *For normal spaces, the following are equivalent: countable paracompactness, countable F_σ -screenability.*

1.13. THEOREM. *Paracompactness is equivalent to screenability plus normality plus F_σ -screenability.*

Proof. Only the backward implication requires proof. By 1.12, the space is countably metacompact. By 1.6, the space is then paracompact.

1.14. DEFINITION. Let \mathcal{A} be a collection of sets. Then

$$\mathcal{A}^* = \bigcup \{A : A \in \mathcal{A}\}.$$

Consider the following situation. $\mathcal{H} = \bigcup_{i < \omega} \mathcal{H}_i$ is a not necessarily open cover of a space X . Then $\mathcal{H}^\# = \{\mathcal{H}_i^*\}_{i < \omega}$ is also a cover. Let \mathcal{W} be a not necessarily open refinement of $\mathcal{H}^\#$. Let $f: \mathcal{W} \rightarrow \omega$ be defined by

$$f(W) = \text{least } i \text{ such that } W \subset \mathcal{H}_i^*.$$

Let $\mathcal{K}_i = \{W \in \mathcal{W} : f(W) = i\}$. Then $i \neq j$ implies $\mathcal{K}_i \cap \mathcal{K}_j = \emptyset$. Let $\mathcal{K} = \bigcup_{i < \omega} \mathcal{K}_i$. Then $\mathcal{K} = \mathcal{W}$, so \mathcal{K} is a cover, and $\mathcal{K}^\# = \{\mathcal{K}_i^*\}_{i < \omega}$ is a cover. Moreover, $\mathcal{K}_i^* \subset \mathcal{H}_i^*$.

1.15. LEMMA. *If \mathcal{W} is point-finite (locally finite), then so is $\mathcal{K}^\#$.*

Proof. If p is in \mathcal{K}_i^* , then there is $W \in \mathcal{K}_i$, p in W . The \mathcal{K}_i are disjoint, so if p is in infinitely many \mathcal{K}_i^* 's, then p is in infinitely many W 's. Similarly, if every neighborhood of p intersects infinitely many \mathcal{K}_i^* 's, then every neighborhood of p intersects infinitely many W 's.

Note that this argument does not work for star-finite.

Let $\mathcal{U}_i = \{H \cap \mathcal{K}_i^* : H \in \mathcal{H}_i\}$. Let $\mathcal{U} = \bigcup_{i < \omega} \mathcal{U}_i$. Then \mathcal{U} is a cover, for if p is in X , then p is in some \mathcal{K}_i^* . $\mathcal{K}_i^* \subset \mathcal{H}_i^*$, so there is an $H \in \mathcal{H}_i$ which contains p . Therefore p is in \mathcal{U}_i^* . Clearly, \mathcal{U} refines \mathcal{H} .

1.16. LEMMA.

- (a) *Suppose $\mathcal{K}^\#$ and each \mathcal{H}_i are point-finite. Then \mathcal{U} is point-finite.*
- (b) *Suppose $\mathcal{K}^\#$ and each \mathcal{H}_i are locally finite. Then \mathcal{U} is locally finite.*
- (c) *Suppose each \mathcal{H}_i is point-countable. Then \mathcal{U} is point-countable.*
- (d) *Suppose each \mathcal{H}_i is locally countable, and $\mathcal{K}^\#$ is locally finite. Then \mathcal{U} is locally countable.*

Proof. (a) p is in only finitely many \mathcal{K}_i^* 's, and for each i , p is in only finitely many H in \mathcal{H}_i . Therefore p is in only finitely many $H \cap \mathcal{K}_i^*$, $H \in \mathcal{H}_i$. Thus \mathcal{U} is point-finite.

(b) $\mathcal{K}^\#$ is locally finite, so for each p there is a neighborhood $N(p)$ which intersects only say $\mathcal{K}_{n_1}^*, \dots, \mathcal{K}_{n_j}^*$. Each \mathcal{H}_i is locally finite, so there exist neighborhoods $N_{n_i(p)}$ about p such that each $N_{n_i(p)}$ intersects only finitely many members of \mathcal{H}_{n_i} . It follows that $N(p) \cap N_{n_1(p)} \cap \dots \cap N_{n_j(p)}$ intersects only finitely many members of \mathcal{U} .

(c) The method of (a) suffices, since $\mathcal{K}^\#$ is countable and so is trivially point-countable.

(d) Same proof as (b). Unfortunately, the intersection of countably many open sets is not necessarily open, so the second hypothesis is necessary.

Having built up machinery, we can now prove some theorems in rapid order. Note that for these, no separation axioms need be assumed.

1.17. THEOREM. *Paracompactness is equivalent to countable paracompactness plus σ -paracompactness.* (Essentially the same theorem appears in Aull [4].)

1.18. THEOREM. *Metacompactness is equivalent to countable metacompactness plus σ -metacompactness.*

1.19. COROLLARY. *Every screenable countably metacompact space is metacompact.*

1.20. THEOREM. *σ -metalindelofness is equivalent to metalindelofness.*

1.21. THEOREM. *Every σ -paralindelof countably paracompact space is paralindelof.*

A final triviality:

1.22. THEOREM. *Every σ -metacompact space is metalindelof.*

Proofs. The proofs should now be evident. For example, to prove 1.17, given an arbitrary open cover, take a σ -locally finite open refinement \mathcal{H} . Then take a locally finite open refinement of $\mathcal{H}^\#$. Chase through the two lemmas to get \mathcal{U} , a locally finite open refinement of the original cover. The other proofs are left to the reader.

We shall extract one more theorem from the two lemmas. First, we state

1.23. THEOREM (Michael [32]). *For regular spaces, paracompactness is equivalent to the property that every open cover has a locally finite, not necessarily open, refinement.*

Now to prove

1.24. THEOREM. *Suppose in a regular, F_σ -screenable space every countable closed cover has a not necessarily open, locally finite refinement. Then the space is paracompact.*

Proof. Let \mathcal{H} be a σ -discrete closed refinement of an arbitrary open cover. Observe that $\mathcal{H}^\#$ is a countable closed cover, proceed as usual, and then use 1.23.

Recall from the introductory chapter

1.25. DEFINITION. A collection \mathcal{A} of sets in a topological space is *closure-preserving* if for each $\mathcal{B} \subset \mathcal{A}$,

$$\overline{\bigcup \{A : A \in \mathcal{B}\}} = \bigcup \{\bar{A} : A \in \mathcal{B}\}.$$

1.26. THEOREM (Michael [34]). *For regular spaces, paracompactness is equivalent to the property that every open cover has a closure-preserving, not necessarily open, refinement.*

Looking at the proof of 1.16 (d) and 1.21, it is easy to see that in a space in which the intersection of countably many open sets is always open, σ -paralindelofness is equivalent to paralindelofness. If the space is also regular, we have the pleasant

1.27. THEOREM. *For regular spaces in which the intersection of countably many open sets is always open, paracompactness is equivalent to σ -paralindelofness.*

Proof. By the above remark and the previous theorem, it suffices to show that in these spaces, locally countable collections are closure-preserving. Our proof is modeled on the corresponding proof for locally finite in Dugundji [13]. Note that if \mathcal{A} is a locally countable collection, then $\{\bar{A} : A \in \mathcal{A}\}$ is also locally countable. For, given any p , there is a neighborhood $N(p)$ such that $A \cap N(p) = \emptyset$ for all but countably many $A \in \mathcal{A}$. But $A \cap N(p) = \emptyset$ implies $A \subset X - N(p)$ implies $\bar{A} \subset X - N(p)$ implies $\bar{A} \cap N(p) = \emptyset$. Let $\mathcal{B} \subset \mathcal{A}$. Let $B = \bigcup \{\bar{A} : A \in \mathcal{B}\}$. For each p not in B , there is a neighborhood $N(p)$ meeting at most countably many \bar{A} 's such that $A \in \mathcal{B}$, say $\{\bar{A}_i\}_{i < \omega}$. But then $N(p) \cap \bigcap_{i < \omega} (X - \bar{A}_i)$ is a neighborhood of p disjoint from B . Hence $X - B$ is open, and B is closed.

2. Covering axioms and collectionwise normality. In this section, relations between various covering axioms and several forms of collectionwise normality are explored. First, the reader should recall the definition of discrete from 0.0.5, and then verify the following

2.1. LEMMA. *A collection \mathcal{Y} of disjoint sets in a topological space X is discrete if and only if $\{\bar{Y} : Y \in \mathcal{Y}\}$ is discrete, if and only if for each p in X , there is a neighborhood $N(p)$ which intersects at most one member of \mathcal{Y} .*

It is sometimes useful to note

2.2. LEMMA. *A space is collectionwise normal if and only if for each discrete collection \mathcal{Y} , there is a discrete collection of open sets U_Y , $Y \in \mathcal{Y}$, $Y \subset U_Y$.*

The lemma follows immediately from

2.3. LEMMA (Dowker [12], also see Michael [34]). *Suppose \mathcal{Y} is a disjoint collection of sets in a normal space, and \mathcal{Y}^* is closed. Suppose there are disjoint open sets V_Y , $Y \in \mathcal{Y}$, $Y \subset V_Y$. Then there is a discrete collection of open sets U_Y , $Y \in \mathcal{Y}$, such that $Y \subset U_Y \subset V_Y$.*

One more easy fact about discreteness.

2.4. LEMMA. *Let \mathcal{Y} be a collection of subsets of F , a subspace of X . Then if \mathcal{Y} is discrete in X , it is discrete in F . If F is a closed subspace of X , and \mathcal{Y} is discrete in F , then \mathcal{Y} is discrete in X .*

For convenience, we make the following

2.5. DEFINITION. Let \mathcal{Y} be a disjoint collection of sets in a topo-

logical space X . \mathcal{Y} is *separated* if there exist disjoint open sets U_Y , $Y \in \mathcal{Y}$, $Y \subset U_Y$.

For discrete \mathcal{Y} , this is of course the same as saying X is collectionwise normal with respect to \mathcal{Y} .

Depending on what kind of discrete collections one wishes to separate, one may define various restricted forms of collectionwise normality. The following two natural ones seem to be the most fruitful.

2.6. DEFINITION. A space X is *collectionwise Hausdorff* if every discrete collection of points is separated.

Note that collectionwise Hausdorff does not imply Hausdorff unless T_1 is assumed.

2.7. DEFINITION. A subset Z of a topological space X is *lindelof* if, as a subspace, Z is a lindelof space. (Equivalently, if every collection of open sets in X which covers Z has a countable subcollection covering Z .)

2.8. DEFINITION. A space X is *L -collectionwise normal* if every discrete collection of lindelof sets is separated.

In the same way as the notion of collectionwise normality of a space leads to the notion of a particular collection being separated in a space, the idea of normality is abstracted and applied to a particular collection as follows:

2.9. DEFINITION. Let \mathcal{Y} be a disjoint collection of sets in a space X . \mathcal{Y} is *normalized* if for each $\mathcal{Z} \subset \mathcal{Y}$, there exist disjoint open sets U , U' such that $\mathcal{Z}^* \subset U$, $(\mathcal{Y} - \mathcal{Z})^* \subset U'$.

As justification for the definition, one easily proves

2.10. LEMMA. *In a normal space, every discrete collection is normalized.*

The question of when does normality imply collectionwise normality, thus gives rise to the larger question of under what conditions is a normalized collection separated. One answer is embodied in the following definition and theorem.

2.11. DEFINITION. Let λ be a cardinal. A space is *λ -additive* if the union of fewer than λ closed sets is closed. Equivalently, if the intersection of fewer than λ open sets is open. Thus, every space is \aleph_0 -additive.

2.12. THEOREM. *In a λ -additive space, every normalized collection of power $\leq \lambda$ is separated.*

Proof. Let $\mathcal{Y} = \{Y_\gamma\}_{\gamma < \lambda}$ be normalized. For each γ , let $U_\gamma \supset Y_\gamma$, $U'_\gamma \supset \bigcup_{\gamma' \neq \gamma} Y_{\gamma'}$, $U_\gamma \cap U'_\gamma = \emptyset$. Let $V_\gamma = U_\gamma \cap \bigcap_{\delta < \gamma} U'_\delta$. Then $\{V_\gamma\}_{\gamma < \lambda}$ is the required separation.

2.13. COROLLARY. *In a normal space, every countable discrete collection is separated.*

Proof. 2.10 plus 2.12.

2.14. COROLLARY. *In a regular space, every countable discrete collection of lindelof sets is separated.*

Proof. It suffices to show that, in a regular space, any two lindelof sets with disjoint closures can be separated. (This is essentially the same as the proof that every regular lindelof space is normal.) For then, if \mathcal{Y} is a countable discrete collection of lindelof sets, and $\mathcal{X} \subset \mathcal{Y}$, then \mathcal{X}^* and $(\mathcal{Y} - \mathcal{X})^*$ are lindelof sets with disjoint closures, since the union of countably many lindelof sets is a lindelof set. It follows that \mathcal{Y} is normalized, and therefore by 2.12, separated. So suppose A, B are lindelof, $\bar{A} \cap \bar{B} = 0$. By regularity, for each p in A , there is an open set U_p about p , $\bar{U}_p \cap B = 0$. Similarly, for each q in B , there is open V_q about q , $\bar{V}_q \cap A = 0$. By lindelofness, there exist $\{p_n\}_{n < \omega}$, $\{q_n\}_{n < \omega}$ such that $A \subset \bigcup_{n < \omega} U_{p_n}$, $B \subset \bigcup_{n < \omega} V_{q_n}$. Then

$$\bigcup_{n < \omega} (U_{p_n} - \bigcup_{i \leq n} \bar{V}_{q_i}), \quad \bigcup_{n < \omega} (V_{q_n} - \bigcup_{i \leq n} \bar{U}_{p_i})$$

are the desired disjoint open sets containing A, B respectively.

If a disjoint collection is covered by an appropriate family of open sets, a separation may be obtained. For example, making the obvious definition of σ -closure-preserving, we have the following

2.15. THEOREM. *Suppose the disjoint collection \mathcal{Y} is covered by a σ -closure-preserving collection of open sets, such that the closure of any one of the open sets intersects at most one member of \mathcal{Y} . Then \mathcal{Y} is separated.*

Proof. This is essentially the proof of Theorem 2 of Bing [5]. Let $\mathcal{U} = \bigcup_{i < \omega} \mathcal{U}_i$ be the hypothesized collection of open sets. For each $Y \in \mathcal{Y}$, and each $i < \omega$, let $U_{Y,i} = \bigcup \{U \in \mathcal{U}_i : U \cap Y \neq \emptyset\}$. Let $U'_{Y,i} = \bigcup \{U \in \mathcal{U} : \text{for some } Y' \in \mathcal{Y}, Y' \neq Y, U \cap Y' \neq \emptyset\}$. Then the desired collection of disjoint open sets V_Y , is given by

$$V_Y = \bigcup_{i < \omega} (U_{Y,i} - \bigcup_{j=1}^{i-1} \bar{U}'_{Y,j}).$$

2.16. COROLLARY. *A strongly screenable or paracompact space is collectionwise normal if it is either normal or regular.*

Proof. Left to the reader.

2.17. DEFINITION. A disjoint collection of sets in a topological space is *countably separated* if every countable subcollection of it is separated.

2.18. THEOREM. *Suppose \mathcal{Y} is a countably separated collection, and suppose there is a star-countable collection \mathcal{S} of open sets covering \mathcal{Y} (more precisely, \mathcal{Y}^*), such that each $S \in \mathcal{S}$ intersects at most one member of \mathcal{Y} . Then \mathcal{Y} is separated.*

Proof. For each $S \in \mathcal{S}$, let

$$\mathcal{S}(S, 0) = \{S\},$$

$$\mathcal{S}(S, n+1) = \{S' \in \mathcal{S} : \text{for some } S'' \in \mathcal{S}(S, n), S'' \cap S' \neq \emptyset\},$$

$$\mathcal{S}(S, \omega) = \bigcup \mathcal{S}(S, n),$$

$$\mathcal{S}^*(S) = \left(\bigcup_{n < \omega} \mathcal{S}(S, n) \right)^* = \bigcup \{S' : \text{for some } n, S' \in \mathcal{S}(S, n)\}.$$

Note that each $\mathcal{S}(S, \omega)$ is countable, since \mathcal{S} is star-countable. Further, note that $S' \in \mathcal{S}(S, \omega)$ (read as “ S' is connected to S ”) is an equivalence relation. Also, it is easy to see that $\mathcal{S}^*(S) \cap \mathcal{S}^*(S') \neq \emptyset$ if and only if $\mathcal{S}^*(S) = \mathcal{S}^*(S')$.

Since $\mathcal{S}(S, \omega)$ is countable, and each S intersects at most one member of \mathcal{Y} , it follows that each $\mathcal{S}^*(S)$ intersects at most countably many members of \mathcal{Y} .

Pick a representative S_t from each equivalence class $\mathcal{S}(S, \omega)$, so that the collection of equivalence classes is indexed as $\{\mathcal{S}(S_t, \omega)\}_{t \in \mathcal{T}}$, $\mathcal{T} \subset \mathcal{S}$. Let $\{Y_{n,t}\}_{n < \omega}$ be the collection of distinct Y 's such that $Y \cap \mathcal{S}^*(S_t) \neq \emptyset$. By hypothesis, for each t , there exist disjoint open sets $U_{n,t}$, $n < \omega$, $U_{n,t} \supset Y_{n,t}$. Let $V_{n,t} = \mathcal{S}^*(S_t) \cap U_{n,t}$. Let $W_{\mathcal{F}} = \bigcup \{V_{n,t} : Y = Y_{n,t} \text{ for some } n, t\}$. Claim $W_{\mathcal{F}}$, $Y \in \mathcal{Y}$, is the desired separation of \mathcal{Y} . First, $W_{\mathcal{F}} \supset Y$, for if $y \in Y$, then since \mathcal{S} covers \mathcal{Y}^* , there is an $S \in \mathcal{S}$, $y \in S$. S is in $\mathcal{S}(S_t, \omega)$ for some t ; therefore $y \in \mathcal{S}^*(S_t)$. Then for some n , $y \in Y_{n,t} = Y$, so $y \in W_{\mathcal{F}}$. Secondly, if $W_{\mathcal{F}} \cap W_{\mathcal{F}'} \neq \emptyset$, then for some n, n' , t, t' , $V_{n,t} \cap V_{n',t'} \neq \emptyset$, and $Y = Y_{n,t}$, $Y' = Y_{n',t'}$. Since $V_{n,t} \cap V_{n',t'} \neq \emptyset$, it follows that $\mathcal{S}^*(S_t) \cap \mathcal{S}^*(S_{t'}) \neq \emptyset$. Therefore $t = t'$. It also follows that $U_{n,t} \cap U_{n',t'} \neq \emptyset$. But since $t = t'$, and for fixed t , $U_{n,t} \cap U_{n',t} = \emptyset$ unless $n = n'$, we indeed have $U_{n,t} = U_{n',t}$ and $Y_{n,t} = Y_{n',t}$. So $Y = Y'$, under the assumption that $W_{\mathcal{F}} \cap W_{\mathcal{F}'} \neq \emptyset$, and the theorem is proved.

We may now rapidly prove a number of corollaries.

2.19. COROLLARY. *Every hypolindelof normal space is collectionwise normal.*

Proof. Given a discrete collection \mathcal{Y} , there certainly exists for each $Y \in \mathcal{Y}$, an open set $U_Y \supset Y$, U_Y intersecting no other member of \mathcal{Y} . Simply take $U_Y = X - \overline{\bigcup \{Y' \in \mathcal{Y} : Y' \neq Y\}}$. Let \mathcal{S} be a star-countable refinement of the cover $\{U_Y\}_{Y \in \mathcal{Y}} \cup \{X - \overline{\mathcal{Y}^*}\}$. Then each $S \in \mathcal{S}$ intersects at most one $Y \in \mathcal{Y}$. By 2.13, \mathcal{Y} is countably separated, so the corollary now follows from the theorem. (One can also prove it by 1.4 plus 2.16.)

2.20. COROLLARY. *Every paralindelof regular space is L -collectionwise normal.*

Proof. As in the previous proof, given a discrete collection \mathcal{Y} of lindelof sets, get open sets U_Y , $Y \in \mathcal{Y}$, $Y \subset U_Y$, U_Y intersects no other member of \mathcal{Y} . Let \mathcal{R} be a locally countable refinement of the cover $\{U_Y\}_{Y \in \mathcal{Y}} \cup \{X - \overline{\mathcal{Y}^*}\}$. For each $y \in \mathcal{Y}^*$, let $N(y)$ be an open set about y

which intersects at most countably many members of \mathcal{R} . For each $y \in \mathcal{Y}^*$, also pick some $R(y)$, a member of \mathcal{R} containing y . By lindelofness, for each $y \in \mathcal{Y}^*$, there is $\{y_n\}_{n < \omega} \subset Y$, such that $\bigcup_{n < \omega} N(y_n) \cap R(y_n) \supset Y$. Let $V_{\mathcal{Y}} = \bigcup_{n < \omega} N(y_n) \cap R(y_n)$. Then $Y \subset V_{\mathcal{Y}} \subset U_{\mathcal{Y}}$; so $V_{\mathcal{Y}}$ intersects no $Y' \neq Y$. The collection $\{V_{\mathcal{Y}}\}_{\mathcal{Y} \in \mathcal{Y}}$ is star-countable, for each $N(y_n)$ intersects at most countably many $V_{\mathcal{Y}}$'s, else it would intersect uncountably many $R(y)$'s. Certainly $N(y_n) \cap R(y_n)$ then intersects at most countably many $V_{\mathcal{Y}}$'s, and finally, a countable union of $N(y_n) \cap R(y_n)$'s can then only intersect at most countably many $V_{\mathcal{Y}}$'s. By 2.14, \mathcal{Y} is countably separated, so the corollary follows.

2.21. COROLLARY. *Suppose \mathcal{Y} is disjoint and countably separated. Suppose for each $Y \in \mathcal{Y}$, there is $U_Y \supset Y$, U_Y open, U_Y intersects no other member of \mathcal{Y} , such that the collection $\{U_Y\}_{Y \in \mathcal{Y}}$ is σ -star-countable. Finally, suppose that if $\{\mathcal{Y}_n\}_{n < \omega}$ is any countable family of subcollections of \mathcal{Y} such that $\{\mathcal{Y}_n^*\}_{n < \omega}$ is disjoint, then $\{\mathcal{Y}_n^*\}_{n < \omega}$ is separated. Then \mathcal{Y} is separated.*

Proof. Let $\{U_Y\}_{Y \in \mathcal{Y}} = \bigcup_{n < \omega} \mathcal{U}_n$, where each \mathcal{U}_n is star-countable.

We may assume that $\{\mathcal{U}_n\}_{n < \omega}$ is disjoint, and hence, letting $\mathcal{Y}_n = \{Y : U_Y \in \mathcal{U}_n\}$, that $\{\mathcal{Y}_n\}_{n < \omega}$ is disjoint, and hence that $\{\mathcal{Y}_n^*\}_{n < \omega}$ is disjoint. By hypothesis then, there exist disjoint open $V_n \supset \mathcal{Y}_n^*$. Since \mathcal{Y} is countably separated, each \mathcal{Y}_n is countably separated. By the theorem then, it follows that each \mathcal{Y}_n is separated. By taking appropriate intersections of open sets, a separation of \mathcal{Y} may now be achieved.

2.22. COROLLARY. *Every screenable, normal, regular space is collectionwise Hausdorff.*

Proof. Let \mathcal{Y} be a discrete collection of points. Since a point is trivially a lindelof set, by 2.14 \mathcal{Y} is countably separated. Since \mathcal{Y} is discrete, for each $y \in \mathcal{Y}^*$, there is an open U_y which contains no other member of \mathcal{Y}^* . By screenability, take a σ -disjoint, a fortiori σ -star-countable, refinement \mathcal{R} of the cover $\{U_y\}_{y \in \mathcal{Y}^*} \cup \{X - \overline{\mathcal{Y}^*}\}$. Then for each $y \in \mathcal{Y}^*$, pick $R_y \in \mathcal{R}$, $y \in R_y \subset U_y$. $\{R_y\}_{y \in \mathcal{Y}^*}$ is then σ -star-countable; each R_y contains y and no other member of \mathcal{Y}^* . Finally, if $\{\mathcal{Y}_n\}_{n < \omega}$ is any collection of subcollections of \mathcal{Y} such that $\{\mathcal{Y}_n^*\}_{n < \omega}$ is disjoint, then in fact $\{\mathcal{Y}_n^*\}_{n < \omega}$ is discrete. By normality, it follows that $\{\mathcal{Y}_n^*\}_{n < \omega}$ is separated. By the previous corollary then, we are done.

Since by 1.6, countable metacompactness is all that a screenable normal space needs in order to be paracompact, and hence collectionwise normal, 2.22 is rather weak. The author has been unable to improve it. If there were a screenable normal Hausdorff space that was not collectionwise normal, its product with the unit interval would not be normal (see Dowker [11]), so such spaces will be hard to find. Fleissner

[18] constructed a normal collectionwise Hausdorff not collectionwise normal space by tinkering with Bing's example of a normal not collectionwise Hausdorff space. However the author knows of no paralindelof regular space which is not paracompact, although such spaces almost certainly exist. The question of what must be added to paralindelof to get paracompact or collectionwise normal has a large bearing on the normal Moore space question, as will be seen later. An optimistic conjecture is that every paralindelof, normal, F_σ -screenable space is paracompact. (The author's investigation of paralindelofness started when R. W. Heath asked him whether every paralindelof normal Moore space is metrizable. This is still open.) The relation between paralindelofness and screenability is also very unclear.

3. Covering axioms and countability conditions. In this section, the interplay between covering axioms and diverse countability conditions, including separability, lindelofness, and their local versions, is explored. The general conclusion is that the combination of weak covering axioms with weak countability conditions produces very strong covering axioms. Besides separability, lindelofness, compactness, and countable compactness, we consider the following "global" countability conditions:

3.1. DEFINITION. A space is said to satisfy the *countable chain condition*, or be *c.c.c.*, if every collection of mutually disjoint open sets is countable. A space is \aleph_1 -compact if every uncountable set has a limit point in the space.

Each of these conditions can be localized:

3.2. DEFINITION. A space has *property P locally*, or is *locally P*, if for each point p , and each open set V about p , there is an open set W containing p , and a set Z , $W \subset Z \subset V$, such that Z has property P.

For the properties we are considering, this definition simplifies to one of two forms. One need only prove

3.3. LEMMA. *An open subspace of a separable (c.c.c.) space is separable (c.c.c.).*

3.4. LEMMA. *A closed subspace of a compact, countably compact, lindelof, \aleph_1 -compact space inherits the respective property.*

The simple proofs are left to the reader. We then have

3.5. THEOREM. *A space is locally separable (locally c.c.c.) if each point has an open separable (open c.c.c.) neighborhood.*

3.6. THEOREM. *A regular space is locally compact, locally countably compact, locally lindelof, locally \aleph_1 -compact, if each point has an open neighborhood, the closure of which possesses the corresponding property.*

It should be mentioned that the only non-trivial relations that hold in general among the countability conditions are the following:

3.7. THEOREM. *Separable implies c.c.c. Lindelof implies \aleph_1 -compact.*

A similar remark applies for the local versions. In the presence of covering axioms, though, the situation is different. For example, the following theorem and corollary are known. The simple proof is left to the reader.

3.8. THEOREM. *A point-countable cover of a separable space is countable.*

3.9. COROLLARY. *Every separable metalindelof space is lindelof.*

Aquaro has proved the following sharp results:

3.10. THEOREM. *A point-countable cover of an \aleph_1 -compact space has a countable subcover.*

3.11. COROLLARY. *A space is lindelof if and only if it is \aleph_1 -compact and metalindelof.*

3.12. COROLLARY. *A space is compact if and only if it is both countably compact and metalindelof.*

Juhász (see Tall [48]) has proved that every locally countable cover of a c.c.c. space is countable, and in particular

3.13. THEOREM. *Every c.c.c. paralindelof space is lindelof.*

A classical result:

3.14. THEOREM (Bing [5]). *Every c.c.c. screenable space is lindelof.*

One can prove various theorems about the inheritance of covering properties by closed sets, open sets, and F_σ -sets, and then go on to establish circumstances under which a space hereditarily satisfies a covering axiom. One can also establish equivalences among various hereditary countability conditions in the presence of suitable covering axioms. However, we shall not do that here. We confine ourselves to mentioning the two weak lemmas we shall need later:

3.15. LEMMA. *A closed subspace of a metalindelof space is metalindelof.*

3.16. LEMMA. *A closed subspace of a collectionwise Hausdorff space is collectionwise Hausdorff.*

Proof. The obvious proof works for the first; the second lemma follows from 2.4.

From 3.6, 3.11, 3.12, and 3.15, it may be deduced that

3.17. THEOREM. *For metalindelof regular spaces, locally lindelof, and locally \aleph_1 -compact are equivalent, as are locally compact and locally countably compact.*

For paralindelof regular spaces, any of these five local properties

implies hypolindelofness. By Smirnov's 1.5, a regular hypolindelof space is hypocompact, hence paracompact. We state the theorem for the weakest property.

3.18. THEOREM. *Every locally \aleph_1 -compact, regular paralindelof space is hypocompact, and hence paracompact.*

Proof. Let \mathcal{U} be a cover of the locally \aleph_1 -compact, regular paralindelof space X . As remarked above, it suffices to find a star-countable refinement of \mathcal{U} . By 3.17, X is locally lindelof, so let \mathcal{V} be a refinement of \mathcal{U} consisting of open sets having lindelof closures. Let \mathcal{R} be a locally countable refinement of \mathcal{V} . Claim \mathcal{R} is star-countable. Suppose not. Then there is an $R \in \mathcal{R}$, R intersects uncountably many members of \mathcal{R} , say $\{R_\alpha\}_{\alpha < \omega_1}$. Pick $r_\alpha \in R_\alpha \cap R$. Then $\{r_\alpha\}_{\alpha < \omega_1}$ is indeed uncountable, else some r_α would be in uncountably many R_α 's. There is a $V \in \mathcal{V}$, $R \subset V$. Since \bar{V} is lindelof, there exists a $p \in \bar{V}$ such that every open neighborhood of p contains uncountably many r_α 's, hence intersects uncountably many R_α 's. This contradicts the assumption that \mathcal{R} was locally countable.

The presence of the other two local countability conditions also assures that a regular paralindelof space is paracompact.

3.19. THEOREM. *Every locally c.c.c. paralindelof space is hypolindelof and hence, if regular, paracompact.*

Proof follows from the result of Juhász quoted before 3.13.

Screenability can be substituted for paralindelofness in 3.21:

3.20. THEOREM. *Every locally c.c.c. screenable space is hypolindelof.*

3.21. COROLLARY. *Every regular, locally c.c.c. screenable space is hypocompact, and hence paracompact.*

Proof. The corollary follows from the theorem, as usual, by 1.5. To prove the theorem, given an arbitrary cover \mathcal{U} , take a refinement \mathcal{V} , consisting of c.c.c. sets. Then let $\mathcal{R} = \bigcup_{i < \omega} \mathcal{R}_i$ be a σ -disjoint refinement of \mathcal{V} . Claim \mathcal{R} is star-countable. For if not, there is an $R \in \mathcal{R}$, and an integer i , such that R intersects uncountably many members of \mathcal{R}_i , say $\{R_{\alpha,i}\}_{\alpha < \omega_1}$. But this is impossible, since \mathcal{R}_i is disjoint, while R , as an open subset of a c.c.c. set, is c.c.c. by 3.3.

Local separability is, as far as the author knows, the only local countability condition that yields hypolindelofness in the presence of merely metalindelofness. The following theorem is essentially proved in Traylor [52].

3.22. THEOREM. *A point-countable collection of separable open sets is star-countable.*

3.23. COROLLARY. *Every locally separable, metalindelof space is hypolindelof.*

The easy proofs are left to the reader.

CHAPTER II

0. Preliminaries. In this introductory section, topological notions necessary for stating the results of the chapter are defined and elucidated. Then the main consistency theorem and its principal corollaries are stated. The next section, almost entirely set-theoretic, contains a proof of the main theorem. Section 2 is devoted to topological derivations of corollaries from this theorem. In Section 3, Bing's example of a normal, non-collectionwise normal space is given, and it is shown that, in one of the parameters, the results of this chapter are "best possible". In the final section, possible set-theoretic and topological approaches toward strengthening these results in other respects are discussed.

0.1. DEFINITION. Let \mathcal{Y} be a disjoint collection of sets in a topological space X . \mathcal{Y} is *relatively discrete* if \mathcal{Y} is discrete in the subspace \mathcal{Y}^* .

Thus, \mathcal{Y} is relatively discrete (by 1.2.1) if for each $y \in Y \in \mathcal{Y}$, there is an open set $N(y)$ about y , such that $N(y)$ intersects no member of \mathcal{Y} other than Y . Thus,

0.2. LEMMA. If \mathcal{Y} is relatively discrete, and $Y \in \mathcal{Y}$, then $X - \overline{\mathcal{Y}^* - Y}$ is an open set containing Y .

0.3. LEMMA. If \mathcal{Y} is normalized, then it is relatively discrete.

Proof. Obvious.

0.4. DEFINITION. Let $\mathcal{Y} = \{Y_\gamma\}_{\gamma < \omega_1}$ be a relatively discrete collection. \mathcal{Y} is said to be *starry* if, given any cover of \mathcal{Y}^* by open sets $\{V_\nu\}_{\nu \in \mathcal{Y}^*}$, such that $y \in V_\nu$, and, if $y \in Y \in \mathcal{Y}$, V_ν intersects no other member of \mathcal{Y} , then there is a $y \in \mathcal{Y}^*$, such that V_ν intersects $V_{\nu'}$'s "arbitrarily far out". More formally, if there is a y such that

$$(\forall \beta < \omega_1)(\exists \gamma > \beta)(\exists y' \in Y_\gamma)(V_\nu \cap V_{\nu'} \neq \emptyset).$$

Intuitively then, a relatively discrete collection is starry when, not only is it not separated, but, also, its elements are so "close" to each other, that given any appropriate cover $\{V_\nu\}_{\nu \in \mathcal{Y}^*}$, there will be at least one y such that V_ν is "badly intertwined" with the other $V_{\nu'}$'s.

The reason for introducing the notion of starriness will be seen in the actual consistency proof, where it is needed to assure that certain

forcing conditions exist. A less esoteric formulation of starriness is given in the following definition and lemma.

0.5. DEFINITION. Let \mathcal{U} be a collection of subsets of the topological space X . Let $Z \subset X$. \mathcal{U} is said to be *Z-locally countable* if, for each $z \in Z$, there is a neighborhood $N(z)$ which intersects at most countably many members of \mathcal{U} .

0.6. LEMMA. Let $\mathcal{Y} = \{Y_\gamma\}_{\gamma < \omega_1}$ be relatively discrete. Then, as in 0.2, $U_\gamma = X - \overline{\mathcal{Y}^* - Y_\gamma}$ is an open set containing Y_γ . \mathcal{Y} is starry if and only if $\mathcal{U} = \{U_\gamma\}_{\gamma < \omega_1}$ has no \mathcal{Y}^* -locally countable refinement. (In this context, by "refinement" is meant a collection of open sets, each contained in a member of \mathcal{U} , covering \mathcal{Y}^* (but not necessarily \mathcal{U}^* .)

The following definition and sublemma will simplify the proof of the lemma.

0.7. DEFINITION. A refinement \mathcal{R} of the collection $\mathcal{U} = \{U_a\}_{a \in A}$ is *precise* if $\mathcal{R} = \{R_a\}_{a \in A}$ and for each $a \in A$, $R_a \subset U_a$.

0.8. SUBLEMMA. \mathcal{U} has a *Z-locally countable refinement* if and only if it has a *precise Z-locally countable refinement*.

Proof. Let $\mathcal{U} = \{U_a\}_{a \in A}$. Let \mathcal{S} be a *Z-locally countable refinement* of \mathcal{U} . Define a function $f: \mathcal{S} \rightarrow A$ by picking for each $S \in \mathcal{S}$, some $f(S) \in A$, such that $S \subset U_{f(S)}$. Let $V_a = \bigcup \{S: f(S) = a\}$. Then $V_a \subset U_a$. Let $\mathcal{V} = \{V_a\}_{a \in A}$. Then $\mathcal{V}^* = \mathcal{S}^*$, and each V_a is open. \mathcal{V} is precise, so it remains to show \mathcal{V} is *Z-locally countable*. But given any $z \in Z$, there is a neighborhood $N(z)$ which meets only countably many S 's, and hence only countably many V_a 's.

Proof of lemma. Suppose first that the canonical $\mathcal{U} = \{U_\gamma\}_{\gamma < \omega_1}$ has a precise \mathcal{Y}^* -locally countable refinement $\mathcal{W} = \{W_\gamma\}_{\gamma < \omega_1}$. Then for each $y \in \mathcal{Y}^*$, there is an open V_y about y , such that V_y hits at most countably many members of \mathcal{W} . Moreover, if $y \in Y_\gamma$, V_y can be assumed to be contained in W_γ . Let $R_\gamma = \bigcup \{V_y: y \in Y_\gamma\}$. Then $R_\gamma \subset W_\gamma$. Therefore, no V_y , $y \in \mathcal{Y}^*$, can intersect uncountably many R_γ 's, else it would intersect uncountably many W_γ 's. It follows that \mathcal{Y} is not starry.

Conversely, if \mathcal{Y} is not starry, there is a collection of open sets $\{V_y\}_{y \in \mathcal{Y}^*}$ such that $y \in V_y$, and if $y \in Y \in \mathcal{Y}$, V_y intersects no other member of \mathcal{Y} . It follows that if $R_\gamma = \bigcup \{V_y: y \in Y_\gamma\}$, then $\mathcal{R} = \{R_\gamma\}_{\gamma < \omega_1}$ is a precise refinement of \mathcal{U} . Further, this collection $\{V_y\}_{y \in \mathcal{Y}^*}$, has the property that

$$(\forall y)(\exists \beta < \omega_1)(\forall \gamma: \beta < \gamma < \omega_1)(\forall y' \in Y_\gamma)(V_y \cap V_{y'} = 0).$$

In other words,

$$(\forall y)(\exists \beta < \omega_1)(\forall \gamma: \beta < \gamma < \omega_1)(V_y \cap R_\gamma = 0).$$

That is, \mathcal{R} is \mathcal{Y}^* -locally countable.

A few more definitions:

0.9. DEFINITION. Let $\langle X, \mathcal{T} \rangle$ be a topological space. Let $Z \subset X$. $\mathcal{T}_Z \subset \mathcal{T}$ is a base at Z if, for each open V containing Z , there is a $U \in \mathcal{T}_Z$ such that $Z \subset U \subset V$. The set character of Z , $\chi[Z]$, is the least cardinal κ , such that there is a base at Z of power κ . If Z is a singleton, $Z = \{z\}$, we write $\chi(z)$ for $\chi[\{z\}]$. The character of Z , $\chi(Z)$, is defined by $\chi(Z) = \sup_{z \in Z} \chi(z)$. The character of the space $\langle X, \mathcal{T} \rangle$ is defined to be $\chi(X)$. If $\chi(X) \leq \aleph_0$, $\langle X, \mathcal{T} \rangle$ is said to satisfy the first axiom of countability, or be first countable.

0.10. DEFINITION. The following abbreviations are useful:

ZFC for "Zermelo-Fraenkel set theory, including the Axiom of Choice",

CH for " $2^{\aleph_0} = \aleph_1$ " (the continuum hypothesis),

CONCH φ for "If ZFC is consistent, then so is ZFC plus CH plus φ ".

Section 1 is devoted to the proof of

0.11. THEOREM. CONCH Suppose $\langle X, \mathcal{T} \rangle$ is a topological space, the cardinality of X is less than 2^{\aleph_1} , and there is a base \mathcal{B} for \mathcal{T} , the cardinality of \mathcal{B} less than 2^{\aleph_1} . Then if \mathcal{U} is starry, and the cardinality of \mathcal{U}^* is $\leq \aleph_1$, \mathcal{U} is not normalized.

The following congeries of results is a topological consequence of 0.11.

0.12. THEOREM. CONCH Suppose X is a normal space, $\mathcal{U} = \{Y_\gamma\}_{\gamma < \omega_1}$ is a discrete collection, and \mathcal{U}^* has character less than 2^{\aleph_1} . Then if any of the following conditions is satisfied, \mathcal{U} is separated.

- (a) Each Y_γ is a lindelof set of cardinality $\leq \aleph_1$.
- (b) Each Y_γ is a hereditarily lindelof set and X is regular.
- (c) Each Y_γ is a lindelof set, X is regular, and each closed subset of X is a G_δ .
- (d) Each Y_γ has set character less than 2^{\aleph_1} .
- (e) Each Y_γ has cardinality $\leq \aleph_1$, X is regular, locally lindelof, and F_σ -screenable, every closed subset of X is a G_δ , and $\chi(X) < 2^{\aleph_1}$.
- (f) Each Y_γ has cardinality $\leq \aleph_1$, and is contained in a separable open set.
- (g) Each Y_γ is contained in a separable open set, X is Hausdorff and first countable.
- (h) Each Y_γ is a lindelof set, X is locally separable, Hausdorff, and first countable.
- (i) Each Y_γ has cardinality $\leq \aleph_1$, X is locally separable, F'_σ -screenable, Hausdorff, and first countable.

0.13. THEOREM. CONCH Suppose X is a locally separable, normal, regular space, and \mathcal{U} is a discrete collection of lindelof subsets of X , $|\mathcal{U}^*| \leq \aleph_1$. Then \mathcal{U} is separated.

Since a collection \mathcal{Y} is discrete if and only if $\{\bar{Y} : Y \in \mathcal{Y}\}$ is discrete (1.2.1), 0.12 and 0.13 can be extended in an obvious manner by replacing clauses hypothesizing that Y_γ has a certain property, by clauses hypothesizing that \bar{Y}_γ is contained in a subset of \bar{Y}_γ which has that property.

Two Moore space metrization corollaries of 0.12 are

0.14. COROLLARY. CONCH *Every c.c.c. normal Moore space is metrizable.*

0.15. COROLLARY. CONCH *Every locally c.c.c. normal Moore space of cardinality $\leq \aleph_1$ is metrizable.*

Since the completion of this thesis, several improvements have been obtained by various authors. We discuss these in more detail in Section 4, but in particular we should mention here the following three results due respectively to B. Šapirovskii [40], W. Fleissner [17], and K. Alster and T. Przymusiński [1].

0.16. THEOREM. $2^{\aleph_0} < 2^{\aleph_1}$ *implies every c.c.c. normal Moore space is metrizable.*

0.17. THEOREM. $V = L$ (Gödel's Axiom of Constructibility) *implies every normal space of character $< 2^{\aleph_1}$ is collectionwise Hausdorff.*

0.18. THEOREM. $V = L$ *implies every locally c.c.c. normal Moore space is metrizable.*

1. The consistency proof. Before getting to the technical details, the reader will no doubt appreciate a discourse on the intuitive motivation behind the proof. Let us consider a simple case. Suppose we wish to show it is consistent with ZFC that every discrete collection of \aleph_1 points in a normal space is separated. We can not quite do this, as Bing's example given in Section 3 establishes, but let us add whatever restrictions are necessary, only when the need for them arises in the proof. So, given a model \mathfrak{U} of ZFC, we want to try to extend \mathfrak{U} to a model \mathfrak{U}' of ZFC in which every discrete collection of \aleph_1 points in a normal space is separated. Equivalently, extend \mathfrak{U} to a model in which every space in which there is an unseparated discrete collection of \aleph_1 points is not normal. As a first step, we try to do this for one space and one collection. So suppose $\langle X, \mathcal{T} \rangle$ is a topological space in \mathfrak{U} , \mathcal{Y} is unseparated in $\langle X, \mathcal{T} \rangle$, but \mathcal{Y} is normalized in $\langle X, \mathcal{T} \rangle$. Let us attempt to make \mathcal{Y} not normalized in $\langle X, \mathcal{T} \rangle$ in \mathfrak{U}' .

Well, the obvious way to make \mathcal{Y} not normalized is to add a new subcollection \mathcal{Z} of \mathcal{Y} that is not in the ground model, and hope that \mathcal{Z}^* , $(\mathcal{Y} - \mathcal{Z})^*$ cannot be separated by disjoint open sets in \mathfrak{U}' . Identifying \mathcal{Y}^* with ω_1 , we see that any new subset of ω_1 induces a new subcollection of \mathcal{Y} .

There are a number of different methods of adding new subsets of ω_1 , so which one to choose? By the Law of Conservation of Energy,

we look at the easiest way first, and our indolence, it turns out, is rewarded. The easiest, or at any rate, most familiar method, is known as the technique of adding Cohen subsets of ω_1 . The forcing conditions are all the countable partial functions from ω_1 into $2 (= \{0, 1\})$, ordered by containment. If the continuum hypothesis is assumed in the ground model, cardinals are preserved, as is the power of the continuum. So let $P = \{p: p \subset \omega_1 \times 2, p \text{ is a function, domain } p \subset \omega_1, \text{ range } p \subset 2, \text{ cardinality of } p \text{ is } \leq \aleph_0\}^{\text{st}}$. Given $p, q \in P, p \leq q$ if $p \supset q$. Let $\mathcal{P} = \langle P, \leq \rangle$.

As usual, a generic subset G of P determines a function G^* from ω_1 into 2 , and sets $G_0, G_1 \subset \omega_1, G_0 = \{\alpha: G^*(\alpha) = 0\}, G_1 = \{\alpha: G^*(\alpha) = 1\}$. Then $G_0 \cup G_1 = \omega_1, G_0 \cap G_1 = \emptyset$. Let $\mathfrak{U}' = \mathfrak{U}[G]$. We want to show that in \mathfrak{U}' , if U_0, U_1 are open sets containing G_0, G_1 respectively (recall \mathcal{P}^* was identified with ω_1), then $U_0 \cap U_1 \neq \emptyset$.

By discreteness, it suffices to consider collections of open sets, $\mathcal{V} = \{V_\alpha\}_{\alpha \in \omega_1}$, where $\alpha \in V_\alpha, V_\alpha$ contains no other member of ω_1 . For any such collection, we want to have that

$$\mathfrak{U}' \models \bigcup_{\alpha \in G_0} V_\alpha \cap \bigcup_{\alpha \in G_1} V_\alpha \neq \emptyset.$$

Let p be any member of P . Then, since domain p is countable, and \mathcal{P} is normalized in $\langle X, \mathcal{T} \rangle$ in \mathfrak{U} , by 1.2.12 there exist mutually disjoint open sets $W_\alpha, \alpha \in \text{domain } p, \alpha \in W_\alpha \subset V_\alpha$. Consider the cover of \mathcal{P} , $\{W_\alpha\}_{\alpha \in \text{domain } p} \cup \{V_\alpha\}_{\alpha \notin \text{domain } p}$. By hypothesis, this fails to separate \mathcal{P} , so there is an α_1 , and an $\alpha_2 \notin \text{domain } p$, such that $V_{\alpha_1} \cap V_{\alpha_2} \neq \emptyset$. If $\alpha_1 \in \text{domain } p$, say $p(\alpha_1) = \varepsilon$, define $q \in P$ by $q = p \cup \{\langle \alpha_2, 1 - \varepsilon \rangle\}$. If $\alpha_1 \notin \text{domain } p$, define $q = p \cup \{\langle \alpha_1, 1 \rangle\} \cup \{\langle \alpha_2, 0 \rangle\}$. In either case, $q \leq p$. For any $r \in P$, define $r_0 = \{\alpha: r(\alpha) = 0\}, r_1 = \{\alpha: r(\alpha) = 1\}$. Then

$$\bigcup_{\alpha \in r_0} V_\alpha \cap \bigcup_{\alpha \in r_1} V_\alpha \neq \emptyset.$$

We have shown that for each $\mathcal{V}, D_{\mathcal{V}} = \{p: \bigcup_{\alpha \in p_0} V_\alpha \cap \bigcup_{\alpha \in p_1} V_\alpha \neq \emptyset\}$ is dense in \mathcal{P} . If $\mathcal{V} \in \mathfrak{U}$, then $D_{\mathcal{V}} \in \mathfrak{U}$, so by genericity, there is a $p \in G \cap D_{\mathcal{V}}$. Since G^* extends p , it follows that

$$\bigcup_{\alpha \in G_0} V_\alpha \cap \bigcup_{\alpha \in G_1} V_\alpha \neq \emptyset.$$

We have slurred over a number of important details, but the foregoing discussion essentially demonstrates that open sets in the ground model cannot separate G_0 from G_1 . The catch is that, in adjoining G , we may inadvertently add new open sets that do separate G_0 from G_1 . Silver's contribution — in effect, since he was using Bing's translation — was to figure out a way, given a cover in the extension, to define a cover in the ground model, so that the fact that the ground cover failed to

separate G_0 from G_1 implied that the extension cover also failed. As will be seen in the actual proof, however, \mathcal{Y} not separated is apparently insufficient in general to make this idea work. In addition, it is necessary to assume that \mathcal{Y} is starry.

Once we have that the adjunction of a subset of ω_1 makes a starry collection (of \aleph_1 points) not normalized, we can proceed to add sufficiently many subsets of ω_1 , getting say $\mathfrak{U}^\# = \mathfrak{U}[G_\alpha: \alpha < \aleph]$, so that each starry collection has some subset of ω_1 "generic over it". That is, so that each pair $\langle \langle X, \mathcal{T} \rangle, \mathcal{Y} \rangle$ appears in an intermediate model $\mathfrak{U}[G_\alpha: \alpha < \lambda]$, some $\lambda < \aleph$. But that can happen only if X and \mathcal{T} , as well as \mathcal{Y} , have cardinality less than $\aleph = (2^{\aleph_1})^{\aleph^\#}$. That is the reason for the restriction on \mathfrak{B} in 0.11, and the character restrictions in 0.12. ($(2^{\aleph_1})^{\aleph^\#}$ can be made to equal anything "reasonable". See Easton, or Shoenfield's article. For simplicity however, we assume $2^{\aleph_1} = \aleph_2$ in the ground model, and adjoin \aleph_2 subsets of ω_1 .)

Now we come to the actual proof.

1.1. DEFINITION. From now on, let \mathfrak{B} be a fixed countable transitive ε -model of ZFC plus CH plus $2^{\aleph_1} = \aleph_2$.

We shall show that in the model obtained by adjoining \aleph_2 Cohen subsets of $\omega_1^{\mathfrak{B}}$, the statement, the consistency of which is asserted in 0.11, holds. The standard conversion of this argument into a formal consistency proof will, as usual, be omitted. Since this model is well-known, we also omit the verification that it — and any model obtained by adding fewer than $\aleph_2^{\mathfrak{B}}$ Cohen subsets of $\omega_1^{\mathfrak{B}}$ — is indeed a model of ZFC plus CH, with the same ordinals as the original model, and that cardinals are preserved. For a proof, see for example Shoenfield's article.

1.2. DEFINITION. Let Z be any set. Then $|Z|$ stands for the cardinality of Z .

1.3. DEFINITION. Let $P = \{p: p \subset \omega_1 \times 2, p \text{ is a function, domain } p \subset \omega_1, \text{ range } p \subset 2, |p| \leq \aleph_0\}^{\mathfrak{B}}$. Given $p, q \in P$, say $p \leq q$ if, in \mathfrak{B} , $p \supset q$. Let $\mathcal{P} = \langle P, \leq \rangle$.

1.4. DEFINITION. Let G be \mathcal{P} -generic over \mathfrak{B} . Let $G_\varepsilon = \{\alpha: G^*(\alpha) = \varepsilon\}$, $\varepsilon = 0$ or 1 .

1.5. LEMMA. G^* is a well-defined function. $G_0 \cup G_1 = \omega_1^{\mathfrak{B}}$, $G_0 \cap G_1 = 0$.

Proof. Left to reader.

1.6. DEFINITION. Suppose $\langle X, \mathcal{T} \rangle$ is a topological space in \mathfrak{B} . Let $\mathcal{T}(G)$ be the topology on X generated by \mathcal{T} in $\mathfrak{B}[G]$, i.e. the result of closing \mathcal{T} under arbitrary unions and finite intersections in $\mathfrak{B}[G]$.

1.7. LEMMA. $\mathcal{T}(G)$ is the result of closing \mathcal{T} under arbitrary unions in $\mathfrak{B}[G]$.

Proof. Obvious.

1.8. THEOREM. *If, in \mathfrak{B} , $|\mathcal{Y}^*| \leq \aleph_1$ and \mathcal{Y} is starry in $\langle X, \mathcal{T} \rangle$, then in $\mathfrak{B}[G]$, \mathcal{Y} is not normalized in $\langle X, \mathcal{T}(G) \rangle$.*

To avoid unnecessary superscripts, from now on we work in \mathfrak{B} , unless stated otherwise. Thus we shall for example use " ω_1 " rather than " $\omega_1^{\mathfrak{B}}$ ".

1.9. DEFINITION. Suppose $p \in P, \gamma \in \omega_1$. Define p^γ as follows:

- (a) If $\gamma \notin \text{domain } p, p^\gamma = p$.
- (b) If $\gamma \in \text{domain } p, p^\gamma = (p - \{\langle \gamma, p(\gamma) \rangle\}) \cup \{\langle \gamma, 1 - p(\gamma) \rangle\}$.

To prove 1.8, which is the crucial theorem, we need

1.10. LEMMA. *Let Ω be an ordinal. Let τ be a term in the forcing language for \mathcal{P} . Suppose $q \in P, q \Vdash \tau \in {}^{\omega_1}\Omega$. Then for each $a, \gamma \in \omega_1$, there is a dense below q set of conditions $D_{a,\gamma}^{\tau,a}$, such that $p \in D_{a,\gamma}^{\tau,a}$ implies*

- (a) $\gamma \in \text{domain } p$.
- (b₁) *If $\gamma \in \text{domain } q$, there is a $\mu \in \Omega$ such that $p \Vdash \tau(a) = \mu$.*
- (b₂) *If $\gamma \notin \text{domain } q$, there is a $\mu \in \Omega$ such that $p \Vdash \tau(a) = \mu$, and there is a $\nu \in \Omega$ such that $p^\gamma \Vdash \tau(a) = \nu$.*

Proof of lemma. Let $r \in P, r \leq q$, be given. Let $a, \gamma \in \omega_1$. We wish to find $p \leq r$ satisfying (a), (b₁), (b₂). $\{p : \gamma \in \text{domain } p\}$ is dense, so without loss of generality, assume $\gamma \in \text{domain } r$. Since $q \Vdash \tau \in {}^{\omega_1}\Omega, q \Vdash (\exists \mu)(\tau(a) = \mu, \mu \in \Omega)$. By the rules for forcing, for each $r \leq q$, there is a $p \leq r$, and a term $\bar{\mu}$, such that $p \Vdash (\tau(a) = \bar{\mu}, \bar{\mu} \in \Omega)$. But forcing with \mathcal{P} adds no new ordinals, so indeed there is an ordinal μ , such that $p \Vdash (\tau(a) = \mu, \mu \in \Omega)$. Since μ, Ω are in \mathfrak{B} , it follows that there is a $\mu \in \Omega$, such that $p \Vdash \tau(a) = \mu$. So, whether or not $\gamma \in \text{domain } q$, we have found a p below an arbitrary $r \leq q$, such that there is a $\mu \in \Omega$, such that $p \Vdash \tau(a) = \mu$. Suppose $\gamma \notin \text{domain } q$. Suppose, as just accomplished, we have $p \leq r$, and there is a $\mu \in \Omega, p \Vdash \tau(a) = \mu$. Then $p^\gamma \leq q$. So, repeating the argument, there is an $s \leq p^\gamma$, and there is a $\nu \in \Omega$ such that $s \Vdash \tau(a) = \nu$. But then $s^\gamma \leq p$, so $s^\gamma \Vdash \tau(a) = \mu$. Let $t = s^\gamma$ if $\gamma \notin \text{domain } q, t = p$ otherwise. Then t is the required condition below r .

Proof of theorem. Identify \mathcal{Y}^* with ω_1 ; thus \mathcal{Y} is a partition of ω_1 into the \aleph_1 disjoint pieces $\{Y_\gamma\}_{\gamma < \omega_1}$. We claim that if, in $\mathfrak{B}[G]$, $U_0 \supset \bigcup_{\gamma \in G_0} Y_\gamma, U_1 \supset \bigcup_{\gamma \in G_1} Y_\gamma$, where $U_0, U_1 \in \mathcal{T}(G)$, then $U_0 \cap U_1 \neq 0$, i.e. \mathcal{Y} is not normalized in $\langle X, \mathcal{T}(G) \rangle$.

For each $a \in \omega_1$, say $a \in Y_\gamma$, let $\mathcal{T}_a = \{V \in \mathcal{T} : a \in V \text{ and } V \cap (\mathcal{Y}^* - Y_\gamma) = 0\}$. Then $\mathcal{T}_a \in \mathfrak{B}$ and can be well-ordered in \mathfrak{B} , say as $\mathcal{T}_a = \{V_{\alpha,\beta}\}_{\beta < \Omega_\alpha}$. Let $\Omega = \sup_{a \in \omega_1} \Omega_a$. Since \mathcal{T} is a basis for $\mathcal{T}(G)$, and \mathcal{Y} is relatively discrete, it suffices to consider only U_0, U_1 's of the form

$$U_0 = \bigcup_{\gamma \in G_0} \bigcup_{a \in Y_\gamma} V_{a,f(a)}, \quad U_1 = \bigcup_{\gamma \in G_1} \bigcup_{a \in Y_\gamma} V_{a,f(a)},$$

where f is a function from ω_1 into Ω in $\mathfrak{B}[G]$, such that $f(\alpha) \in \Omega_\alpha$. So, suppose τ is a term, and there is a $q \in G$,

$$q \Vdash \tau \in {}^{\omega_1}\Omega, \quad (\forall \alpha)(\tau(\alpha) \in \Omega_\alpha).$$

It is sufficient to find a $p \in G$, such that

$$p \Vdash \bigcup_{\gamma \in G_0} \bigcup_{\alpha \in Y_\gamma} V_{\alpha, \tau(\alpha)} \cap \bigcup_{\gamma \in G_1} \bigcup_{\alpha \in Y_\gamma} V_{\alpha, \tau(\alpha)} \neq 0.$$

For any $p \in P$, let $p_\varepsilon = \{\alpha: p(\alpha) = \varepsilon\}$, $\varepsilon = 0$ or 1 . Then it is sufficient to find a $p \in G$ such that

$$p \Vdash \bigcup_{\gamma \in P_0} \bigcup_{\alpha \in Y_\gamma} V_{\alpha, \tau(\alpha)} \cap \bigcup_{\gamma \in P_1} \bigcup_{\alpha \in Y_\gamma} V_{\alpha, \tau(\alpha)} \neq 0,$$

since $G_\varepsilon \supset p_\varepsilon$. We shall prove that

$$\{p: p \Vdash \bigcup_{\gamma \in P_0} \bigcup_{\alpha \in Y_\gamma} V_{\alpha, \tau(\alpha)} \cap \bigcup_{\gamma \in P_1} \bigcup_{\alpha \in Y_\gamma} V_{\alpha, \tau(\alpha)} \neq 0\}$$

is dense below q , for then, by genericity, there is a $p \in G$ which works.

Let r be an arbitrary member of P below q . By the lemma, and the fact that any descending sequence of fewer than \aleph_1 conditions has a lower bound in P , we may define a sequence of conditions $\{p_\alpha\}_{\alpha \in \omega_1}$ such that

(a) $\alpha < \alpha'$ implies $p_{\alpha'} \leq p_\alpha$,

(b) for each α , $p_\alpha \leq r$,

(c) $\alpha \in Y_\gamma$ implies $p_\alpha \in D_{\alpha, \gamma}^{r, q}$.

Define a function $h \in {}^{\omega_1}\Omega$ as follows: let $\alpha \in Y_\gamma$, then

$h(\alpha) = \mu_\alpha$ if $\gamma \in \text{domain } q$, and $p_\alpha \Vdash \tau(\alpha) = \mu_\alpha \in \Omega_\alpha$,

$h(\alpha) =$ that $\varrho_\alpha \in \Omega_\alpha$ such that $V_{\alpha, \varrho_\alpha} = V_{\alpha, \mu_\alpha} \cap V_{\alpha, \nu_\alpha}$, if $\gamma \notin \text{domain } q$,

and $p \Vdash \tau(\alpha) = \mu_\alpha \in \Omega_\alpha$, $p' \Vdash \tau(\alpha) = \nu_\alpha \in \Omega_\alpha$.

Then h is well-defined, since \mathcal{F}_α is closed under finite intersections. Moreover, h is in \mathfrak{B} by the definability of forcing.

This is Silver's trick — we now have a cover in the ground model, $\mathcal{V} = \{V_{\alpha, h(\alpha)}\}_{\alpha \in \omega_1}$, such that for each α , $\alpha \in V_{\alpha, h(\alpha)}$, and if $\alpha \in Y_\gamma$, then $V_{\alpha, h(\alpha)} \cap (\mathcal{Q}^* - Y_\gamma) = 0$. Since \mathcal{Q} is starry,

$$(\exists \gamma_1)(\exists \alpha_1 \in Y_{\gamma_1})(\forall \beta \in \omega_1)(\exists \gamma_2 > \beta)(\exists \alpha_2 \in Y_{\gamma_2})(V_{\alpha_1, h(\alpha_1)} \cap V_{\alpha_2, h(\alpha_2)} \neq 0).$$

Confining ourselves to the uncountably many γ_2 's which "work" and are greater than $\sup\{\alpha: \alpha \in \text{domain } p_{\alpha_1}\}$, we see that it is possible to pick a pair γ_2, α_2 such that $\alpha_2 > \alpha_1$. So in sum, there exist $\gamma_1, \alpha_1, \gamma_2, \alpha_2$, such that

(d) $\alpha_1 \in Y_{\gamma_1}, \alpha_2 \in Y_{\gamma_2}$,

(e) $\gamma_2 > \sup\{\alpha: \alpha \in \text{domain } p_{\alpha_1}\}$,

(f) $\alpha_2 > \alpha_1$,

(g) $V_{\alpha_1, h(\alpha_1)} \cap V_{\alpha_2, h(\alpha_2)} \neq 0$.

By the definition of $D_{\alpha, \gamma}^{\tau, \alpha}$, $\gamma_2 \in \text{domain } p_{\alpha_2}$, $\gamma_1 \in \text{domain } p_{\alpha_1}$. Since $\alpha_2 > \alpha_1$, $p_{\alpha_2} \leq p_{\alpha_1}$. Therefore $\gamma_1 \in \text{domain } p_{\alpha_2}$.

Since $p_{\alpha_2} \leq p_{\alpha_1}$, p_{α_2} "decides" $\tau(\alpha_1)$ as well as $\tau(\alpha_2)$. By the crucial clause (e), which starriness assures, $p_{\alpha_2}^{\gamma_2}$ is also $\leq p_{\alpha_1}$ and hence also decides $\tau(\alpha_1)$ and $\tau(\alpha_2)$.

Either $p_{\alpha_2}(\gamma_2) \neq p_{\alpha_2}(\gamma_1)$, or $p_{\alpha_2}^{\gamma_2}(\gamma_2) \neq p_{\alpha_2}^{\gamma_2}(\gamma_1)$. Let the one that has different values at γ_1, γ_2 be called p . If $p = p_{\alpha_2}$, then $p \Vdash (\tau(\alpha_1) = \mu_{\alpha_1}, \tau(\alpha_2) = \mu_{\alpha_2})$. But

$$0 \neq V_{\alpha_1, h(\alpha_1)} \cap V_{\alpha_2, h(\alpha_2)} = V_{\alpha_1, \mu_{\alpha_1}} \cap V_{\alpha_2, \mu_{\alpha_2}}.$$

Therefore, $p \Vdash V_{\alpha_1, \tau(\alpha_1)} \cap V_{\alpha_2, \tau(\alpha_2)} \neq 0$. Similarly, if $p = p_{\alpha_2}^{\gamma_2}$, then $p \Vdash (\tau(\alpha_1) = \mu_{\alpha_1}, \tau(\alpha_2) = \nu_{\alpha_2})$, and

$$0 \neq V_{\alpha_1, h(\alpha_1)} \cap V_{\alpha_2, h(\alpha_2)} = V_{\alpha_1, \mu_{\alpha_1}} \cap V_{\alpha_2, \nu_{\alpha_2}}.$$

So again, $p \Vdash V_{\alpha_1, \tau(\alpha_1)} \cap V_{\alpha_2, \tau(\alpha_2)} \neq 0$. In either case,

$$p \Vdash \bigcup_{\gamma \in \mathcal{P}_0} \bigcup_{\alpha \in \mathcal{Y}_\gamma} V_{\alpha, \tau(\alpha)} \cap \bigcup_{\gamma \in \mathcal{P}_1} \bigcup_{\alpha \in \mathcal{Y}_\gamma} V_{\alpha, \tau(\alpha)} \neq 0,$$

which was to be proved.

1.11. DEFINITION. Let β be an ordinal in \mathfrak{B} . Let $P_\beta = \{p: p \subset \beta \times \omega_1 \times 2, p \text{ is a function, domain } p \subset \beta \times \omega_1, \text{ range } p \subset 2, |p| \leq \aleph_0\}^\mathfrak{B}$. Given $p, q \in P_\beta$, say $p \leq q$ if, in \mathfrak{B} , $p \supset q$. Let $\mathcal{P}_\beta = \langle P_\beta, \leq \rangle$.

1.12. THEOREM. Suppose, in \mathfrak{B} , that β is an ordinal $\leq \omega_2$. Let G be \mathcal{P}_β -generic over \mathfrak{B} . Then $\mathfrak{B}[G]$ is a model of ZFC plus CH, with the same ordinals as \mathfrak{B} , and cardinals are preserved. Moreover, if in \mathfrak{B} , $\beta = \omega_2$, then in $\mathfrak{B}[G]$, $2^{\aleph_1} = \aleph_2$, and also, if \mathcal{Y} is a starry collection in a topological space $\langle X, \mathcal{T} \rangle$, such that $|\mathcal{Y}^*|$ and $|X|$ both are $\leq \aleph_1$, and there is a base \mathcal{B} for \mathcal{T} , $|\mathcal{B}| \leq \aleph_1$, then \mathcal{Y} is not normalized.

Once the theorem is proved, 0.11 follows as an immediate corollary. The first sentence of the conclusion of 1.12, as well as the calculation of the cardinality in $\mathfrak{B}[G]$ of 2^{\aleph_1} , are well-known, and could indeed be stated in far greater generality than is necessary for our purposes. See Section 4. The rest of the theorem, as will be seen, follows from 1.8 without too much difficulty.

Proof. Suppose $\beta = \omega_2^\mathfrak{B}$, and G is \mathcal{P}_β -generic over \mathfrak{B} . Since \mathcal{Y}^* , X , \mathcal{B} all have cardinality $\leq \aleph_1$, it can be shown that there is a $\delta \in \omega_2^\mathfrak{B}$, such that $\mathcal{Y}, X, \mathcal{B}$ are all in $\mathfrak{B}_\delta = \mathfrak{B}[G \cap (\delta \times \omega_1 \times 2)]$. It can be verified that $G \cap (\delta \times \omega_1 \times 2)$ is \mathcal{P}_δ -generic over \mathfrak{B} , and hence the first part of 1.12 applies to \mathfrak{B}_δ . It is also known that \mathfrak{B}_δ is a transitive ε -submodel of the transitive ε -model $\mathfrak{B}[G]$.

1.13. LEMMA. If $C, D \in \mathfrak{B}_\delta$, then $\mathfrak{B}_\delta \Vdash C \cap D \neq 0$ if and only if $\mathfrak{B}[G] \Vdash C \cap D \neq 0$.

Proof. Follows from transitivity.

Let $\mathcal{T}_{\mathcal{B}}$ be the topology on X generated by \mathcal{B} in \mathfrak{B}_δ . Using 1.13, we can show that, since \mathcal{Y} is starry in $\langle X, \mathcal{T} \rangle$ in $\mathfrak{B}[G]$, \mathcal{Y} is starry in $\langle X, \mathcal{T} \rangle$ in \mathfrak{B}_δ . For suppose that \mathcal{Y}^* is identified with ω_1 in \mathfrak{B}_δ , and, also in \mathfrak{B}_δ , $\mathcal{V} = \{V_\alpha\}_{\alpha < \omega_1}$ is a cover of \mathcal{Y}^* such that $\alpha \in V_\alpha$, each V_α is a member of $\mathcal{T}_{\mathcal{B}}$, and if $\alpha \in Y_\gamma$, then $V_\alpha \cap (\mathcal{Y}^* - Y_\gamma) = \emptyset$. Then in $\mathfrak{B}[G]$ it is also true that $\alpha \in V_\alpha$, $V_\alpha \in \mathcal{T}_{\mathcal{B}}$, and if $\alpha \in Y_\gamma$, $V_\alpha \cap (\mathcal{Y}^* - Y_\gamma) = \emptyset$. \mathcal{Y} is starry in $\mathfrak{B}[G]$, so

$$\mathfrak{B}[G] \models (\exists \gamma_1)(\exists \alpha_1 \in Y_{\gamma_1})(\forall \beta \in \omega_1)(\exists \gamma_2 > \beta)(\exists \alpha_2 \in Y_{\gamma_2})(V_{\alpha_1} \cap V_{\alpha_2} \neq \emptyset).$$

By transitivity, 1.13, and the fact that \mathfrak{B}_δ has the same ordinals as $\mathfrak{B}[G]$, it follows that

$$\mathfrak{B}_\delta \models (\exists \gamma_1)(\exists \alpha_1 \in Y_{\gamma_1})(\forall \beta \in \omega_1)(\exists \gamma_2 > \beta)(\exists \alpha_2 \in Y_{\gamma_2})(V_{\alpha_1} \cap V_{\alpha_2} \neq \emptyset).$$

Recall some other elementary facts about the adjunction of Cohen subsets of ω_1 . Namely

1.14. LEMMA. Let $Q = \{q: q \subset \omega_1 \times 2, q \text{ is a function, domain } q \subset \omega_1, \text{ range } q \subset 2, |q| \leq \aleph_0\}^{\mathfrak{B}_\delta}$. Given $q, r \in Q$, say $q \leq_\delta r$ if, in \mathfrak{B}_δ , $q \supset r$. Let $\mathcal{Q} = \langle Q, \leq_\delta \rangle$. Then $\mathcal{Q} = \mathcal{P}$.

Let $G_\eta = \{\langle \alpha, \varepsilon \rangle: \langle \eta, \alpha, \varepsilon \rangle \in G\}$. Then if $\delta \leq \eta < \beta$, G_η is \mathcal{P} -generic over \mathfrak{B}_δ .

Using this lemma, we can now apply 1.8 to \mathfrak{B}_δ to get that \mathcal{Y} is not normalized in $\langle X, \mathcal{T}_{\mathcal{B}}(G_\eta) \rangle$ in $\mathfrak{B}_\delta[G_\eta]$. So, for example, \mathcal{Y} is not normalized in $\langle X, \mathcal{T}_{\mathcal{B}}(G_\delta) \rangle$ in $\mathfrak{B}_\delta[G_\delta]$. All that remains to complete the proof of 1.12, is to show that \mathcal{Y} is not normalized in $\langle X, \mathcal{T} \rangle$ in $\mathfrak{B}[G]$, i.e. that the adjunction of more Cohen subsets will not undo the “unnormalizing” of \mathcal{Y} . As was pointed out to me by K. Kunen, this can be proved in a simple manner, using the following

1.15. LEMMA (Kunen). Suppose \mathfrak{U} is a countable transitive ε -model of ZFC, and \mathcal{R} is a partial order in \mathfrak{U} , such that any countable descending sequence in \mathfrak{U} of conditions in \mathcal{R} has a lower bound in \mathcal{R} . In \mathfrak{U} , let $\langle X, \mathcal{T} \rangle$ be a topological space, and \mathcal{Y} a non-normalized collection of disjoint sets in $\langle X, \mathcal{T} \rangle$, $|\mathcal{Y}^*| \leq \aleph_1$. Let H be \mathcal{R} -generic over \mathfrak{U} . Then in $\mathfrak{U}[H]$, \mathcal{Y} is not normalized in $\langle X, \mathcal{T}(H) \rangle$.

Proof. In \mathfrak{U} , let \mathcal{Z} be a subcollection of \mathcal{Y} , such that \mathcal{Z}^* , $(\mathcal{Y} - \mathcal{Z})^*$ cannot be separated by disjoint open sets. Identify \mathcal{Y}^* with ω_1 as usual. Also, as in the proof of 1.8, well-order $\mathcal{T}_\alpha = \{V \in \mathcal{T}: \alpha \in V\}$ as $\mathcal{T}_\alpha = \{V_{\alpha, \beta}\}_{\beta < \Omega_\alpha}$. Since \mathcal{T} is a basis for $\mathcal{T}(H)$, if there exist disjoint open sets about \mathcal{Z}^* , $(\mathcal{Y} - \mathcal{Z})^*$ in $\langle X, \mathcal{T}(H) \rangle$ in $\mathfrak{U}[H]$, then there is a function $f: \omega_1 \rightarrow \Omega_\alpha$ such that $\bigcup_{\alpha \in \mathcal{Z}^*} V_{\alpha, f(\alpha)}$, $\bigcup_{\alpha \in (\mathcal{Y} - \mathcal{Z})^*} V_{\alpha, f(\alpha)}$ are disjoint open sets about \mathcal{Z}^* , $(\mathcal{Y} - \mathcal{Z})^*$ respectively. But if τ is a term in the forcing

language, such that for some $q \in H$, $q \Vdash (\tau \text{ is a function from } \omega_1, (\forall \alpha)(\tau(\alpha) \in \Omega_\alpha))$, then as in the proof of 1.8, get a descending sequence of conditions $\{p_\alpha\}_{\alpha < \omega_1}$ below an arbitrary $r \leq q$, such that for some $\mu_\alpha \in \Omega_\alpha$, $p_\alpha \Vdash \tau(\alpha) = \mu_\alpha$. Then the function g , defined by $g(\alpha) = \mu_\alpha$, is in \mathfrak{A} . Therefore there is an $\alpha_1 \in \mathcal{E}^*$, and an $\alpha_2 \in (\mathcal{Y} - \mathcal{E})^*$, such that $V_{\alpha_1, \mu_{\alpha_1}} \cap V_{\alpha_2, \mu_{\alpha_2}} \neq 0$. But then

$$p_{\max(\alpha_1, \alpha_2)} \Vdash V_{\alpha_1, \tau(\alpha_1)} \cap V_{\alpha_2, \tau(\alpha_2)} \neq 0.$$

We have shown that

$$\{p: p \Vdash V_{\alpha_1, \tau(\alpha_1)} \cap V_{\alpha_2, \tau(\alpha_2)} \neq 0\}$$

is dense below q . It follows that there can be no such f .

To complete the proof of 1.12, let

$$\mathcal{P}^{\delta+1} = \{p: p \subset (\omega_2 - (\delta+1)) \times \omega_1 \times 2, p \text{ is a function, domain } p \subset (\omega_2 - (\delta+1)) \times \omega_1, \text{ range } p \subset 2, |p| \leq \aleph_0\}^{\mathfrak{B}_\delta[G_\delta]},$$

ordered by containment in $\mathfrak{B}_\delta[G_\delta]$.

Then standard arguments establish

1.16. LEMMA. (a) *In $\mathfrak{B}_\delta[G_\delta]$, every descending countable sequence in $\mathcal{P}^{\delta+1}$ has a lower bound.*

(b) $G \cap ((\omega_2 - (\delta+1)) \times \omega_1 \times 2)^{\mathfrak{B}}$ is $\mathcal{P}^{\delta+1}$ -generic over $\mathfrak{B}_\delta[G_\delta]$.

(c) $\mathfrak{B}[G] = (\mathfrak{B}_\delta[G_\delta]) (G \cap ((\omega_1 - (\delta+1)) \times \omega_1 \times 2)^{\mathfrak{B}})$.

Recapitulating, we start with \mathcal{Y} starry in $\langle X, \mathcal{T} \rangle$ in $\mathfrak{B}[G]$. \mathcal{T} has a base \mathcal{B} , and $X, \mathcal{Y}, \mathcal{B}$ are all in \mathfrak{B}_δ . We then conclude first that \mathcal{Y} is starry in the topology $\mathcal{T}_{\mathcal{B}}$ on X generated by \mathcal{B} in \mathfrak{B}_δ , and then that \mathcal{Y} is not normalized in the topology $\mathcal{T}_{\mathcal{B}}(G_\delta)$ on X in $\mathfrak{B}_\delta[G_\delta]$. By 1.15, it follows that \mathcal{Y} is not normalized in the topology

$$(\mathcal{T}_{\mathcal{B}}(G_\delta)) (G \cap ((\omega_2 - (\delta+1)) \times \omega_1 \times 2)^{\mathfrak{B}})$$

on X in $(\mathfrak{B}_\delta[G_\delta]) (G \cap ((\omega_2 - (\delta+1)) \times \omega_1 \times 2)^{\mathfrak{B}})$. But by 1.16 (c), this is just the topology \mathcal{T} on X in $\mathfrak{B}[G]$. This completes the consistency proof.

2. Topological consequences. Our first theorem enables us to replace the cardinality restrictions on X and \mathcal{B} in 0.11 by a restriction merely on the character of \mathcal{Y}^* .

2.1. THEOREM. *Let $\langle X, \mathcal{T} \rangle$ be a topological space, $\mathcal{Y} = \{Y_\gamma\}_{\gamma < \omega_1}$ a relatively discrete collection of subsets of X , such that $|\mathcal{Y}^*| \leq \aleph_1$, and $\chi(\mathcal{Y}^*) = \kappa$. Then there is a space $\langle X', \mathcal{T}' \rangle$, a relatively discrete collection $\mathcal{Y}' = \{Y'_\gamma\}_{\gamma < \omega_1}$, $|\mathcal{Y}'^*| \leq \aleph_1$, and a basis \mathcal{C} for \mathcal{T}' , $|X'|$ and $|\mathcal{C}| \leq \max(\kappa, \aleph_1)$, such that \mathcal{Y}' is normalized, separated, or starry in $\langle X', \mathcal{T}' \rangle$ if and only if \mathcal{Y} is respectively normalized, separated, or starry in $\langle X, \mathcal{T} \rangle$.*

2.2. COROLLARY. CONCH *Suppose $\langle X, \mathcal{T} \rangle$ is a topological space, and \mathcal{Y} is a starry collection in X such that $|\mathcal{Y}^*| \leq \aleph_1$, and $\chi(\mathcal{Y}^*) < 2^{\aleph_1}$. Then \mathcal{Y} is not normalized.*

Proof. The corollary follows immediately from the theorem plus 0.11.

Identify \mathscr{Y}^* with ω_1 . For each $\alpha \in \omega_1$, let $\mathscr{B}^\alpha = \{B_\beta^\alpha\}_{\beta < \aleph}$ be a base at α in $\langle X, \mathscr{T} \rangle$. Without loss of generality, assume \mathscr{B}^α is closed under finite intersection, and — by relative discreteness — that if $\alpha \in Y_\gamma$, then $(\mathscr{B}^\alpha)^* \cap \langle \mathscr{Y}^* - Y_\gamma \rangle = 0$. Let

$$\mathscr{D} = \{B_\beta^\alpha \cap B_{\beta'}^{\alpha'} : B_\beta^\alpha \cap B_{\beta'}^{\alpha'} \neq 0, \text{ and } \alpha, \alpha' \text{ are not in the same } Y_\gamma\}.$$

Note that ω_1 is disjoint from \mathscr{D} . Let $X' = \omega_1 \cup \mathscr{D}$, with the following basis \mathscr{C} for a topology \mathscr{T}' :

- (a) if $D \in \mathscr{D}$, then $\{D\} \in \mathscr{C}$,
- (b) if $\alpha \in \omega_1$, $\beta \in \aleph$, then $C_\beta^\alpha \in \mathscr{C}$, where

$$C_\beta^\alpha = \{\alpha\} \cup \{D \in \mathscr{D} : D \subset B_\beta^\alpha\}.$$

Let $\mathscr{Y}' = \mathscr{Y}$, considered as a collection of subsets of ω_1 in X' .

To verify that \mathscr{C} is a basis, it suffices to show that if $p \in X'$, $C_1, C_2 \in \mathscr{C}$, $p \in C_1 \cap C_2$, then there is a $C_3 \in \mathscr{C}$, $p \in C_3 \subset C_1 \cap C_2$. If $p \in \mathscr{D}$, simply take $C_3 = \{p\}$. If $p = \alpha \in \omega_1$, then $C_1 = C_{\beta_1}^\alpha$, $C_2 = C_{\beta_2}^\alpha$, for some $\beta_1, \beta_2 \in \aleph$. $C_{\beta_1}^\alpha \cap C_{\beta_2}^\alpha = \{\alpha\} \cup \{D \in \mathscr{D} : D \subset B_{\beta_1}^\alpha \cap B_{\beta_2}^\alpha\}$. But by assumption, there is a $B_{\beta_3}^\alpha = B_{\beta_1}^\alpha \cap B_{\beta_2}^\alpha$, $C_{\beta_3}^\alpha = C_{\beta_1}^\alpha \cap C_{\beta_2}^\alpha$.

Clearly, \mathscr{Y}' is relatively discrete (indeed discrete!) in $\langle X', \mathscr{T}' \rangle$. It is also obvious that $|X'|$ and $|\mathscr{C}|$ are as required.

The crucial point of this construction is that if $\alpha_1 \in Y_{\gamma_1}$, $\alpha_2 \in Y_{\gamma_2}$, $\gamma_1 \neq \gamma_2$, then for any β_1, β_2 ,

$$C_{\beta_1}^{\alpha_1} \cap C_{\beta_2}^{\alpha_2} \neq 0 \quad \text{if and only if} \quad B_{\beta_1}^{\alpha_1} \cap B_{\beta_2}^{\alpha_2} \neq 0.$$

For both are equivalent to the existence of a $D \in \mathscr{D}$, $D \subset B_{\beta_1}^{\alpha_1} \cap B_{\beta_2}^{\alpha_2}$.

It is clear that the separation properties of \mathscr{Y} and \mathscr{Y}' depend only on those open sets that are unions of B_β^α 's or C_β^α 's respectively. The natural correspondence between B_β^α and C_β^α extends in a natural way to establish a correspondence between those open sets, which "preserves disjointness" wherever it matters. From these remarks, the reader may finish the proof of the theorem.

The plan of proof of the various parts of 0.12 — and their stronger versions to be obtained by appropriate replacement of "discrete collection in normal space" by "normalized collection" — is typically as follows: We consider a certain class of collections of disjoint sets in a certain class of spaces, for example, collections of disjoint lindelof sets in regular spaces in which every closed set is a G_δ . We show that whenever such a collection \mathscr{Y} in such a space has a \mathscr{Y}^* -locally countable cover, each element of which intersects at most one element of \mathscr{Y} , then \mathscr{Y} is separated. If we assume that \mathscr{Y} is normalized, and that $|\mathscr{Y}^*| \leq \aleph_1$, $\chi(\mathscr{Y}^*) < 2^{\aleph_1}$, then it is consistent that \mathscr{Y} be separated. For if \mathscr{Y} is normalized, it is relatively discrete by 0.3

and countably separated by 1.2.12. By 2.2 it is consistent that \mathcal{Y} not be starry, i.e. by 0.6 have such a \mathcal{Y}^* -locally countable cover.

Typically, the step from " \mathcal{Y} is countably separated and has a \mathcal{Y}^* -locally countable cover, each element of which intersects at most one element of \mathcal{Y} ", to " \mathcal{Y} is separated", is accomplished by showing that the cover is actually star-countable, or at least has a star-countable refinement covering \mathcal{Y}^* . By 1.2.18, it follows that \mathcal{Y} is separated. One could also, in theory, try to show that such a \mathcal{Y}^* -locally countable cover has a closure-preserving refinement, and then use a version of 1.2.15, but the author has been unable to find any applications of this idea. A first thought might be to assume that the space is \aleph_1 -additive, so that locally countable covers would automatically be closure-preserving (see the proof of 1.1.27), but by 1.2.12, any normalized collection of power \aleph_1 in any \aleph_1 -additive space is separated, so this approach is unpromising.

Look back at the proof of 1.2.20, and observe that what is proved is that if \mathcal{Y} is a countably separated collection of lindelof sets, and there is a \mathcal{Y}^* -locally countable cover of \mathcal{Y}^* , such that each element of the cover intersects at most one member of \mathcal{Y} , then \mathcal{Y} is separated. It follows that

2.3. THEOREM. CONCH *Suppose \mathcal{Y} is a normalized collection of lindelof sets, such that $|\mathcal{Y}^*| \leq \aleph_1$, and $\chi(\mathcal{Y}^*) < 2^{\aleph_1}$. Then \mathcal{Y} is separated.*

2.4. COROLLARY. 0.12(a).

Isbell has shown

2.5. LEMMA. *The cardinality of a regular, hereditarily lindelof space does not exceed 2^{\aleph_0} .*

Since we may assume the continuum hypothesis, it follows that

2.6. COROLLARY. CONCH *Suppose X is a regular space, and \mathcal{Y} is a normalized collection of no more than \aleph_1 hereditarily lindelof subsets of X , such that $\chi(\mathcal{Y}^*) < 2^{\aleph_1}$. Then \mathcal{Y} is separated.*

2.7. COROLLARY. 0.12(b).

The following is well known:

2.8. LEMMA. *Let X be a lindelof space in which every closed set is a G_δ . Then X is hereditarily lindelof.*

2.9. COROLLARY. CONCH *Suppose X is a regular space in which every closed set is a G_δ , and \mathcal{Y} is a normalized collection of no more than \aleph_1 lindelof subsets of X , such that $\chi(\mathcal{Y}^*) < 2^{\aleph_1}$. Then \mathcal{Y} is separated.*

2.10. COROLLARY. 0.12(c).

Proof. To prove 2.9, and then 2.10, one need only observe that the property of having every closed set be a G_δ is hereditary.

The next theorem follows from 2.3 in a less trivial fashion.

2.11. THEOREM. CONCH *Suppose \mathcal{Y} is a normalized collection of*

no more than \aleph_1 subsets of X , such that each member of \mathcal{U} has set character $< 2^{\aleph_1}$. Then \mathcal{U} is separated.

2.12. COROLLARY. 0.12(d).

Proof. The idea of the proof is simple, but there are some details to verify. The trick is to identify each member of \mathcal{U} to a point, and then check that in the resulting quotient space, the projection of \mathcal{U} has the same separation properties as \mathcal{U} had in the original space, and that the character of $Y \in \mathcal{U}$ in the quotient space is equal to the set character of Y in the original space. Then, by 2.3, the proof is complete, once one observes that the character of the union of \aleph_1 points, each of character $< 2^{\aleph_1}$, is itself $< 2^{\aleph_1}$.

If the reader is not familiar with quotient spaces, he may refer to the section in Engelking on the subject. Briefly, if X is a topological space, and \mathcal{U} is a disjoint collection of subsets of X , then the identification of each member of \mathcal{U} to a point, while leaving $X - \mathcal{U}^*$ fixed, is clearly an equivalence relation R on X . Let $f(x) = x/R$ be the natural map. The quotient topology is defined to consist of those and only those sets $U \in X/R$ for which $f^{-1}(U)$ is open in X . From this definition and the properties of inverse images, the reader can easily verify that \mathcal{U} is normalized or separated in X , if and only if \mathcal{U}/R is respectively normalized or separated in X/R .

Further, the reader should check that \mathcal{U} is relatively discrete in X if and only if \mathcal{U}/R is relatively discrete in X/R . It is then easy to see that if \mathcal{U} is relatively discrete in X/R , and $Y \in \mathcal{U}$ then $\chi[Y] = \chi(Y/R)$. Since \mathcal{U} normalized implies \mathcal{U} relatively discrete, the theorem is proved.

Our next

2.13. THEOREM. 0.12(e)

has no convenient normalized variant. Recall the definition of F_σ -screenability from 1.0.6. As will be seen in the proof, F_σ -screenability, in conjunction with a local property, enables one to go from a discrete, unseparated collection to a "smaller" or "nicer" discrete, unseparated collection.

Proof. Let X be a normal, regular, locally lindelof, F_σ -screenable space, such that $\chi(X) < 2^{\aleph_1}$, and every closed set is a G_δ . Let \mathcal{U} be a discrete collection in X , $|\mathcal{U}^*| \leq \aleph_1$. Suppose \mathcal{U} is unseparated. By regularity plus discreteness, for each $y \in \mathcal{U}^*$, there is an open set V_y , such that if $y \in Y \in \mathcal{U}$, then $\overline{V_y}$ intersects no other member of \mathcal{U} . Further, by local lindelofness, $\overline{V_y}$ may be assumed to be a lindelof set. Let $\mathcal{F} = \bigcup_{i < \omega} \mathcal{F}_i$ be a σ -discrete closed refinement of the cover $\{V_y\}_{y \in \mathcal{U}^*} \cup \{X - \overline{\mathcal{U}^*}\}$. Let $\mathcal{H}_i = \{F \in \mathcal{F}_i: \text{for some } y \in \mathcal{U}^*, F \subset V_y\}$. Then each \mathcal{H}_i is a discrete collection of closed sets. Claim some \mathcal{H}_i is not separated. For suppose

each \mathcal{H}_i is separated. Then by 1.2.3, each \mathcal{H}_i may be assumed to have a discrete separation. But then by 1.2.15, \mathcal{Y} is separated. So let \mathcal{H}_i be unseparated. Each $F \in \mathcal{H}_i$ is contained in some \overline{V}_y , and hence, as a closed subset of a lindelof set, is lindelof. Since every closed subset of X is a G_δ , each \overline{V}_y is hereditarily lindelof, and so by 2.5 plus the continuum hypothesis, has cardinality $\leq \aleph_1$. $|\mathcal{Y}^*|$ was assumed $\leq \aleph_1$, so $|\bigcup_{y \in \mathcal{Y}^*} \overline{V}_y| \leq \aleph_1$. Therefore \mathcal{H}_i is a discrete, unseparated collection of lindelof sets, $|\mathcal{H}_i^*| \leq \aleph_1$, contradicting 2.4 = 0.12(a).

Next, we turn to separability.

2.14. THEOREM. CONCH *Suppose \mathcal{Y} is a normalized collection, $|\mathcal{Y}^*| \leq \aleph_1$, $\chi(\mathcal{Y}^*) < 2^{\aleph_1}$, such that each member of \mathcal{Y} is contained in a separable open set. Then \mathcal{Y} is separated.*

2.15. COROLLARY. 0.12(f).

Proof. By 2.2 \mathcal{Y} is not starry, so it follows that there is a \mathcal{Y}^* -locally countable collection of open sets, $\mathcal{W} = \{W_\gamma\}_{\gamma < \omega_1}$, such that $W_\gamma \supset Y_\gamma$, W_γ intersects no other member of \mathcal{Y} . For each γ , let S_γ be a separable open set containing Y_γ . Since \mathcal{W} is \mathcal{Y}^* -locally countable, we can choose for each $y \in \mathcal{Y}^*$, say $y \in Y_\gamma$, an open set V_y , $y \in V_y \subset W_\gamma \cap S_\gamma$, such that V_y intersects at most countably many members of \mathcal{W} . Let $V_\gamma = \bigcup_{y \in Y_\gamma} V_y$. Then it suffices to show that the collection $\{V_\gamma\}_{\gamma < \omega_1}$ is star-countable, whence by 1.2.18, \mathcal{Y} is separated. $V_\gamma \subset S_\gamma$, and so by 1.3.3 is separable. By 1.3.22, it follows that there is a p in some V_γ , such that p is in uncountably many V_γ 's, if $\{V_\gamma\}_{\gamma < \omega_1}$ is not star-countable. But $p \in V_y$ for some y , so V_y intersects uncountably many V_γ 's, and hence uncountably many W_γ 's, a contradiction.

2.16. THEOREM. CONCH *Suppose \mathcal{Y} is a normalized collection of no more than \aleph_1 subsets of the first countable Hausdorff space X . Then if each member of \mathcal{Y} is contained in a separable open set, \mathcal{Y} is separated.*

2.17. COROLLARY. 0.12(g).

The theorem and corollary follow immediately from 2.14 and the continuum hypothesis, in conjunction with

2.18. LEMMA. *The cardinality of a separable, first countable, Hausdorff space does not exceed 2^{\aleph_0} .*

Proof. Let D be a countable dense subset of the first countable Hausdorff space X . By first countability, for each $x \in X$, there is a sequence, i.e. a function from ω into D , converging to x . By Hausdorff, no two sequences converge to the same point. Thus $|X| \leq |{}^\omega D| = 2^{\aleph_0}$.

2.19. THEOREM. CONCH *Suppose \mathcal{Y} is a normalized collection of no more than \aleph_1 lindelof subsets of the locally separable, first countable, Hausdorff space X . Then \mathcal{Y} is separated.*

2.20. COROLLARY. 0.12(h).

The theorem and corollary follow immediately from 2.16 as soon as we establish

2.21. LEMMA. *If X is locally separable, and $Z \subset X$ is a lindelof set, then there is a separable open set containing Z .*

Proof. Left to reader.

The proof of the next theorem employs the method used in the proof of 2.13 = 0.12(e).

2.22. THEOREM. 0.12(i).

Proof. Let X be a normal, Hausdorff, locally separable, first countable, F_σ -screenable space. Let \mathcal{Y} be a discrete collection in X , $|\mathcal{Y}^*| \leq \aleph_1$. As in the proof of 2.13, if \mathcal{Y} is not separated, then there exists a discrete unseparated collection \mathcal{H}_i , $|\mathcal{H}_i| = \aleph_1$, each member of \mathcal{H}_i contained in a separable open set. But this violates 2.17 = 0.12(g).

Local separability is so strong, that we need not list a character restriction, as will be seen.

2.23. LEMMA (de Groot). *Every separable regular space has a basis of cardinality $\leq 2^{\aleph_0}$.*

2.24. THEOREM. *Every locally separable regular space has character $\leq 2^{\aleph_0}$.*

Proof. It suffices to show each point has character not exceeding 2^{\aleph_0} . But this follows from the lemma.

2.25. COROLLARY. CONCH *Suppose X is a locally separable regular space, and \mathcal{Y} is a normalized collection of no more than \aleph_1 lindelof sets, $|\mathcal{Y}^*| \leq \aleph_1$. Then \mathcal{Y} is separated.*

2.26. COROLLARY. 0.13.

2.27. COROLLARY. CONCH *Suppose X is a locally separable regular space, in which every closed set is a G_δ , and \mathcal{Y} is a normalized collection of no more than \aleph_1 lindelof sets. Then \mathcal{Y} is separated.*

Proofs. All follow from 2.24 in conjunction with the appropriate earlier theorems.

2.28. THEOREM. CONCH *No separable regular space contains an uncountable normalized collection.*

Proof. Let X be a separable regular space, \mathcal{Y} an uncountable normalized collection in X . Since a subcollection of a normalized collection is normalized, assume without loss of generality that $\mathcal{Y} = \{Y_\gamma\}_{\gamma < \omega_1}$. Pick $y_\gamma \in Y_\gamma$, each $\gamma < \omega_1$. Then the collection of points, $\{y_\gamma\}_{\gamma < \omega_1}$, is normalized. By 2.25, it is separated. But the existence of an uncountable collection of mutually disjoint open sets contradicts separability.

2.28 should be compared with Jones' result that

2.29. THEOREM. $2^{\aleph_0} < 2^{\aleph_1}$ implies every separable normal space is \aleph_1 -compact.

Now we shall quickly dispose of the applications of this section to the metrization of normal Moore spaces. Moore spaces are obviously first countable. We will not need the following two classical results:

2.30. THEOREM (Moore). Every closed subset of a Moore space is a G_δ .

2.31. THEOREM (Bing [5]). Every Moore space is F_σ -screenable.

To prove

2.32. THEOREM. 0.14,

observe that by 0.12(a), the space in question will be \aleph_1 -compact. But a normal \aleph_1 -compact space is collectionwise normal by 1.2.13, and by 0.0.6 a collectionwise normal Moore space is metrizable. An extension of this argument establishes

2.33. THEOREM. 0.15.

By what we have just remarked, the space is locally metrizable, hence locally separable, since c.c.c., separability, and for that matter, lindelofness, \aleph_1 -compactness, and the possession of a countable basis are all equivalent in metric spaces. By the cardinality restriction, it follows from 0.13 that the space is collectionwise normal, and hence metrizable.

3. Bing's example. Bing's [5] example of a normal space containing a discrete collection of \aleph_1 points which is not separated, was, until Fleissner's, the only known "genuine" example of a normal, non-collectionwise normal space. In certain models of set theory, other examples exist. See Chapter III, Charlesworth et al. [8], Fleissner [18].

Engelking has an excellent discussion of Bing's example in his book; the reader is referred there, or to Bing's paper, for a proof that the space is normal and Hausdorff, but not collectionwise normal. Here, we confine ourselves to establishing that the space has character $2^{2^{\aleph_0}}$, or if the continuum hypothesis is assumed, 2^{\aleph_1} .

Following Engelking, let R be the set of real numbers, \mathcal{S} the collection of all subsets of R . For each $S \in \mathcal{S}$, let $D_S = \{0, 1\}$ with the discrete topology, and let $Y = \prod_{S \in \mathcal{S}} D_S$ with the usual product topology.

For every $r \in R$, let the function $f_r \in Y$ be defined by

$$f_r(S) = \begin{cases} 0 & \text{if } r \notin S, \\ 1 & \text{if } r \in S. \end{cases}$$

Let $M = \{f_r\}_{r \in R}$. The intersection of the neighborhood $U_{f_r} = p_{\{r\}}^{-1}(1)$ of the point f_r with the set M^* — where p_S is the projection function,

onto the S th coordinate — consists only of the point f_r , hence M is a relatively discrete collection.

Put a stronger topology on the set Y to get the space $X = \mathbf{Y}_M$, by defining all sets of the form $U \cup K$, U open in the original space Y , $K \subset Y - M^*$, to be open. Then it can be verified that X is normal and Hausdorff, M is discrete in X , and that no uncountable subcollection of M can be separated in X . It suffices to show that each point $f_r \in M^*$ has character $2^{2^{\aleph_0}}$ in X . Then either X — or if $\aleph_1 < 2^{\aleph_0}$, the subspace obtained by “throwing out some points of M ” — will be the desired normal space of character $2^{2^{\aleph_0}}$ containing a discrete collection of \aleph_1 points that is not separated. In the model of set theory considered in the earlier part of this chapter, $2^{\aleph_0} = \aleph_1$; thus the existence of the space X shows that the character restrictions in the consistency theorems cannot be improved.

To show that each f_r has character $2^{2^{\aleph_0}}$ in X , we shall show this to be the case in Y , by a method found in Engelking ([15], p. 79). Assume this has been done; let us see how this implies that $\chi(f_r) = 2^{2^{\aleph_0}}$ in X . The point is that if V is an open set in X , $V = U \cup K$, U open in Y , $K \subset Y - M^*$, then $f_r \in V$ if and only if $f_r \in U$. It follows that the character of f_r is the same in both spaces.

Now to establish that $\chi(f_r) = 2^{2^{\aleph_0}}$ in Y . From the definition of the product topology, it follows that there is a base for Y of power $2^{2^{\aleph_0}}$, namely the collection \mathcal{B} of all sets $\prod_{S \in \mathcal{S}} W_S$, where $W_S \subset D_S$, and $W_S \neq D_S$ only for S belonging to a finite subset $\mathcal{S}' \subset \mathcal{S}$. A fortiori, the character of Y does not exceed $2^{2^{\aleph_0}}$. Call the elements of the set \mathcal{S}' the *distinguished indices* of the given element of the base \mathcal{B} . Let y be any point in Y , for example an f_r . Suppose $\chi(y) < 2^{2^{\aleph_0}}$. Let $\mathcal{B}(y)$ be a base at y such that $\mathcal{B}(y) \subset \mathcal{B}$, and $|\mathcal{B}(y)| = \kappa < 2^{2^{\aleph_0}}$. Then the cardinality of the set of distinguished indices of all elements of $\mathcal{B}(y)$ does not exceed κ . Therefore there exists an $S_0 \in \mathcal{S}$ which is not distinguished for any element of $\mathcal{B}(y)$. Suppose y , as a function from \mathcal{S} into $\{0, 1\}$ has value ε at S_0 . Consider the open set $\prod_{S \in \mathcal{S}} W_S$, where $W_S = D_S$, $S \neq S_0$, and $W_S = \{\varepsilon\}$, $S = S_0$. $\prod_{S \in \mathcal{S}} W_S$ is an open neighborhood of y , but $\mathcal{B}(y)$ does not distinguish S_0 , so no element of $\mathcal{B}(y)$ is contained in $\prod_{S \in \mathcal{S}} W_S$, a contradiction.

W. Fleissner has pointed out to me that by starting with a set of \aleph_1 reals, the variant of Bing's example obtained has character 2^{\aleph_1} regardless of whether the continuum hypothesis holds.

4. To be continued ... In this section, some of the obstacles toward improving the results of Sections 1 and 2 are outlined, and areas where progress can be made are briefly discussed.

For the reader unfamiliar with Easton's Theorem, it should be mentioned that our results are consistent with any values of 2^κ , $\kappa \geq \aleph_1$, that do not violate a suitably generalized König's Lemma. So in particular, our results are consistent with the generalized continuum hypothesis (GCH), viz. for every infinite cardinal κ , $2^\kappa = \kappa^+$. We cannot violate the continuum hypothesis by starting out with a model \mathfrak{M} in which $2^{\aleph_0} > \aleph_1$, since the adjunction of Cohen subsets of ω_1 "collapses" 2^{\aleph_0} down to \aleph_1 . However, if one starts out with a model \mathfrak{M} , adjoins say \aleph_2 Cohen subsets of ω_1 , and then adjoins \aleph_2 Cohen subsets of ω , our results may perhaps still hold, although 2^{\aleph_0} would be greater than \aleph_1 . If sufficiently many subsets of ω are adjoined so that $2^{\aleph_0} = 2^{\aleph_1}$, then by Heath [22] there is a separable normal Hausdorff space (by 2.24 having character $\leq 2^{\aleph_0}$) containing a discrete collection of \aleph_1 points which is not separated.

There are a number of possible approaches, both topological and set-theoretic, toward improving Sections 1 and 2. Fleissner's recent example [18] of a normal space of character 2^{\aleph_0} containing an unseparated collection of \aleph_1 closed sets, each of cardinality \aleph_1 , shows that "starry" cannot in general be replaced by "unseparated". However, this may still be possible for Moore spaces. One can certainly add to the list of properties which produce starry collections from unseparated collections. If one could prove that every paralinkelof normal Moore space is metrizable, the same proof would probably establish that if there is an unseparated \mathscr{Y} , $|\mathscr{Y}^*| \leq \aleph_1$ in a normal Moore space, then there is a starry \mathscr{Y}' , $|\mathscr{Y}'^*| \leq \aleph_1$ in the space.

Is the " $\aleph_1 - \aleph_1$ " restriction necessary? That is, is it consistent that any starry collection \mathscr{Y} of \aleph_1 disjoint sets, \mathscr{Y}^* having appropriate character, is not normalized, even though $|\mathscr{Y}^*| = \kappa > \aleph_1$? The author's efforts in this direction have been frustrating. For example, if one tries the obvious expedient of identifying each member of \mathscr{Y} to a point, then the character becomes too high. If one wishes to have a sequence of conditions deciding the values of the function τ on the members of κ , a partial order in which any descending sequence of fewer than κ conditions has a lower bound is needed. If $\kappa > \aleph_1$, such a partial order adds no new subsets of ω_1 , yet it seems clear that a generic subset of ω_1 is required to non-normalize \mathscr{Y} .

There is one direction in which the results of this chapter can be extended. For simplicity, assume in what follows that the ground model \mathfrak{B} satisfies GCH. If the reader will think back on the proofs contained in Sections 1 and 2, he will see that if to \mathfrak{B} are adjoined \aleph_2 Cohen sub-

sets of ω_2 , then any relatively discrete $\mathcal{Y} = \{Y_\gamma\}_{\gamma < \omega_2}$, $|\mathcal{Y}^*| \leq \aleph_2$, $\chi(\mathcal{Y}^*) < 2^{\aleph_2}$, having the property that the canonical cover has no " \mathcal{Y}^* -locally \aleph_1 " refinement, is not normalized. Note also that 1.2.18 can be generalized to prove that if a collection is "star- \aleph_1 " and " \aleph_1 -separated", then it is separated. Thus if in this model we have an \aleph_1 -additive normal space of character $< 2^{\aleph_2}$, then in that space any discrete collection of \aleph_2 points will be separated. A suitable generalization of Bing's example shows the character restriction again is best possible. Better yet, to \mathfrak{B} first add \aleph_2 Cohen subsets of ω_1 , and then add \aleph_3 Cohen subsets of ω_2 , *Cohen over the second model rather than the ground model*. Then in this new model, discrete collections of \aleph_2 points can be separated in *any* normal space of character $< 2^{\aleph_1}$. Simply show the collection is \aleph_1 -separated in the second model, and then use that fact to get it separated in the third model.

There is no reason to stop at \aleph_2 , so try to perform this construction for all cardinals, "one after another". Needless to say, there are a large number of technical problems involved in formalizing this intuitive idea. Fortunately, many of them were already solved by Easton. First of all, the best one can hope to do is to add subsets of every regular cardinal. If one tries to add Cohen subsets of a singular cardinal, cardinals will be collapsed. Easton adjoined Cohen subsets of each regular cardinal, working "from the top down" in the sense that subsets of larger cardinals were adjoined "before" subsets of smaller cardinals. Our intent is to do a "reverse Easton", working "from the bottom up", adjoining κ^+ subsets of each regular cardinal κ . Since we add only κ^+ subsets of κ , the GOH is preserved at each stage, which should be sufficient to insure that cardinals are preserved.

This intuitive idea of course has to be formalized and coded up in the ground model. As the thesis deadline approached, the author had not finished verifying that this model preserved cardinals and had all the other properties it ought to have, so he decided not to discuss it formally here. However the following (where collections of points are used for simplicity) should be true in this model:

If X is a normal space of character $< 2^{\aleph_1}$, \mathcal{Y} is a discrete collection of points in X , and $|\mathcal{Y}^*| = \kappa$, a regular cardinal, then if for each $\lambda < \kappa$, every subcollection of \mathcal{Y} of power λ is separated, it follows that \mathcal{Y} is separated.

It is then evident that if one could conclude in this model that if a collection were λ -separated for all λ less than a singular cardinal σ , then it would be σ -separated, then one could also conclude that in this model every normal space of character $< 2^{\aleph_1}$ would be collectionwise Hausdorff. The former conclusion indeed holds for $\sigma = \aleph_\omega$, using the fact that countable discrete collections are separated in a normal space,

but \aleph_{ω_1} is the sticking point. The author considered a number of possible ways around this difficulty, but none has succeeded. However, recently Fleissner [17] proved that the generalized continuum hypothesis enabled the induction to be carried out at singular cardinals, while Menas worked out the details of the reverse Easton model. Thus one has in this model that every normal space of character $< 2^{\aleph_1}$ is collectionwise Hausdorff, and various generalizations of the sort considered earlier.

This reverse Easton model has the property — which could prove useful in a variety of set-theoretic applications — that for each set of cardinality κ , κ regular, there is a subset of κ that — roughly speaking — is Cohen-generic over the first set.

Kunen raised the possibility that the separation properties holding in the reverse Easton model might hold in the constructible universe as well. This was verified by Fleissner [17]. All of Fleissner's results hold in the reverse Easton model. This can be seen by using our methods at regular cardinals, plus GCH at singular ones. Alternatively, one can verify that Fleissner's combinatorial principles hold in the reverse Easton model. It is likely that all our results hold in the constructible universe, but Fleissner's methods do not obviously yield those that cannot be obtained by collapsing closed sets to points (for example 2.6). However, using Arhangel'skii's result that the cardinality of Lindelöf first countable spaces is $\leq 2^{2^{\aleph_0}}$, Fleissner does obtain a number of results in this manner. Arhangel'skii's theorem may also be used to improve our conclusions. That is left as an exercise for the reader. The fact remains that Fleissner's example [18] shows that no straightforward extension of either his methods or ours can prove the consistency of the normal Moore space conjecture. At this point in time we do not understand how (or if) collectionwise normality is easier to achieve in Moore spaces than in spaces of character 2^{\aleph_0} .

CHAPTER III

0. Prehistory. Texas topology and Polish set theory have a totally unexpected meeting point in the question of whether

0.1. PROPOSITION P_0 . *There exists a separable, normal, non-metrizable Moore space.*

This question has so many disparate topological and set-theoretic formulations that it must truly be fundamental.

Back in the golden days of *Fundamenta Mathematicae*, Hausdorff [21] asked whether

0.2. PROPOSITION P_1 . *Let E be a set of power \aleph_1 . Then there exists a countable family \mathcal{F} of subsets of E , such that every subset of E is of the form $\bigcap_{i < \omega} \bigcup_{i < j < \omega} E_j$, where each $E_j \in \mathcal{F}$.*

Sierpiński [42] considered the following propositions, and then proved them all equivalent to P_1 .

0.3. PROPOSITION P_2 . *Let E be a set of power \aleph_1 . Then there exists a countable family \mathcal{F} of subsets of E , such that every subset of E is of the form $\bigcap_{i < \omega} \bigcup_{i < j < \omega} E_j$, and also of the form $\bigcup_{i < \omega} \bigcap_{i < j < \omega} E_j$, where each $E_j \in \mathcal{F}$.*

0.4. PROPOSITION P_3 . *Let E be a set of power \aleph_1 . Then there exists a countable family \mathcal{F} of subsets of E , such that every subset of E is of the form $\bigcap_{i < \omega} \bigcup_{j < \omega} E_{ij}$, where each $E_{ij} \in \mathcal{F}$.*

0.5. PROPOSITION P_4 . *There is an uncountable set B of real numbers, such that every subset of B is F_σ relative to B .*

0.6. THEOREM (Sierpiński). P_1, P_2, P_3, P_4 are all equivalent.

In 1937, as mentioned in Chapter 0, Jones conjectured that every normal Moore space was metrizable, and proved

0.7. THEOREM (0.0.3). $2^{\aleph_0} < 2^{\aleph_1}$ implies every separable normal Moore space is metrizable.

0.8. DEFINITION. A *denumerable base* for a family \mathcal{F} of functions is a sequence of functions such that every function belonging to \mathcal{F} is a limit of some subsequence of that sequence.

0.9. PROPOSITION. *There is a set of reals of power \aleph_1 such that there is a denumerable base for the family of all real functions defined on it.*

In 1948, Rothberger introduced P_6 and proved

0.10. THEOREM (Rothberger). P_4 is equivalent to P_6 .

Another formulation of P_6 can be found in Tall [48].

0.11. DEFINITION. An uncountable set of reals, every subset of which is relative F_σ is called a Q -set.

0.12. DEFINITION. A partial order $\mathcal{P} = \langle P, \leq \rangle$ is said to contain an ω_1 -limit if there is $\{p_\alpha\}_{\alpha < \omega_1+1} \subset P$, such that $\alpha < \beta$ implies $p_\alpha \leq p_\beta$; no p_α , $\alpha < \omega_1$, is equivalent to p_{ω_1} in the sense that $p_\alpha \leq p_{\omega_1} \leq p_\alpha$; and further, there is no $p \in P$ such that p is not equivalent to p_{ω_1} , yet for each $\alpha < \omega_1$, $p_\alpha \leq p \leq p_{\omega_1}$.

Rothberger considered two more propositions, among others:

0.13. PROPOSITION R_0 . Let the family of all subsets of ω be partially ordered by the relation $C \leq D$ if $C - D$ is finite (read " C is almost contained in D "). Then this partial order has no ω_1 -limits.

0.14. PROPOSITION R_1 . Suppose $\{E_\alpha\}_{\alpha < \omega_1}$ is a collection of subsets of ω , such that the common intersection of any finite number of E_α 's is infinite. Then there is an infinite subset D of ω , such that D is almost contained in each E_α .

0.15. THEOREM (Rothberger). R_0 is equivalent to R_1 .

0.16. THEOREM (Rothberger). R_1 implies every set of reals of power \aleph_1 is a Q -set.

1. History. The "bubble space" or "tangent disk space" is a familiar example in elementary topology. It is obtained by taking the upper right quadrant of the plane, including the non-negative portions of the x and y axes, and making the x axis discrete. More formally, a neighborhood of a point p on the x axis consists of p together with the interior of a disk tangent to the axis at p . Other points in the space receive the usual topology inherited from the plane. It is easy to check that the space is a separable Moore space. It is somewhat harder to verify that the space is not normal. We shall do so later. Variations of this space may be obtained by including only a portion of the x axis in the space.

1.1. DEFINITION. Let B be a set of real numbers. $M(B)$, the Moore space derived from B , is that subspace of the bubble space obtained by removing all reals on the x axis that are not in B .

1.2. THEOREM. For any B , $M(B)$ is a separable Moore space, and B is a discrete collection of points in $M(B)$.

Proof. Left to reader.

Bing [5] announced the remarkable

1.3. THEOREM. If every subset of B is F_σ relative to B , then $M(B)$ is normal.

The proof is an interesting geometric exercise, which we omit.

Since $M(B)$ is separable and B is discrete, it follows that if B is uncountable, it cannot be separated. Hence

1.4. COROLLARY (Bing). P_4 implies P_0 .

We shall later prove the converse of 1.3, namely

1.5. THEOREM. If $M(B)$ is normal, then every subset of B is F_σ relative to B .

By a simple cardinality argument, the reader can then prove that the bubble space is not normal (and hence is not metrizable).

R. W. Heath [22] proved the converse of 1.4:

1.6. THEOREM. P_0 implies P_4 .

The reader can easily prove

1.7. THEOREM. $2^{\aleph_0} < 2^{\aleph_1}$ implies the negation of P_4 .

This yields, in conjunction with 1.6, a proof of 0.7.

Since Gödel in 1939 proved in effect that

1.8. THEOREM. If ZFC is consistent, then so is ZFC plus GCH.

It follows from 0.7 that

1.9. THEOREM (0.0.8). If ZFC is consistent, then so is ZFC plus "every separable normal Moore space is metrizable".

1.10. DEFINITION. Let $\mathcal{P} = \langle P, \leq \rangle$ be a partial order. $p, q \in P$ are compatible if there is an $r \in P$ such that $r \leq p$ and $r \leq q$. \mathcal{P} is said to satisfy the countable chain condition if every collection of mutually incompatible elements of P is countable. (Confusion with the c.c.c. defined earlier should be minimal.) $D \subset P$ is dense if for each $p \in P$, there is a $d \in D$, such that $d \leq p$. $G \subset P$ is a filter if the members of G are all compatible, and if whenever $p \geq q \in G$, then $p \in G$.

1.11. MARTIN'S AXIOM. Let \mathcal{P} be a partial order satisfying the countable chain condition. Let $\{D_\alpha\}_{\alpha < \omega_1}$ be a collection of dense subsets of P . Then there is a filter $G \subset P$, such that for each α , $G \cap D_\alpha \neq \emptyset$.

1.12. DEFINITION. Two sets are said to be almost disjoint if their intersection has finite cardinality.

1.13. PROPOSITION S_0 . Let $\{A_\alpha\}_{\alpha < \omega_1}$ be a collection of mutually almost disjoint subsets of ω . Then for each $C \subset \omega_1$, there is a $c \subset \omega$, such that $C = \{\alpha: |C_\alpha \cap c| < \aleph_0\}$.

Around 1967, Solovay proved

1.14. THEOREM. If ZFC is consistent, then so is ZFC plus Martin's Axiom plus $2^{\aleph_0} = \aleph_2$.

He also established that

1.15. THEOREM. Martin's Axiom implies S_0 .

From similar considerations, a number of people noticed that

1.16. THEOREM. *Martin's Axiom implies R_1 .*

For details, see the paper by Solovay and Martin.

It is a well-known classical result that

1.17. THEOREM. *There exists a family of 2^{\aleph_0} mutually almost disjoint subsets of ω .*

A proof can be found in Monk's book, among other places.

1.18. PROPOSITION P_6 . *There exists a collection $\{A_\alpha\}_{\alpha < \omega_1}$ of mutually almost disjoint subsets of ω , such that for each $C \subset \omega_1$, there is a $c \subset \omega$, such that $C = \{\alpha: |C_\alpha \cap c| < \aleph_0\}$.*

Silver, having been informed of Solovay's work and Bing's 1.4, proved

1.19. THEOREM. P_6 implies P_4 .

Hence

1.20. THEOREM (Silver). *If ZFC is consistent, then so is ZFC plus "there is a separable, normal, non-metrizable Moore space".*

We shall give his proof of 1.19 in the next section along with a proof of its converse. Shortly after Silver's proof was announced, D. Booth "discovered" Rothberger's paper. It was immediately evident that 1.20 followed from 1.14, 1.16, 0.16, and 1.4. Indeed a stronger result is true:

1.21. THEOREM. *If ZFC is consistent, then so is ZFC plus $2^{\aleph_0} = \aleph_2$ plus "if B is a set of reals of power less than 2^{\aleph_0} , then every subset of B is relative F_σ ".*

Another way of stating it:

1.22. THEOREM. *If ZFC is consistent, then so is ZFC plus $2^{\aleph_0} = \aleph_2$ plus "every Moore space derived from a set B of power less than 2^{\aleph_0} is separable, normal, and if B is uncountable, non-metrizable".*

Proofs. One need only put together the previous theorems with the trivial fact that every subset of a countable set of reals is a relative F_σ .

We noted earlier (1.7) that $2^{\aleph_0} < 2^{\aleph_1}$ implies there are no Q -sets. It follows from a theorem of Bukovsky that this implication cannot be reversed.

1.23. THEOREM (Bukovsky). *If ZFC is consistent, then so is ZFC plus $2^{\aleph_0} = 2^{\aleph_1}$ plus "in every separable uncountable metric space, there is a set which is not a Borel set".*

Since a Q -set with the topology inherited from the real line is a separable uncountable metric space, it follows that

1.24. THEOREM. *If ZFC is consistent, then so is ZFC plus $2^{\aleph_0} = 2^{\aleph_1}$ plus "there are no Q -sets" (equivalently, "every separable normal Moore space is metrizable").*

It is worth mentioning that a weaker version of the problem considered by Bukovsky is equivalent to the P_i 's:

1.25. PROPOSITION P_7 . *There is a separable uncountable metric space in which every set is F_σ .*

The proof of the equivalence is postponed until the next section.

1.26. PROPOSITION P_8 . *There is a separable, normal, first countable Hausdorff space which is not \aleph_1 -compact.*

A glance at Heath's proof of 1.6 will establish that he has actually proved a stronger result:

1.27. THEOREM. P_8 implies P_4 .

It is easy to see that a separable, normal, non-metrizable Moore space is not \aleph_1 -compact, so

1.28. THEOREM. P_0 implies P_8 .

Heath [56] proved

1.29. THEOREM. $2^{\aleph_0} = 2^{\aleph_1}$ implies there is a separable, normal, Hausdorff space which is not \aleph_1 -compact (and hence not collectionwise normal).

We may now restate Bukovsky's theorem as

1.30. THEOREM. *If ZFC is consistent, then so is ZFC plus "there is a separable, normal, Hausdorff space which is not \aleph_1 -compact, but every separable, normal, first countable, Hausdorff space is \aleph_1 -compact".*

Perhaps we should state explicitly a lemma we have used implicitly:

1.31. LEMMA. *A separable normal space is \aleph_1 -compact if and only if it is collectionwise normal.*

Proof. Left to reader.

We mentioned in Chapter 0 Heath's [22]

1.32. THEOREM. *If there is a separable, normal, non-metrizable Moore space, then there is a metacompact normal, non-metrizable Moore space.*

Fitzpatrick and Traylor proved

1.33. THEOREM. *If there is a separable, normal, non-metrizable Moore space, then there is a separable, normal, locally compact, non-metrizable Moore space.*

Consistency results may then be obtained from 1.20.

Some topologists have thought there is a connection between the normal Moore space problem and the Souslin problem. For example, Traylor [55] asks whether the existence of a normal non-metrizable Moore space implies the existence of a Souslin space. The answer is no, since Solovay and Tennenbaum proved

1.34. THEOREM. *Martin's Axiom implies there are no Souslin spaces.*

The results of Jensen [25] and of Fleissner [17] on the constructible universe provide evidence against another possible connection between the two problems.

2. Everything is equivalent. In this section we establish those equivalences which have not appeared – or are not about to appear – in print elsewhere, and enlarge the list of P_i 's.

Proof of 1.5. Let $C \subset B$. We wish to show C is F_σ relative to B in the usual topology on the real line. By the normality of $M(B)$, there is an open set U of $M(B)$, such that $C \subset U$, and $\bar{U} \cap (B - C) = \emptyset$. Using the usual ordered pair notation for points in the plane, for each $n < \omega$ let

$$C_n = \{\langle \omega, 0 \rangle : \langle \omega, 0 \rangle \in C \text{ and } \langle \omega, 1/i \rangle \in U \text{ for all } i > n\}.$$

Note that $\{\langle \omega, 1/i \rangle\}_{i < \omega} \rightarrow \langle \omega, 0 \rangle$ in $M(B)$. Therefore $C = \bigcup_{n < \omega} C_n$, since $\{\langle \omega, 1/i \rangle\}_{i < \omega}$ must eventually lie in U . Claim $C = \bigcup_{n < \omega} \bar{C}_n \cap B$, where \bar{C}_n denotes the closure of C_n in the topology on the real line, in other words, C is F_σ relative to B . Certainly $C \subset \bigcup_{n < \omega} \bar{C}_n \cap B$ so let $\langle \omega, 0 \rangle \in B$ be a real line limit point of some C_n . Claim $\langle \omega, 0 \rangle \in C$. For let $\{\langle \omega_j, 0 \rangle\}_{j < \omega} \rightarrow \langle \omega, 0 \rangle$ in the real line topology, $\langle \omega_j, 0 \rangle \in C_n$. Then for each i , $\{\langle \omega_j, 1/i \rangle\}_{j < \omega} \rightarrow \langle \omega, 1/i \rangle$ in the topology of the plane, and hence in $M(B)$, since above the axis these latter two topologies agree. Thus for each $i > n$, $\langle \omega, 1/i \rangle \in \bar{U}$. But $\{\langle \omega, 1/i \rangle\}_{i < \omega} \rightarrow \langle \omega, 0 \rangle$ in $M(B)$. So $\langle \omega, 0 \rangle \in \bar{U}$. Therefore $\langle \omega, 0 \rangle \in C$, since $\bar{U} \cap (B - C) = \emptyset$.

Proof of 1.19 (Silver). Recall that the Cantor set, which is a closed subset of the reals, is homeomorphic to the topological product of countably many copies of the two point space with the discrete topology. This product space can also be considered as the natural topology on the power set of ω , having a clopen subbasis consisting of sets of the form $\{r \subseteq \omega : n \in r\}$, $\{r \subseteq \omega : n \notin r\}$, each $n \in \omega$. Thus the collection $\mathcal{A} = \{A_\alpha\}_{\alpha < \omega_1}$ can be considered as a subset of a closed set of reals. It follows that we need only show that every subset of \mathcal{A} is F_σ relative to \mathcal{A} in this natural topology, in order to obtain a Q -set. So let $\mathcal{C} \subset \mathcal{A}$. Let $C = \{\alpha : A_\alpha \in \mathcal{C}\}$. Then there is a $c \subset \omega$ such that $C = \{\alpha : |A_\alpha \cap c| < \aleph_0\}$. Hence $\mathcal{C} = \{A_\alpha : |A_\alpha \cap c| < \aleph_0\}$. For each finite subset $F \subset \omega$, let $c_F = \{r \subset \omega : r \cap c = F\}$. Then $\mathcal{C} = \mathcal{A} \cap \bigcup_{\text{finite } F \subset \omega} c_F$. So it suffices to show each c_F is closed in the topology we are considering.

$$\begin{aligned} c_F &= \{r \subset \omega : r \cap c = F\} \\ &= \{r : r \cap c \subset F\} \cap \{r : r \cap c \supset F\} \\ &= \{r : \text{for each } n, \text{ if } n \notin F, \text{ then } n \notin r \text{ or } n \notin c\} \cap \\ &\quad \cap \{r : \text{for each } n, \text{ if } n \in F, \text{ then } n \in r \text{ and } n \in c\} \\ &= \bigcap_{n \notin F} (\{r : n \notin r\} \cup \{r : n \notin c\}) \cap \bigcap_{n \in F} (\{r : n \in r\} \cap \{r : n \in c\}). \end{aligned}$$

Each of the bracketed sets is closed, therefore c_F is closed.

2.1. PROPOSITION P_9 . *There exists a collection $\{A_\alpha\}_{\alpha < \omega_1}$ of mutually almost disjoint subsets of ω , such that for each $C \subset \omega_1$, there is a $c \subset \omega$, such that*

$$C = \{\alpha: |A_\alpha \cap c| < \aleph_0\} \quad \text{and} \quad \omega_1 - C = \{\alpha: |A_\alpha - c| < \aleph_0\}.$$

We shall prove the converse of 1.19 by showing that this apparently stronger version of P_6 is actually equivalent to P_8 . Since Bing proved P_4 implies P_0 , and a fortiori P_8 , and P_9 trivially implies P_6 , this will suffice.

2.2. THEOREM. P_8 is equivalent to P_9 .

Proof that P_8 implies P_9 . Let X be a first countable, separable, normal Hausdorff space containing an uncountable set Y with no limit point. Without loss of generality, we may assume there is a countable set D dense in X and disjoint from Y . Since X is first countable, for each $y \in Y$, there is a sequence D_y of points in D which converges to y . Since X is Hausdorff, if $y \neq y'$, then D_y and $D_{y'}$ eventually differ, indeed their ranges are almost disjoint. Identify Y with ω_1 , D with ω . (To be absolutely precise, one should identify D with say $\omega \times \{0\}$, so that it will be disjoint from ω_1 .) Then we have an uncountable family $\{A_\alpha\}_{\alpha < \omega_1}$ of mutually almost disjoint subsets of ω . Let $C \subset \omega_1$. Then by normality, there is an open set $U \supset \omega_1 - C$, $\bar{U} \cap C = \emptyset$. Let $c = \bar{U} \cap \omega$. Claim c is the desired subset of ω . For if $\alpha \in \omega_1 - C$, then since U is an open set about α , any sequence approaching α must have all but a finite number of terms inside U . It follows that $A_\alpha - c$ is finite, and $A_\alpha \cap c$ is infinite. On the other hand, if $\alpha \in C$, $A_\alpha \cap \bar{U}$ must be finite, because every infinite subset of A_α has α as its (only) limit point, and $C \cap \bar{U} = \emptyset$. So $\alpha \in C$ implies $A_\alpha \cap c$ is finite, and hence $A_\alpha - c$ is infinite. Therefore, as claimed, $C = \{\alpha: |A_\alpha \cap c| < \aleph_0\}$, $\omega_1 - C = \{\alpha: |A_\alpha - c| < \aleph_0\}$.

Proof that P_9 implies P_8 . Define a space X to be the disjoint union of ω and ω_1 with the following topology: each $\{n\}$, $n \in \omega$, is open; well-order each A_α , say as $\{a_{\alpha n}\}_{n \in \omega}$, and let $\{N(\alpha, n)\}_{n \in \omega}$ be a base at α , where $N(\alpha, n) = \{\alpha\} \cup \{a_{\alpha k}: n < k \in \omega\}$. Then X is certainly first countable and separable, and ω_1 has no limit point in X . The almost disjointness of the A_α 's makes X Hausdorff. One may easily check that X is locally compact, hence regular, but that is unnecessary. To verify normality, it suffices to show $C \subset \omega_1$ can be separated from $\omega_1 - C$. Claim $U = (\omega_1 - C) \cup c$ is open and closed, completing the proof. If $\alpha \in \omega_1 - C$, then $A_\alpha - c$ is finite, so for n sufficiently large, $N(\alpha, n) \subset (\omega_1 - C) \cup c$, so U is open. On the other hand, if $\alpha \in \bar{U}$, then $A_\alpha \cap U$ is infinite, hence $A_\alpha \cap c$ is infinite, so $\alpha \in \omega_1 - C$.

The author has been unable to prove directly that P_6 implies P_8 . Another open question is whether P_6 implies S_0 .

The only remaining proof postponed from Section 1 is that of

2.3. THEOREM. P_7 is equivalent to P_4 .

Proof. One direction is trivial; for the other, let X be an uncountable separable metric space, in which every set is an F_σ . By well-known arguments, it follows that X has cardinality less than 2^{\aleph_0} . Therefore (Kuratowski [27], p. 286) X is 0-dimensional and (ibid., p. 285) homeomorphic to a subset of the Cantor set. A fortiori, X is homeomorphic to a subspace X' of the reals. The relative topology on X' coincides with the topology on X , so X' is a Q -set.

2.4. PROPOSITION P_{10} . There is a countable $\mathcal{G} \subset {}^{\omega_1}\omega$ such that

$$(a) (\forall \alpha \in \omega_1)(\forall n \in \omega)(\exists g \in \mathcal{G})(g(\alpha) > n),$$

$$(b) (\forall p \in {}^{\omega_1}2)(\exists f \in {}^{\omega_1}\omega)(\forall g \in \mathcal{G})(\forall \alpha_0, \alpha_1 \in \omega_1) \\ (g(\alpha_0) > f(\alpha_0) \text{ and } g(\alpha_1) > f(\alpha_1) \text{ imply } p(\alpha_0) = p(\alpha_1)).$$

2.5. THEOREM. P_{10} is equivalent to P_8 .

Proof. The proof is an application of techniques developed in Bing [6] and generalized in Tall [46]. First we prove the forward implication. Let $X = \mathcal{G} \cup \omega_1$ with the following topology: for each $g \in \mathcal{G}$, $\{g\}$ is open; a base at $\alpha \in \omega_1$ is given by $\{N(\alpha, n)\}_{n < \omega}$, where $N(\alpha, n) = \{\alpha\} \cup \{g : g(\alpha) > n\}$. Then X is first countable and T_1 . By (a), \mathcal{G} is dense in X , so X is separable. It remains to show X is normal, hence Hausdorff. It suffices to show any subset C of ω_1 can be separated from $\omega_1 - C$ by disjoint open sets. Let $p \in {}^{\omega_1}2$ be defined by $p(\alpha) = 0$ if and only if $\alpha \in C$. Let f be given by (b). Then the open sets $U_0 = \bigcup_{\alpha \in C} N(\alpha, f(\alpha))$ and $U_1 = \bigcup_{\alpha \in \omega_1 - C} N(\alpha, f(\alpha))$ are as required.

Conversely, let X be a separable, normal, first countable, Hausdorff space containing an uncountable set Y with no limit point. Without loss of generality, assume there is a countable set D dense in X and disjoint from Y . Identify Y with ω_1 , and let $\{N(\alpha, n)\}_{n < \omega}$ be a base at α , such that $N(\alpha, n)$ contains no other member of ω_1 , and such that $N(\alpha, n) \supset N(\alpha, n+1)$, for each n . For each $d \in D$, define $g_d \in {}^{\omega_1}\omega$ by

$$g_d(\alpha) = 1 \text{ plus the largest } n \text{ such that } d \in N(\alpha, n),$$

$$g_d(\alpha) = 0 \text{ if for no } n \text{ is } d \in N(\alpha, n).$$

Then g_d is well-defined since X is T_1 . Claim $\mathcal{G} = \{g_d\}_{d \in D}$ satisfies (a), (b). The first condition is clear, since D is dense. As for (b), let $p_0 = \{\alpha : p(\alpha) = 0\}$, $p_1 = \{\alpha : p(\alpha) = 1\}$. By normality, there exist disjoint open sets $U_0 \supset p_0$, $U_1 \supset p_1$. Since $\{N(\alpha, n)\}_{n < \omega}$ is a base at α , we may define $f \in {}^{\omega_1}\omega$

so that $N(\alpha, f(\alpha)) \subset U_0$ or U_1 , according to whether $\alpha \in p_0$ or p_1 . Then $\bigcup_{\alpha \in p_0} N(\alpha, f(\alpha)) \cap \bigcup_{\alpha \in p_1} N(\alpha, f(\alpha)) = \emptyset$. $g(\alpha) > f(\alpha)$ if and only if $g \in N(\alpha, f(\alpha))$, so we have (b).

If we add the additional condition

$$(c) (\forall g \in \mathcal{S})(\exists n \in \omega)(\forall \alpha \in \omega_1)(g(\alpha) \leq n),$$

then we can prove directly that the existence of a countable \mathcal{S} satisfying (a), (b), (c) is equivalent to P_0 . That is left as an exercise for the reader.

Many of the propositions considered in this chapter have generalizations to higher cardinals, and appropriate equivalences hold.

Bibliography

- [1] K. Alster and T. Przymusiński, *Normality and Martin's axiom*, Fund. Math. 91(1976), pp. 123–131.
- [2] G. Aquaro, *Point countable open coverings in countably compact spaces*, pp. 39–41 in *General Topology and its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, 1966*, Academic Press, New York 1967.
- [3] A. V. Arhangel'skiĭ, *On the cardinality of bicompaeta satisfying the first axiom of countability*, Soviet Math. Dokl. 10 (1969), pp. 951–955.
- [4] C. E. Aull, *Paracompact subsets*, pp. 45–51 in *General Topology and its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, 1966*, Academic Press, New York 1967.
- [5] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), pp. 175–186.
- [6] — *A translation of the normal Moore space conjecture*, Proc. Amer. Math. Soc. 16 (1965), pp. 612–619.
- [7] L. Bukovsky, *Borel subsets of metric separable spaces*, pp. 83–86 in *General Topology and its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, 1966*, Academic Press, New York 1967.
- [8] A. Charlesworth, R. Hodel and F. Tall, *On a theorem of Jones and Heath concerning discrete subspaces*, Colloq. Math. 34 (1975), pp. 33–37.
- [9] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, W. A. Benjamin, New York 1966.
- [10] J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. 23 (1944), pp. 65–76.
- [11] C. H. Dowker, *On countably paracompact spaces*, Canad. J. Math. 3 (1951), pp. 219–224.
- [12] — *On a theorem of Hanner*, Ark. for Mat. 2 (1952), pp. 307–313.
- [13] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.
- [14] W. Easton, *Powers of regular cardinals*, Thesis, Princeton University, 1964.
- [15] R. Engelking, *Outline of General Topology*, North-Holland and Polish Scientific Publ., Amsterdam – Warszawa 1968.
- [16] B. Fitzpatrick and D. R. Traylor, *Two theorems on metrizability of Moore spaces*, Pacific J. Math. 19 (1966), pp. 259–264.
- [17] W. G. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. 46 (1974), pp. 294–298.
- [18] — *A normal collectionwise Hausdorff not collectionwise normal space*, Gen. Top. Appl. 6 (1976), pp. 57–64.
- [19] K. Gödel, *Consistency-proof for the generalized continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. 25 (1939), pp. 220–224.
- [20] J. de Groot, *Discrete subspaces of Hausdorff spaces*, Bull. Acad. Polon. Sci, Sér. Sci. Math. Astronom. Phys. 13 (1965), pp. 537–544.
- [21] F. Hausdorff, *Problème 53*, Fund. Math. 20 (1933), p. 286.

- [22] R. W. Heath, *Screenability, pointwise paracompactness and metrization of Moore spaces*, *Canad. J. Math.* 16 (1964), pp. 763–770.
- [23] J. Isbell, *Remarks on spaces of large cardinal number*, *Czechoslovak Math. J.* 14 (1964), pp. 383–385.
- [24] T. J. Jech, *Lectures in set theory with particular emphasis on the method of forcing*, *Lect. Notes Math.* 217, Springer-Verlag, Berlin 1971.
- [25] R. B. Jensen, *The fine structure of the constructible hierarchy*, *Ann. Math. Logic* 4 (1972), pp. 229–308.
- [26] F. B. Jones, *Concerning normal and completely normal spaces*, *Bull. Amer. Math. Soc.* 43 (1937), pp. 671–677.
- [27] K. Kuratowski, *Topology*, vol. I, Academic Press, PWN—Polish Scientific Publishers, New York, London, Warszawa 1966.
- [28] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, *Ann. Math. Logic* 2 (1970), pp. 143–178.
- [29] M. J. Mansfield, *On countably paracompact normal spaces*, *Canad. J. Math.* 9 (1957), pp. 443–449.
- [30] L. F. McAuley, *A note on complete collectionwise normality and paracompactness*, *Proc. Amer. Math. Soc.* 9 (1958), pp. 796–799.
- [31] T. Menas, *On strong compactness and supercompactness*, Thesis, University of California, Berkeley 1973.
- [32] E. Michael, *A note on paracompact spaces*, *Proc. Amer. Math. Soc.* 4 (1953), pp. 831–838.
- [33] — *Point-finite and locally finite coverings*, *Canad. J. Math.* 7 (1955), pp. 275–279.
- [34] — *Another note on paracompact spaces*, *Proc. Amer. Math. Soc.* 8 (1957), pp. 822–828.
- [35] J. D. Monk, *Introduction to Set Theory*, McGraw-Hill, New York 1969.
- [36] R. L. Moore, *Foundations of Point Set Theory*, Amer. Math. Soc. Coll. Publ. 13, 1932. Rev. ed., Providence 1962.
- [37] K. Morita, *Star-finite coverings and the star-finite property*, *Math. Jap.* 1 (1948), pp. 60–68.
- [38] K. Nagami, *Paracompactness and strong screenability*, *Nagoya Math. J.* 8 (1955), pp. 83–88.
- [39] F. Rothberger, *On some problems of Hausdorff and of Sierpiński*, *Fund. Math.* 35 (1948), pp. 29–46.
- [40] B. Šapirovič, *On separability and metrizability of spaces with Souslin's condition*, *Soviet Math. Dokl.* 13 (1972), pp. 1633–1638.
- [41] J. R. Shoenfield, *Unramified forcing*, *Proc. Sympos. Pure Math.* 13, Part I (1971), pp. 357–382.
- [42] W. Sierpiński, *Sur un problème de M. Hausdorff*, *Fund. Math.* 30 (1938), pp. 1–7.
- [43] Yu. M. Smirnov, *On strongly paracompact spaces* (in Russian), *Izv. Akad. Nauk SSSR ser. mat.* 20 (1956), pp. 253–274.
- [44] R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, *Ann. of Math.* 94 (1971), pp. 201–245.
- [45] F. D. Tall, *Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems*, Thesis, University of Wisconsin, Madison 1969.
- [46] — *A set-theoretic proposition implying the metrizability of normal Moore spaces*, *Proc. Amer. Math. Soc.* 33 (1972), pp. 195–198.
- [47] — *On the existence of non-metrizable metacompact normal Moore spaces*, *Canad. J. Math.* 26 (1974), pp. 1–6.
- [48] — *How separable is a space? That depends on your set theory!*, *Proc. Amer. Math. Soc.* 46 (1974), pp. 310–314.

- [49] F. D. Tall, *The countable chain condition versus separability — applications of Martin's Axiom*, Gen. Top. Appl. 4 (1974), pp. 315–340.
 - [50] D. R. Traylor, *A note on metrization of Moore spaces*, Proc. Amer. Math. Soc. 14 (1963), pp. 804–805.
 - [51] — *Normal, separable Moore spaces and normal Moore spaces*, Duke Math. J. 30 (1963), pp. 485–494.
 - [52] — *Concerning metrization of pointwise paracompact Moore spaces*, Canad. J. Math. 16 (1964), pp. 407–411.
 - [53] — *Metrizability and completeness in normal Moore spaces*, Pacific J. Math. 17 (1966), pp. 381–390.
 - [54] — *Metrizability in normal Moore spaces*, *ibid.* 19 (1966), pp. 175–181.
 - [55] — *On normality, pointwise paracompactness and the metrization question*, pp. 286–289 in *Topology Conference Arizona State University 1967*, Tempe, Arizona, 1968.
 - [56] R. W. Heath, *Separability and \aleph_1 -compactness*, Colloq. Math. 12 (1964), pp. 11–14.
-