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ON THE DERIVATIVE OF INDEFINITE RC-INTEGRAL

BY

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In [1], Henstock defined a Riemann-complete integral as follows. A division of an interval $[a, b]$, denoted by $([a, b], \mathscr{D})$, consists of two finite sequences $\{x_j\}_{j=0}^n$ and $\{z_j\}_{j=1}^n$ with conditions $a = x_0 < x_1 < \ldots < x_n = b$ and $x_{j-1} \leqslant z_j \leqslant x_j$ $(j = 1, \ldots, n)$. If $\delta(z)$ is a positive function defined in $[a, b]$, $([a, b], \mathscr{D}, \delta(z))$ denotes a division $([a, b], \mathscr{D})$ compactible with $\delta(z)$. That is, in addition to the above-mentioned conditions, $x_j - z_j < \delta(z_j)$ and $z_j - x_{j-1} < \delta(z_j)$ for $j = 1, \ldots, n$. A real-valued function f is said to be *Riemann-complete integrable* in $[a, b]$ if there is a real number $I(f, [a, b])$, which is called the *integral* of f in $[a, b]$, such that to each $\varepsilon > 0$ there corresponds a positive function $\delta(z)$ defined in $[a, b]$ with $|S - I(f, [a, b])| < \varepsilon$ for all sums

$$S = \sum_{1}^{n} f(z_j)(x_j - x_{j-1}) \quad \text{over } \big([a, b], \mathscr{D}, \delta(z)\big),$$

where $(\{x_j\}_{j=0}^n, \{z_j\}_{j=1}^n)$ forms the division $([a, b], \mathscr{D}, \delta(z))$. Henstock pointed out that this integral is equivalent to the Perron integral. It follows that the theorem stated below holds. However, we shall prove it in this paper without using the Perron integral.

THEOREM. *If f is Riemann-complete integrable in $[a, b]$, then its indefinite integral $I(f, [a, x])$ has a derivative and the derivative $DI(f, [a, x]) = f(x)$ for almost all x of $[a, b]$, and $f(x)$ is measurable in $[a, b]$.*

The proof for the theorem is based on several lemmas. These lemmas can be found in [1] and [3] if they are stated without proof.

LEMMA 1 (cf. [3]). *If f and g are Riemann-complete integrable in $[a, b]$ and α, β are real numbers, then $\alpha f + \beta g$ is also Riemann-complete integrable, and*

$$I(\alpha f + \beta g, [a, b]) = \alpha I(f, [a, b]) + \beta I(g, [a, b]).$$

LEMMA 2 (cf. [3]). *A real-valued function f is Riemann-complete integrable in $[a, b]$ if and only if, for $\varepsilon > 0$, there corresponds a function $\delta(z) > 0$*

defined in $[a, b]$ *with* $|S'-S''|<\varepsilon$ *whenever* S' *and* S'' *are sums over* $([a, b], \mathscr{D}', \delta(z))$ *and* $([a, b], \mathscr{D}'', \delta(z))$, *respectively.*

LEMMA 3. *If* f *is Riemann-complete integrable in* $[a, b]$ *and* $c \in (a, b)$, *then* f *is Riemann-complete integrable in both* $[a, c]$ *and* $[c, b]$, *and*

$$I(f, [a, b]) = I(f, [a, c]) + I(f, [c, b]).$$

Proof. The integrability of f in both $[a, c]$ and $[c, b]$ follows from Lemma 2. By definition of integrability, for $\varepsilon > 0$, there are three positive functions $\delta_0(z)$, $\delta_1(z)$, $\delta_2(z)$ defined in $[a, b]$, $[a, c]$, $[c, b]$, respectively, such that

$$|S-I(f, [a, b])| < \frac{\varepsilon}{3} \quad \text{for sum } S \text{ over any } ([a, b], \mathscr{D}, \delta_0(z)),$$

$$|S_1-I(f, [a, c])| < \frac{\varepsilon}{3} \quad \text{for sum } S_1 \text{ over any } ([a, c], \mathscr{D}_1, \delta_1(z)),$$

$$|S_2-I(f, [c, b])| < \frac{\varepsilon}{3} \quad \text{for sum } S_2 \text{ over any } ([c, b], \mathscr{D}_2, \delta_2(z)).$$

Let $\delta(z)$ be defined as $\delta_0(z) \wedge \delta_1(z)$ if $z \in [a, c)$, $\delta_0(z) \wedge \delta_2(z)$ if $z \in (c, b]$ and $\delta(c) = \delta_0(c) \wedge \delta_1(c) \wedge \delta_2(c)$; then the above-mentioned three inequalities hold if $\delta_0(z)$, $\delta_1(z)$ and $\delta_2(z)$ are replaced by $\delta(z)$. If S_1 and S_2 are sums over two fixed divisions $([a, c], \mathscr{D}_1, \delta(z))$ and $([c, b], \mathscr{D}_2, \delta(z))$, respectively, then S_1+S_2 is the sum over $([a, b], \mathscr{D}, \delta(z))$ which is the union of our two fixed divisions. Hence

$$\begin{aligned} &\big|I(f, [a, b]) - \big(I(f, [a, c]) + I(f, [c, b])\big)\big| \\ &\quad \leqslant |I(f, [a, b]) - (S_1+S_2)| + |S_1 - I(f, [a, c])| + |S_2 - I(f, [c, b])| \\ &\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

The equality follows clearly.

LEMMA 4 (cf. [1]). *Let* f *be Riemann-complete integrable in* $[a, b]$ *and* $\varepsilon > 0$ *be given, let* $\delta(z) > 0$ *be defined in* $[a, b]$ *such that* $|S-I(f, [a, b])| < \varepsilon$ *for all sums* S *over* $([a, b), \mathscr{D}, \delta(z))$. *If* p *is any partial sum of terms* $\{f(z)(v-u) - I(f, [u, v])\}$ *corresponding to distinct intervals* $[u, v]$ *of* $([a, b], \mathscr{D}, \delta(z))$, *then* $|p| \leqslant \varepsilon$.

LEMMA 5. *If* f *is Riemann-complete integrable in* $[a, b]$, *then* $I(f, [a, x])$ *is a continuous real-valued function in* $[a, b]$.

Proof. By Lemma 3, $I(f, [a, x])$ is defined in $[a, b]$. Let z_0 be a fixed point with $a \leqslant z_0 < b$. Given $\varepsilon > 0$, choose ε' such that

$$0 < \varepsilon' < \frac{\varepsilon}{1+|f(z_0)|}.$$

Since f is Riemann-complete integrable in $[a, b]$, for $\varepsilon' > 0$, there is $\delta(z) > 0$ defined in $[a, b]$ such that

$$\text{(i)} \qquad |S - I(f, [a, b])| < \varepsilon'$$

for all sums S over $\big([a, b], \mathscr{D}, \delta(z)\big)$. Let δ be such that $0 < \delta < \varepsilon' \wedge \delta(z_0)$ and let x be any point with $0 < x - z_0 < \delta$.

Case 1. $z_0 = a$. Let S be the sum over $\big([x, b], \mathscr{D}', \delta(z)\big)$, $\big([a, b], \mathscr{D}, \delta(z)\big)$ be the union of $\big([x, b], \mathscr{D}', \delta(z)\big)$ and the division of $[a, x]$ consisting of $\{a, x\}$ and $\{z_0 = a\}$; then $S + f(z_0)(x - z_0)$ is the sum over $\big([a, b], \mathscr{D}, \delta(z)\big)$. By (i), we have

$$|S + f(z_0)(x - z_0) - I(f, [a, b])| < \varepsilon'.$$

Then, by Lemma 4,

$$|f(z_0)(x - z_0) - I(f, [z_0, x])| < \varepsilon'.$$

Case 2. $a < z_0 < b$. Let S_1 and S_2 be the sums over $\big([a, z_0], \mathscr{D}_1, \delta(z)\big)$ and $\big([x, b], \mathscr{D}_2, \delta(z)\big)$, respectively, let $\big([a, b], \mathscr{D}, \delta(z)\big)$ be the union of $\big([a, z_0], \mathscr{D}_1, \delta(z)\big)$, $\big([x, b], \mathscr{D}_2, \delta(z)\big)$ and the division of $[z_0, x]$ consisting of $\{z_0, x\}$ and $\{z_0\}$; then $S_1 + f(z_0)(x - z_0) + S_2$ is the sum over $\big([a, b], \mathscr{D}, \delta(z)\big)$. Consequently,

$$|S_1 + f(z_0)(x - z_0) + S_2 - I(f, [a, b])| < \varepsilon'.$$

By Lemma 4 again,

$$|f(z_0)(x - z_0) - I(f, [z_0, x])| < \varepsilon'.$$

In either case, we have

$$\begin{aligned} |I(f, [z_0, x])| &< \varepsilon' + |f(z_0)|\,|x - z_0| \\ &< \varepsilon' + |f(z_0)|\,\delta < \varepsilon'\big(1 + |f(z_0)|\big) < \varepsilon. \end{aligned}$$

Thus $I(f, [a, x])$ is right continuous at z_0. Similarly, we can prove that $I(f, [a, x])$ is left continuous at z_0 if $a < z_0 \leqslant b$. Hence, $I(f, [a, x])$ is continuous in $[a, b]$.

LEMMA 6. *If f is Riemann-complete integrable in $[a, b]$, then, for $\varepsilon > 0$, there is a monotone increasing function G defined in $[a, b]$ such that $0 \leqslant G(b) \leqslant \varepsilon$ and the upper derivate*

$$\bar{D}\big(I(f, [a, x]) - G(x)\big) \leqslant f(x) \quad \text{in } [a, b].$$

Proof. By hypothesis, there is a $\delta(z) > 0$ defined in $[a, b]$ such that $|S - I(f, [a, b])| < \varepsilon/2$ for all sums S over $\big([a, b], \mathscr{D}, \delta(z)\big)$. Let

$$S = \sum_{1}^{n} f(z_j)(x_j - x_{j-1})$$

be a sum over $([a, b], \mathscr{D}, \delta(z))$; then, by Lemma 4, we have

$$\sum \{|f(z_j)(x_j - x_{j-1}) - I(f, [x_{j-1}, x_j])|: \\ f(z_j)(x_j - x_{j-1}) - I(f, [x_{j-1}, x_j]) \geqslant 0\} < \frac{\varepsilon}{2}$$

and

$$\sum \{|f(z_j)(x_j - x_{j-1}) - I(f, [x_{j-1}, x_j])|: \\ f(z_j)(x_j - x_{j-1}) - I(f, [x_{j-1}, x_j]) < 0\} < \frac{\varepsilon}{2}.$$

Hence we have

$$\sum_{1}^{n} |f(z_j)(x_j - x_{j-1}) - I(f, [x_{j-1}, x_j])| < \varepsilon$$

for the sum S over $([a, b], \mathscr{D}, \delta(z))$.

For $I = [u, v] \subset [a, b]$, we write

$$G(I) = \sup\Big\{\sum |f(z_j)(x_j - x_{j-1}) - I(f, [x_{j-1}, x_j])|: \\ (\{x_j\}_{j=0}^{n}, \{z_j\}_{j=1}^{n}) \text{ is a division } ([u, v], \mathscr{D}, \delta(z))\Big\}.$$

Clearly, $G(I) \leqslant \varepsilon$. Also if $I_1 = [u, t]$, $I_2 = [t, v]$ and $I = I_1 \cup I_2$, then $G(I_1) + G(I_2) \leqslant G(I)$.

Let $G(x) = G([a, x])$ for $a < x \leqslant b$ and $G(a) = 0$; then $G(x)$ in $[a, b]$ is, clearly, non-negative, monotone increasing and $G(b) \leqslant \varepsilon$. To show that

$$\bar{D}\big(I(f, [a, x]) - G(x)\big) \leqslant f(x) \quad \text{in } [a, b],$$

we fix $z_0 \in [a, b]$. Consider x with $0 < |x - z_0| < \delta(z_0)$. If $0 < z_0 - x < \delta(z_0)$, then $(\{x, z_0\}, \{z_0\})$ is a division $([x, z_0], \mathscr{D}, \delta(z))$ and

$$G([x, z_0]) \geqslant |f(z_0)(z_0 - x) - I(f, [x, z_0])|.$$

It follows that

$$\frac{\big(I(f, [a, x]) - G(x)\big) - \big(I(f, [a, z_0]) - G(z_0)\big)}{x - z_0} \leqslant f(z_0).$$

Similarly, the same inequality holds if $0 < x - z_0 < \delta(z_0)$. Thus we have this inequality for $0 < |x - z_0| < \delta(z_0)$. Consequently,

$$\bar{D}\big(I(f, [a, z_0]) - G(z_0)\big) \leqslant f(z_0).$$

The proof of the lemma is complete.

Proof of the theorem. Let

$$A = \{t \in [a, b]: \bar{D}I(f, [a, t]) > f(t)\}.$$

We want to show that $m_e(A) = 0$, where m_e is the Lebesgue outer measure. If $m_e(A) > 0$, then there is an $r > 0$ and a set A_r with $m_e(A_r) = \mu > 0$ in which $\bar{D}I(f, [a, t]) - f(t) > r$. Let ε be such that $0 < \varepsilon < \frac{1}{2}\mu r$ and G be a function in $[a, b]$ as stated in Lemma 6; then

$$(\mathscr{L})\int_a^b G' \leqslant G(b) - G(a) = G(b) \leqslant \varepsilon.$$

Set $B \equiv \{x \in [a, b]: G'(x) > \frac{1}{2}r\}$. Since G is monotone increasing,

$$G'(x) \geqslant 0 \quad \text{and} \quad (\mathscr{L})\int_B G' \leqslant (\mathscr{L})\int_a^b G' \leqslant \varepsilon.$$

It follows that $m_e(B) < \mu$. Thus there exists $t \in A_r$ with $0 \leqslant G'(t) \leqslant \frac{1}{2}r$. For this t, we have

$$\begin{aligned}\bar{D}I(f, [a, t]) - \tfrac{1}{2}r &\leqslant \bar{D}I(f, [a, t]) - G'(t)\\ &\leqslant \bar{D}\big(I(f, [a, t]) - G(t)\big) \leqslant f(t),\end{aligned}$$

a contradiction to the fact that $t \in A_r$. Therefore, $m_e(A) = 0$ and $\bar{D}I(f, [a, x]) \leqslant f(x)$ for almost all x in $[a, b]$.

Applying this fact to $-f$ which, by Lemma 1, is Riemann-complete integrable in $[a, b]$ and $I(-f, [a, x]) = -I(f, [a, x])$, we have

$$\bar{D}I(-f, [a, x]) \leqslant -f(x) \quad \text{for almost all } x \text{ in } [a, b].$$

Since

$$\underline{D}I(f, [a, x]) = -\bar{D}\big(-I(f, [a, x])\big) = -\bar{D}I(-f, [a, x]),$$

we see that

$$\underline{D}I(f, [a, x]) \geqslant f(x) \quad \text{for almost all } x \text{ in } [a, b].$$

Hence

$$f(x) \leqslant \underline{D}I(f, [a, x]) \leqslant \bar{D}I(f, [a, x]) \leqslant f(x)$$

or

$$DI(f, [a, x]) = f(x)$$

for almost all x in $[a, b]$. By Lemma 5 and a well-known theorem (cf. [2], p. 194), $f(x) = DI(f, [a, x])$ is measurable in $[a, b]$.

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