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The product-decomposability of probability measures on Abelian metrizable groups

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Introduction

Let G be an Abelian metrizable group. A probability measure μ on G is said to be *product-decomposable* if there exist nondegenerate measures μ_1, μ_2, Borel subgroups G_1, G_2 of G and an element x in G such that

(i) $\mu = \mu_1 * \mu_2 * \delta_x$;
(ii) $\mu_i(G_i) = 1$ for $i = 1, 2$;
(iii) $G_1 \cap G_2 = \{0\}$.

Otherwise, μ is said to be *product-indecomposable*. The measures μ_1 and μ_2 in (i) are said to be the *product-factors* of the measure μ. This notion of product-decomposability is a particular case of the classical convolution decomposability of measures. However, the results are essentially different and obtained by new methods. In the paper we introduce the notion of a Borel decomposability semigroup of a probability measure that generalizes the Urbanik decomposability semigroup (see [23] and [24]). It allows us to present measure problems on Abelian metrizable groups (no characteristic functions!) in terms of algebraic and topological properties.

The thesis consists of six chapters. In Chapter I, we quote basic definitions and theorems from topology, theory of groups and functional analysis, used later on. Chapter II contains some basic facts about measures. The most crucial part is Section 3 devoted to Radon measures on Abelian metrizable groups. In particular, we investigate measures with nontrivial idempotent factors and so-called infinitely divisible measures.

In Chapter III we define the fundamental notion of a Borel decomposability semigroup consisting of additive measurable operators on the group. It forms a metrizable right semigroup. Furthermore, we study properties of the additive projectors from the decomposable semigroup. The main result of this chapter (Theorem 3.4.11) shows that the set of all those projectors in the Borel decomposability semigroup of a probability measure without an idempotent factor forms a complete lattice with respect to the natural order. This is the key that enables us to find a complete description of the product-decomposability of measures without idempotent factors.

The main part of the whole thesis is Chapter IV. Here we first observe that there is a close connection between the existence of nontrivial additive projectors (in the Borel decomposability semigroup of a probability measure) and the product-decomposability of the measure in question. Then, we introduce new notions of purely product-atomic measures, product-atomless measures and Gaussian measures in the sense of Gnedenko. For the sake of completeness of our Introduction we quote all of them below.

A Radon probability measure μ on an Abelian metrizable group is said to be *purely product-atomic* if every product-factor of μ has a product-indecomposable product-factor. If every product-factor of μ is product-decomposable, then μ is said to be *product-atomless*. A Radon probability measure μ on G is said to be a *Gaussian measure in the sense of Gnedenko* (*G-Gaussian*) if there exist a Radon probability measure ν on G, Radon probability measures λ_1, λ_2 on $G \times G$, Borel subgroups F_1, F_2 of $G \times G$ and an element x in G such that

(i) $(\mu * \delta_x) \otimes \nu = \lambda_1 * \lambda_2$;
(ii) $F_1 \cap F_2 = \{0\}$;
(iii) $(G \times \{0\} \cup \{0\} \times G) \cap (F_1 \cup F_2) = \{0\}$;
(iv) $\lambda_i(F) = 1$ for $i = 1, 2$.

Our terminology is patterned on Gnedenko's results (see [8]) who proved that characterization for Gaussian measures on $\mathbb{R}^2$. Chapter IV ends with the following main result of the thesis:

THEOREM. *Let G be an Abelian metrizable group and μ be a Radon probability measure on G without an idempotent factor. Then there exist Radon probability measures μ_0, μ_1, μ_2 on G, Borel subgroups G_0, G_1, G_2 of G and an element x_0 in G such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;
(ii) μ_0 *is a Gaussian measure in the sense of Gnedenko*;
(iii) μ_1 *is a product-atomless measure without a G-Gaussian product-factor*;
(iv) μ_2 *is a purely product-atomic measure without a G-Gaussian product-factor*;
(v) $\mu_i(G_i) = 1$ *for* $i = 0, 1, 2$;
(vi) $G_0 \cap G_1 = \{0\}$ *and* $(G_0 + G_1) \cap G_2 = \{0\}$.

Moreover, the measures μ_0, μ_1 and μ_2 are uniquely determined up to degenerate convolution factors.

This is an analogue of the classical Khinchin factorization theorem (see [14]).

In Chapter V we consider the product-decomposability on locally convex metrizable spaces. In particular, we prove that a measure is G-Gaussian iff it is Gaussian and that every product-atomless measure is a generalized Poisson measure.

Chapter VI specializes the notion of product-decomposability to measures on LCA metrizable groups. Again, we investigate properties of purely product-atomic measures, product-atomless measures and G-Gaussian measures.

I. Preliminaries

In this chapter we present the necessary results from topology, group theory and functional analysis. Our intention is to fix the terminology and notation (to avoid possible confusions) and to list the facts most often used in the farther chapters.

1.1. Semigroups. A *semigroup* is a nonempty set S together with an associative binary operation from $S \times S$ into S, called *multiplication*. If S is a semigroup, then for each $t \in S$ the maps $\varrho_t : S \to S$, $\varrho_t(s) = st$, and $\lambda_t : S \to S$, $\lambda_t(s) = ts$ are called, respectively, *right* and *left multiplication maps* (*by* t).

A nonempty subset T of a semigroup S is called a *subsemigroup* of S if $TT \subset T$. Given a subset F of S, we denote by $\mathrm{Sem}(F)$ the subsemigroup of S generated by F.

An element $s \in S$ is called an *idempotent* if $s^2 = s$, an *identity* if $st = ts = t$ for all $t \in S$, or a *zero* if $st = ts = s$ for all $t \in S$.

Let S be a semigroup with a Hausdorff topology. S is said to be *right* (*left*) *topological semigroup* if for each $s \in S$ the mapping ϱ_s (λ_s) is continuous. If S is both left and right topological then S is called *semitopological*. Thus S is semitopological iff multiplication is separately continuous. If this mapping is (jointly) continuous then S is said to be *topological*.

1.2. Algebraic groups. In this section, we establish the terminology and notation concerning algebraic groups that will be used throughout the work.

Let G be an algebraic group. The *order* of the group G is the cardinal number G of the set of its elements. If G is finite (countable) cardinal, G is called a finite (countable) group. If A is a subset of G, the symbol $\langle A\rangle$ will denote the subgroup of G generated by A. Moreover, if $\langle A\rangle = G$, A is said to be a *generating system* of G; the elements of A are *generators* of G. The order of a group $\langle a\rangle$ ($a \in G$) is also called the *order of the element* a, in notation $o(a)$.

If every element of G is of finite order, G is called a *torsion* or *periodic group*, while G is *torsion-free* if all its elements, except 0, are of infinite order. A *primary group* or *p-group* is defined to be a group the orders of whose elements are powers of a prime p. A group G is said to be an *elementary p-group* if every element of G is of order p. For a possibly infinite collection of subgroups F_i of G, the subgroup F they generate consists of all finite sums $b_{i_1} + \ldots + b_{i_k}$ with $b_{i_j} \neq 0$ belong to same F_{i_j}. We shall write $F = \sum_{i\in I} F_i$. Let F_i ($i \in I$) be a family of subgroups of G, subject to the following two conditions:

(i) $\sum F_i = G$ (i.e. the F_i together generate G);
(ii) for every $i \in I$, $F_i \cap (\sum_{i\neq j} F_j) = \{0\}$.

Then G is said to be the *direct sum* of its subgroups F_i, in symbols $G = \bigoplus_{i\in I} F_i$ or $G = F_1 \oplus \ldots \oplus F_n$, if $I = \{1, \ldots, n\}$. Again, every $a \in G$ can be written in a unique form $a = b_{i_1} + \ldots + b_{i_k}$ with $b_{i_j} \neq 0$ belonging to different components F_{i_j} ($j = 1, \ldots, k$, where $k \geq 0$). Since every element of $\sum F_i$ is contained in a subgroup generated by a finite number of the F_i, condition (ii) can replaced by the apparently weaker postulate

$$F_i \cap (F_{i_1} + \ldots + F_{i_k}) = \{0\}$$

where $i_j \neq i$ and k is a positive integer.

A subset A of an Abelian groups G is said to be *independent* (respectively, *p-independent* for some prime p) if whenever $x_1, \ldots, x_n$ are distinct elements of A and $m_1, \ldots, m_n$ are positive integers (respectively, $m_1, \ldots, m_n \in \mathbb{Z}_p$), the equality $m_1x_1 + \ldots + m_nx_n = 0$ implies that $m_1 = \ldots = m_n = 0$.

Lemma 1.2.1. *Let $p > 2$ be a prime. Then*

(i) $\mathbb{Z}_p \otimes \mathbb{Z}_p = G_1 \oplus G_2$, *where* $G_1 = \langle (1,1) \rangle$, $G_2 = \langle (1, p-1) \rangle$;
(ii) $((\mathbb{Z}_p \times \{0\}) \cup (\{0\} \times \mathbb{Z}_p)) \cap (G_1 \cup G_2) = \{(0,0)\}$.

1.3. Additive operators in Abelian groups and linear operators in linear spaces. Let G be an algebraic Abelian group. A mapping A transforming a subgroup $D(A)$ of G into G is called an *algebraic additive operator in* G if $A(x+y) = Ax + Ay$ for all $x, y \in D(A)$. The set $D(A)$ is said to be the *domain* of the operator A.

If A and B are two algebraic additive operators in G, their *sum* $A + B$ is defined by $(A+B)x = Ax + Bx$ for all $x \in D(A) \cap D(B)$, and it is an algebraic additive operator in G with $D(A+B) = D(A) \cap D(B)$. It may happen that $D(A+B)$ consists of a single element $x = 0$.

The *product* AB of two algebraic additive operators A, B in G is defined by $(AB)x = A(Bx)$ for all $x \in B^{-1}(D(A))$, and it is again an algebraic additive operator in G with $D(AB) = B^{-1}(D(A))$. It is possible that $D(AB)$ consists of a single element 0.

Let G be an algebraic Abelian group. A subgroup F of G is said to be:

(i) *invariant* under an algebraic additive operator A in G (or A-*invariant*) if $F \subset D(A)$ and $A(F) \subset F$;
(ii) *invariant under a family* $\mathcal{A}$ of algebraic additive operators in G (or $\mathcal{A}$-*invariant*) if F is A-invariant for each $A \in \mathcal{A}$.

An algebraic additive operator P in G is said to be an *algebraic additive projection in* G if $P^2 = P$.

We now state some lemmata.

Lemma 1.3.1. *Let G be an algebraic Abelian group and A be an algebraic additive operator in G. Then*

(i) *A is an algebraic additive projection iff $D(A) = \operatorname{im} A \oplus \ker A$ and $Ax = x$ for each $x \in \operatorname{im} D(A)$*;
(ii) *a subgroup F of G is A-invariant iff $F \subset D(A)$ and $F = A(F) + (I - A)(F)$*;
(iii) *if A is an algebraic additive projection then a subgroup F of G is A-invariant iff $F \subset D(A)$ and $F = (F \cap \operatorname{im} A) \oplus (F \cap \ker A)$*;

Lemma 1.3.2. *Let G be an algebraic Abelian group, H be a subgroup of G and P be an additive projection on G to itself. Then $P(H) \subset H$ iff $G/H = \pi_H(\operatorname{im} P) \oplus \pi_H(\ker P)$, where π_H is the canonical map from G onto the quotient group G/H.*

Lemma 1.3.3. *Let G be an algebraic Abelian group and P, Q be additive projections on G to itself. Then*

(i) $PQ(I-P) = -P(I-Q)(I-P)$;
(ii) *for every positive integer n and $k = 1, \ldots, n$,*

$$(PQ)^n (I-P) = -(PQ)^{n-k} P((I-Q)(I-P))^k = (PQ)^{n-k}(I-P)((I-Q)(I-P))^k;$$

(iii) *for every positive integer n and $k = 1, \ldots, n$,*

$$(I-P)(QP)^n = -((I-P)(I-Q))^k P(QP)^{n-k} = ((I-P)(I-Q))^k (I-P)(QP)^{n-k}.$$

Let G be an algebraic Abelian group and Q be an additive operator on G to itself. Set

$$Q^{(i)} = \begin{cases} Q & \text{if } i = 0, \\ I - Q & \text{if } i = 1. \end{cases}$$

Let P and Q be additive projections on G to itself and $\underline{k}_n \in \{0,1\}^n$. Put

$$(PQ)_{\underline{k}_n} = \begin{cases} P^{(k_1)}Q^{(k_2)} \dots P^{(k_{n-1})}Q^{(k_n)} & \text{if } n \text{ is odd;} \\ P^{(k_1)}Q^{(k_2)} \dots P^{(k_{n-2})}Q^{(k_{n-1})}P^{(k_n)} & \text{if } n \text{ is even.} \end{cases}$$

Let $v, u \in \{0,1\}$ and $n \in \mathbb{N}$. We will denote by $\underline{uv}_n$ the sequence $(x_1, \dots, x_n)$ such that

$$x_k = \begin{cases} u & \text{if } k \text{ is odd,} \\ v & \text{if } k \text{ is even.} \end{cases}$$

Lemma 1.3.4. *Let G be an algebraic Abelian group and P, Q be additive projections on G to itself. Then for every $n \in \mathbb{N}$,*

(i) $(PQ)_{\underline{00}_{n+1}} = P(QP)_{\underline{00}_n}$;
(ii) $(PQ)_{\underline{11}_{n+1}} = P^{(1)}(QP)_{\underline{11}_n}$;
(iii) $(PQ)_{\underline{10}_{n+1}} = P^{(1)}(QP)_{\underline{01}_n}$;
(iv) $(PQ)_{\underline{01}_{n+1}} = P(QP)_{\underline{10}_n}$.

Set $J_n = \{0,1\}^n \setminus \{\underline{00}_n, \underline{11}_n, \underline{01}_n, \underline{10}_n\}$. The next result will be used below.

Theorem 1.3.5. *Let G be an algebraic Abelian group and P, Q be additive projections on G to itself. Suppose that $(\underline{k}_n)$ is a sequence such that $\underline{k}_n \in J_n$ for each $n \in \mathbb{N}$. Then*

$$\limsup_{n\to\infty} \operatorname{card}\{\underline{i}_n \in J_n : (PQ)_{\underline{i}_n} = (PQ)_{\underline{k}_n}\} = \infty.$$

Proof. Set $l(\underline{k}_n) = \operatorname{card}\{k_i : 1 < i < n, k_{i-1} \neq k_{i+1}\}$. We now consider two cases:

(a) $\sup l(\underline{k}_n) = \infty$;
(b) $\sup l(\underline{k}_n) < \infty$.

The first case follows by Lemma 1.3.3(i). For the second case it suffices to apply Lemma 1.3.3(ii), (iii). ■

Let E be an algebraic linear space. A linear mapping A transforming a subspace $D_l(A)$ of E into E is called an *algebraic linear operator in E*. The set $D_l(A)$ is said to be the *linear domain* of the operator A.

Corollary 1.3.6. *The algebraic linear operators in E form a subclass of the algebraic additive operators in E.*

Corollary 1.3.7. *Let E be a linear space and A, B be algebraic linear operators in E. Then $A + B$ and AB are algebraic linear operators in E.*

Let E be a linear space and A be a linear operator in E. A subspace F of E is said to be:

(i) *invariant* under an algebraic linear operator A in E (or A-*invariant*) if $F \subset D(A)$ and $A(F) \subset F$;

(ii) *invariant under a family* $\mathcal{A}$ of algebraic linear operators in G (or $\mathcal{A}$-*invariant*) if F is A-invariant for each $A \in \mathcal{A}$.

An algebraic linear operator P in E is called an *algebraic linear projection in E* if $P^2 = P$.

1.4. Abelian metrizable groups. An Abelian topological group G is said to be *metrizable* if there exists a metric ϱ on G such that the original topology coincides with the topology defined by the metric ϱ. An Abelian T_0 topological group G is metrizable iff there is a countable basis at zero. In this case, the metric can be taken to be invariant.

A *component* of an Abelian metrizable group G is the connected component of zero, which is always a closed subgroup of G.

By a *closed neighborhood* we shall mean the closure of a nonempty open set.

LEMMA 1.4.1. *Let G be an Abelian metrizable group and G_1, G_2 be σ-compact subgroups of G with $G_1 \cap G_2 = \{0\}$. Then*

(i) $G_0 = G_1 + G_2$ *is a σ-compact subgroup of G*;

(ii) *the mapping P of G_0 into itself defined by $Px = x_1$, where $x = x_1 + x_2$ and $x_i \in G_i$ for $i = 1, 2$, is an algebraic additive Borel-measurable projection.*

Proof. (i) is obvious.

(ii) Let ϕ be the mapping from $G_1 \times G_2$ into G_0 defined by $\phi(x_1, x_2) = x_1 + x_2$. Since ϕ is continuous, Corollary 1.3.3 of [16] implies that ϕ^{-1} is Borel-measurable. Hence, P is Borel-measurable. ∎

LEMMA 1.4.2. *Let G be an Abelian metrizable complete nondiscrete group containing only elements of finite order. There there exist a neighborhood U of 0 in G and some positive integer k_0 such that for every $x \in U \setminus \{0\}$, $o(x) \le k_0$.*

Proof. We denote $D_k = \{x \in G : o(x) \le k\} \cup \{0\}$. Thus $G = \bigcup_{i=1}^{\infty} D_k$. Let (x_n) be a sequence in D_k. We assume that $x_n \to x$ for some $x \in G$. It is easy to see that there exist a positive integer l with $l \le k$ and a subsequence (n_m) of positive integers with $o(x_{n_m}) = l$. Since $0 = l x_{n_m} \to l x$ as $m \to \infty$, we conclude that $o(x) \le l$ and thus D_k is closed. By the Baire theorem $\operatorname{Int} D_k \ne \emptyset$ for some positive integer k. But this implies that $0 \in \operatorname{Int} D_k - \operatorname{Int} D_k \subset \operatorname{Int} D_{k^2}$. ∎

THEOREM 1.4.3. *Let G be an Abelian metrizable complete nondiscrete group such that there exists a neighborhood of 0 in G containing only elements of finite order. Then G contains a nondiscrete closed primary p-group H for some prime p.*

Proof. Let U be a neighborhood of 0 in G containing only elements of finite order and let G_0 be the group generated algebraically by U. Then G_0 is an open-closed nondiscrete subgroup of G containing only elements of finite order.

Let Π be the set of all primes. For every $p \in \Pi$ we define

$$G_p = \{x \in G_0 : o(x) = p^k \text{ for some } k \in \mathbb{N}\}.$$

Hence, G_p is a subgroup of G_0. Moreover, Theorem 8.4 of [7] implies that

$$G = \bigoplus_{p \in \Pi} G_p. \tag{1}$$

We claim that G_p is closed. Let (x_n) be a sequence in G_p such that $x_n \to x$ for some $x \in G_0$. Since $o(x) = m$ for some positive integers m we obtain $mx_n \to 0$ as $n \to \infty$. Hence, by Lemma 1.4.2 the set $\{o(x_n) : n \in \mathbb{N}\}$ is bounded. But this implies that there exist a subsequence (n_m) of positive integers and $k_0 \in \mathbb{N}$ such that $o\,(x_{n_m}) = p^{k_0}$. Since $0 = p^{k_0} x_{n_m} \to p^{k_0} x$ as $m \to \infty$, we obtain $o(x) \le p^{k_0}$ and thus $o(x) = p^l$ for some $l \in \mathbb{N}$. That proves that G_p is closed. Hence (1) together with the Baire theorem yields that there exists $p_0 \in \Pi$ such that G_{p_0} is nondiscrete.

For $n \in \mathbb{N}$ let $G_{p_0,n} = \{x \in G_{p_0} : o(x) \le p_0^n\}$. Thus $G_{p_0,n}$ is a closed subgroup of G_{p_0} and $G_{p_0} = \bigcup_{n=1}^{\infty} G_{p_0,n}$. From the Baire theorem we conclude that for some $n_0 \in \mathbb{N}$, $\operatorname{Int} G_{p_0,n_0} \ne \emptyset$, which implies that G_{p_0,n_0} is nondiscrete.

Let f be the mapping from G_{p_0,n_0} into itself defined by $f(x) = p^{n_0-1}x$ for $x \in G_{p_0,n_0}$. Since f is continuous and $f^{-1}(G_{p_0,n_0} \setminus \{0\}) \subset G_{p_0,1}$ we conclude that $f^{-1}(G_{p_0,n_0} \setminus \{0\})$ is open, which implies $\operatorname{Int} G_{p_0,1} \ne \emptyset$ and thus $G_{p_0,1}$ is nondiscrete. ■

An Abelian metrizable group G is said to be an *I-group* if every neighborhood of 0 in G contains an element of infinite order.

LEMMA 1.4.4. *Let G be an Abelian metrizable complete nondiscrete group and $m_1, \ldots$ $\ldots, m_k$ be integers with $|m_1| + \ldots + |m_k| > 0$. Set $Q = \{(x_1, \ldots, x_k) \in G : \sum_{i=1}^{k} m_i x_i = 0\}$. Suppose that either*

(i) *G is an I-group, or*

(ii) *G is a primary p-group for some prime p and $m_1, \ldots, m_k \in \mathbb{Z}_p$.*

Then Q is a dense open subset of G^k.

The proof is similar to the proof of Lemma 1 of [17] and will be omitted.

THEOREM 1.4.5. *Let G be an Abelian metrizable complete nondiscrete group and U be a neighborhood of* 0 *in G. Then*

(i) *if G is an I-group then U contains an independent set homeomorphic to Cantor's ternary set*;

(ii) *if some neighborhood of* 0 *in G contains only elements of finite order*, *then there exists some prime p such that U contains a p-independent set homeomorphic to Cantor's ternary set.*

Proof. Application of Theorem 1.4.3 shows that it will be sufficient to prove the theorem in the case where either G is an I-group, or G is a primary p-group for some prime p. Define

$$p^* = \begin{cases} \infty & \text{if } G \text{ is an } I\text{-group}, \\ p & \text{if } G \text{ is a primary } p\text{-group}, \end{cases}$$

$$\mathbb{Z}_{p^*} = \begin{cases} \mathbb{Z} & \text{if } p^* = \infty, \\ \mathbb{Z}_p & \text{if } p^* \text{ is a prime.} \end{cases}$$

Let $E^{(0)}$ be a closed neighborhood in U.

We claim that there exists a sequence $(E^{(n)})$ of closed subsets of G such that

(a) $E^{(n)} \subset E^{(0)}$;

(b) $E^{(n)} = \bigcup_{s \in \{0,1\}^n} E_s^{(n)}$;

(c) $E_s^{(n)}$, for $s \in \{0,1\}^n$, are disjoint closed neighborhoods with $\operatorname{diam}(E_s^{(n)}) < 1/n$;

(d) $E_{s,0}^{(n)} \cup E_{s,1}^{(n)} \subset E_s^{(n-1)}$ for $s \in \{0,1\}^n$;

(e) if $x_s \in E_s^{(n)}$, $m_s \in \mathbb{Z}_{p^*}$, $|m_s| \le \min(p^*, n)$ and $\sum_{s\in\{0,1\}^n} |m_s| > 0$ then $\sum_{s\in\{0,1\}^n} m_s x_s \neq 0$.

Suppose $E^{(j)}$, $j = 1, \ldots, k$, have been constructed. Let Q_{k+1} be the set of all points $(x_1, \ldots, x_{2^{k+1}}) \in G^{2^{k+1}}$ such that the conditions: $n_1, \ldots, n_{2^{k+1}} \in \mathbb{Z}$, $|n_i| \le \min(k+1, p^*)$ for $i = 1, \ldots, 2^{k+1}$ and $|n_1| + \ldots + |n_{2^{k+1}}| > 0$ imply $n_1x_1 + \ldots + n_{2^{k+1}}x_{2^{k+1}} \neq 0$. Applying Lemma 1.4.4, we see that Q_{k+1} is a dense open subset of $G^{2^{k+1}}$. Thus the set

$$(E_{s_1}^{(1)} \times E_{s_1}^{(1)} \times E_{s_2}^{(1)} \times E_{s_2}^{(1)} \times \ldots \times E_{s_{2^k}}^{(1)} \times E_{s_{2^k}}^{(1)}) \cap Q_{k+1},$$

where $\{0,1\}^n = \{s_1, s_2, \ldots, s_{2^k}\}$, contains an open set

$$V_{s_1,0} \times V_{s_1,1} \times V_{s_2,0} \times V_{s_2,1} \times \ldots \times V_{s_{2^{k+1}},0} \times V_{s_{2^{k+1}},1}.$$

Moreover, there are disjoint closed neighborhoods $E_s^{(k+1)} \subset V_s$ for all $s \in \{0,1\}^{k+1}$ whose diameters are less than $1/(k+1)$. Put $E^{(k+1)} = \bigcup_{s\in\{0,1\}^{k+1}} E_s^{(k+1)}$. That proves that the sequence $(E^{(n)})$ exists.

Consequently, if $x_1, \ldots, x_j$ are disjoint elements of $E^{(k)}$ for some $k \in \mathbb{N}$, if no two of these elements lie in the same set $E_s^{(k)}$, if $n_1, \ldots, n_j \in \mathbb{Z}_{p^*}$, $|n_i| \le \min(p^*, k)$ for $i = 1, \ldots, j$ and $|n_1| + \ldots + |n_j| > 0$, then $n_1x_1 + \ldots + n_jx_j \neq 0$. Putting $E = \bigcap_{n=1}^{\infty} E^{(n)}$, we obtain $E = \bigcup_{s\in\{0,1\}^\infty} \bigcap_{k=1}^{\infty} E_{t_1,\ldots,t_k}^{(k)}$, where $s = (t_1, t_2, \ldots)$, and $\operatorname{card}(\bigcap_{k=1}^{\infty} E_{t_1,\ldots,t_k}^{(k)}) = 1$. Moreover, if $p^* = \infty$ then E is an independent compact set and if p^* is a prime then E is a p-independent compact set.

Consider the mapping f from E into $\{0,1\}^\infty$ defined by the formula $f(x) = (t_1, t_2, \ldots)$ if $\bigcap_{k=1}^{\infty} E_{t_1,\ldots,t_k}^{(k)} = x$. Then f is a homeomorphism. ∎

THEOREM 1.4.6. *Let G_1, G_2, be Abelian metrizable groups and D_i be a compact subset of G_i for $i = 1, 2$. Assume that D_1 and D_2 are homeomorphic and either D_1 and D_2 are independent, or D_1 and D_2 are p-independent for some prime p. Let F_i be a subgroup of G_i generated algebraically by D_i for $i = 1, 2$. Then*

(i) *F_1 and F_2 are σ-compact;*

(ii) *if ϕ is a homeomorphism from F_1 onto F_2 then the mapping Φ from F_1 into F_2 defined by*

$$\Phi\Big(\sum_{k=1}^{n} m_k x_k\Big) = \sum_{k=1}^{n} m_k \phi(x_k),$$

where $x_1, \ldots, x_n \in D_1$ and $m_1, \ldots, m_n \in \mathbb{Z}$, is a Borel isomorphism.

Proof. For every positive integer n and $i = 1, 2$ we define

$$F_i^{(n)} = \Big\{\sum_{k=1}^{n} m_k x_k : x_1, \ldots, x_n \in D_i,\ |m_i| \le n\Big\}.$$

Thus $F_1^{(n)}$ and $F_2^{(n)}$ are compact and $\Phi(F_1^{(n)}) = F_2^{(n)}$. Moreover, $\Phi|_{F^{(n)}}$ is a homeomorphism. Since $F_i = \bigcup_{n=1}^{\infty} F_i^{(n)}$ for $i = 1.2$, Φ is a Borel isomorphism. ∎

1.5. Locally compact Abelian groups. A *character* on a locally compact Abelian (LCA) group G is a continuous complex-valued function ξ on G satisfying $|\xi(x)| = 1$ for each $x \in G$ and $\xi(x+y) = \xi(x)\xi(x)$ for all $x, y \in G$.

The set G' of all the characters on G is clearly an Abelian group under pointwise multiplication. We write the group operation of G' as addition and replace $\xi(x)$ by $\langle x, \xi\rangle$.

G' equipped with the topology of uniform convergence on compact subsets of G is a LCA group. We call it the *dual group* of G.

For each $x \in G$, the mapping $\xi \to \langle x, \xi\rangle$ defines a character on G'. The Pontryagin duality theorem states that every character on G' has this form and that the topology of uniform convergence on compact subsets of G' coincides with the original topology on G. In other words, if G' is the dual group of G, then G is the dual group of G'.

Let G_1 and G_2 be LCA groups with dual groups G_1' and G_2', respectively. To every continuous homomorphism ϕ from G_1 into G_2 there corresponds the *adjoint mapping* ϕ' from G_2' into G_1' defined by $\langle \phi(x), \gamma\rangle = \langle x, \phi'(\gamma)\rangle$ for $x \in G_1$ and $\gamma \in G_2'$.

It is easy to see that ϕ' is a continuous homomorphism from G_2' into G_1'. We now list without proofs some additional properties of adjoint homomorphisms.

(a) If ϕ is an open continuous homomorphism from G_1 into G_2 then ϕ' is an open homomorphism from G_2' into G_1'.

(b) $(\phi')' = \phi$.

(c) ϕ' is injective iff $\phi(G_1)$ is dense in G_2.

(d) ϕ' is a topological isomorphism from G_2' onto G_1' iff ϕ is a topological isomorphism from G_1 onto G_2.

THEOREM 1.5.1. *Let G be a LCA metrizable group and $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ be a family of additive continuous projections from G into itself such that for every $n \in \mathbb{N}$,*

(i) $P_i^{(n)} P_j^{(n)} = P_j^{(n)} P_i^{(n)} = 0$ *for* $i \neq j$;

(ii) $\sum_{i=1}^{k_n} P_i^{(n)} = I$;

(iii) *for every positive integer $j \in \{1, \ldots, k_n\}$ there exist $m_1, \ldots, m_p \in \{1, \ldots, k_{n+1}\}$ such that*

$$P_j^{(n)} = \sum_{i=1}^{p} P_{m_i}^{(n+1)}.$$

Assume that either

(a) *G has a nontrivial component, or*

(b) *G is a compactly generated nondiscrete group.*

Then

$$\bigcap_{n=1}^{\infty} \Big(\bigcup_{i=1}^{k_n} \operatorname{im} P_i^{(n)} \Big) \neq \{0\}.$$

The proof of this theorem is prepared by two lemmata.

LEMMA 1.5.2. *Let G be a nontrivial compact metrizable group and $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ be a family of additive continuous projections from G into itself satis-*

fying the conditions (i)–(iii) *of Theorem* 1.5.1. *Then*

$$\bigcap_{n=1}^{\infty}\Big(\bigcup_{i=1}^{k_n}\operatorname{im}P_i^{(n)}\Big)\neq\{0\}.$$

P r o o f. We assume on the contrary that $\bigcap_{n=1}^{\infty}(\bigcup_{i=1}^{k_n}\operatorname{im}P_i^{(n)})=\{0\}$. Let γ be a nonconstant continuous character. Set $U=\{\exp(it):|t|<1/13\}$ and $V=\gamma^{-1}(U)$. Since $\bigcap_{n=1}^{\infty}(\bigcup_{i=1}^{k_n}(\operatorname{im}P_i^{(n)}\setminus V))=\emptyset$, we conclude that there exists $n_0\in\mathbb{N}$ such that $\bigcup_{i=1}^{k_{n_0}}\operatorname{im}P_i^{(n_0)}\subset V$, which implies $\bigcup_{i=1}^{k_{n_0}}\gamma(\operatorname{im}P_i^{(n_0)})\subset U$, and thus $\bigcup_{i=1}^{k_{n_0}}\gamma(\operatorname{im}P_i^{(n_0)})=\{1\}$. But this implies that $G=\operatorname{im}P_1^{(n_0)}\oplus\ldots\oplus\operatorname{im}P_{k_{n_0}}^{(n_0)}\subset\ker\gamma\neq G$. The contradiction proves the lemma. ∎

LEMMA 1.5.3. *Let* $\{P_i^{(n)}:i=1,\ldots,k_n;\ n\in\mathbb{N}\}$ *be a family of additive continuous projections from* $\mathbb{R}^p$ *into itself satisfying the conditions* (i)–(iii) *of Theorem* 1.5.1. *Then*

$$\bigcap_{n=1}^{\infty}\Big(\bigcup_{i=1}^{k_n}\operatorname{im}P_i^{(n)}\Big)\neq\{0\}.$$

P r o o f. Since $\{P_i^{(n)}:i=1,\ldots,k_n;\ n\in\mathbb{N}\}$ is a family of linear projections there is $n_0\in\mathbb{N}$ such that for every $n\geq n_0$, $\{P_1^{(n_0)},\ldots,P_{k_{(n_0)}}^{(n_0)}\}=\{P_1^{(n)},\ldots,P_{k_n}^{(n)}\}$. The rest of the proof is clear. ∎

P r o o f o f T h e o r e m 1.5.1. Assume on the contrary that $\bigcap_{n=1}^{\infty}(\bigcup_{i=1}^{k_n}\operatorname{im}P_i^{(n)})=\{0\}$. Let G_0 be the nontrivial component of G. It is easy to see that for every positive integers n and $i\in\{1,\ldots,k_n\}$, $P_i^{(n)}(G_0)\subset G_0$. Without loss of generality we may assume that there exist an Abelian compact connected metrizable group K and a nonnegative integer p such that $G_0=K\times\mathbb{R}^p$.

Let n be some positive integer and $i\in\{1,\ldots,k_n\}$. Since $P_i^{(n)}(K\times\{0\})$ is a compact subgroup of G_0 we conclude that $P_i^{(n)}(K\times\{0\})\subset K\times\{0\}$. Application of Lemma 1.5.2 now implies that $K=\{0\}$ and finally that $G=\mathbb{R}^p$. Hence, by Lemma 1.5.3, $G_0=\{0\}$ and thus G is totally disconnected. This contradicts the assumption G has the nontrivial component.

Let G be a compactly generated nondiscrete group. Without loss of generality we may assume that G has the trivial component. The Pontryagin theorem now implies $G=K\times D$, where K is a compact nontrivial group and D is discrete group. Since $P_i^{(n)}(K\times\{0\})$ is a compact subgroup of G we conclude that $P_i^{(n)}(K\times\{0\})\subset K\times\{0\}$. Application of Lemma 1.5.2 now implies that $K=\{0\}$. This contradicts the assumption that K is nontrivial. ∎

On every LCA group G there exists a nonnegative regular measure ω_G, the so-called *Haar measure* on G, which is not identically 0 and which is translation-invariant. Haar measure is unique, up to a multiplicative positive constant. If G is compact, it is customary to normalize ω_G so that $\omega_G(G)=1$. If G is discrete, any set consisting of a single point is assigned the measure 1. These requirements are of course a restriction if G is a finite group, but this causes no difficulty.

The *characteristic function* of a Radon probability measure μ on G is defined by the formula

$$\widehat{\mu}(\xi) = \int_G \langle x, \xi \rangle \, \mu(dx) \qquad \text{for each } \xi \in G'.$$

The Bochner theorem states that a complex-valued function ϕ on G' is the characteristic function on G iff it is continuous, positive definite and $\phi(0) = 1$.

1.6. Transformation groups. By a *topological transformation group* we mean a triple (F, X, Θ), where F is a topological group, X is a metrizable space and Θ is a continuous mapping from $F \times X$ into X such that

(i) $\Theta(f, \Theta(h, x)) = \Theta(fh, x)$ for all $f, h \in F$ and $x \in X$;
(ii) $\Theta(e, x) = x$ for each $x \in X$, where e is the identity of F.

The map Θ is called an *action* of F on X. The space X, together with a given action Θ on F, is called an *F-space*. When Θ is understood from the context we shall often use the notation $f(x)$ or fx for $\Theta(f, x)$ so that (i) and (ii) become $f(h(x)) = (fh)(x)$ and $e(x) = x$. For $C \subset F$ and $A \subset X$ we put $C(A) = \{f(x) : f \in C,\ x \in A\}$. For $f \in F$ let Θ_f be the mapping from X into itself defined by $\Theta_f(x) = f(x) = \Theta(f, x)$. Then $\Theta_f \Theta_h = \Theta_{fh}$ and $\Theta_e = I_X$, by (i) and (ii). Thus $\Theta_f \Theta_{f^{-1}} = \Theta_{f^{-1}} \Theta_f = I$ for each $f \in F$, which shows that each Θ_f is a homeomorphism of X.

We shall now give some examples of actions.

Example 1.6.1. Let G be an Abelian metrizable group and F be a compact subgroup of G. We define the action of F on G by $f(x) = f + x$ for $f \in F$ and $x \in G$.

Example 1.6.2. Let E be a linear metrizable space and F be a subgroup of $\mathrm{Aut}(E)$. Then F acts on $E \setminus \{0\}$.

The following theorem will be useful in next chapters.

Theorem 1.6.1. *Let $\Theta : F \times X \to X$ be an action of a compact group F on X. Then*

(i) *Θ is a closed mapping*;
(ii) *$F(A)$ is closed for each closed subset A of X*;
(iii) *if C is a compact subset of X then $F(C)$ is compact.*

If X is an F-space and $x \in X$, then the set $F(x) = \{f(x) : f \in F\}$ is called the *orbit* of x (under F). The orbits $F(x)$ and $F(y)$ of any two points $x, y \in X$ are either equal or disjoint.

We will denote by X/F the set of all orbits. Let $\pi = \pi_F : X \to X/F$ denote the canonical mapping. Then X/F endowed with the quotient topology is called the *orbit space* of X (with respect to F). Thus π is a continuous open mapping.

For actions of compact groups, the orbit space has the following properties.

Theorem 1.6.2. *Let X be an F-space with F compact. Then*

(i) *X/F is metrizable*;
(ii) *π_F is closed*;

(iii) *If C is compact subset of X/F then $\pi_F^{-1}(C)$ is compact*;
(iv) *X is compact iff X/F is compact.*

For the proof see, for example, [4].

1.7. Locally convex spaces. A topological linear space E is said to be *locally convex* if it has a base of neighborhoods of 0 consisting of convex sets. A complete metrizable locally convex space is called a *Fréchet space*.

Suppose that E is locally convex and metrizable and that $(\|\cdot\|_n)$ is a sequence of seminorms which defines the original topology T on E. Then T is also defined by the F-norm

$$\|\cdot\| = \sum_{n=1}^{\infty} 2^{-n} \min(1, \|\cdot\|_n).$$

The topological product $\prod_i E_i$ of locally convex spaces is again locally convex. This product is complete iff each E_i is complete. The topological product of metrizable locally convex spaces is metrizable iff the product has finitely or countably many factors. The topological product of countably many Fréchet spaces is again a Fréchet space.

The *direct sum* $E = \bigoplus_{i\in I} E_i$ of the vector spaces E_i is defined to be the subspace of $\prod_i E_i$ consisting of those (x_i) which have finitely many nonzero x_i. We denote the embedding from E_i into E by I_i. This mapping sends $x_i \in E_i$ to the element $(x_j) \in E$ whose ith coordinate is x_i, and other coordinates vanish.

The *locally convex direct sum* $E = \bigoplus_i E_i$ of the locally convex spaces E_i is defined to be the direct sum E of the spaces E_i equipped with the finest locally convex topology T for which each embedding I_i from E_i into E is continuous. The locally convex direct sum $\bigoplus_i E_i$ of the locally convex spaces E_i is complete iff each E_i is complete.

Every locally convex space E is topologically isomorphic to a linear subspace $\widehat{E}$ of a topological product of Banach spaces. E is complete iff $\widehat{E}$ is closed. If, further, E is metrizable (separable) then $\widehat{E}$ is a linear subspace of some topological product of countably many Banach (separable) spaces.

The set of all continuous linear functionals on a locally convex space forms a vector space. We call it the *dual space* E' of E. If E is a locally convex space, then E and its dual E' form a dual pair $\langle E, E'\rangle$, when a bilinear form $\langle\cdot,\cdot\rangle$ is defined by $\langle x, x'\rangle = x'(x)$ for $x \in E$, $x' \in E'$.

Suppose that a family of locally convex spaces (E_i) is given. The dual of the topological product $\prod_i E_i$ is algebraically isomorphic to the direct sum $\bigoplus_i E_i'$ of the duals. The dual of the locally convex direct sum $\bigoplus_i E_i$ is algebraically isomorphic to the product $\prod_i E_i'$. In the dual pairs $\langle \prod_i E_i, \bigoplus_i E_i'\rangle$ and $\langle \bigoplus_i E_i, \prod_i E_i'\rangle$ which arise in this way, the bilinear form is given by $\langle (x_i), (x_i')\rangle = \sum_i \langle x_i, x_i'\rangle_i$ for all $(x_i), (x_i')$, where $x_i \in E_i, x_i' \in E$ and $\langle\cdot,\cdot\rangle_i$ is a bilinear form on $\langle E_i, E_i'\rangle$.

A subset T of E' is called *total* if $\{x \in E : \langle x, x'\rangle = 0 \text{ for all } x' \in T\} = \{0\}$. If E is a metrizable separable locally convex space then E' has a countable total subset.

If E is a locally convex space, the original topology T is finer than the *weak topology* $\sigma(E, E')$ and coarser than the *Mackey topology* $\tau(E, E')$. Moreover, if E is metrizable then the original topology T coincides with the Mackey topology.

We call a locally convex topology T_1 on E_1 *compatible* with the dual pair $\langle E_1, E_2\rangle$ when the dual space of E_1 equipped with the topology T_1 is E_2.

Let $\langle E_1, E_2\rangle$ be a dual pair. The topology of uniform convergence on all weakly bounded subsets of E_2 is called the *strong topology* $\beta(E_1, E_2)$ on E_1. In the general case the strong topology $\beta(E_1, E_2)$ on E_1 is not compatible, and has a larger dual space than E_2. We call the space E' with the strong topology $\beta(E', E)$ the *strong dual* of E.

A closed absorbent absolutely convex subset of a locally convex space E is called a *barrel*. If $\langle E_1, E_2\rangle$ is a dual pair then the barrels in E_1 form a base of neighborhoods of 0 for the strong topology $\beta(E_1, E_2)$ on E_1. A locally convex space E is said to be *barrelled* if the barrels form a base of neighborhoods of 0 for the original topology on E. Thus E is barrelled iff the original topology on E coincides with the strong topology $\beta(E_1, E_2)$. Moreover, if E is barrelled then $\tau(E_1, E_2) = \beta(E_1, E_2)$. The topological product and the locally convex direct sum of barrelled spaces are again barrelled. All Fréchet spaces are barrelled.

Let E be a locally convex space. We call the dual of the strong dual E' the *bidual space* of E and denote it by E''. A locally convex space E is said to be *reflexive* if $E = E''$ and if the topology $\beta(E'', E')$ coincides with the original topology on E. If E is reflexive then the strong dual E' is also reflexive.

A barrelled space E is called a *Montel space* if every bounded subset of E is relatively compact. Every Montel space is reflexive. The strong dual of a Montel space E is again a Montel space. The weak and the strong topology coincide on the bounded subset of a Montel space. The topological product and the locally convex direct sum of Montel spaces are again both Montel spaces.

A Fréchet space which is also a Montel space is called an *FM-space*. Every *FM*-space is separable.

Let E and F be locally convex spaces. Suppose that E is the locally convex direct sum of the locally convex spaces (E_i). Then

(i) A linear mapping A from E into F is continuous iff for every finite partial sum E_1 of (E_i) the restriction of A to E_1 is continuous.

(ii) A bilinear mapping B from $E \times E$ into F is continuous iff for every finite partial sum E_1 of (E_i) the restriction of B to $E_1 \times E_1$ is continuous.

Every continuous linear mapping A from a locally convex space E into a locally convex space F is weakly continuous.

Suppose that two dual pairs $\langle E_1, E_2\rangle$ and $\langle F_1, F_2\rangle$ are given. Then E_2 and F_2 are linear subspaces of the algebraic dual spaces E_1^* and F_1^*, respectively. To every linear mapping A from E_1 into F_1 there corresponds the *adjoint mapping* A' from F_1^* into E_1^*, defined by $\langle Ax, y^*\rangle = \langle x, A'y^*\rangle$ for $x \in E_1$ and $y^* \in F_1^*$. In what follows, A' will always mean the restriction of A' to $F_2 \subset F_1^*$. A linear mapping from E_1 into F_1 is weakly continuous iff A' maps F_2 into E_2. A is weakly continuous iff A' is. If a linear mapping A from E_1 into F_1 is weakly continuous then the mapping $(A')'$ adjoint to A' is equal to A, and so maps E_1 into F_1.

Let E be a Montel space and A a weakly continuous linear mapping from E into itself. Then A is strongly continuous iff A' is strongly continuous.

1.8. The space $\mathbb{R}^\infty$. The Cartesian power $\mathbb{R}^\infty$ is a linear space over $\mathbb{R}$ if for (x_n), (y_n) and $\lambda \in \mathbb{R}$, addition and multiplication are defined by $(x_n) + (y_n) = (x_n + y_n)$, $\lambda(x_n) = (\lambda x_n)$.

THEOREM 1.8.1. *The space $\mathbb{R}^\infty$ equipped with the product topology is an FM-space. The product topology coincides with the topology defined by the sequence (p_n) of seminorms such that $p_n((x_k)) = |x_n|$ for all $(x_k) \in \mathbb{R}^\infty$ and $n \in \mathbb{N}$.*

We define the *length* of $(x_k) \in \mathbb{R}^\infty$ by

$$l((x_k)) = \begin{cases} \inf\{n : x_m = 0 \text{ for all } m > n\} & \text{if } (x_k) \neq 0, \\ 0 & \text{if } (x_k) = 0. \end{cases}$$

Set $\mathbb{R}_0^\infty = \{(x_k) \in \mathbb{R}^\infty : l((x_k)) < \infty\}$.

THEOREM 1.8.2. *The space $\mathbb{R}_0^\infty$ can be identified with the dual space of $\mathbb{R}^\infty$, under the duality $\langle (x_k), (y_k) \rangle = \sum_{k=1}^\infty x_k y_k$ for all $(x_k) \in \mathbb{R}^\infty$, $(y_k) \in \mathbb{R}_0^\infty$.*

The proof is obvious.

THEOREM 1.8.3. *The space $\mathbb{R}_0^\infty$ equipped with the strong topology $\beta(\mathbb{R}_0^\infty, \mathbb{R}^\infty)$ is a locally convex direct sum. The strong topology $\beta(\mathbb{R}_0^\infty, \mathbb{R}^\infty)$ coincides with the Mackey topology $\tau(\mathbb{R}_0^\infty, \mathbb{R}^\infty)$. Moreover, $\mathbb{R}_0^\infty$ is a Montel space.*

COROLLARY 1.8.4. (i) *If A is a linear mapping from $\mathbb{R}_0^\infty$ into a locally convex space E then A is continuous.*

(ii) *If B is a real bilinear form on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ then B is continuous.*

(iii) *If A is a linear mapping from $\mathbb{R}_0^\infty$ into itself then the adjoint mapping A' from $\mathbb{R}^\infty$ into itself is continuous.*

Let ε_n be the element of $\mathbb{R}^\infty$ whose nth coordinate is equal to 1, and all other coordinates vanish.

We now state some lemmata.

LEMMA 1.8.5. *The sequence (ε_n) is a Schauder basis of $\mathbb{R}^\infty$ and a Hamel basis of $\mathbb{R}_0^\infty$. Moreover, $\langle \varepsilon_n, \varepsilon_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.*

LEMMA 1.8.6. *Let $(\underline{e}'_n)$ and $(\underline{f}'_n)$ be Hamel bases of $\mathbb{R}_0^\infty$. Then the linear mapping A from $\mathbb{R}_0^\infty$ into itself defined by the formula $A\underline{e}'_n = \underline{f}'_n$, for each $n \in \mathbb{N}$, is a topological automorphism of the locally convex space $\mathbb{R}_0^\infty$.*

LEMMA 1.8.7. *If $(\underline{e}'_n)$ is a Hamel basis of $\mathbb{R}_0^\infty$ then there exists exactly one Schauder basis $(\underline{e}_n)$ of $\mathbb{R}^\infty$ with $\langle \underline{e}_n, \underline{e}'_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.*

Proof. By Lemma 1.8.6 there exists a topological automorphism A of $\mathbb{R}_0^\infty$ such that $A\underline{e}'_n = \underline{\varepsilon}_n$ for all $n \in \mathbb{N}$. Let $\underline{e}_n = A'\underline{\varepsilon}_n$. Since A' is a topological automorphism of $\mathbb{R}^\infty$ we conclude that $(\underline{e}_n)$ is a Schauder basis of $\mathbb{R}^\infty$. Clearly,

$$\langle \underline{e}_n, \underline{e}'_m \rangle = \langle A'\underline{\varepsilon}_n, \underline{e}'_m \rangle = \langle \underline{\varepsilon}_n, A\underline{e}'_m \rangle = \langle \underline{\varepsilon}_n, \underline{\varepsilon}_m \rangle = \delta_{n,m}.$$

The rest of the assertion can be proved in a similar way. ■

LEMMA 1.8.8. *Let E be a metrizable separable locally convex space. Then there exists a continuous injective linear mapping T from E into $\mathbb{R}^\infty$. Moreover, if F is a σ-compact* *subspace of E then $T(F)$ is σ-compact and the mapping $(T|_F)^{-1}$ from $T(F)$ onto F is Borel-measurable.*

Proof. Let (x'_n) be a total sequence of continuous linear functionals defined on E. Thus the mapping T from E into $\mathbb{R}^\infty$ defined by $T(x) = (\langle x, x'_n\rangle$ for $x \in E$ is continuous linear injective. ∎

II. Basic properties of probability measures

2.1. Probability measures on metrizable spaces. Let X be a metrizable space. We denote by $\mathrm{Bo}(X)$ the smallest σ-field of subsets of X which contains all open subsets of X. $\mathrm{Bo}(X)$ is called the *Borel σ-field* of X and elements of $\mathrm{Bo}(X)$ are called *Borel sets*. By a finite positive measure μ on X we shall understand a finite positive measure on $\mathrm{Bo}(X)$. A positive finite measure μ on X is said to be *Radon* if for each $\varepsilon > 0$ there exists a compact subset K of X such that $\mu(X \setminus K) < \varepsilon$.

The collection of positive finite Radon measures on a metrizable space X will be abbreviated by $M^+(X)$. We will denote by $M^1(X)$ the set of all probability Radon measures on X.

Given two metrizable spaces X and Y, $\mu \in M^1(X)$, $A \in \mathrm{Bo}(X)$ with $\mu(X \setminus A) = 0$ and a Borel-measurable mapping f from A into Y, we denote by $f(\mu)$ the measure on Y defined by $f(\mu)(B) = \mu(f^{-1}(B))$ for $B \in \mathrm{Bo}(Y)$.

LEMMA 2.1.1. *Let μ be a probability measure on a metrizable space X. If $\mu(Z) = 1$ for some σ-compact subset Z of X then $\mu \in M^1(X)$.*

We define the *support* $\mathrm{supp}(\mu)$ of any positive finite measure μ on a metrizable space X as the complement of the union of all open subsets U of X with $\mu(U) = 0$. If $\mu \in M^+(X)$ then $\mathrm{supp}(\mu)$ is separable and $\mu(\mathrm{supp}(\mu)) = \mu(X)$.

For any element x of a metrizable space X we denote by δ_x the probability measure concentrated at x.

We denote by $\mu_t \Rightarrow \mu$ the weak convergence of a net (μ_t) of positive finite measures on a metrizable space X to a positive finite measure μ on X.

THEOREM 2.1.2. *Let X and Y be metrizable spaces, f a Borel-measurable mapping from X into Y and $\mu \in M^+(X)$. Then $f(\mu) \in M^+(Y)$.*

Proof. This follows immediately from Proposition 1.1.11 of [26] together with Theorem I.1.5 of [20]. ∎

Let X and Y be two metrizable spaces, $\mu \in M^+(Y)$ and f be a Borel-measurable mapping from X into Y with $\mu^*(Y \setminus f(X)) = 0$. A Borel subset A of X is said to be a *μ-cross-section* for the mapping f if

(i) $f(A)$ is a Borel subset of Y;
(ii) the mapping $f|_A$ from A onto $f(A)$ is a Borel isomorphism;
(iii) $\mu(Y \setminus f(A)) = 0$.

2.2. Probability measures on transformation groups

LEMMA 2.2.1. *Let X be an F-space with F compact and let K be a compact subset of X. Then there exists a Borel subset B of K such that*

(i) $\pi_F(B) = \pi_F(K)$;
(ii) *the mapping $\pi_F|_B$ from B onto $f(B)$ is a Borel isomorphism.*

Proof. Since X/F is a metrizable spaces and $\pi_F(K)$ is a compact subset of X/F, Theorem 1.4.1 of [16] proves the lemma. ∎

THEOREM 2.2.2. *Let X be an F-space with F compact and let $\mu \in M^1(X/F)$. Then there exists a μ-cross-section for the canonical mapping π_F.*

Proof. An easy computation shows that there exists a sequence (C_n) of compact subsets of X/F such that $C_n \cap C_k = \emptyset$ for $n \neq k$ and $\mu(\bigcup_{n=1}^{\infty} C_n) = 1$. Put $K_n = \pi_F^{-1}(C_n)$. By Theorem 1.6.2, K_n is compact for each $n \in \mathbb{N}$. Moreover,

(a) $K_n \cap K_k = \emptyset$ for $n \neq k$;
(b) $\pi_F(K_n) = C_n$;
(c) $F(K_n) = K_n$.

Lemma 2.2.1 now implies that for every positive integer n there exists a Borel subset B_n of K_n with

(d) $\pi_F(B_n) = K_n$;
(e) the mapping $\pi_F|_{B_n}$ from B_n onto $\pi_F(K_n)$ is a Borel isomorphism.

Hence, $\bigcup_{n=1}^{\infty} B_n$ is a μ-cross-section for the canonical mapping π_F. ∎

Let X be an F-space with F compact. A measure $\mu \in M^1(X)$ is said to be F-*invariant* if $f(\mu) = \mu$ for each $f \in F$. We denote by $M_F^1(X)$ the set of all Radon probability F-invariant measures on X.

THEOREM 2.2.3. *Let X be an F-space with F compact and $\mu \in M_F^1(X)$. Let B be a μ-cross-section for the canonical mapping π_F. Then*

$$\mu = (\Theta|_{F\times B})(\omega_F \otimes \phi_B(\mu)),$$

where ω_F is the Haar measure of F and ϕ_B is the mapping from $F(B)$ onto B defined by $\phi_B = (\pi_F|_B)^{-1}(\pi_F|_{F(B)})$. Moreover, for each $A \in \mathrm{Bo}(X)$,

$$\mu(A) = \int_B \Big[\int_F I_A(\Theta(f,x))\, \omega_F(df) \Big] \phi_B(\mu)(dx)$$
$$= \int_{\pi_F(B)} \Big[\int_F I_A(\Theta(f,(\pi|_B)^{-1}z))\, \omega_F(df) \Big] \pi_F(\mu)(dx).$$

Remark. $\phi_B(\mu)(A) = \mu(F(A))$ for all $A \in \mathrm{Bo}(B)$.

Proof of Theorem 2.2.3. Let A be a Borel subset of $F(B)$. Since for each $x \in F(B)$,

$$\int_F I_A(\Theta(f,x))\, \omega_F(df) = \int_F I_A(\Theta(f,\phi_B(x)))\, \omega_F(df)$$

we conclude that

$$\begin{aligned}\mu(A) = f(\mu)(A) &= \int_F \mu(\Theta_f^{-1}(A))\,\omega_F(df)\\ &= \int_F \Big[\int_X I_A(\Theta(f,x))\,\mu(dx)\Big]\omega_F(df)\\ &= \int_{F(B)} \Big[\int_F I_A(\Theta(f,\phi_B(x)))\,\omega_F(df)\Big]\mu(dx)\\ &= \int_B \Big[\int_F I_A(\Theta(f,x))\,\omega_F(df)\Big]\phi_B(\mu)(dx).\end{aligned}$$

Hence, $\mu = (\Theta|_{F\times B})(\omega_F \otimes \phi_B(\mu))$. ∎

The next two corollaries follow immediately from Theorem 2.2.3.

COROLLARY 2.2.4. *Let $\mu_1, \mu_2 \in M_F^+(X)$. Then $\mu_1 = \mu_2$ iff $\pi_F(\mu_1) = \pi_F(\mu_2)$.*

COROLLARY 2.2.5. *Let $\mu \in M^1(X)$. Then $\mu \in M_F^1(X)$ iff there exists $\nu \in M^1(X)$ such that*

$$\mu(A) = \int_X \Big[\int_F I_A(\Theta(f,x))\,\omega_F(df)\Big]\nu(dx) \quad \textit{for each } A \in \mathrm{Bo}(X).$$

COROLLARY 2.2.6. *Let $\nu \in M^1(X/F)$. Then there exists $\mu \in M_F^1(X)$ with $\pi_F(\mu) = \nu$.*

P r o o f. Let B be a μ-cross-section for the canonical mapping π_F and let μ be the measure defined by

$$\mu(A) = \int_C \Big[\int_F I_A(\Theta(f,(\pi_F|_B)^{-1}x))\,\omega_F(df)\Big]\nu(dx) \quad \text{for } A \in \mathrm{Bo}(X),$$

where $C = \pi_F(B)$. It is easy to see that $\mu \in M_F^1(X)$. ∎

THEOREM 2.2.7. *Let X be an F-space with F compact and (F_n) be a sequence of compact subgroups of F such that $F_n \subset F_{n+1}$ and $F \subset \overline{\bigcup_{n=1}^{\infty} F_n}$. Assume that (μ_n) is a sequence in $M^1(X)$ with $\mu_n \in M_{F_n}^1(X)$ for each $n \in \mathbb{N}$. If $\pi_F(\mu_n) \Rightarrow \nu$ for some $\nu \in M^1(X/F)$ then there exists $\mu \in M^1(X)$ such that*

(i) $\mu_n \Rightarrow \mu$;
(ii) $\pi_F(\mu) = \nu$.

P r o o f. Theorem A.III.8 of [3] implies that the sequence $(\pi_F(\mu_n))$ is uniformly tight. Hence, by Theorem 1.6.2 the sequence (μ_n) is uniformly tight. Let λ be a cluster point of (μ_n). Clearly, $\lambda \in M_F^1(X)$. Application of Corollary 2.2.4 shows that $\mu_n \Rightarrow \lambda$. ∎

COROLLARY 2.2.8. *The mapping $M_F^1(X) \ni \mu \to \pi_F(\mu) \in M^1(X/F)$ is a homeomorphism.*

Let X be an F-space and D be a subset of F. A subset A of X is said to be *invariant under D* if $D(A) = A$.

LEMMA 2.2.9. *Let X be a compact F-space with F compact. Suppose that a Borel subset Y of X is invariant under F. Then $\pi_F(Y)$ is a Borel subset of X/F.*

Proof. By Lemma 2.2.1 there exists $Z \in \mathrm{Bo}(X)$ such that $\pi_F(Z) = X/F$ and $\pi_F|_Z : Z \to X/F$ is a Borel isomorphism. Clearly, $\pi_F(Y) = \pi_F(Y \cap Z)$. ∎

THEOREM 2.2.10. *Let X be an F-space with F compact and H be a compact subgroup of F. Let $\nu \in M^1(X/F)$. Assume that a Borel subset Y of X has the following properties*:

(i) *Y is invariant under H*;
(ii) $\nu_*(\pi_F(Y)) = 1$;
(iii) *for all $y_1, y_2 \in Y$, if $F(y_1) = F(y_2)$ then $H(y_1) = H(y_2)$.*

Then there exists $\mu \in M_H^1(X)$ such that $\pi_F(\mu) = \nu$ and $\mu(Y) = 1$.

Proof. By assumption there exists a sequence (C_n) of compact subsets of $\pi_F(Y)$ such that $C_n \cap C_m = \emptyset$ for $n \neq m$ and $\nu(\bigcup_{n=1}^\infty C_n) = 1$.

Let $K_n = \pi_F^{-1}(C_n)$ and $Y_n = K_n \cap Y$. Then (K_n) and (Y_n) are two sequences of Borel subsets of X such that

(a) K_n is compact;
(b) $K_n \cap K_m = \emptyset$ for $n \neq m$;
(c) K_n is invariant under F;
(d) $Y_n \subset K_n$;
(e) $\pi_F(K_n) = \pi_F(Y_n) = C_n$;
(f) Y_n is invariant under H.

Let π_n be the canonical mapping from K_n onto K_n/H. The property (f), together with Lemma 2.2.9, yields $\pi_n(Y_n) = Y_n/H \in \mathrm{Bo}(K_n/H)$.

We define the mapping ϕ_n from Y_n/H onto $\pi_F(Y_n)$ by the formula $\phi_n(H(y)) = F(y)$ for each $y \in Y$. By property (iii), ϕ_n is bijective. Since $\phi_n(\pi_n|_{Y_n}) = \pi_F|_{Y_n}$ we conclude that ϕ_n is continuous. Hence, by Lemma 1.3.1(iii) of [16], ϕ_n^{-1} is Borel-measurable.

Set $Y_0 = \bigcup_{n=1}^\infty Y_n$. Then Y_0 is invariant under H, $Y_0 \subset Y$ and $\nu(\pi_F(Y_0)) = 1$. Let ϕ_0 be the mapping from Y_0/H onto $\pi_F(Y_0)$ such that $\phi_0|_{Y_n/H} = \phi_n$ for each $n \in \mathbb{N}$. Then ϕ_0 is a Borel isomorphism. Put $\lambda = (\phi_0^{-1})(\nu)$. It is easy to see that $\lambda \in M^1(Y/H)$, $Y_0/H \in \mathrm{Bo}(Y/H)$ and $\lambda(Y_0/H) = 1$. According to Corollary 2.2.6 there is $\mu \in M_H^1(Y)$ such that $\pi_H(\mu) = \lambda$. Clearly, $\pi_F(\mu) = \nu$. ∎

2.3. Probability measures on Abelian metrizable groups. Let G be an Abelian metrizable group. For any not necessarily bounded Borel measure μ on G we denote by $\overline{\mu}$ the measure defined by $\overline{\mu}(A) = \mu(-A)$ for $A \in \mathrm{Bo}(G)$.

Let $\mu \in M^+(G)$. The measure $\mu^{\mathrm{s}} = \mu * \overline{\mu}$ is called the *symmetrization* of μ.

We now state some lemmata.

LEMMA 2.3.1. *Let G be an Abelian metrizable group, F a subgroup of G and $\mu \in M^1(G)$ with $\mu_*(F) = 1$. Then there exists a σ-compact subgroup H of F with $\mu(H) = 1$.*

LEMMA 2.3.2. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and let (H_n) be a sequence of Borel subgroups of G. Assume that for each $n \in \mathbb{N}$ there exists $x_n \in G$ such that $\mu(H_n + x_n) = 1$. Then*

(i) *if $x \in G$ then $\mu(H_1 + x) = 1$ iff $x - x_1 \in H_1$*;
(ii) $\bigcap_{n=1}^\infty (H_n + x_n) = \bigcap_{n=1}^\infty H_n + x$ *for each* $x \in \bigcap_{n=1}^\infty (H_n + x_n)$;

(iii) $\mu(\bigcap_{n=1}^{\infty}(H_n + x_n)) = 1$.

The proof is immediate and thus is omitted.

LEMMA 2.3.3. *Let G be an Abelian metrizable group and $\mu, \mu_1, \mu_2 \in M^1(G)$ such that $\mu = \mu_1 * \mu_2$. Assume that $\mu(H) = 1$ for a Borel subgroup H of G. Then*

(i) *if $\mu_1 * \delta_x(H) = 1$ for some $x \in H$ then $\mu_2 * \delta_{-x}(H) = 1$;*
(ii) *if $\mu_1(H) = 1$ then $\mu_2(H) = 1$;*
(iii) *there is an $x \in G$ such that $\mu_1 * \delta_x(H) = \mu_2 * \delta_{-x}(H) = 1$;*
(iv) *if $\mu_1 = \nu^{\mathrm{s}}$ for some $\nu \in M^1(G)$ then $\mu_1(H) = \mu_2(H) = 1$.*

Proof. (i) Since $\mu_1(H - x) = 1$ we conclude that

$$1 = \mu(H) = \int\limits_{H-x} \mu_2(H - y)\,\mu_1(dy) = \mu_2(H + x).$$

(ii) is obvious.

(iii) The equality $1 = \mu(H) = \int_G \mu_1(H - y)\mu_2(dy)$ implies

$$\int\limits_G [1 - \mu_1(H - y)]\,\mu_2(dy) = 0. \tag{1}$$

Since $1 - \mu_1(H - y) \geq 0$ for every $x \in G$, (1) shows that $\mu_1(H - y) = 1$ μ_2-a.s. Hence, there is $x \in G$ with $\mu_1(H - x) = 1$.

(iv) The statement (iii) implies $\nu(H - x) = 1$ for some $x \in G$, hence $\mu_1(H) = 1$. Application of (ii) completes the proof of the lemma. ∎

Let G be an Abelian metrizable group. With every $\mu \in M^1(G)$ one associates the system $\mathcal{F}_\mu$ of all closed subgroups F of G for which exists an element x_F of G with the property that $\mu(F + x_F) = 1$. It is easy to verify that

(i) $\mathrm{supp}(\mu) \subset F + x_F$;
(ii) $F \in \mathcal{F}_\mu$ iff $\mathrm{supp}(\mu^{\mathrm{s}}) \subset F$;
(iii) $\bigcap \mathcal{F}_\mu \in \mathcal{F}_\mu$.

The closed subgroup $\bigcap F_\mu$ is called the *reduced group support* of the measure μ. We write $\mathrm{supp}_{\mathrm{g}}(\mu) = \bigcap F_\mu$. A measure $\mu \in M^1(G)$ is called *full* if $\mathrm{supp}_{\mathrm{g}}(\mu) = G$.

COROLLARY 2.3.4. *Let $\mu \in M^1(G)$. Then*

(i) $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is separable*;
(ii) $\mathrm{supp}_{\mathrm{g}}(\mu) = \mathrm{supp}_{\mathrm{g}}(\mu^{\mathrm{s}}) = \mathrm{supp}_{\mathrm{g}}(\mu * \delta_x)$ *for each $x \in G$.*

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. We denote by $\mathcal{G}_\mu = \mathcal{G}(\mu)$ the family of all Borel subgroups F of G with $\mu^{\mathrm{s}}(F) = 1$.

LEMMA 2.3.5. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then*

(i) *there exists a σ-compact group $F \in \mathcal{G}_\mu$;*
(ii) *if $F \in \mathcal{G}_\mu$ then* $\mathrm{supp}_{\mathrm{g}}(\mu) \subset \overline{F}$;
(iii) *if $\bigcap \mathcal{G}_\mu = \{\mathrm{supp}_{\mathrm{g}}(\mu)\}$ then* $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is σ-compact*;
(iv) *if $\bigcap \mathcal{G}_\mu = \{\mathrm{supp}_{\mathrm{g}}(\mu)\}$ and* $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is complete then* $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is a LCA group.*

LEMMA 2.3.6. *Let G be an Abelian metrizable group and X be an arbitrary nonempty subset of G. Let (μ_n) be a sequence in $M^1(G)$ and $\mu \in M^1(G)$. Then the following conditions are equivalent*:

(i) *there exists a sequence (x_n) in X such that $\mu_n * \delta_{x_n} \Rightarrow \mu$ as $n \to \infty$;*

(ii) *every increasing sequence of positive integers contains a subsequence (n_k) such that $\mu_{n_k} * \delta_{y_k} \Rightarrow \mu$ as $k \to \infty$ for some sequence (y_k) in X.*

THEOREM 2.3.7. *Let G be an Abelian metrizable group and (μ_n) and (ν_n) be two sequences in $M^1(G)$ with*

$$\mu_n = \mu_{n+1} * \nu_n \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

Then the following statements are satisfied:

1. *There exist two sequences $(x_n), (y_n)$ in G and $\lambda, \eta \in M^1(G)$ such that*

   (i) $\mu_n * \delta_{x_n} \Rightarrow \lambda$ *as* $n \to \infty$;
   (ii) $\ast_{k=0}^{n} \nu_k * \delta_{y_n} \Rightarrow \eta$ *as* $n \to \infty$;
   (iii) $\mu_0 = \lambda * \eta$;
   (iv) $x_n + y_n \in I(\mu_0)$.

2. *If $\mu_n \Rightarrow \lambda$ for some measure $\lambda \in M^1(G)$ then there exist a measure η in $M^1(G)$ and a sequence (x_n) in $I(\mu_0)$ such that*

   (i) $\ast_{k=0}^{n} \nu_k * \delta_{x_n} \Rightarrow \eta$ *as* $n \to \infty$;
   (ii) $\mu_0 = \lambda * \eta$.

The proof of the theorem is prepared by two lemmata.

LEMMA 2.3.8. *Let (μ_n) be a sequence in $M^1(G)$ weakly converging to a probability measure μ on G. Assume that (μ_n) is shift uniformly tight. Then (μ_n) is uniformly tight and $\mu \in M^1(G)$.*

Proof. Since (μ_n^{s}) is uniformly tight we conclude that $\mu^{\mathrm{s}} \in M^1(G)$ and finally $\mu \in M^1(G)$. Hence, by Theorem A.III.8 of [3], (μ_n) is uniformly tight. ∎

LEMMA 2.3.9. *Let G be an Abelian metrizable group and (μ_n) be a sequence in $M^1(G)$. Then either $\sup\{(\ast_{k=1}^{n} \mu_k)(K + x) : x \in G\} \to 0$ as $n \to \infty$ for every compact subset K of G, or there exists a sequence (x_n) in G such that for each positive integer k there exists a measure λ_k in $M^1(G)$ such that $\ast_{i=k}^{n} \mu_i * \delta_{x_n} \Rightarrow \lambda_k$ as $n \to \infty$.*

Proof. See [21]. ∎

Proof of Theorem 2.3.7. Since $\mu_0 = \mu_{n+1} * \ast_{k=0}^{n} \nu_k$ for each positive integer, we conclude that the sequence $(\ast_{k=0}^{n} \nu_k)$ is shift uniformly tight. Hence, by Lemma 2.3.9 there exist a measure η in $M^1(G)$ and a sequence (y_n) in G such that $\ast_{k=0}^{n} \nu_k * \delta_{y_n} \Rightarrow \eta$ as $n \to \infty$. Clearly, $(\mu_n * \delta_{-y_n})$ is uniformly tight.

We now show 1. Put $\mu'_n = \mu_n * \delta_{-y_n}$. Let λ_1 and λ_2 be cluster points of (μ'_n). It is easy to see that there exist two increasing sequences (n_k) and (m_k) of positive integers such that $m_{k-1} < n_k \le m_k$, $\mu'_{n_k} \Rightarrow \lambda_1$ and $\mu'_{m_k} \Rightarrow \lambda_2$.

By (2) there are $\varrho_1, \varrho_2 \in M^1(G)$ with $\lambda_1 = \lambda_2 * \varrho_1$ and $\lambda_2 = \lambda_1 * \varrho_2$. Hence, by Theorem 3.5.2 of [16], $\lambda_1 = \lambda_2 * \delta_x$ for some $x \in I(\mu_0)$. Application of Lemma 2.3.6 proves 1.

2. Lemma 2.3.8 implies that (μ_n) is uniformly tight. By Theorem 3.2.1 of [16], (y_n) is conditionally compact. Let y_0 be a cluster point of (y_n). An easy computation shows that there exists a sequence (x_n) in $I(\mu_0)$ such that $\lim_{n\to\infty}(y_n - x_n) = 0$. The rest of the proof is clear. ∎

LEMMA 2.3.10. *Let G be an Abelian metrizable group, F a compact subgroup of G and $\mu \in M^1(G)$. Assume that A is a subset of G such that*

(i) $A + F = A$;

(ii) $\mu_*(A) = 1$.

Then there exists a μ-cross-section B for the canonical mapping π_F such that $B \subset A$.

Proof. Since $\mu_*(A) = 1$ we conclude that there exists a Borel subset X of G such that $X + F = X$ and $\mu(X) = 1$. Thus the compact group F acts on X through translation. Hence, by Theorem 2.2.2 there is a μ-cross-section B for π_F such that $B \subset A$. ∎

Let G be an Abelian metrizable group. A measure $\mu \in M^1(G)$ is called *weakly infinitely divisible* if for every positive integers n there exist a measure $\mu_n \in M^1(G)$ and $x_n \in G$ such that $\mu = \mu_n^{*n} * \delta_{x_n}$. The set of all weakly infinitely divisible measures in $M^1(G)$ will be denoted by $I_0(G)$. Obviously, $I_0(G)$ is a closed subsemigroup of $M^1(G)$.

A measure $\mu \in M^1(G)$ is called *infinitely divisible* if for every positive integer n there exists a measure $\mu_n \in M^1(G)$ such that $\mu = \mu_n^{*n}$. The collection of all infinitely divisible measures in $M^1(G)$ will be denoted by $I(G)$. Of course, $I(G)$ is a subsemigroup of $I_0(G)$.

Let G be an Abelian metrizable group. If N is any bounded nonnegative Radon measure on G, the measure $e(N)$ is defined as follows:

$$e(N) = \exp(-N(G)) \sum_{k=0}^{\infty} \frac{1}{k!} N^{*k},$$

where $N^{*0} = \delta_0$. Clearly, $e(N) \in M^1(G)$. The measure $e(N)$ is called a *Poisson measure*.

Let M be a not necessarily bounded Borel measure on G vanishing at 0. If there exists a representation $M = \sup M_n$, where (M_n) is a sequence of bounded Radon measures and the sequence $(e(M_n))$ of the associated Poisson measures is shift uniformly tight, then each cluster point of the sequence $(e(M_n) * \delta_{x_n})$ $(x_n \in G)$ is called a *generalized Poisson measure* and denoted by $\widetilde{e}(M)$. Clearly, $\widetilde{e}(M) \in M^1(G)$ and $\widetilde{e}(M)$ is uniquely determined up to translation, i.e. for any two cluster points, say μ_1 and μ_2 of $(e(M_n)*\delta_{x_n})$ and $(e(M'_n) * \delta_{y_n})$ respectively, we have $\mu_1 = \mu_2 * \delta_x$ for some $x \in G$ ([22], p. 313). In the sequel, the measure M will be called a *Lévy measure*.

LEMMA 2.3.11. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$ be a generalized Poisson measure. Then $\mu \in I_0(G)$.*

LEMMA 2.3.12. *Let G be an Abelian metrizable group and $\widetilde{e}(M)$ be a generalized Poisson measure on G. Suppose that F be a Borel subgroup of G such that $\widetilde{e}(M)^{\mathrm{s}}(F) = 1$. Then $M(G \setminus F) = 0$.*

P r o o f. Putting $N = M + \overline{M}$, we obtain $\widetilde{e}(M)^{\mathrm{s}} = \widetilde{e}(N)$. Let (U_n) be the decreasing sequence of open neighborhoods of 0 in G with $\bigcap_{n=1}^{\infty} U_n = \{0\}$. Set $M_n(A) = N(A \cap U_n)$ and $N_n(A) = N(A \cap (G \setminus U_n))$ for $n \in \mathbb{N}$ and $A \in \mathrm{Bo}(G)$. Since $e(N_n)(F) = 1$ we conclude that $N_n(G) = N_n(F)$, which implies $N((G \setminus F) \cap (G \setminus U_n)) = 0$, and finally $(M + \overline{M})(G \setminus F) = 0$. ■

Let G be an Abelian metrizable group. A measure $\mu \in M^1(G)$ is called a *Gaussian measure in the sense of Parthasarathy* (*P-Gaussian*) if

(i) $\mu \in I_0(G)$;

(ii) for any factorization of μ of the form $\mu = e(M) * \lambda$ with $M \in M^+(G)$ and $\lambda \in I_0(G)$ one has $M = c\delta_0$ for some $c \in \mathbb{R}_+$.

The class of all P-Gaussian measures in $M^1(G)$ will be abbreviated by $\Gamma_P(G)$.

Let G be an Abelian metrizable group and let η be the mapping from $G \times G$ into itself defined by $\eta(x, y) = (x + y, x - y)$ for $x, y \in G$. A measure $\mu \in M^1(G)$ is said to be a *Gaussian measure in the sense of Bernstein* (*B-Gaussian*) if $\eta(\mu \otimes \mu) = (\mu * \mu) \otimes (\mu * \overline{\mu})$. The class of all B-Gaussian measures on G will be denoted by $\Gamma_B(G)$.

2.4. Invariant subgroups of probability measures. Let G be an Abelian metrizable group. For every $\mu \in M^1(G)$ we define the *invariant subgroup*

$$I(\mu) = \{x \in G : \mu = \mu * \delta_x\}.$$

LEMMA 2.4.1. *Let G be an Abelian metrizable group and $\mu, \nu \in M^1(G)$. Then*

(i) *$I(\mu)$ is compact*;

(ii) *$I(\mu) = I(\mu * \delta_x)$ for every $x \in G$*;

(iii) *$\mu * \nu = \mu$ iff $\mathrm{supp}(\nu) \subset I(\mu)$*;

(iv) *if $F \in \mathcal{G}_\mu$ then $I(\mu) \subset F$.*

P r o o f. (i) See Theorem 1.2.4 of [11].

(iii) See Theorem 1.2.7 of [11].

(iv) Without loss of generality we may assume that $\mu(F) = 1$. If $x \in I(\mu)$ then $\mu(F - x) = 1$. Lemma 2.3.2(ii) now implies that $x \in F$. ■

LEMMA 2.4.2. *Let G be an Abelian metrizable group and $\mu, \mu_1, \mu_2 \in M^1(G)$ such that $\mu = \mu_1 * \mu_2$. Then*

(i) *$I(\mu_1) \subset I(\mu)$*;

(ii) *if there exist Borel subgroups G_1 and G_2 of G with $G_1 \cap G_2 = \{0\}$ and $\mu_i(G_i) = 1$ for $i = 1, 2$ then $I(\mu) = I(\mu_1) \oplus I(\mu_2)$.*

Let G be an Abelian metrizable group. A measure $\mu \in M^1(G)$ is said to be a *measure without idempotent factors* if $I(\mu) = \{0\}$. The class of all measures without idempotent factors is denoted by $M_0^1(G)$.

An Abelian metrizable group G is called *aperiodic* if the only compact subgroup of G is $\{0\}$.

R e m a r k. We note that every aperiodic Abelian topological group (with a countable basis) is a subgroup of a (finite-dimensional) topological linear space.

The following fact is an immediate consequence of the definition of an aperiodic Abelian group.

Corollary 2.4.3. *If G is an aperiodic Abelian group then $M^1(G) = M^1_0(G)$.*

Lemma 2.4.4. *$\Gamma_P(G) \subset M^1_0(G)$.*

Proof. See Remark 5.2.2 of [11]. ∎

Lemma 2.4.5. *Let G be an Abelian metrizable group and $(\mu_i)_{i\in I}$ be a net in $M^1(G)$ such that*

(i) *μ_i is symmetric*;

(ii) *$(\mu_i)_{i\in I}$ is uniformly tight*;

(iii) *for all $i_1, i_2 \in I$ with $i_1 \le i_2$ there exists a symmetric measure $\lambda_{i_1,i_2} \in M^1(G)$ such that $\mu_{i_1} = \mu_{i_2} * \lambda_{i_1,i_2}$*;

(iv) *$\mu_i \in M^1_0(G)$ for some $i \in I$.*

Then there exists a symmetric measure $\mu \in M^1_0(G)$ such that $\mu_i \Rightarrow \mu$.

Proof. Let μ_1 and μ_2 be any two cluster points of $(\mu_i)_{i\in I}$. By (iv), $I(\mu_1) = I(\mu_2) = \{0\}$. From (iii) we conclude that there exist two symmetric measures ν_1, ν_2 in $M^1(G)$ such that $\mu_1 = \mu_2 * \nu_1$ and $\mu_2 = \mu_1 * \nu_2$. Hence, by Lemma 2.4.1(iii), $\nu_1 * \nu_2 = \delta_0$. Since ν_1 and ν_2 are symmetric we see that $\mu_1 = \mu_2$. ∎

Lemma 2.4.6. *Let G be an Abelian metrizable group, $\mu \in M^1_0(G)$ and let (ν_n) be a sequence in $M^1(G)$. Suppose that there exist an increasing sequence (k_n) of positive integers and $(x_n) \subset G$ such that $\nu_n^{*k_n} * \delta_{x_n} \Rightarrow \mu$. Then $\nu_n^{\mathrm{s}} \Rightarrow \delta_0$.*

Lemma 2.4.7. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and F be a compact subgroup of $I(\mu)$. Then*

(i) *F acts on G through the translation $\theta : F \times G \to G$ defined by $\theta(f, x) = \theta_f(x) = f + x$ for $f \in F$ and $x \in G$*;

(ii) *μ is θ_f-invariant for each $f \in F$*;

(iii) *$\pi_F(I(\mu)) = I(\pi_F(\mu))$.*

Proof. We only prove (iii). Let $x \in I(\mu)$. Since $\mu = \mu * \delta_x$ we conclude that $\pi_F(\mu) = \pi_F(\mu * \delta_x) = \pi_F(\mu) * \delta_{\pi_F(x)}$. Hence, $\pi_F(x) \in I(\pi_F(\mu))$.

Let $y \in I(\pi_F(\mu))$. This implies that there exists $x \in G$ such that $\pi_F(x) = y$. Since $\pi_F(\mu) = \pi_F(\mu) * \delta_y = \pi_F(\mu * \delta_x)$, Corollary 2.2.4 implies that $\mu = \mu * \delta_x$. ∎

Theorem 2.4.8. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and F be a compact subgroup of $I(\mu)$. Assume that B is a μ-cross-section for the canonical mapping π_F. Then*

(i) *$\mu = \omega_F * \nu$, where ν is the probability measure defined by*

$$\nu(A) = ((\pi_F|_B)^{-1}\pi_F)(\mu)(A \cap B) \quad \text{for } A \in \mathrm{Bo}(G);$$

(ii) *if $F = I(\mu)$ then $\nu \in M^1_0(G)$.*

Proof. (i) follows by Theorem 2.2.3.

(ii) Let $x \in I(\nu)$. Since $\nu(B) = \nu(B - x)$, we conclude that $B \cap (B - x) \neq \emptyset$, which implies that there is $z \in B$ such that $z - x \in B$. Clearly, $z + I(\mu) = (z - x) + I(\mu)$. Since $z, z - x \in B \cap (z + I(\mu))$, we obtain $z = z + x$. In particular, $I(\nu) = \{0\}$. ∎

THEOREM 2.4.9. *Let G be an Abelian metrizable group, F be a compact subgroup of G and $\nu \in M^1(G/F)$. Assume that there exists a Borel subgroup G_0 of G such that*

(i) *$G_0 \cap F$ is compact*;
(ii) *$\nu_*(\pi_F(G_0)) = 1$.*

Then there exists $\mu \in M^1(G)$ such that

(a) *$\pi_F(\mu) = \nu$*;
(b) *$\mu(G_0) = 1$*;
(c) *$G_0 \cap F \subset I(\mu) \subset G_0 \cap \pi_F^{-1}(I(\nu))$*;
(d) *if $\nu \in M_0^1(G/F)$ then $I(\mu) = G_0 \cap F$.*

Proof. By Theorem 2.2.10, there exists $\mu \in M^1(G)$ satisfying (a) and (b). Lemma 2.4.7(iii) implies $\pi_F(I(\mu)) = I(\nu)$. Hence, $I(\mu) \subset \pi_F^{-1}(I(\nu))$. Clearly, $I(\mu) \subset G_0$. ∎

THEOREM 2.4.10. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Let (ν_n) be a sequence in $M^1(G/I(\mu))$ such that $\pi_{I(\mu)}(\mu) = \ast_{n=1}^{\infty} \nu_n$. Assume that there exists a sequence (G_0) of Borel subgroups of G such that*

(i) *$G_n \cap G$ is compact*;
(ii) *$(\nu_n)_*(\pi_{I(\mu)}(G_n)) = 1$*;
(iii) *$I(\mu) = \overline{\bigcup_{n=1}^{\infty} (G_1 \cap I(\mu) + \ldots + G_n \cap I(\mu))}$.*

Then there exists a sequence (μ_n) in $M^1(G)$ such that

(a) *$\pi_{I(\mu)}(\mu_n) = \nu_n$*;
(b) *$\mu_n(G_n) = 1$*;
(c) *$I(\mu_n) = G_n \cap I(\mu)$*;
(d) *$\mu = \ast_{n=1}^{\infty} \mu_n$.*

Proof. By Theorem 2.4.9, for each $n \in \mathbb{N}$ there exists $\mu_n \in M^1(G)$ such that

(1) $\pi_{I(\mu)}(\mu_n) = \nu_n$;
(2) $\mu_n(G_n) = 1$;
(3) $G_n \cap I(\mu) \subset I(\mu_n)$.

Since $G_1 \cap I(\mu) + \ldots + G_n \cap I(\mu) \subset I(\ast_{k=1}^{\infty} \mu_k)$ and $\pi_{I(\mu)}(\ast_{k=1}^{\infty} \mu_k) \Rightarrow \pi_{I(\mu)}(\mu)$, Theorem 2.2.7 implies that $\mu = \ast_{n=1}^{\infty} \mu_n$. We observe that $I(\mu_n) \subset I(\mu) \cap G_n$ and hence $I(\mu_n) = I(\mu) \cap G$. ∎

COROLLARY 2.4.11. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then*

(i) *$\mu \in I_0(G)$ iff $\pi_{I(\mu)}(\mu) \in I_0(G/I(\mu))$*;
(ii) *$\mu \in I(G)$ iff $\pi_{I(\mu)}(\mu) \in I(G/I(\mu))$.*

III. Borel decomposability semigroups of probability measures

3.1. Additive measurable operators in Abelian metrizable groups. Let G be an Abelian metrizable group and $\mu \in M^1(G)$. An algebraic additive operator A in G is called an *additive measurable operator in* G (or simply a *measurable operator*) if

(i) $D(A) \in \mathrm{Bo}(G)$;
(ii) A is Borel-measurable;
(iii) $\mu(D(A) + x_0) = 1$ for some $x_0 \in G$.

We denote by $\mathrm{Add}(G;\mu)$ the set of all additive measurable operators in G.

Corollary 3.1.1. *$A(\mu * \delta_{-x_0}) \in M^1(G)$.*

The proof is a direct application of Theorem 2.1.2.

We now state some lemmata.

Lemma 3.1.2. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $A \in \mathrm{Add}(G;\mu)$. Let x_0 be an element G with $\mu(D(G) + x_0) = 1$. Then $\mu(D(G) + y) = 1$ for some $y \in G$ iff $x_0 - y \in D(A)$.*

Lemma 3.1.3. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $A, B \in \mathrm{Add}(G;\mu)$. Then*

(i) $A + B \in \mathrm{Add}(G;\mu)$;
(ii) $-A \in \mathrm{Add}(G;\mu)$;
(iii) $AB \in \mathrm{Add}(G;\mu)$ *iff* $\mu^{\mathrm{s}}(B^{-1}(D(A))) = 1$.

Lemma 3.1.4. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $A \in \mathrm{Add}(G;\mu)$. Then*

(i) $I(\mu) \subset D(A)$;
(ii) $A(I(\mu)) \subset I(A(\mu * \delta_{-x_0}))$ *for all* $x_0 \in G$ *with* $\mu(D(A) + x_0) = 1$;
(iii) $A|_{I(\mu)}$ *is continuous.*

Proof. (i) follows from Lemma 2.4.1(iv).

(ii) Let $\mu \in A(I(\mu))$. Clearly, there exists $z \in I(\mu)$ with $Az = y$. Since $\mu * \delta_{-x_0} = \mu * \delta_{-x_0} * \delta_z$, we see that $A(\mu * \delta_{-x_0}) = A(\mu * \delta_{-x_0}) * \delta_y$, and thus $y \in I(A(\mu * \delta_{-x_0}))$. ∎

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. A Borel subgroup F of G is said to be a *subdomain* of an operator $A \in \mathrm{Add}(G;\mu)$ if $F \subset D(A)$ and there exists $x \in G$ such that $\mu(F + x) = 1$.

Corollary 3.1.5. *A Borel subgroup F of G with $F \subset D(A)$ is a subdomain of A iff $\mu^{\mathrm{s}}(F) = 1$.*

Corollary 3.1.6. *If $A \in \mathrm{Add}(G;\mu)$ and F is a subdomain of A then $A|_F \in \mathrm{Add}(G;\mu)$.*

A Borel subgroup F of G is said to be a *common subdomain* of a family $\mathcal{A}$ in $\mathrm{Add}(G;\mu)$ if F is the subdomain of A for each $A \in \mathcal{A}$.

Corollary 3.1.7. *If $\mathcal{A}$ is a countable family in $\mathrm{Add}(G;\mu)$ then $\mathcal{A}$ has a common subdomain.*

THEOREM 3.1.8. *Let G be an Abelian metrizable group, μ be a measure in $M^1(G)$ and $A, B \in \mathrm{Add}(G;\mu)$. Then the following conditions are equivalent*:

(i) *there is an element x_0 in G such that $A = B$ $\mu * \delta_{-x_0}$-a.s.*;
(ii) *$A = B$ μ^{s}-a.s.*;
(iii) *there exists a common subdomain F of A and B such that $Ax = Bx$ for each $x \in F$.*

Proof. (i)⇒(ii) and (iii)⇒(i) are evident.

(i)⇒(iii). By assumption, there is $X \in \mathrm{Bo}(G)$ such that $\mu*\delta_{-x_0}(X) = 1$ and $Ax = Bx$ for all $x \in X$. An easy computation shows that there exists a common subdomain F of A and B such that $A|_F = B|_F$. ∎

COROLLARY 3.1.9. *Let $A, B \in \mathrm{Add}(G;\mu)$ such that $A = B$ $\mu * \delta_{-x_0}$-a.s. for some $x_0 \in G$. Then $A(\mu * \delta_{-x_0}) = B(\mu * \delta_{-x_0})$.*

COROLLARY 3.1.10. *The relation "$A = B$ μ^{s}-a.s. for $A, B \in \mathrm{Add}(G;\mu)$" is an equivalence relation in $\mathrm{Add}(G;\mu)$.*

Remark. $\mathrm{Add}(G;\mu)$ denotes the set of all additive Borel-measurable operators in G. At the same time the elements of $\mathrm{Add}(G;\mu)$ are understood as equivalence classes of the relation "equality μ^{s}-a.s.", but this should not lead to any confusion.

LEMMA 3.1.11. *Let G be an Abelian metrizable group with a metric ϱ and $\mu \in M^1(G)$. Define*

$$d(A,B) = \int\limits_{D(A)\cap D(B)} \frac{\varrho(Ax,Bx)}{1+\varrho(Ax,Bx)}\,\mu^{\mathrm{s}}(dx)$$

for $A, B \in \mathrm{Add}(G;\mu)$. Then d is a metric in $\mathrm{Add}(G;\mu)$.

THEOREM 3.1.12. *Let G be an Abelian metrizable group and $\mu \in M^1(G;\mu)$. Assume that $A \in \mathrm{Add}(G;\mu)$ and (A_n) is a sequence in $\mathrm{Add}(G;\mu)$. Then the following conditions are equivalent*:

(i) *$A_n \to A$ in d*;
(ii) *$A_n \to A$ in μ^{s}*;
(iii) *for every increasing sequence (n_m) of positive integers there exist a subsequence (n_k) of (n_m) and a common subdomain F of A and (A_{n_k}) such that $A_{n_k}x \to Ax$ for each $x \in F$.*

Proof. (i)⇔(ii) follows immediately.

(ii)⇒(iii). Let (n_m) be an increasing sequence of positive integers. Then there exists a subsequence (n_k) of (n_m) such that $A_{n_k} \to A$ μ^{s}-a.s. Hence, there exists a common subdomain F of (A_{n_k}) and A such that $A_{n_k}x \to Ax$ as $k \to \infty$, for all $x \in F$.

(iii)⇒(ii) is obvious. ∎

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. A Borel subgroup F of G is said to be

(i) an *invariant subdomain* of an operator $A \in \mathrm{Add}(G;\mu)$ if F is a subdomain of A and $A(F) \subset F$;

(ii) an *invariant common subdomain* of a family $\mathcal{A}$ in $\mathrm{Add}(G;\mu)$ if F is the invariant subdomain of A for each $A \in \mathcal{A}$.

THEOREM 3.1.13. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and (A_n) be a sequence in $\mathrm{Add}(G;\mu)$. Then the following conditions are equivalent*:

(i) *there exists an invariant common subdomain F of the sequence (A_n)*;
(ii) $\mathrm{Sem}((A_n)) \subset \mathrm{Add}(G;\mu)$.

Proof. (i)⇒(ii) is evident.
(ii)⇒(i). Set

$$F_1^{(1)} = D(A_1);$$
$$F_1^{(2)} = A_1^{-1}(F_1^{(1)}) \cap F_1^{(1)}, \quad F_2^{(2)} = A_2^{-1}(F_1^{(2)}) \cap F_1^{(2)};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F_1^{(n+1)} = A_1^{-1}(F_n^{(n)}) \cap F_n^{(n)}, \quad \ldots, \quad F_{n+1}^{(n+1)} = A_{n+1}^{-1}(F_n^{(n+1)}) \cap F_n^{(n+1)}.$$

Then $\{F_k^{(n)} : k = 1,\ldots,n;\ n \in \mathbb{N}\}$ is a family of Borel subgroups of G such that $F_{k+1}^{(n)} \subset F_k^{(n)}$ for all $k = 1,\ldots,n-1$, $n = 2,3,\ldots$ and $F_1^{(n+1)} \subset F_n^{(n)}$ for $n \in \mathbb{N}$. Since for every $n \in \mathbb{N}$ and $k = 1,\ldots,n$ there exist a positive integer m and $B_1,\ldots,B_m \in \mathrm{Sem}((A_n))$ such that $F_k^{(n)} = \bigcap_{i=1}^m D(B_i)$ we conclude that

$$\mu^{\mathrm{s}}(F_k^{(n)}) = 1. \tag{1}$$

Let $F = \bigcap_{n=1}^\infty \bigcap_{k=1}^n F_k^{(n)}$. By (1), $\mu^{\mathrm{s}}(F) = 1$. We have

$$A_1(F) \subset \bigcap_{n=1}^\infty \bigcap_{k=1}^n A_1(F_k^{(n)}) = \bigcap_{n=1}^\infty A_1(F_1^{(n)}) \subset \bigcap_{n=2}^\infty F_{n-1}^{(n-1)} = F.$$

Let $m > 1$. Then

$$A_m(F) \subset \bigcap_{n=1}^\infty \bigcap_{k=1}^n A_m(F_k^{(n)}) = \bigcap_{n=1}^\infty A_m(F_m^{(n)}) \subset \bigcap_{n=m}^\infty F_{m-1}^{(n-1)} = F.$$

This completes the proof of the theorem. ∎

COROLLARY 3.1.14. *Let $A \in \mathrm{Add}(G;\mu)$. Then $A^n \in \mathrm{Add}(G;\mu)$ for $n = 1,2,\ldots$ iff A has an invariant subdomain.*

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. An operator $P \in \mathrm{Add}(G;\mu)$ is called an *additive measurable projection* (or simply a *projection*) if there exists a subdomain F of P such that $P|_F$ is an algebraic additive projection.

THEOREM 3.1.15. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \mathrm{Add}(G;\mu)$. Then the following statements are equivalent*:

(i) *P is a projection*;
(ii) *$P^2 \in \mathrm{Add}(G;\mu)$ and $P = P^2$ μ^{s}-a.s.*;
(iii) *there exists a σ-compact invariant subdomain H of P such that*

 (a) *$Px = P^2x$ for each $x \in H$*;
 (b) *$H \cap \operatorname{im} P$ and $H \cap \ker P$ are σ-compact.*

Proof. (i)⇒(ii) and (iii)⇒(i) are obvious.

(ii)⇒(iii). By assumption together with Theorem 3.1.8 there exists a common subdomain F of P and P^2 such that

$$Px = P^2x \quad \text{for each } x \in F. \tag{2}$$

In particular, $P(F) \subset D(P)$. We conclude from (2) that $Px = x$ for each $x \in P(F)$, hence for all $n \in \mathbb{N}$ and $x \in P(F)$, $P^n x = x$, and finally $P^n x = Px$ for all $n \in \mathbb{N}$ and $x \in F$. But this implies that $(P^n) \subset \mathrm{Add}(G;\mu)$. By Corollary 3.1.14 there exists an invariant subdomain H_0 of P. Set

$$H' = \{x \in H_0 : Px = P^2x\}. \tag{3}$$

Clearly, $P(H') \subset H_0$ and $\mu^{\mathrm{s}}(H') = 1$.

We now show that $P(H') \subset H'$. Let $x \in P(H')$. We conclude from (3) that $Px = x$, hence that $P^2x = Px$. But this implies that $x \in H'$.

By assumption, $P(\mu^{\mathrm{s}}), (I-P)(\mu^{\mathrm{s}}) \in M^1(G)$. Consequently, there exist σ-compact subgroups H_1 and H_2 such that

$$H_1 \subset H' \cap (\operatorname{im} P) \quad \text{and} \quad H_2 \subset H' \cap (\ker P). \tag{4}$$

Moreover, $P(\mu^{\mathrm{s}})(H_1) = (I-P)(\mu^{\mathrm{s}})(H_2) = 1$. We conclude from (4) that $(P|_{H'})^{-1}(H_1) = H_1 + H' \cap (\ker P)$ and $((I-P)|_{H'}^{-1})(H_2) = H_2 + H' \cap (\operatorname{im} P)$. Since $\mu^{\mathrm{s}}(H_1 + H' \cap (\ker P)) = \mu^{\mathrm{s}}(H_2 + H' \cap (\operatorname{im} P)) = 1$ it follows that $\mu^{\mathrm{s}}(H_1 + H_2) = 1$. Put $H = H_1 + H_2$. It is easy to see that H has the required properties. ■

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. A Borel subgroup F of G is said to be a

(i) a *proper subdomain* of a projection $P \in \mathrm{Add}(G;\mu)$ if F is an invariant subdomain of P with $Px = P^2x$ for each $x \in F$;

(ii) a *proper common subdomain* of a family of projections $\mathcal{P}$ in $\mathrm{Add}(G;\mu)$ if F is a proper subdomain of P for each $P \in \mathcal{P}$.

COROLLARY 3.1.16. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then every projection $P \in \mathrm{Add}(G;\mu)$ has a σ-compact proper subdomain F such that $F \cap (\operatorname{im} P)$ and $F \cap (\ker P)$ are σ-compact.*

This follows immediately from Theorem 3.1.15.

LEMMA 3.1.17. *Let G be an Abelian metrizable group, H be a compact subgroup of G and $\mu \in M^1(G)$. Let $Q_1, Q_2 \in \mathrm{Add}(G/H; \pi_H(\mu))$ be projections such that*

$$Q_1\pi_H = Q_2\pi_H \quad \mu^{\mathrm{s}}\text{-a.s.} \tag{5}$$

Then $Q_1 = Q_2$ $\pi_H(\mu^{\mathrm{s}})$-a.s.

Proof. From (5) together with Theorem 3.1.8 we conclude that there exists a σ-compact subgroup F of G such that $\mu^{\mathrm{s}}(F) = 1$ and $Q_1\pi_H(x) = Q_2\pi_H(x)$ for each $x \in F$, which implies $Q_1 y = Q_2 y$ for each $y \in \pi_H(F)$. Clearly, $\pi_H(F) \in \mathrm{Bo}(G/H)$ and $(\pi_H(\mu^{\mathrm{s}})(\pi_H(F)) = 1$. ■

Let G be an Abelian metrizable group, H be a compact subgroup of G, $\mu \in M^1(G)$ and $P \in \mathrm{Add}(G;\mu)$. A projection $Q \in \mathrm{Add}(G/H; \pi_H(\mu))$ is called a *quotient projection* *corresponding to* P if $Q\pi_H = \pi_H P$ $\mu^{\rm s}$-a.s. We infer from Lemma 3.1.17 that Q is uniquely determined by P; it will be denoted by P_H.

THEOREM 3.1.18. *Let* G *be an Abelian metrizable group*, $\mu \in M^1(G)$, $P \in \mathrm{Add}(G;\mu)$ *be a projection and* H *be a compact subgroup* H *of* $D(P)$ *with* $H = (H\cap \mathrm{im}\, P)\oplus(H\cap \ker P)$. *Then the quotient projection* $P_H \in \mathrm{Add}(G/H;\pi_H(\mu))$ *corresponding to* P *exists.*

Proof. By assumption, $P(H) \subset H$, hence Lemma 3.1.4(iii) implies that $P(H)$ and $(I-P)(H)$ are compact. Let F_0 be a σ-compact proper subdomain of P such that $F_0 \cap \mathrm{im}\, P$ and $F_0 \cap \ker P$ are σ-compact. Set $F = ((F_0 \cap \mathrm{im}\, P) + (H \cap \mathrm{im}\, P)) \oplus ((F_0 \cap \ker P) + (H \cap \ker P))$. An easy computation shows that F is a σ-compact proper subdomain of P such that $F \cap \mathrm{im}\, P$ and $F \cap \ker P$ are σ-compact. Moreover, $H \subset F$.

Since π_H is continuous we conclude that $\pi_H(P(F))$ and $\pi_H((I-P)(F))$ are σ-compact. Hence, by Lemma 1.3.2, $\pi_H(P(F)) \cap \pi_H((I-P)(F)) = \{[0]_H\}$.

Put $F_0 = \pi_H(P(F)) + \pi_H((I-P)(F))$. Clearly, F_0 is σ-compact and $\pi_H(\mu)(F_0) = 1$. From Lemma 1.4.1 we conclude that there exists a projection $Q \in \mathrm{Add}(G/H;\pi_H(\mu))$ such that $D(Q) = F_0$, $\mathrm{im}\, Q = \pi_H(P(H))$ and $\ker Q = \pi_H((I-P)(F))$. Moreover, $Q\pi_H(x) = \pi_H P(x)$ for each $x \in F$. ∎

COROLLARY 3.1.19. *Let* $P \in \mathrm{Add}(G;\mu)$ *be a projection with* $P(I(\mu)) \subset I(\mu)$. *Then the quotient projection* $P_{I(\mu)} \in \mathrm{Add}(G/I(\mu);\pi_H(\mu))$ *corresponding to* P *exists.*

Proof. Let F be a σ-compact proper subdomain of P such that $F \cap (\mathrm{im}\, P)$ and $F \cap (\ker P)$ are σ-compact. By Lemma 2.4.1, $I(\mu) \subset F$. ∎

LEMMA 3.1.20. *Let* G *be an Abelian metrizable group*, $\mu \in M^1(G)$ *and* $A \in \mathrm{Add}(G;\mu)$. *Then*

(i) *if* $\mu \in I_0(G)$ *then* $A(\mu * \delta_{-x}) \in I_0(G)$ *for each* $x \in G$ *with* $\mu(D(A)+x) = 1$;
(ii) *if* $D(A)$ *is a divisible group and* $\mu \in I(G)$ *such that* $\mu(D(A)) = 1$ *then* $A\mu \in I(G)$.

3.2. Borel decomposability semigroups of probability measures. Let G be an Abelian metrizable group and $\mu \in M^1(G)$. We denote by $\mathbb{D}_B(\mu)$ the subset of $\mathrm{Add}(G;\mu)$ consisting of those operators A for which there exist a measure $\nu_A \in M^1(G)$ and an element $x_A \in G$ such that $\mu(D(A)+x_A) = 1$ and

$$\mu = A(\mu * \delta_{-x_A}) * \nu_A. \tag{1}$$

We denote by $\mathbb{S}_B(\mu)$ the subset of $\mathbb{D}_B(\mu)$ consisting of those operators A for which in (1) we may take $\nu_A = \delta_y$ for some $y \in G$.

COROLLARY 3.2.1. (i) $0, I \in \mathbb{D}_B(\mu)$;
(ii) $I \in \mathbb{S}_B(\mu)$.

LEMMA 3.2.2. *Let* G *be an Abelian metrizable group*, $\mu \in M^1(G)$ *and* $A \in \mathbb{D}_B(\mu)$. *Then for every* $y \in G$ *with* $\mu(D(G)+y) = 1$ *there exists a measure* $\nu \in M^1(G)$ *such that* $\mu = A(\mu * \delta_{-y}) * \nu$.

The proof is immediate from Lemma 3.1.2.

LEMMA 3.2.3. *Let* G *be an Abelian metrizable group and* $\mu \in M^1(G)$. *If* $A \in \mathrm{Add}(G;\mu)$ *and* $B \in \mathbb{D}_B(\mu)$ *then* $AB \in \mathrm{Add}(G;\mu)$.

Proof. By assumption, $\mu^{\rm s}(D(A)) = 1$ and there exists $\nu_B \in M^1(G)$ such that $\mu^{\rm s} = B\mu^{\rm s} * \nu_B^{\rm s}$. From Lemma 2.3.3(iii) we conclude that $\mu^{\rm s}(B^{-1}(D(A))) = B\mu^{\rm s}(D(A)) = 1$. Application of Lemma 3.1.3(iii) ends the proof. ∎

THEOREM 3.2.4. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then*:

(i) *$\mathbb{D}_B(\mu)$ with the metric d is a right metrizable semigroup under multiplication of operators*;

(ii) *the set $\mathbb{D}_B(\mu)$ is closed in* $\mathrm{Add}(G;\mu)$.

Proof. (i) Let $A, B \in \mathbb{D}_B(\mu)$. By Lemma 3.2.3, $\mu(D(AB) + x_{AB}) = 1$ for some $x_{AB} \in G$. Since $D(AB) + x_{AB} \subset D(B) + x_{AB}$ we conclude that $\mu(D(B) + x_{AB}) = 1$, which implies that

$$\mu * \delta_{-x_A} = B(\mu * \delta_{-x_{AB}}) * \nu'_B * \delta_{-x_A} \tag{2}$$

for some $\nu'_B \in M^1(G)$. It follows that $B(\mu * \delta_{-x_{AB}})(D(A)) = 1$. By Lemma 2.3.3(ii),

$$\nu'_B * \delta_{-x_A}(D(A)) = 1. \tag{3}$$

From (2) together with (3) we conclude that $\mu = AB(\mu * \delta_{-x_{AB}}) * A(\nu'_B * \delta_{-x_A}) * \nu_A$. This ends the proof of (i).

(ii) Let (A_n) be a sequence in $\mathbb{D}_B(\mu)$ such that $A_n \to A$ in d for some $A \in \mathrm{Add}(G;\mu)$. Hence, by Theorem 3.1.12 there exist an increasing sequence (n_k) of positive integers, a Borel subgroup H of $D(A) \cap \bigcap_{k=1}^\infty D(A_{n_k})$ and $x_0 \in G$ with $\mu(H + x) = 1$ such that

$$A_{n_k} x \to Ax \quad \text{as } k \to \infty, \tag{4}$$

for each $x \in H$. Moreover, there is a sequence (ν_k) in $M^1(G)$ with $\mu = A_{n_k}(\mu * \delta_{-x_0}) * \nu_k$. From (4) we conclude that $A_{n_k}(\mu * \delta_{-x_0}) \Rightarrow A(\mu * \delta_{-x_0})$. Clearly, $A(\mu * \delta_{-x_0}) \in M^1(G)$. Application of Theorem A.III.8 of [3] yields the sequence $(A_{n_k}(\mu * \delta_{-x_0}))$ is uniformly tight. Theorem 3.2.1 of [16] now implies that the sequence (ν_k) is uniformly tight. Without loss of generality we may assume that there exists $\nu \in M^1(G)$ such that $\nu_k \Rightarrow \nu$ as $k \to \infty$. But this implies that $\mu = A(\mu * \delta_{-x_0}) * \nu$. ∎

COROLLARY 3.2.5. *$\mathbb{S}_B(\mu)$ is a closed subgroup of $\mathbb{D}_B(\mu)$.*

The semigroup $\mathbb{D}_B(\mu)$ is called the *Borel decomposability semigroup* of the measure μ, and the subsemigroup $\mathbb{S}_B(\mu)$ is called the *Borel symmetry semigroup* of μ.

LEMMA 3.2.6. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $A \in \mathbb{D}_B(\mu)$. Then*

(i) *$I(\mu) \subset D(A)$*;

(ii) *$A(I(\mu)) \subset I(A(\mu * \delta_{-x_0})) \subset I(\mu)$ for each $x_0 \in G$ with $\mu(D(A) + x_0) = 1$*;

(iii) *$A|_{I(\mu)} : I(\mu) \to I(\mu)$ is continuous.*

Proof. (i) follows from Lemma 2.4.1.

(ii) See Lemma 3.1.4(ii) and Lemma 2.4.2(i).

(iii) follows from Lemma 3.1.4(iii). ∎

THEOREM 3.2.7. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and (A_n) be a sequence in $\mathbb{D}_B(\mu)$. Assume that F is a Borel subgroup of G with $\mu^{\rm s}(F) = 1$. Then there exists an invariant common subdomain H of (A_n) such that $H \subset F$.*

P r o o f. We denote by I_F the additive operator from F into G defined by $I_F x = x$ for $x \in F$. Clearly, $I_F \in \mathbb{D}_B(\mu)$. By Theorem 3.1.13 there exists an invariant common subdomain H of (A_n) and I_F. ■

LEMMA 3.2.8. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and (A_n) be a sequence in $\mathbb{D}_B(\mu)$. Let x_0 be an element of G such that $\mu(D(B) + x_0) = 1$ for each $B \in \mathrm{Sem}((A_n))$. Assume that (ν_n) is a sequence in $M^1(G)$ with $\mu = A_n(\mu*\delta_{-x_0})*\nu_n*\delta_{x_0}$. Then*

(i) $A_1 \ldots A_n(\mu * \delta_{-x_0}) = A_1 \ldots A_{n+1}(\mu * \delta_{-x_0}) * A_1 \ldots A_n \nu_{n+1}$;
(ii) $\mu = A_1 \ldots A_n(\mu * \delta_{-x_0}) * A_1 \ldots A_{n-1}\nu_n * \ldots * A_1\nu_2 * \nu_1$.

P r o o f. (i) Since $A_{n+1}(\mu * \delta_{-x_0})(D(A_1 \ldots A_n)) = \mu(D(A_1 \ldots A_{n+1}) + x_0) = 1$ we conclude that (i) follows from Lemma 2.3.3(i).

(ii) follows at once from (i). ■

THEOREM 3.2.9. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and (A_n) be a sequence in $\mathbb{D}_B(\mu)$. Then there exist two measures $\nu, \lambda \in M^1(G)$, two sequences $(z_n), (y_n)$ in G and an element $x_0 \in G$ such that*

(i) $\mu(D(A_1 \ldots A_n) + x_0) = 1$;
(ii) $A_1 \ldots A_n(\mu * \delta_{-x_0}) * \delta_{z_n} \Rightarrow \nu$;
(iii) $A_1 \ldots A_{n-1}\nu_n * \ldots * A_1\nu_2 * \nu_1 * \delta_{x_0+y_n} \Rightarrow \lambda$;
(iv) $\mu = \nu * \lambda$;
(v) $z_n + y_n \in I(\mu)$.

P r o o f. Follows from Lemma 3.2.8 together with Theorem 2.3.7. ■

LEMMA 3.2.10. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$ with $\bigcap \mathcal{G}_\mu = \{\mathrm{supp}_{\mathrm{g}}(\mu)\}$. Then for every $A \in \mathbb{D}_B(\mu)$,*

(i) $\mathrm{supp}_{\mathrm{g}}(\mu) \subset D(A)$;
(ii) $A(\mathrm{supp}_{\mathrm{g}}(\mu)) \subset \mathrm{supp}_{\mathrm{g}}(\mu)$;
(iii) *if* $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is complete then* $A|_{\mathrm{supp}_{\mathrm{g}}(\mu)}$ *is continuous.*

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. We denote by $\mathbb{D}(\mu)$ the subset of $\mathbb{D}_B(\mu)$ consisting all continuous operators.

If $A \in \mathbb{D}(\mu)$ then without loss of generality we may assume that $D(A) = \mathrm{supp}_{\mathrm{g}}(\mu)$.

LEMMA 3.2.11. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then*

(i) $I, 0 \in \mathbb{D}(\mu)$;
(ii) $\mathbb{D}(\mu)$ *is a subsemigroup of* $\mathbb{D}_B(\mu)$;
(iii) *there exists $x \in G$ such that for every $A \in \mathbb{D}(\mu)$ there exists $\nu \in M^1(G)$ such that $\mathrm{supp}(\nu) \subset \mathrm{supp}_{\mathrm{g}}(\mu)$ and $\mu * \delta_{-x} = A(\mu * \delta_{-x}) * \nu$.*

Set $\mathbb{S}(\mu) = \mathbb{S}_B(\mu) \cap \mathbb{D}(\mu)$.

The semigroup $\mathbb{D}(\mu)$ is called the *decomposability semigroup* of the measure μ, and the subsemigroup $\mathbb{S}(\mu)$ is called the *symmetry semigroup* of μ.

3.3. Additive projections in Borel decomposability semigroups of probability measures. Let G be an Abelian metrizable group and $\mu \in M^1(G)$. We denote by

$\Pi_B(\mu)$ the subset of $\mathbb{D}_B(\mu)$ consisting of all projections. The study of $\Pi_B(\mu)$ will be the central aim of this section.

Let $\Pi(\mu) = \Pi_B(\mu) \cap \mathbb{D}(\mu)$.

LEMMA 3.3.1. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \Pi_B(\mu)$. Then*

(i) $P(I(\mu)) \subset I(\mu)$;
(ii) $P|_{I(\mu)} : I(\mu) \to I(\mu)$ *is continuous*;
(iii) $P(I(\mu))$ *and* $(I-P)(I(\mu))$ *are compact*;
(iv) $I(\mu) = P(I(\mu)) \oplus (I-P)(I(\mu))$;
(v) $P(I(\mu)) = I(P(\mu * \delta_{-x}))$ *for each* $x \in G$ *with* $\mu(D(P)+x) = 1$.

Proof. We only prove (v). Let x be an element G with $\mu(D(P)+x) = 1$. By Lemma 3.1.4(ii), $P(I(\mu)) \subset I(P(\mu * \delta_{-x}))$. Clearly, $I(P(\mu * \delta_{-x})) \subset I(\mu) \cap (\operatorname{im} P) = P(I(\mu))$. ∎

The following result extends Proposition 1.4 of [23].

THEOREM 3.3.2. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \Pi_B(\mu)$. Assume that x is an element of G such that $\mu(D(P)+x) = 1$. Then*

(i) *if* $\mu * \delta_{-x} = P(\mu * \delta_{-x}) * \nu$ *for some* $\nu \in M^1(G)$ *then* $\nu(I(P(\mu * \delta_{-x})) + \ker P) = 1$;
(ii) *if* $\mu * \delta_{-x} = P(\mu * \delta_{-x}) * \nu$ *for some* $\nu \in M^1(G)$ *and* $I(P(\mu * \delta_{-x})) \in M_0^1(G)$ *then* $\nu = (I-P)(\mu * \delta_{-x})$;
(iii) $\mu * \delta_{-x} = P(\mu * \delta_{-x}) * (I-P)(\mu * \delta_{-x})$.

Proof. (i) Let $\nu \in M^1(G)$ such that $\mu * \delta_{-x} = P(\mu * \delta_{-x}) * \nu$. Since $P(\mu * \delta_{-x}) = P(\mu * \delta_{-x}) * P\nu$ we see that $\operatorname{supp}(P\nu) \subset I(P(\mu * \delta_{-x}))$. But this implies that

$$1 = (P\nu)(I(P(\mu * \delta_{-x}))) = \nu(I(P(\mu * \delta_{-x})) + \ker P).$$

(ii) If $I(P(\mu * \delta_{-x})) = \{0\}$ then $\nu(\ker P) = 1$. Hence,

$$(I-P)(\mu * \delta_{-x}) = (I-P)(P(\mu * \delta_{-x}) * \nu) = (I-P)\nu = \nu.$$

(iii) Corollary 3.1.16 implies that there exists a σ-compact proper subdomain F of P such that $(\operatorname{im} P) \cap F$ and $(\ker P) \cap F$ are σ-compact. Putting $Q = P|_F$ we obtain $Q \in \Pi_B(\mu)$ and $P = Q$ μ^{s}-a.s. Hence there exist $z \in G$ with $\mu(F+z) = 1$ and $\nu \in M^1(G)$ such that $\mu * \delta_{-z} = Q(\mu * \delta_{-z}) * \nu$. Clearly, $(I-Q)(I(\mu)) = I(\mu) \cap (\ker Q)$ is compact.

The statement (i) implies that $(\pi_H \nu)(\pi_H(\ker Q)) = 1$, where $H = I(\mu)$. By Theorem 2.4.9 there exists $\lambda \in M^1(F)$ with $\lambda(\ker Q) = 1$, $\pi_H(\lambda) = \pi_H(\nu)$ and $(I-Q)(I(\mu)) \subset I(\lambda)$. Since $I(\pi_H(\nu)) = \{0\}$ one has $I(\lambda) = (I-Q)(I(\mu))$. Putting $\mu_0 = Q(\mu * \delta_{-z}) * \lambda$ we obtain $I(\mu_0) = I(\mu)$ and $\pi_H(\mu * \delta_{-z}) = \pi_H(\mu_0)$. Application of Corollary 2.2.4 now implies that $\mu * \delta_{-z} = \mu_0$. An easy computation shows that $Q(\mu * \delta_{-z}) = \lambda$. ∎

COROLLARY 3.3.3. *If $P \in \Pi_B(\mu)$ then $I - P \in \Pi_B(\mu)$.*

In the following we extend Theorem 3.2.7 and Corollary 3.1.16.

THEOREM 3.3.4. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and G_0 be a Borel subgroup of G with $\mu^{\mathrm{s}}(G_0) = 1$. Assume that (P_n) is a sequence in $\Pi_B(\mu)$. Then there exists a σ-compact proper common subdomain H of (P_n) such that*

(i) $(\operatorname{im} P_n) \cap H$ *and* $(\ker P_n) \cap H$ *are* σ*-compact for* $n \in \mathbb{N}$;
(ii) $H \subset G_0$.

We prepare the proof of the theorem with the following crucial lemma.

LEMMA 3.3.5. *Let* $(\mu_{n,1})$ *and* $(\mu_{n,2})$ *be two sequences in* $M^1(G)$ *with*

$$\mu_{1,1}^{\mathrm{s}} * \mu_{1,2}^{\mathrm{s}} = \mu_{n,1}^{\mathrm{s}} * \mu_{n,2}^{\mathrm{s}} \quad \textit{for each } n \in \mathbb{N}.$$

Assume that there exist two sequences $(G_{n,1})$ *and* $(G_{n,2})$ *of* σ*-compact subgroups of* G *such that*

(i) $G_{n,1} \cap G_{n,2} = \{0\}$ *for each* $n \in \mathbb{N}$;
(ii) $\mu_{n,j}^{\mathrm{s}}(G_{n,j}) = 1$ *for* $j = 1, 2$ *and* $n \in \mathbb{N}$.

Then there exist two sequences $(H_{n,1})$ *and* $(H_{n,2})$ *of* σ*-compact subgroups of* G *such that*

(a) $H_{n,j} \subset G_{n,j}$ *for* $j = 1, 2$ *and* $n \in \mathbb{N}$;
(b) $\mu_{n,j}^{\mathrm{s}}(H_{n,j}) = 1$ *for* $j = 1, 2$ *and* $n \in \mathbb{N}$;
(c) $H_{1,1} + H_{1,2} = H_{n,1} + H_{n,2}$ *for each* $n \in \mathbb{N}$.

P r o o f. We define two families $\{F_{k,1}^{(n)} : n \in \mathbb{N};\ k = 1, \ldots, n\}$ and $\{F_{k,2}^{(n)} : n \in \mathbb{N};\ k = 1, \ldots, n\}$ inductively by

$$F_{1,j}^{(1)} = G_{1,j} \quad \text{for } j = 1, 2;$$

$$F_{1,j}^{(2)} = F_{1,j}^{(1)} \cap (F_{1,1}^{(1)} + F_{1,2}^{(1)}), \quad F_{2,j}^{(2)} = G_{2,j} \cap (F_{1,1}^{(2)} + F_{1,2}^{(2)}) \quad \text{for } j = 1, 2;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_{n+1,j}^{(n+1)} = F_{1,j}^{(n)} \cap (F_{n,1}^{(n)} + F_{n,2}^{(n)}), \quad \ldots, \quad F_{n,j}^{(n+1)} = F_{n,j}^{(n)} \cap (F_{n-1,1}^{(n+1)} + F_{n-1,2}^{(n+1)}),$$

$$F_{1,j}^{(2)} = G_{n+1,j} \cap (F_{n,1}^{(n+1)} + F_{n,2}^{(n+1)}) \quad \text{for } j = 1, 2.$$

We have

(1) $F_{m,j}^{(n+1)} \subset F_{m,j}^{(n)}$ for $j = 1, 2$ and $n \geq m$;
(2) $F_{1,1}^{(n+1)} + F_{1,2}^{(n+1)} \subset F_{n,1}^{(n)} + F_{n,2}^{(n)} \subset \ldots \subset F_{1,1}^{(n)} + F_{1,2}^{(n)}$;
(3) $F_{k,j}^{(n)}$ is σ-compact.

Moreover, for all $n \geq m$ and $j = 1, 2$, $\mu_{m,j}^{\mathrm{s}}(F_{m,j}^{(n)}) = 1$ as is easily seen by induction. Putting $H_{m,j} = \bigcap_{n=m}^{\infty} F_{m,j}^{(n)}$ for $j = 1, 2$ and $m \in \mathbb{N}$, we obtain $\mu_{m,j}^{\mathrm{s}}(H_{m,j}) = 1$, $H_{m,1} \cap H_{m,2} = \{0\}$ and $H_{1,1} \oplus H_{1,2} = H_{m,1} \oplus H_{m,2}$. ■

P r o o f o f T h e o r e m 3.3.4. By Theorem 3.2.7 together with Corollary 3.1.16 for each $n \in \mathbb{N}$ there exists a σ-compact proper subdomain F_n of P_n such that

(a) $(\operatorname{im} P) \cap F_n$ and $(\ker P) \cap F_n$ are σ-compact;
(b) $F_n \subset G_0$.

Moreover, Theorem 3.3.2 implies that

$$P_n \mu^{\mathrm{s}}((\operatorname{im} P_n) \cap F_n) = (I - P_n)\mu^{\mathrm{s}}((\ker P_n) \cap F_n) = 1.$$

Application of Lemma 3.3.5 shows that there exist two sequences $(H_{n,1})$ and $(H_{n,2})$ of σ-compact subgroups of G such that for $n \in \mathbb{N}$,

(c) $H_{n,1} \subset F_n \cap (\operatorname{im} P_n), H_{n,2} \subset F_n \cap (\ker P_n)$;
(d) $P_n\mu^{\mathrm{s}}(H_{n,1}) = (I - P_n)\mu^{\mathrm{s}}(H_{n,2}) = 1$;
(e) $H_{1,1} + H_{1,2} = H_{n,1} + H_{n,2}$.

Set $H = H_{1,1} + H_{1,2}$. It is easy to verify that H has the required properties. ■

THEOREM 3.3.6. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \operatorname{Add}(G;\mu)$ be an projection. Then $P \in \Pi_B(\mu)$ iff $P(I(\mu)) \subset I(\mu)$ and $P_{I(\mu)} \in \Pi_B(\pi_{I(\mu)}(\mu))$.*

Proof. Let $P \in \Pi_B(\mu)$. By Theorem 3.3.2 there is $x \in G$ such that

$$\mu * \delta_{-x} = P(\mu * \delta_{-x}) * (I - P)(\mu * \delta_{-x}).$$

Application of Lemma 3.3.1 shows that $P(I(\mu)) \subset I(\mu)$. Hence, by Corollary 3.1.19 there exists a quotient projection $P_{I(\mu)} \in \operatorname{Add}(G/I(\mu); \pi_{I(\mu)}(\mu))$ corresponding to P. Since $P_{I(\mu)}\pi_{I(\mu)} = \pi_{I(\mu)}P$ μ^{s}-a.s. we conclude that

$$\begin{aligned}\pi_{I(\mu)}(\mu) * \delta_{-\pi_{I(\mu)}(x)} &= \pi_{I(\mu)}(\mu * \delta_{-x}) \\ &= \pi_{I(\mu)}P(\mu * \delta_{-x}) * \pi_{I(\mu)}(I - P)(\mu * \delta_{-x}) \\ &= P_{I(\mu)}(\pi_{I(\mu)}(\mu) * \delta_{-\pi_{I(\mu)}(x)}) * (I_{I(\mu)} - P_{I(\mu)})(\pi_{I(\mu)}(\mu) * \delta_{-\pi_{I(\mu)}(x)}).\end{aligned}$$

But this implies that $P_{I(\mu)} \in \Pi_B(\pi_{I(\mu)}(\mu))$.

Let $P(I(\mu)) \subset I(\mu)$ and $P_{I(\mu)} \in \Pi_B(\pi_{I(\mu)}(\mu))$. By Corollary 3.1.16 there exists a σ-compact proper subdomain F of P such that $F \cap (\operatorname{im} P)$ and $F \cap (\ker P)$ are σ-compact. Clearly, $I(\mu) \subset F$ and $I(\mu) = I(\mu) \cap (\operatorname{im} P) \cap F \oplus I(\mu) \cap (\ker P) \cap F$. Moreover, $I(\mu) \cap (\operatorname{im} P) \cap F$ and $I(\mu) \cap (\ker P) \cap F$ are compact.

Let $x \in G$ be such that $\mu(F + x) = 1$. Since $\pi_{I(\mu)}(\mu)(\pi_{I(\mu)}(F) + \pi_{I(\mu)}(x)) = 1$ we conclude that

$$\pi_{I(\mu)}(\mu) * \delta_{-\pi_{I(\mu)}(x)} = P_{I(\mu)}(\pi_{I(\mu)}(\mu) * \delta_{-\pi_{I(\mu)}(x)}) * (I_{I(\mu)} - P_{I(\mu)})(\pi_{I(\mu)}(\mu) * \delta_{-\pi_{I(\mu)}(x)}),$$

which implies $\pi_{I(\mu)}(\mu * \delta_{-x}) = \pi_{I(\mu)}(P(\mu * \delta_{-x}) * \pi_{I(\mu)}(I - P)(\mu * \delta_{-x})$. Application of Theorem 2.4.9 now shows that there exist ν, λ in $M^1(G)$ such that

(a) $\nu(F \cap (\operatorname{im} P)) = \lambda(F \cap (\ker P)) = 1$;
(b) $\pi_{I(\mu)}(P(\mu * \delta_{-x})) = \pi_{I(\mu)}(\nu)$;
(c) $\pi_{I(\mu)}((I - P)(\mu * \delta_{-x}) = \pi_{I(\mu)}(\lambda)$;
(d) $I(\nu) = I(\mu) \cap (\operatorname{im} P) \cap F,\ I(\lambda) = I(\mu) \cap (\ker P) \cap F$.

Moreover,

(e) $I(\nu * \lambda) = I(\mu)$;
(f) $\pi_{I(\mu)}(\mu * \delta_{-x}) = \pi_{I(\mu)}(\nu * \lambda)$.

From Corollary 2.2.4 we have $\mu * \delta_{-x} = \nu * \lambda$. ■

COROLLARY 3.3.7. *Let $\mu \in M^1(G)$ and $Q \in \Pi_B(\pi_{I(\mu)}(\mu))$. Assume that there exists a σ-compact subgroup F of G such that*

(i) $F \cap I(\mu) = \{0\}$;
(ii) $(Q\pi_{I(\mu)}(\mu))^{\mathrm{s}}(\pi_{I(\mu)}(F)) = 1$.

Then there exists $P \in \Pi_B(\mu)$ such that $P_{I(\mu)} = Q$ and $\operatorname{im} P \subset F$.

LEMMA 3.3.8. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P, Q \in \Pi_B(\mu)$. Then the following statements are equivalent*:

(i) $QP = P$ μ^{s}*-a.s.*
(ii) *there exists a common subdomain F of P and Q with* $(\operatorname{im} P) \cap F \subset (\operatorname{im} Q) \cap F$;
(iii) *there exists a σ-compact proper common subdomain H of P and Q such that*
  (a) $(\operatorname{im} P) \cap H \subset (\operatorname{im} Q) \cap H$;
  (b) $(\operatorname{im} P) \cap H$ *and* $(\operatorname{im} Q) \cap H$, $(\ker P) \cap H$ *and* $(\ker Q) \cap H$ *are σ-compact.*

This is a consequence of Theorem 3.3.4.

THEOREM 3.3.9. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P, Q \in \Pi_B(\mu)$. Assume that $QP = P$ μ^{s}-a.s. Then*

(i) $P(I(\mu)) \subset Q(I(\mu))$;
(ii) $P \in \Pi_B(Q(\mu * \delta_{-x}))$ *for every $x \in G$ with* $\mu(D(Q) + x)) = 1$;
(iii) *there exists $P' \in \Pi_B(\mu)$ such that $PP' = P'$ μ^{s}-a.s., $P'P = P$ μ^{s}-a.s. and $QP' = P'Q = P'$ μ^{s}-a.s.*;
(iv) *if $P(I(\mu)) = \{0\}$ then $PQ = QP = P$ μ^{s}-a.s.*

Proof. By Lemma 3.3.8 there exists a σ-compact proper common subdomain F of P and Q such that

(a) $(\operatorname{im} P) \cap F \subset (\operatorname{im} Q) \cap F$;
(b) $(\operatorname{im} P) \cap F$, $(\operatorname{im} Q) \cap F$, $(\ker P) \cap F$, $(\ker Q) \cap F$ are σ-compact.

Let $x \in G$ be such that $\mu(F + x) = 1$. Hence

$$\mu * \delta_{-x} = P(\mu * \delta_{-x}) * (I - P)(\mu * \delta_{-x}) = Q(\mu * \delta_{-x}) * (I - Q)(\mu * \delta_{-x}).$$

(ii) By assumption

$$\begin{aligned} Q(\mu * \delta_{-x}) &= P(\mu * \delta_{-x}) * Q(I - P)(\mu * \delta_{-x}) \\ &= PQ(\mu * \delta_{-x}) * P(I - Q)(\mu * \delta_{-x}) * Q(I - P)(\mu * \delta_{-x}), \end{aligned}$$

which implies $P \in \Pi_B(Q(\mu * \delta_{-x}))$.

(iii) Since $P(Q(F)) \subset P(F) = (\operatorname{im} P) \cap F \subset (\operatorname{im} Q) \cap F = Q(F)$ we obtain $Q(F) = P(F) \oplus (I - P)(Q(F))$. We conclude from (ii) that $P \in \Pi_B(Q(\mu * \delta_{-x}))$ and hence by Theorem 3.3.2, $Q(\mu * \delta_{-x}) = P(Q(\mu * \delta_{-x})) * (I - P)(Q(\mu * \delta_{-x}))$. It is easy to see that $P(Q(\mu * \delta_{-x}))(P(F)) = 1$ and $(I - P)(Q(\mu * \delta_{-x}))((I - P)(Q(F))) = 1$.

Put $F_1 = P(F)$ and $F_2 = (I-P)(Q(F)) + (I-Q)(F)$. Thus F_1 and F_2 are σ-compact. Set $F = F_1 \oplus F_2$. Let P' be the map from F into itself defined by $P'x = x_1$, where $x = x_1 + x_2$, $x_1 \in F_1$, $x_2 \in F_2$. Clearly, $P' \in \operatorname{Add}(G; \mu)$, $(\operatorname{im} P) \cap F = (\operatorname{im} P') \cap F$ and $QP' = P'Q = P'$ μ^{s}-a.s. Since $\mu * \delta_{-x} = PQ(\mu * \delta_{-x}) * (I - P)Q(\mu * \delta_{-x}) * (I - Q)(\mu * \delta_{-x})$, we conclude that $P'(\mu * \delta_{-x}) = PQ(\mu * \delta_{-x})$ and

$$(I - P')(\mu * \delta_{-x}) = (I - P)Q(\mu * \delta_{-x}) * (I - Q)(\mu * \delta_{-x}),$$

which implies $P' \in \Pi_B(\mu)$. This ends the proof of (iii).

We now prove (iv). Let $P(I(\mu)) = \{0\}$. Hence,

$$\begin{aligned} P(\mu * \delta_{-x}) &= PQ(\mu * \delta_{-x}) * P(I - Q)(\mu * \delta_{-x}) \\ &= PQ(\mu * \delta_{-x}) * (I - P)(\mu * \delta_{-x}) * P(I - Q)(\mu * \delta_{-x}) \end{aligned}$$

$$\begin{aligned}&= PQP(\mu * \delta_{-x}) * PQ(I-P)(\mu * \delta_{-x}) * P(I-Q)(\mu * \delta_{-x})\\ &= P(\mu * \delta_{-x}) * P(Q-P)(\mu * \delta_{-x}) * P(I-Q)(\mu * \delta_{-x})\\ &= P(\mu * \delta_{-x}) * P(I-Q(\bar{\mu} * \delta_x) * P(I-Q)(\mu * \delta_{-x})\\ &= P(\mu * \delta_{-x}) * (P-PQ)\mu^{\mathrm{s}}.\end{aligned}$$

From Lemma 2.4.1(iii) we conclude that $(P-PQ)\mu^{\mathrm{s}} = \delta_0$. But this implies that $P = PQ$ μ^{s}-a.s., which is the desired assertion. ■

COROLLARY 3.3.10. *Let $P, Q \in \Pi_B(\mu)$. Assume that $QP = PQ$ μ^{s}-a.s. Then $P \in \Pi_B(Q(\mu * \delta_{-x}))$ for every $x \in G$ with $\mu(D(Q)+x)) = 1$.*

THEOREM 3.3.11. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P, Q \in \Pi_B(\mu)$. Then the following conditions are equivalent*:

(i) *$PQ = Q$ $\mu * \delta_{-x}$-a.s. and $QP = P$ $\mu * \delta_{-x}$-a.s for some $x \in G$ with $\mu(D(P) \cap D(Q)+x) = 1$;*
(ii) *$PQ = Q$ μ^{s}-a.s. and $QP = P$ μ^{s}-a.s.*;
(iii) *there exists a common subdomain F of P and Q such that* $(\operatorname{im} P) \cap F = (\operatorname{im} Q) \cap F$;
(iv) *there exists a σ-compact proper common subdomain F of P and Q such that*

(a) $(\operatorname{im} P) \cap F = (\operatorname{im} Q) \cap F$;
(b) $(\operatorname{im} P) \cap F$, $(\operatorname{im} Q) \cap F$, $(\ker P) \cap F$ *and* $(\ker Q) \cap F$ *are σ-compact*;

(v) *$P(\mu * \delta_{-x}) = Q(\mu * \delta_{-x})$ for some $x \in G$ with $\mu(D(P) \cap D(Q) + x) = 1$;*
(vi) *$P\mu^{\mathrm{s}} = Q\mu^{\mathrm{s}}$.*

Proof. The equivalence (i) of (ii) is obvious.

(ii)⇔(iii) and (iii)⇔(iv) follow at once from Lemma 3.3.8.

(iv)⇒(v). Let F be a σ-compact proper common subdomain of P and Q such that $\operatorname{im} P \cap F = \operatorname{im} Q \cap F$. Let $x \in G$ be such that $\mu(F+x) = 1$. Hence,

$$\mu * \delta_{-x} = P(\mu * \delta_{-x}) * (I-P)(\mu * \delta_{-x}) = Q(\mu * \delta_{-x}) * (I-Q)(\mu * \delta_{-x}).$$

Since $Q(\mu * \delta_{-x}) = P(\mu * \delta_{-x}) * Q(I-P)(\mu * \delta_{-x})$ and $P(\mu * \delta_{-x}) = Q(\mu * \delta_{-x}) * P(I-Q)(\mu * \delta_{-x})$, Theorem 3.5.2 of [16] implies that $P(\mu * \delta_{-x}) = Q(\mu * \delta_{-x}) * \delta_y$ for some $y \in F$. Clearly,

$$y \in (\operatorname{im} P) \cap F. \tag{1}$$

From Theorem 3.3.9(iv) we conclude that $P_{I(\mu)} = Q_{I(\mu)}$ $\pi_{I(\mu)}(\mu * \delta_{-x})$-a.s., which implies $\pi_{I(\mu)} Q(\mu * \delta_{-x}) = \pi_{I(\mu)} P(\mu * \delta_{-x}) = \pi_{I(\mu)} Q(\mu * \delta_{-x}) * \delta_{-\pi_{I(\mu)}(y)}$ and thus

$$y \in I(\mu). \tag{2}$$

By (1) together with (2) we see that $y \in (\operatorname{im} P) \cap F \cap I(\mu) = I(P(\mu * \delta_{-x}))$. But this implies that $P(\mu * \delta_{-x}) = Q(\mu * \delta_{-x})$.

(v)⇒(vi) is obvious.

(vi)⇒(iii). By Theorem 3.3.4 there exists a σ-compact proper common subdomain F of P and Q such that $(\operatorname{im} P) \cap F$, $(\operatorname{im} Q) \cap F$ and $(\ker P) \cap F$, $(\ker Q) \cap F$ are σ-compact. Since $P(\mu^{\mathrm{s}})((\operatorname{im} P) \cap F) = Q(\mu^{\mathrm{s}})((\operatorname{im} Q) \cap F) = 1$, we conclude that

$$P(\mu^{\mathrm{s}})((\operatorname{im} P) \cap (\operatorname{im} Q) \cap F) = Q(\mu^{\mathrm{s}})((\operatorname{im} P) \cap (\operatorname{im} Q) \cap F) = 1.$$

Set $F_1 = (\operatorname{im} P) \cap (\operatorname{im} Q) \cap F + (\ker P) \cap F$ and $F_2 = (\operatorname{im} P) \cap (\operatorname{im} Q) \cap F + (\ker Q) \cap F$. Then F_1 and F_2 are σ-compact subgroups of G such that $\mu^{\mathrm{s}}(F_1) = \mu^{\mathrm{s}}(F_2) = 1$. Putting $H = F_1 \cap F_2$ we obtain $\mu^{\mathrm{s}}(H) = 1$ and $(\operatorname{im} P) \cap H = (\operatorname{im} Q) \cap H$. ∎

COROLLARY 3.3.12. *Let P, Q be projections in $\Pi_B(\mu)$ such that $QP = P$ μ^{s}-a.s. and $PQ = Q$ μ^{s}-a.s. Then*

(i) $P(I(\mu)) = Q(I(\mu))$;
(ii) *if* $P(I(\mu)) = \{0\}$ $(Q(I(\mu)) = \{0\})$ *then* $P = Q$ μ^{s}*-a.s.*;
(iii) $P_{I(\mu)} = Q_{I(\mu)}$ $\pi_{I(\mu)}(\mu)$*-a.s.*;
(iv) $\ker P \cap F + P(I(\mu)) = \ker Q \cap F + P(I(\mu))$ *for some proper common subdomain F of P and Q.*

Let G be an Abelian metrizable group and $\mu \in M^1(G)$. We define the relation $\le$ on $\Pi_B(\mu)$ by

$$P \le Q \quad \text{iff} \quad PQ = QP = P \ \mu^{\mathrm{s}}\text{-a.s.}$$

It is easy to verify that $\le$ is a partial ordering in $\Pi_B(\mu)$. Moreover, $\Pi_B(\mu)$ has a smallest element, namely 0, and a largest element, namely I.

Let $\mathcal{M}$ be a nonempty subset of $\Pi_B(\mu)$. We set $I - \mathcal{M} = \{I - P : P \in \mathcal{M}\}$.

LEMMA 3.3.13. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $\mathcal{M}$ be a nonempty subset of $\Pi_B(\mu)$. Then*

(i) $\sup \mathcal{M}$ *exists iff* $\inf(I - \mathcal{M})$ *exists*;
(ii) $\inf \mathcal{M}$ *exists iff* $\sup(I - \mathcal{M})$ *exists.*

Moreover, $\sup \mathcal{M} = I - \inf(I - \mathcal{M})$ *and* $\inf \mathcal{M} = I - \sup(I - \mathcal{M})$.

LEMMA 3.3.14. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and (P_n) be a sequence in $\Pi_B(\mu)$. Then*

(i) *if* $\inf_n P_n$ *exists then there exists a σ-compact proper common subdomain F of* $\inf_n P_n$ *and* (P_n) *such that* $(\operatorname{im} Q) \cap F \subset \bigcap_{n=1}^{\infty} (\operatorname{im} P_n) \cap F$;
(ii) *if there exists $Q \in \Pi_B(\mu)$ such that $QP_n = P_n Q$ μ^{s}-a.s. for each $n \in \mathbb{N}$ and* $(\operatorname{im} Q) \cap F = \bigcap_{n=1}^{\infty} (\operatorname{im} P_n) \cap F$ *for a Borel subgroup F of G with $\mu^{\mathrm{s}}(F) = 1$ then $Q = \inf P_n$.*

3.4. Additive projections in Borel decomposability semigroups of probability measures without idempotent factors. We now study the properties of the set of all projections in Borel decomposability semigroups of Radon probability measures without idempotent factors.

The next result is an immediate consequence of Theorem 3.3.9(iv).

THEOREM 3.4.1. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $P, Q \in \Pi_B(\mu)$. Then the following conditions are equivalent*:

(i) $P \le Q$;
(ii) $QP = P$ μ^{s}*-a.s.*;
(iii) *there exists a proper common subdomain F of P and Q such that* $(\operatorname{im} P) \cap F \subset (\operatorname{im} Q) \cap F$.

Corollary 3.4.2. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $P, Q \in \Pi_B(\mu)$. Then the following conditions are equivalent*:

(i) $P = Q$ $(\mu * \delta_{-x})$-*a.s. for some* $x \in G$ *with* $\mu(D(P) \cap D(Q) + x) = 1$;

(ii) $P = Q$ μ^{s}-*a.s.*;

(iii) *there exists a common subdomain F of P and Q such that* $(\operatorname{im} P) \cap F = (\operatorname{im} Q) \cap F$;

(iv) *there exists a σ-compact proper common subdomain F of P and Q such that*

  (a) $(\operatorname{im} P) \cap F = (\operatorname{im} Q) \cap F$;

  (b) $(\operatorname{im} P) \cap F$, $(\operatorname{im} Q) \cap F$, $(\ker P) \cap F$, $(\ker Q) \cap F$ *are σ-compact*;

(v) $P(\mu * \delta_{-x}) = Q(\mu * \delta_{-x})$ *for some* $x \in G$ *with* $\mu(D(P) \cap D(Q) + x) = 1$;

(vi) $P(\mu^{\mathrm{s}}) = Q(\mu^{\mathrm{s}})$.

The proof is a direct application of Theorem 3.3.11 together with Theorem 3.4.1.

Theorem 3.4.3. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $P, Q \in \Pi_B(\mu)$. Then there exists $P \wedge Q \in \Pi_B(\mu)$ such that*

(i) *there exists a σ-compact proper common subdomain F of P, Q and $P \wedge Q \in \Pi_B(\mu)$ such that*

$$\operatorname{im}(P \wedge Q) \cap F = (\operatorname{im} P) \cap (\mathrm{imQ}) \cap F;$$

(ii) $(PQ)^n \to P \wedge Q$, $(PQ)^n P \to P \wedge Q$, $(QP)^n \to P \wedge Q$ *and* $(QP)^n Q \to P \wedge Q$ *in* μ^{s}.

We prove two lemmata before proving the theorem.

Lemma 3.4.4. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $P, Q \in \Pi_B(\mu)$. Let $(\underline{k}_n)$ be a sequence with $\underline{k}_n \in J_n$ and let $x \in G$ be such that $\mu(D(P) \cap D(Q) + x) = 1$. Then there exists a sequence (x_n) in G such that $(PQ)_{\underline{k}_n}(\mu * \delta_{-x}) * \delta_{x_n} \Rightarrow \delta_0$.*

The proof is immediate from Lemma 2.3.13 together with Theorem 1.3.5.

Lemma 3.4.5. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $P, Q \in \Pi_B(\mu)$. Assume that x is an element of G such that $\mu(D(P) \cap D(Q) + x) = 1$. Then*

(i) *for every positive integer n*

$$\mu * \delta_{-x} = \mathop{\ast}_{\underline{k}_n \in J_n} (PQ)_{\underline{k}_n}(\mu * \delta_{-x}) * \mathop{\ast}_{i,j=0}^{1} (PQ)_{\underline{ij}_n}(\mu * \delta_{-x});$$

(ii) $\{\ast_{\underline{k}_n \in J_n} (PQ)_{\underline{k}_n}(\mu^{\mathrm{s}}) : n \in \mathbb{N}\}$ *is shift uniformly tight*;

(iii) *there exist measures η, μ_{ij}^{PQ}, $(i, j = 0, 1)$ in $M^1(G)$ and sequences (y_n), $(x_n^{(ij)})$ $(i, j = 0, 1)$ in G such that*

  (a) $(PQ)_{\underline{ij}_n}(\mu * \delta_{-x}) * \delta_{x_n^{(ij)}} \Rightarrow \mu_{ij}^{PQ}$ *for* $i, j = 0, 1$;

  (b) $\ast_{\underline{k}_n \in J_n} (PQ)_{\underline{k}_n}(\mu * \delta_{-x}) * \delta_{y_n} \Rightarrow \eta$, *where* $y_n = -\sum_{i,j=0}^{1} x_n^{(ij)}$;

  (c) $\mu * \delta_{-x} = \eta * \ast_{i,j=0}^{1} \mu_{ij}^{PQ}$.

Proof. Statements (i) and (ii) are evident. The proof of (iii) is immediate from Theorem 3.2.9. ∎

P r o o f o f T h e o r e m 3.4.3. By Theorem 3.3.4, there exists a σ-compact proper common subdomain F of P and Q. Let $x \in G$ be such that $\mu(F + x) = 1$. Hence, by Lemma 3.4.7 there exist $\mu_{PQ}, \mu_{QP}, \eta, \lambda \in M^1(G)$ and two sequences $(x_n), (y_n)$ in G such that

(a) $\mu * \delta_{-x} = (PQ)_{\underline{00}_n}(\mu * \delta_{-x}) * \delta_{x_n} * \lambda_n = (QP)_{\underline{00}_n}(\mu * \delta_{-x}) * \delta_{y_n} * \eta_n$;
(b) $(PQ)_{\underline{00}_n}(\mu * \delta_{-x}) * \delta_{x_n} \Rightarrow \mu_{PQ}$ as $n \to \infty$;
(c) $(QP)_{\underline{00}_n}(\mu * \delta_{-x}) * \delta_{y_n} \Rightarrow \mu_{QP}$ as $n \to \infty$;
(d) $\lambda_n \Rightarrow \lambda$ and $\eta_n \Rightarrow \eta$;
(e) $\mu * \delta_{-x} = \mu_{PQ} * \lambda = \mu_{QP} * \eta$.

We conclude from (a) that for all $n, m \in \mathbb{N}$,

$$(PQ)_{\underline{00}_n}(\mu^{\mathrm{s}}) = (PQ)_{\underline{00}_{n+m}}(\mu^{\mathrm{s}}) * (PQ)_{\underline{00}_m}(\lambda^{\mathrm{s}}_m),$$

hence for each $n \in \mathbb{N}$ there exists $\nu_n \in M^1(G)$ such that $(PQ)_{\underline{00}_n}(\mu^{\mathrm{s}}) = \mu^{\mathrm{s}}_{PQ} * \nu_n$ and finally that

$$Q^{(1)}(PQ)_{\underline{00}_n}(\mu^{\mathrm{s}}) = Q^{(1)}(\mu^{\mathrm{s}}_{PQ}) * Q^{(1)}\nu_n. \tag{1}$$

Lemma 3.4.4 shows that

$$Q^{(1)}(PQ)_{\underline{00}_n}(\mu^{\mathrm{s}}) \Rightarrow \delta_0 \quad \text{as } n \to \infty. \tag{2}$$

From (1) and (2) we conclude that $(I - Q)(\mu^{\mathrm{s}}_{PQ}) = \delta_0$. This implies that

$$\mu^{\mathrm{s}}_{PQ}((\operatorname{im} Q) \cap F) = 1. \tag{3}$$

From (a) it follows that $P(\mu^{\mathrm{s}}) = (PQ)_{\underline{00}_n}(\mu^{\mathrm{s}}) * P\lambda^{\mathrm{s}}_n$. Hence, there exists $\lambda_P \in M^1(G)$ such that $P\mu^{\mathrm{s}} = \mu^{\mathrm{s}}_{PQ} * \lambda_P$. Thus, by Lemma 2.3.3(iv),

$$\mu^{\mathrm{s}}_{PQ}((\operatorname{im} P) \cap F) = 1. \tag{4}$$

From (3) and (4) we have

$$\mu^{\mathrm{s}}_{PQ}((\operatorname{im} P) \cap (\operatorname{im} Q) \cap F) = 1. \tag{5}$$

In the same manner we can see that

$$\mu^{\mathrm{s}}_{QP}((\operatorname{im} P) \cap (\operatorname{im} Q) \cap F) = 1. \tag{6}$$

Conditions (5) and (6) imply that

$$\mu^{\mathrm{s}}_{PQ} = P\mu^{\mathrm{s}}_{PQ} = Q\mu^{\mathrm{s}}_{PQ} \quad \text{and} \quad \mu^{\mathrm{s}}_{QP} = P\mu^{\mathrm{s}}_{QP} = Q\mu^{\mathrm{s}}_{QP}. \tag{7}$$

We conclude from (f) and (7) that for each $n \in \mathbb{N}$,

$$(QP)_{\underline{00}_n}(\mu^{\mathrm{s}}) = \mu^{\mathrm{s}}_{PQ} * (QP)_{\underline{00}_n}\lambda^{\mathrm{s}} \quad \text{and} \quad (PQ)_{\underline{00}_n}(\mu^{\mathrm{s}}) = \mu^{\mathrm{s}}_{QP} * (PQ)_{\underline{00}_n}\eta^{\mathrm{s}},$$

hence there exist two symmetric measures $\lambda', \eta' \in M^1_0(G)$ such that

$$\mu^{\mathrm{s}}_{PQ} = \mu^{\mathrm{s}}_{QP} * \lambda', \quad \mu^{\mathrm{s}}_{QP} = \mu^{\mathrm{s}}_{PQ} * \eta'. \tag{8}$$

Hence, $\mu^{\mathrm{s}}_{PQ} = \mu^{\mathrm{s}}_{PQ} * \lambda' * \eta'$. By Theorem 1.2.13 of [11], $\lambda' * \eta' = \delta_0$. Since the measures λ', η' are symmetric we obtain $\lambda' = \eta' = \delta_0$. From (8) we have $\mu^{\mathrm{s}}_{PQ} = \mu^{\mathrm{s}}_{QP}$. The equality $\mu^{\mathrm{s}} = \mu^{\mathrm{s}}_{PQ} * \lambda^{\mathrm{s}} = \mu^{\mathrm{s}}_{QP} * \lambda^{\mathrm{s}}$, implies that $(QP)_{\underline{00}_n}(\mu^{\mathrm{s}}) = \mu^{\mathrm{s}}_{PQ} * (PQ)_{\underline{00}_n}\lambda^{\mathrm{s}}$ and $(QP)_{\underline{00}_n}(\mu^{\mathrm{s}}) = \mu^{\mathrm{s}}_{QP} * (QP)_{\underline{00}_n}\lambda^{\mathrm{s}}$, hence that $(QP)_{\underline{00}_n}\lambda^{\mathrm{s}} \Rightarrow \delta_0$ and $(PQ)_{\underline{00}_n}\lambda^{\mathrm{s}} \Rightarrow \delta_0$. Clearly, $(QP)_{\underline{00}_n} \to 0$ in λ^{s} and $(PQ)_{\underline{00}_n} \to 0$ in λ^{s}.

Let (n_m) be an increasing sequence of positive integers. Then there exists a subsequence (n_k) of (n_m) such that $(QP)_{\underline{00}_{n_k}} \to 0$ λ^{s}-a.s. and $(PQ)_{\underline{00}_{n_k}} \to 0$ λ^{s}-a.s. Put $F_1 = (\operatorname{im} P) \cap (\operatorname{im} Q) \cap F$, $F_2 = \{x \in F : (QP)_{\underline{00}_{n_k}} x \to 0 \text{ and } (PQ)_{\underline{00}_{n_k}} x \to 0\}$. Then $F_1 \cap F_2 = \{0\}$, $\mu^{\mathrm{s}}_{PQ}(F_1) = 1$, $\lambda^{\mathrm{s}}(F_2) = 1$. Let H_1 be a σ-compact subgroup of F_1 with $\mu^{\mathrm{s}}_{PQ}(H_1) = 1$ and let H_2 be a σ-compact subgroup of F_2 with $\lambda^{\mathrm{s}}(H_2) = 1$. By Lemma 1.4.1, there exists a Borel-measurable projection R from $H = H_1 \oplus H_2$ into itself such that $\operatorname{im} R = H_1$ and $\ker R = H_2$. Clearly, $R \in \operatorname{Add}(G;\mu)$. Since for each $x \in H$, $(QP)_{\underline{00}_{n_k}} x \to Rx$ as $k \to \infty$, and $(PQ)_{\underline{00}_{n_k}} x \to Rx$ as $k \to \infty$, we conclude from Theorem 3.1.12 that $(QP)_{\underline{00}_n} \to R$ in μ^{s} and $(PQ)_{\underline{00}_n} \to R$ in μ^{s}. Hence, by Theorem 3.2.4(ii), $R \in \Pi_B(\mu)$. It is easy to see that $R = P \wedge Q$. ∎

Corollary 3.4.6. *Let $\mu \in M^1_0(G)$ and $P_1, \ldots, P_n \in \Pi_B(\mu)$. Then there exists a projection $P_1 \wedge \ldots \wedge P_n \in \Pi_B(\mu)$ such that*

$$\operatorname{im}(P_1 \wedge \ldots \wedge P_n) \cap F = \bigcap_{i=1}^{n} \operatorname{im}(P_i) \cap F$$

for some σ-compact proper common subdomain F of $\{P_1, \ldots, P_n\}$ and $P_1 \wedge \ldots \wedge P_n$.

Theorem 3.4.7. *Let G be an Abelian metrizable group, $\mu \in M^1_0(G)$ and (P_n) be a sequence in $\Pi_B(\mu)$ such that $P_{n+1} \le P_n$. Then there exists* $\inf P_n \in \Pi_B(\mu)$ *such that*

(i) $P_n \to \inf_n P_n$ *in* μ^{s};

(ii) $\operatorname{im}(\inf_n P_n) \cap F = \bigcap_{n=1}^{\infty} \operatorname{im}(P_n) \cap F$ *for some σ-compact proper common subdomain F of (P_n) and* $\inf_n P_n$.

Proof. By Theorem 3.3.4, there exists a σ-compact proper common subdomain F of (P_n). Moreover, by Theorem 3.3.2, $\mu^{\mathrm{s}} = P_n\mu^{\mathrm{s}} * (I - P_n)\mu^{\mathrm{s}}$. But this implies that for $1 \le k \le n$, $P_k\mu^{\mathrm{s}} = P_n\mu^{\mathrm{s}} * (P_k - P_n)\mu^{\mathrm{s}}$.

We deduce from Theorem 3.2.9 that there exist $\nu_0, \nu_1 \in M^1_0(G)$ and a sequence (λ_n) in $M^1_0(G)$ such that $P_n\mu^{\mathrm{s}} \Rightarrow \nu_0$, $(I - P_n)\mu^{\mathrm{s}} \Rightarrow \nu_1$ and $(P_k - P_n)\mu^{\mathrm{s}} \Rightarrow \lambda_k$ as $n \to \infty$. Hence,

$$\mu^{\mathrm{s}} = \nu_0 * \nu_1, \tag{9}$$

$$P_k\mu^{\mathrm{s}} = \nu_0 * \lambda_k. \tag{10}$$

Now, by (10) we conclude via Lemma 2.3.3(iv) that $\nu_0((\operatorname{im} P_k) \cap F) = 1$. In particular,

$$P_k\nu_0 = \nu_0. \tag{11}$$

We conclude from (9) together with (11) that $P_k\mu^{\mathrm{s}} = \nu_0 * P_k\nu_1$, which implies that $P_k\nu_1 \Rightarrow \delta_0$ as $k \to \infty$, and thus $P_k \to 0$ in ν_1.

Let (n_m) be an increasing sequence of positive integers. Then there exists a subsequence (n_k) of (n_m) such that $P_{n_k} \to 0$ ν_1-a.s. Put $F_1 = \bigcap_{n=1}^{\infty} (\operatorname{im} P_n) \cap F$ and $F_2 = \{x \in F : P_{n_k}x \to 0\}$. Then $F_1 \cap F_2 = \{0\}$ and $\nu_i(F_i) = 1$ for $i = 1, 2$. Let H_i be a σ-compact subgroup of F_i with $\nu_i(F_i) = 1$ for $i = 1, 2$. Lemma 1.4.1 now shows that there exists $R \in \operatorname{Add}(G;\mu)$ such that $\operatorname{im} R = H_1$ and $\ker R = H_2$. Since for each $x \in H_1 \oplus H_2$, $P_{n_k}x \to Rx$ as $k \to \infty$, we conclude from Theorem 3.1.12 that $P_n \to R \in \mu^{\mathrm{s}}$. Hence, by Theorem 3.2.4, $R \in \Pi_B(\mu)$. It is easy to see that $R = \inf_n P_n$. ∎

COROLLARY 3.4.8. *Let* (P_n) *be a sequence in* $\Pi_B(\mu)$ *such that* $P_n \le P_{n+1}$. *Then there exists* $\sup_n P_n \in \Pi_B(\mu)$ *such that*

(i) $P_n \to \sup_k P_k$ *in* μ^{s};

(ii) $\ker(\sup_n P_n) \cap F = \bigcap_{n=1}^{\infty} (\ker P_n) \cap F$ *for some* σ*-compact proper common subdomain* F *of* (P_n) *and* $\sup_n P_n$.

Proof. Putting $Q_n = I - P_n$ we obtain $Q_n \le Q_{n+1}$. Hence, by Theorem 3.4.7 there exists $\inf_n Q_n \in \Pi_B(\mu)$ such that

(i) $Q_n \to \inf_n Q_n$ in μ^{s};

(ii) $\mathrm{im}(\inf_n Q_n) \cap F = \bigcap_{n=1}^{\infty} \mathrm{im}(Q_n) \cap F$ for some σ-compact proper common subdomain of (P_n).

Putting $P = I - Q$ we obtain $P = \sup_n P_n$. The rest of the proof is clear. ∎

THEOREM 3.4.9. *Let* G *be an Abelian metrizable group*, $\mu \in M_0^1(G)$ *and* (P_n) *be a sequence in* $\Pi_B(\mu)$. *Then there exists* $\inf_n P_n \in \Pi_B(\mu)$ *with*

$$\mathrm{im}(\inf_n P_n) \cap F = \bigcap_{n=1}^{\infty} \mathrm{im}(P_n) \cap F$$

for some σ*-compact proper common subdomain* F *of* (P_n) *and* $\inf_n P_n$.

Proof. Putting $Q_n = P_1 \wedge \ldots \wedge P_n$ we see that $Q_n \le Q_{n+1}$ and there exists a proper common subdomain F_0 of (P_n) such that $\mathrm{im}(Q_n) \cap F_0 = \bigcap_{k=1}^{\infty} \mathrm{im}(P_k) \cap F_0$. Moreover, by Theorem 3.4.7 there exists $\inf_n Q_n \in \Pi_B(\mu)$ with $\mathrm{im}(\inf_k Q_k) \cap F_1 = \bigcap_{n=1}^{\infty} \mathrm{im}(Q_n) \cap F_1$ for some proper common subdomain F_1 of (Q_n). Put $F = F_1 \cap F_2$. Since $\mathrm{im}(\inf_k Q_k) \cap F = \bigcap_{n=1}^{\infty} \mathrm{im}(P_n) \cap F$ we conclude that $\inf_n Q_n = \inf P_n$. ∎

THEOREM 3.4.10. *Let* G *be an Abelian metrizable group*, $\mu \in M_0^1(G)$ *and* $(P_a)_{a \in \mathcal{A}}$ *be a family in* $\Pi_B(\mu)$. *Then there exists* $\inf_{a \in \mathcal{A}} P_a \in \Pi_B(\mu)$ *such that* $\inf P_a = \inf P_n$ μ^{s}*-a.s. for some countable subfamily* (P_n) *of* (P_a).

Proof. We will denote by $\mathcal{B}$ the family of all finite subsets of $\mathcal{A}$. We define the relation $\le$ on $\mathcal{B}$ by

$$b_1 \le b_2 \quad \text{iff} \quad b_1 \subset b_2.$$

Set $Q_b = \inf_{a \in b} P_a$ for each $b \in \mathcal{B}$. Then $(Q_b \mu^{\mathrm{s}})_{b \in \mathcal{B}}$ is a net in $M_0^1(G)$. Moreover, $\mu^{\mathrm{s}} = Q_b \mu^{\mathrm{s}} * (I - Q_b) \mu^{\mathrm{s}}$ for each $b \in \mathcal{B}$. If $b_1,\ b_2 \in \mathcal{B}$, $b_1 \le b_2$ then $Q_{b_1} \le Q_{b_2}$ and finally $Q_{b_1} \mu^{\mathrm{s}} = Q_{b_2} \mu^{\mathrm{s}} * (Q_{b_1} - Q_{b_2}) \mu^{\mathrm{s}}$.

Application of Lemma 2.4.5 implies that there exist $\nu_0, \nu_1 \in M_0^1(G)$ and a net $(\lambda_b)_{b \in \mathcal{B}}$ in $M_0^1(G)$ such that

$$Q_b \mu^{\mathrm{s}} \Rightarrow \nu_0, \qquad (I - Q_b) \mu^{\mathrm{s}} \Rightarrow \nu_1 \tag{12}$$

and $(Q_c - Q_b) \mu^{\mathrm{s}} \Rightarrow \lambda_c$ for each $c \in \mathcal{B}$. Hence, $\mu^{\mathrm{s}} = \nu_0 * \nu_1$ and for each $c \in \mathcal{B}$ we have

$$Q_c \mu^{\mathrm{s}} = \nu_0 * \lambda_c. \tag{13}$$

By Lemma 2.3.3(iv) together with (13) we now conclude that $\nu_0(\mathrm{im}\, Q_c) = 1$, so that $Q_c(\nu_0) = \nu_0$ is satisfied for each $c \in \mathcal{B}$.

Now (12) implies, via Theorem A.III.5 of [3], the existence of an increasing sequence (b_n) in B such that $Q_{b_n}\mu^{\rm s} \Rightarrow \nu_0$. By $Q_{b_{n+1}} \le Q_{b_n}$ we conclude via Theorem 3.4.7 that there exists $\inf Q_{b_n} \in \varPi_B(\mu)$ such that $Q_{b_n} \Rightarrow \inf_n Q_{b_n}$ in $\mu^{\rm s}$, hence that $Q_{b_n}\mu^{\rm s} \Rightarrow \inf_n Q_{b_n}(\mu^{\rm s})$ and finally that $\nu_0 = (\inf_n Q_{b_n})(\mu^{\rm s})$. So for each $b \in \mathcal{B}$, $Q_b(\inf Q_{b_n}(\mu^{\rm s})) = \inf Q_{b_n}(\mu^{\rm s})$. Application of Corollary 3.4.2 now yields $Q_b(\inf Q_{b_n}) = \inf_n Q_{b_n}$ $\mu^{\rm s}$-a.s. for each $b \in \mathcal{B}$. Therefore,

$$\inf Q_{b_n} \le P_a \tag{14}$$

for each $a \in \mathcal{A}$.

Let R be a projection in $\varPi_B(\mu)$ such that $R \le P_a$ for each $a \in \mathcal{A}$. But this implies that $R \le Q_{b_n}$ for each $n \in \mathbb{N}$, and finally

$$R \le \inf Q_{b_n}. \tag{15}$$

From (14) together with (15) we conclude that $\inf_n Q_{b_n} = \inf_{a\in\mathcal{A}} P_a$. ∎

We finish the discussion of this section with the formulation of the following theorem.

THEOREM 3.4.11. *$\varPi_B(\mu)$ forms a complete lattice. Moreover*,

(i) *for every family $(P_a)_{a\in\mathcal{A}}$ in $\varPi_B(\mu)$ there exists a countable subfamily (P_n) of (P_a) such that* $\inf_{a\in\mathcal{A}} P_a = \inf_n P_n$ *$\mu^{\rm s}$-a.s.*;

(ii) $\inf \varPi_B(\mu) = 0$;

(iii) $\sup \varPi_B(\mu) = I$.

Proof. This follows from Theorem 3.4.10. ∎

IV. Product-decomposability of probability measures

4.1. Basic definitions and results. Let G be an Abelian metrizable group. A measure μ in $M^1(G)$ is said to be *product-decomposable* if there exist two nondegenerate measures μ_1, μ_2 on G, two Borel subgroups G_1, G_2 of G and an element x in G such that

(i) $\mu = \mu_1 * \mu_2 * \delta_x$;

(ii) $\mu_i(G_i) = 1$ for $i = 1, 2$;

(iii) $G_1 \cap G_2 = \{0\}$.

In the opposite case, μ is said to be *product-indecomposable*.

THEOREM 4.1.1. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then the following statements are equivalent*:

(i) μ *is product-decomposable*;

(ii) $\varPi_B(\mu) \ne \{0, I\}$.

A nondegenerate measure ν in $M^1(G)$ is said to be a *product-factor* of a measure μ in $M^1(G)$ if there exist $P \in \varPi_B(\mu)$ and $x \in G$ with $\mu(D(P) + x) = 1$ such that

$$\nu = P(\mu * \delta_{-x}).$$

LEMMA 4.1.2. *Let an Abelian Polish group G have no nontrivial continuous additive projections from G into itself. Let μ be a measure in $M^1(G)$ with $\bigcap \mathcal{G}_\mu = \{G\}$. Then μ is product-indecomposable.*

The proof is immediate from Lemma 3.2.10.

A projection $P \in \Pi_B(\mu) \setminus \{0\}$ is said to be a *product-atom* if

$$\{Q \in \Pi_B(\mu) : 0 \leq Q \leq P\} = \{0, I\}.$$

We denote by $\mathcal{A}_B(\mu)$ the set of all product-atoms from $\Pi_B(\mu)$ and let $\mathcal{A}(\mu) = \Pi(\mu) \cap \mathcal{A}_B(\mu)$.

The following facts are immediate consequences of the definition of a product-atom.

COROLLARY 4.1.3. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \Pi_B(\mu) \setminus \{0\}$. Then P is a product-atom iff the measure $P(\mu * \delta_{-x})$ is product-indecomposable for some $x \in G$ with $\mu(D(P) + x) = 1$.*

COROLLARY 4.1.4. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P, Q \in \Pi_B(\mu)$ with $PQ = QP$ μ^{s}-a.s. Then*

(i) *if $P \in \mathcal{A}_B(\mu)$ then either $P \leq Q$ or $P \leq I - Q$;*
(ii) *if $P, Q \in \mathcal{A}_B(\mu)$ then $P \leq I - Q$.*

LEMMA 4.1.5. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \mathcal{A}_B(\mu)$. Assume that Q is a projection in $\Pi_B(\mu)$ such that* $\operatorname{im} P \cap F = \operatorname{im} Q \cap F$ *for some common subdomain F of the projections P and Q. Then $Q \in \mathcal{A}_B(\mu)$.*

Proof. This follows immediately from Theorem 3.3.9. ■

COROLLARY 4.1.6. *Let $P \in \mathcal{A}_B(\mu)$ and $Q \in \Pi_B(\mu)$ with $QP = P$ μ^{s}-a.s. Then there exists $P' \in \mathcal{A}_B(\mu)$ such that*

(i) *$PP' = P'$ μ^{s}-a.s. and $P'P = P$ μ^{s}-a.s.;*
(ii) *$QP' = P'Q = P'$ μ^{s}-a.s.*

A measure μ in $M^1(G)$ is said to be

(i) *purely product-atomic* if for every $P \in \Pi_B(\mu) \setminus \{0\}$ there exists $Q \in \mathcal{A}_B(\mu)$ such that $0 < Q \leq P$;
(ii) *product-atomless* if it has no product-atoms.

4.2. Gaussian measures in the sense of Gnedenko. We start with the introduction of the class of Gaussian measures in the sense of Gnedenko.

Let G be an Abelian metrizable group. A measure μ in $M^1(G)$ is said to be a *Gaussian measure in the sense of Gnedenko* (*G-Gaussian*) if there exist $\nu \in M^1(G)$ and $P \in \Pi_B(\mu \otimes \nu)$ such that

$$((G \times \{0\}) \cup (\{0\} \times G)) \cap ((\operatorname{im} P) \cup (\ker P)) \cap F = \{(0,0)\}$$

for some Borel subgroup F of $G \times G$ with the property that $(\mu \otimes \nu)^{\mathrm{s}}(F) = 1$. The class of all G-Gaussian measures on G will be denoted by $\Gamma_G(G)$.

EXAMPLE 4.2.1. Lemma 1.2.1 implies that for every prime p the Haar measure $\omega_{\mathbb{Z}_p}$ on $\mathbb{Z}_p$ is G-Gaussian.

Lemma 4.2.1. *Let G and F be Abelian metrizable groups. Then*

(i) $\{\delta_x : x \in G\} \subset \Gamma_G(G)$;
(ii) *if* $\mu \in \Gamma_G(G)$ *then* $\mu * \delta_x \in \Gamma_G(G)$ *for each* $x \in G$;
(iii) *if* Φ *is a Borel isomorphism from* G *onto* F *and* $\mu \in \Gamma_G(G)$ *then* $\Phi(\mu) \in \Gamma_G(G)$.

Theorem 4.2.2. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then the following statements are equivalent*:

(i) $\mu \in \Gamma_G(G)$;
(ii) *there exist an Abelian metrizable group F, $\nu \in M^1(F)$ and $P \in \Pi_B(\mu \otimes \nu)$ such that*

$$((G \times \{0\}) \cup (\{0\} \times G)) \cap ((\operatorname{im} P) \cup (\ker P)) \cap H = \{(0,0)\}$$

for some Borel subgroup H of $G \times F$ with the property that $(\mu \otimes \nu)^{\mathrm{s}}(H) = 1$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(i) Let Q be the mapping from $G \times F$ to itself defined by $Q(x,y) = (x,0)$ for $(x,y) \in G \times F$. Hence, $Q \in \Pi_B(\mu \otimes \nu)$. Now, by Theorem 3.3.4 there exists a σ-compact proper common subdomain H of projections P and Q such that

$$((\operatorname{im} P) \cup (\ker P)) \cap ((\operatorname{im} Q) \cup (\ker Q)) \cap H = \{(0,0)\}.$$

Moreover, there exist a σ-compact subgroup G_1 of G and a σ-compact subgroup F_1 of F such that

(a) $H = G_1 \times F_1$;
(b) $(\operatorname{im} Q) \cap H = G_1 \times \{0\}$ and $(\ker Q) \cap H = \{0\} \times F_1$.

Set $R = (Q|_{\operatorname{im} P \cap H})(P|_{\ker Q \cap H})$. It is easy to see that R is an injective Borel-measurable mapping. Since $R : \{0\} \times F_1 \to G_1 \times \{0\}$ we conclude that there exists an injective Borel-measurable linear mapping $R_1 : F_1 \to G_1$ such that $R(0,x) = (R_1(x), 0)$ for each $x \in F_1$. Putting $G_2 = R_1(F_1)$, we find that G_2 is a Borel subgroup of G.

Let Φ be the mapping from $G_1 \times F_1$ onto $G_1 \times G_2$ defined by $\Phi(x,y) = (x, R_1 y)$ for $(x,y) \in G_1 \times F_1$. Clearly, Φ is a bijective Borel-measurable linear mapping. Putting $\lambda = R_1(\nu)$, we obtain $\Phi(\mu \otimes \nu) = \mu \otimes \lambda$. We define $P_1 = \Phi(P|_H)\Phi^{-1}$. An easy computation shows that $P_1 \in \Pi_B(\mu \otimes \lambda)$ and

$$((G_1 \times \{0\}) \cup (\{0\} \times G_2)) \cap ((\operatorname{im} P_1) \cup (\ker P_1)) = \{(0,0)\}. \ \blacksquare$$

Corollary 4.2.3. *Let $\mu \in \Gamma_G(G)$ and H be a Borel subgroup of G such that $\mu(H) = 1$. Then $\mu \in \Gamma_G(H)$.*

Theorem 4.2.4. *Let G be an Abelian metrizable group, $\mu \in M^1(G)$ and $P \in \Pi_B(\mu)$. We assume that $P(\mu * \delta_{-x}), (I-P)(\mu * \delta_{-x}) \in \Gamma_G(G)$ for some $x \in G$ with $\mu(D(P)+x) = 1$. Then $\mu \in \Gamma_G(G)$.*

Proof. We define the mapping R on $G \times G$ into itself by $R(x,y) = (x,0)$ for $(x,y) \in G \times G$.

By Theorem 3.3.4 there exist a σ-compact proper subdomain F' of P and $x \in G$ such that $\mu(F + x) = 1$. Set $\mu_1' = P(\mu * \delta_{-x})$, $\mu_2' = (I - P)(\mu * \delta_{-x})$, $G_1' = (\operatorname{im} P) \cap F'$ and $G_2' = (\ker P) \cap F'$.

Let $i = 1, 2$. Since $\mu'_i \in \Gamma_G(G'_i)$ we conclude that there exist $\nu'_i \in \Gamma_G(G)$ and $Q_i \in \Pi_B(\mu'_i \otimes \nu'_i)$ such that for some σ-compact proper common subdomain D_i of projections $R|_{G'_i \times G'_i}$ and Q_i we have

$$((\operatorname{im} R|_{G'_i \times G'_i}) \cup (\ker R|_{G'_i \times G'_i})) \cap ((\operatorname{im} Q_i) \cup (\ker Q_i)) \cap D_i = \{(0,0)\}.$$

Without loss of generality we may assume that $(\ker Q_i) \cap D_i$, $(\operatorname{im} Q_i) \cap D_i$, $(\ker R|_{G'_i \times G'_i}) \cap D_i$ and $(\operatorname{im} R|_{G'_i \times G'_i}) \cap D_i$ are σ-compact. Let (u_i, v_i) be an element of $G'_i \times G'_i$ such that $\mu'_i \otimes \nu'_i(D_i + (u_i, v_i)) = 1$. Put $\mu_i = \mu'_i * \delta_{-u_i}$ and $\nu_i = \nu'_i * \delta_{-v_i}$. Since $(R|_{G'_i \times G'_i})(D_i) \subset D_i$ we conclude that there exist σ-compact subgroups K_i and H_i of G'_i such that

(a) $D_i = K_i \times H_i$;
(b) $(\operatorname{im}(R|_{G'_i \times G'_i}) \cap D_i = K_i \times \{0\}$ and $(\ker R|_{G'_i \times G'_i}) \cap D_i = \{0\} \times H_i$;
(c) $\mu_i(K_i) = \nu_i(H_i) = 1$.

Put $\lambda_i = Q_i(\mu_i \otimes \nu_i), \eta_i = (I - Q_i)(\mu_i \otimes \nu_i)$, $F_i = (\operatorname{im} Q_i) \cap D_i$ and $G_i = (\ker Q_i) \cap D_i$. Since $Q_i(D_i) \subset D_i$ we conclude that $D_i = F_i \oplus G_i$ and $\lambda_i(F_i) = \eta_i(G_i) = 1$. Moreover, $((K_i \times \{0\}) \cup (\{0\} \times H_i)) \cap (F_i \cup G_i) = \{(0,0)\}$.

Set $\mu_0 = \mu_1 * \mu_2$, $\nu_0 = \nu_1 * \nu_2$, $\lambda_0 = \lambda_1 * \lambda_2$, $\eta_0 = \eta_1 * \eta_2$, $K_0 = K_1 + K_2$, $H_0 = H_1 + H_2$, $F_0 = F_1 + F_2$ and $G_0 = G_1 + G_2$. It is easy to see that $\mu_0(K_0) = \nu_0(H_0) = \lambda_0(F_0) = \eta_0(G_0) = 1$ and $K_0 \times H_0 = F_0 \oplus G_0$. By Lemma 1.4.1, there exists $Q \in \Pi_B(\mu_0 \otimes \nu_0)$ such that $\operatorname{im} Q = F_0$ and $\ker Q = G_0$. A trivial verification shows that

$$((\operatorname{im} R) \cup (\ker R)) \cap ((\operatorname{im} Q) \cup (\ker Q)) \cap (K_0 \times H_0) = \{(0,0)\}.$$

Since $\mu = \mu_0 * \delta_{-z}$ for some $z \in G$, we conclude that $\mu \in \Gamma_G(G)$. ■

Corollary 4.2.5. *Let $\mu \in M^1(G)$ and $P_1, \ldots, P_n$ be projections in $\Pi_B(\mu)$ such that*

(i) $P_i P_j = 0$ *μ^{s}-a.s. if $i \neq j$;*
(ii) $P_1 \vee \ldots \vee P_n = I$;
(iii) $P_1(\mu * \delta_{-x}), \ldots, P_n(\mu * \delta_{-x}) \in \Gamma_G(G)$ *for some $x \in G$ with* $\mu(\bigcap_{i=1}^n D(P_i + x)) = 1$.

Then $\mu \in \Gamma_G(G)$.

Corollary 4.2.6. *Let μ be a measure in $M^1(G)$. Assume that there exist two projections P, Q in $\Pi_B(\mu)$ such that*

$$((\operatorname{im} P) \cup (\ker P)) \cap ((\operatorname{im} Q) \cup (\ker Q)) \cap F = \{0\}$$

for some Borel subgroup F of G with $\mu^{\mathrm{s}}(F) = 1$. Then $\mu \in \Gamma_G(G)$.

Theorem 4.2.7. *Let G be an Abelian metrizable group and $\mu \in \Gamma_P(G)$. Assume that there exists a Borel subgroup G_0 of G such that*

(i) $\mu(G_0 + x) = 1$ *for some $x \in G$;*
(ii) G_0 *has no elements of order* 2.

Then $\mu \in \Gamma_G(G)$.

This follows at once from Corollary 4.2.5.

4.3. Gaussian measures in the sense of Gnedenko without idempotent factors. In this section we concentrate on a detailed study of Gaussian measures in the sense of Gnedenko without idempotent factors. The class of all G-Gaussian measures without idempotent factors will be denoted by $\Gamma_{G,0}(G)$.

THEOREM 4.3.1. *Let G, H be Abelian metrizable groups and $\mu \in M^1(G)$, $\nu \in M^1(H)$. Assume that there exists $Q \in \Pi_B(\mu \otimes \nu)$ such that*

$$((G \times \{0\}) \cup (\{0\} \times G)) \cap ((\operatorname{im} Q) \cup (\ker Q)) \cap F = \{(0,0)\}$$

for some Borel subgroup F of $G \times H$ with $(\mu \otimes \nu)^{\mathrm{s}}(F) = 1$. Then $I(\mu) = \{0\}$ iff $I(\nu) = \{0\}$.

Proof. By Lemma 2.4.2(ii) we have $I(\mu \otimes \nu) = I(\mu) \otimes I(\nu)$. Since $I(\mu \otimes \nu) \subset D(Q)$ and $Q(I(\mu \otimes \nu)) \subset I(\mu \otimes \nu)$ we conclude that

$$I(\mu \otimes \nu) = ((\operatorname{im} Q) \cap I(\mu \otimes \nu)) \oplus ((\ker Q) \cap I(\mu \otimes \nu)). \tag{1}$$

Let F be a Borel subgroup of $G \times H$ such that $(\mu \otimes \nu)^{\mathrm{s}}(F) = 1$ and

$$((G \times \{0\}) \cup (\{0\} \times G)) \cap ((\operatorname{im} Q) \cup (\ker Q)) \cap F = \{(0,0)\}.$$

This implies that

$$((I(\mu) \times \{0\}) \cup (\{0\} \times I(\mu))) \cap ((\operatorname{im} Q \cap I(\mu \otimes \nu)) \cup (\ker Q \cap I(\mu \otimes \nu))) = \{(0,0)\}.$$

Let $I(\mu) = \{0\}$. Hence,

$$\begin{aligned} \{(0,0)\} &= (\{0\} \times I(\mu)) \cap ((\operatorname{im} Q \cap I(\mu \otimes \nu)) \cup (\ker Q \cap I(\mu \otimes \nu))) \\ &= ((\operatorname{im} Q \cap I(\mu \otimes \nu)) \cup (\ker Q \cap I(\mu \otimes \nu))) \end{aligned} \tag{2}$$

We now conclude from (1) together with (2) that $I(\mu \otimes \nu) = \{(0,0)\}$. ■

COROLLARY 4.3.2. *Let G be an Abelian metrizable group and $\mu \in \Gamma_{G,0}(G)$. Let ν be a measure in $M^1(G)$ such that there exists $Q \in \Pi_B(\mu \otimes \nu)$ with*

$$((G \times \{0\}) \cup (\{0\} \times G)) \cap ((\operatorname{im} Q) \cup (\ker Q)) \cap F = \{(0,0)\}$$

for some Borel subgroup F of $G \times G$ with $(\mu \otimes \nu)^{\mathrm{s}}(F) = 1$. Then $I(\nu) = \{0\}$.

THEOREM 4.3.3. *Let G be an Abelian metrizable group and $\mu \in M_0^1(G)$. Assume that there exists a sequence (R_n) in $\Pi_B(\mu)$ such that*

(i) $R_n R_m = R_m R_n = 0$ *μ^{s}-a.s.*;
(ii) $\sup R_n = I$ *μ^{s}-a.s.*;
(iii) *there exists a sequence $(x_n) \subset G$ such that for each $n \in \mathbb{N}$, $\mu(D(R_n) + x_n) = 1$ and $(R_n(\mu * \delta_{-x_n})) \subset \Gamma_G(G)$.*

Then $\mu \in \Gamma_{G,0}(G)$.

Proof. By Theorem 3.3.4 there exist a σ-compact proper common subdomain F' of (R_n) and $x \in G$ such that $\mu(F' + x) = 1$. Set $\mu'_n = R_n(\mu * \delta_{-x})$ and $G'_n = \operatorname{im} R_n \cap F'$. Further, we define the mapping Φ_n from $G'_n \times G'_n$ into itself by $\Phi_n(x,y) = (x,0)$ for $(x,y) \in G'_n \times G'_n$.

Let $n \in \mathbb{N}$. Since $\mu'_n \in \Gamma_G(G'_n)$ we conclude that there exist $\nu'_n \in M^1(G'_n)$ and $\Theta_n \in \Pi_B(\mu'_n \otimes \nu'_n)$ such that for some σ-compact proper common subdomain D'_n of Φ_n and Θ_n we have $(\operatorname{im} \Phi_n \cup \ker \Phi_n) \cap (\operatorname{im} \Theta_n \cup \ker \Theta_n) \cap D'_n = \{(0,0)\}$. Let $(u_n, v_n) \in G'_n \times G'_n$ be such that $\mu'_n \otimes \nu'_n(D' + (u_n, v_n)) = 1$. Put $\mu_n = \mu'_n * \delta_{-u_n}$ and $\nu_n = \nu'_n * \delta_{-v_n}$. Since

$\Phi_n(D_n') \subset D_n'$ we conclude that there exist σ-compact subgroups K_n' and H_n' of G_n' such that

(a) $D_n' = K_n' \times H_n'$;
(b) $\operatorname{im} \Phi_n \cap D_n' = K_n' \times \{0\}$ and $\ker \Phi_n \cap D_n' = \{0\} \times H_n'$;
(c) $\mu_n(K_n') = \nu_n(H_n') = 1$;
(d) $((K_n' \times \{0\}) \cup (\{0\} \times H_n')) \cap ((\operatorname{im} \Theta_n \cap D_n') \cup (\ker \Theta_n \cap D_n')) = \{(0,0)\}$.

Set $F_n' \oplus (\ker R_n \cap F_n')$ and $R_n' = R_n|_{F_n'}$. It is easy to see that $R_n' \in \Pi_B(\mu)$, $R_n' = R_n$ μ^{s}-a.s. and $R_n'(\mu * \delta_{-x-u_n}) = \mu_n$.

Put $H' = \prod_{n=1}^{\infty} H_n'$ and $\nu = *_{n=1}^{\infty} \nu_n$. Clearly, $\nu(H') = 1$ and $I(\nu) = \{0\}$. We now define the projection p_n from H' into H_n' by $p_n((x_k)) = x_n$ for $(x_k) \in H'$ and the embedding i_n of H_n' in H' by $i_n(x_n) = (x_k)$, where (x_k) is the element of H' whose nth coordinate is equal to x_n and all other coordinates vanish.

Let r_n be the mapping from $F_n' \times H'$ into $K_n' \times H_n'$ defined by $r_n(x, (y_n)) = (R_n' x, p_n(y_k))$ for $(x, (y_n)) \in F_n' \times H'$ and let j_n be the mapping from $K_n' \times H_n'$ into $F_n' \times H'$ defined by $j_n(x, y) = (x, i_n(y))$ for $(x, y) \in K_n' \times H_n'$.

Putting $Q_n = j_n \Theta_n r_n$, $Q' = j_n(I - \Theta_n) r_n$, $P_n = j_n \Phi_n r_n$, $P' = j_n(I - \Phi_n) r_n$ and $T_n = j_n r_n$ one obtains

(e) $P_n, P_n', Q_n, Q_n', T_n \in \Pi_B(\mu \otimes \nu)$;
(f) $P_n + P_n' = Q_n + Q_n' = T_n$;
(g) $P_n P_m = P_n' P_m' = Q_n Q_m = Q_n' Q_m' = T_n T_m = 0$ $(\mu \otimes \nu)^{\mathrm{s}}$-a.s. for $i \neq j$.

Theorem 3.4.9 implies that there exist $P, P', Q, Q' \in \Pi_B(\mu \otimes \nu)$ such that $P = \sup P_n$, $P' = \sup P_n'$, $Q = \sup Q_n$ and $Q' = \sup Q_n'$. Since $\sup T_n = I$ $(\mu \otimes \nu)^{\mathrm{s}}$-a.s. we conclude that $P = I - P'$ $(\mu \otimes \nu)^{\mathrm{s}}$-a.s. and $Q = I - Q'$ $(\mu \otimes \nu)^{\mathrm{s}}$-a.s. Hence, by Theorem 3.3.4 there exists a σ-compact proper common subdomain D of $(P_n), (P_n'), (Q_n), (Q_n')$ and $\{P, P', Q, Q'\}$ such that $P|_D = (I - P')|_D$ and $Q|_D = (I - Q')|_D$. Moreover,

$$
\begin{aligned}
&\ker P \cap D = \bigcap_{n=1}^{\infty} \ker P_n \cap D, \qquad \ker P' \cap D = \bigcap_{n=1}^{\infty} \ker P_n' \cap D,\\
&\ker Q \cap D = \bigcap_{n=1}^{\infty} \ker Q_n \cap D, \qquad \ker Q' \cap D = \bigcap_{n=1}^{\infty} \ker Q_n' \cap D,
\end{aligned} \tag{3}
$$

and $P_n(D) \subset D$.

Since $P(D) \subset D$ we conclude that there exist σ-compact subgroups: $F \subset \bigcap_{n=1}^{\infty} F_n'$, $H \subset H'$ and $K_n \subset K_n'$, such that $D = F \times H$, $\operatorname{im} P \cap D = F \times \{0\}$, $\ker P \cap D = \{0\} \times H$ and $\operatorname{im} P_n \cap D = K_n \times \{0\}$. Set $H_n = p_n(H)$ and $D_n = K_n \times H_n$. An easy computation shows that

(h) $H = \prod_{n=1}^{\infty} H_n$;
(i) $\ker P_n \cap D = \{0\} \times i_n(H_n)$;
(j) $\Phi_n(D_n) \subset D_n$ and $\Theta_n(D_n) \subset D_n$;
(k) $(\operatorname{im} \Phi_n \cup \ker \Phi_n) \cap (\operatorname{im} \Theta_n \cup \ker \Theta_n) \cap D_n = \{(0,0)\}$.

We now have

$$\text{(4)}\qquad \begin{aligned} \ker Q_n\cap D&=r_n^{-1}(\ker\Theta_n\cap D_n),\quad &\ker Q_n'\cap D&=r_n^{-1}(\operatorname{im}\Theta_n\cap D_n),\\ \ker P_n\cap D&=r_n^{-1}(\ker\Phi_n\cap D_n),\quad &\ker P_n'\cap D&=r_n^{-1}(\operatorname{im}\Phi_n\cap D_n).\end{aligned}$$

From (3) and (4) we obtain

$$\ker Q\cap D=\bigcap_{n=1}^{\infty} r_n^{-1}(\ker\Theta_n\cap D_n),\quad \operatorname{im}Q\cap D=\bigcap_{n=1}^{\infty} r_n^{-1}(\operatorname{im}\Theta_n\cap D_n),$$

$$\ker P\cap D=\bigcap_{n=1}^{\infty} r_n^{-1}(\ker\phi_n\cap D_n),\quad \operatorname{im}P\cap D=\bigcap_{n=1}^{\infty} r_n^{-1}(\operatorname{im}\Phi_n\cap D_n).$$

Thus we get

$$\begin{aligned}(\operatorname{im}P\cup\ker P)&\cap(\operatorname{im}Q\cup\ker Q)\cap D\\ &=\bigcap_{n=1}^{\infty}\{r_n^{-1}(\operatorname{im}\Phi_n\cap D_n)\cap r_n^{-1}(\operatorname{im}\Theta_n\cap D_n)\}\\ &\quad\cup\bigcap_{n=1}^{\infty}\{r_n^{-1}(\operatorname{im}\Phi_n\cap D_n)\cap r_n^{-1}(\ker\Theta_n\cap D_n)\}\\ &\quad\cup\bigcap_{n=1}^{\infty}\{r_n^{-1}(\ker\Phi_n\cap D_n)\cap r_n^{-1}(\operatorname{im}\Theta_n\cap D_n)\}\\ &\quad\cup\bigcap_{n=1}^{\infty}\{r_n^{-1}(\ker\Phi_n\cap D_n)\cap r_n^{-1}(\ker\Theta_n\cap D_n)\}\\ &=\bigcap_{n=1}^{\infty} r_n^{-1}(\{(0,0)\})=\{(0,0)\}.\end{aligned}$$

Application of Theorem 4.2.2 implies that $\mu\in\Gamma_{G,0}(G)$. ∎

LEMMA 4.3.4. *Let G be an Abelian metrizable group, $\mu\in M_0^1(G)$ and $P_1,P_2\in\Pi_B(\mu)$. Define $Q=I-\sum_{i,j=0}^{1}P_1^{(i)}\wedge P_2^{(j)}$. Then*

(i) $Q\in\Pi_B(\mu)$;
(ii) $QP_i^{(j)}=P_i^{(j)}Q$ *μ^{s}-a.s. for $i,j=1,2$;*
(iii) $QP_i^{(j)}\wedge QP_l^{(k)}=0$ *$Q(\mu^{\mathrm{s}})$-a.s. for $(i,j)\neq(k,l)$;*
(iv) $Q(\mu*\delta_{-x})\in\Gamma_{G,0}(G)$ *for some $x\in G$ with $\mu(D(Q)+x)=1$.*

LEMMA 4.3.5. *Let G be an Abelian metrizable group, $\mu\in M_0^1(G)$ and $P,R\in\Pi_B(\mu)$. Assume that $P\wedge R=0$ μ^{s}-a.s. and $P\wedge(I-R)=0$ μ^{s}-a.s. Then $P(\mu*\delta_{-x})\in\Gamma_{G,0}(G)$ for every $x\in G$ with $\mu(D(P)+x)=1$.*

P r o o f. Putting $Q=I-(I-P)\wedge R-(I-P)\wedge(I-R)$ we obtain $QP=PQ=P$. Let $x\in G$ be such that $\mu(D(Q)+x)=1$. Hence, by Theorem 3.3.9(ii), $P\in\Pi_B(Q(\mu*\delta_{-x}))$. We now conclude from Lemma 4.3.4 that

(a) $QR=RQ$ μ^{s}-a.s.;
(b) $QR\in\Pi_B(Q(\mu*\delta_{-x}))$;
(c) $(\operatorname{im}P\cup\ker P)\cap(\operatorname{im}QR\cup\ker QR)\cap F=\{0\}$ for some Borel subgroup F of G with $\mu^{\mathrm{s}}(F)=1$. ∎

THEOREM 4.3.6. *Let G be an Abelian metrizable group, $\mu \in \Gamma_{G,0}(G)$ and $P \in \Pi_B(\mu)$. Then $P(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for every $x \in G$ with $\mu(D(P)+x) = 1$.*

Proof. Let $x \in G$ be such that $\mu(D(P)+x) = 1$. We define the mapping R on $G\times G$ into itself by $R(x,y) = (x,0)$ for $(x,y) \in G\times G$. Since $\mu \in \Gamma_{G,0}(G)$ we see that there exist $\nu \in M^1(G)$ and $Q \in \Pi_B(\mu\otimes\nu)$ such that $(\operatorname{im} R \cup \ker R) \cap (\operatorname{im} Q \cup \ker Q) \cap F = \{(0,0)\}$ for some Borel subgroup F of $G\times G$ with $(\mu\otimes\nu)^{\mathrm{s}}(F) = 1$.

Let P' be the mapping on $D(P)\times G$ into itself defined by $P'(x,y) = (Px,0)$ for $(x,y) \in D(P)\times G$. Clearly, $P' \in \Pi_B(\mu\otimes\nu)$, $P' \wedge Q = 0$ $(\mu\otimes\nu)^{\mathrm{s}}$-a.s. and $P' \wedge (I-Q) = 0$ $(\mu\otimes\nu)^{\mathrm{s}}$-a.s. Moreover, $\mu\otimes\nu(D(P') + (x,0)) = 1$. Lemma 4.3.5 now implies that $P'(\mu\otimes\nu * \delta_{-(x,0)}) \in \Gamma_G^0(G\times G)$. A trivial verification shows that $P'(\mu\otimes\nu * \delta_{-(x,0)}) = P(\mu * \delta_{-x}) \otimes \delta_0$. ■

COROLLARY 4.3.7. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and (P_n) be a sequence in $\Pi_B(\mu)$. Assume that*

(i) $P_n \le P_{n+1}$;

(ii) *for every $n \in \mathbb{N}$ there exists $x_n \in G$ such that $\mu(D(P_n)+x_n) = 1$ and $P_n(\mu*\delta_{-x_n}) \in \Gamma_{G,0}(G)$.*

Then $(\sup P_n)(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for every $x \in G$ with $\mu(D(\sup P_n) + x) = 1$.

Proof. Without loss of generality we may assume that $\sup P_n = I$ μ^{s}-a.s. We define $R_1 = P_1$ and $R_{n+1} = P_n(I - P_{n+1})$ for $n \in \mathbb{N}$. Theorem 4.3.6 implies that $R_n(\mu*\delta_{-y_n}) \in \Gamma_{G,0}(G)$ for every $y_n \in G$ with $\mu(D(R_n) + y_n) = 1$. Therefore, Theorem 4.3.3 yields $\mu \in \Gamma_{G,0}(G)$. ■

THEOREM 4.3.8. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $(P_t)_{t\in T}$ be a family in $\Pi_B(\mu)$. Assume that for every $t \in T$ there exists $x_t \in G$ such that $\mu(D(P_t) + x_t) = 1$ and $P_n(\mu * \delta_{-x_n}) \in \Gamma_{G,0}(G)$. Then $(\sup P_t)(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for every $x \in G$ with $\mu(D(\sup P_t) + x) = 1$.*

We prepare the proof of the theorem with the following lemma.

LEMMA 4.3.9. *Let $\mu \in M_0^1(G)$ and $P_1,\ldots,P_n \in \Pi_B(\mu)$. Assume that for $k = 1,\ldots,n$ there exists $x_k \in G$ such that $\mu(D(P_k) + x_k) = 1$ and $P_k(\mu * \delta_{-x_k}) \in \Gamma_{G,0}(G)$. Then $(P_1 \vee \ldots \vee P_n)(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for every $x \in G$ with $\mu(D(P_1 \vee \ldots \vee P_n) + x) = 1$.*

Proof. Without loss of generality we may assume that $n = 2$. We set

$$Q = I - \sum_{i,j=0}^{1} P_1^{(i)} \wedge P_2^{(j)}.$$

Hence, by Lemma 4.3.4,

(i) $Q(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for every $x \in G$ with $\mu(D(Q) + x) = 1$;
(ii) $P_1 \vee P_2 = Q + P_1^{(0)} \wedge P_2^{(0)} + P_1^{(0)} \wedge P_2^{(1)} + P_1^{(1)} \wedge P_2^{(0)}$ μ^{s}-a.s.;
(iii) $QP_1^{(i)} \wedge P_2^{(j)} = P_1^{(i)} \wedge P_2^{(j)} Q = 0$ μ^{s}-a.s. for $i \ne j$;
(iv) $(P_1^{(i)} \wedge P_2^{(j)})(P_1^{(k)} \wedge P_2^{(l)}) = (P_1^{(k)} \wedge P_2^{(l)})(P_1^{(i)} \wedge P_2^{(j)}) = 0$ μ^{s}-a.s. for $(i,j) \ne (k,l)$;
(v) $P_1^{(i)} \wedge P_2^{(j)} = P_1^{(i)}(P_1^{(i)} \wedge P_2^{(j)}) = (P_1^{(i)} \wedge P_2^{(j)})P_1^{(i)} = 0$ μ^{s}-a.s. for $i,j = 0,1$;
(vi) $P_1^{(i)} \wedge P_2^{(j)} = P_2^{(j)}(P_1^{(i)} \wedge P_2^{(j)}) = (P_1^{(i)} \wedge P_2^{(j)})P_2^{(j)} = 0$ μ^{s}-a.s. for $i,j = 0,1$.

Conditions (v) and (vi) together with Theorem 4.3.6 yield

(5) $$P_1^{(0)} \wedge P_2^{(0)}(\mu * \delta_{-x}), P_1^{(1)} \wedge P_2^{(0)}(\mu * \delta_{-x}), P_1^{(0)} \wedge P_2^{(1)}(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$$

for every $x \in G$ with $\mu(\bigcap_{i,j=0}^{1} D(P_1^{(i)} \wedge P_2^{(j)}) + x) = 1$.

Conditions (i), (ii) and (5) together with Theorem 4.3.3 imply

$$(P_1 \vee P_2)(\mu * \delta_{-z}) \in \Gamma_{G,0}(G)$$

for each $z \in G$ with $\mu(D(P_1 \vee P_2) + z) = 1$. ∎

Proof of Theorem 4.3.8. By Theorem 3.4.10 there exists a sequence $(t_n) \subset T$ such that $\sup_{t \in T} P_t = \sup_n P_{t_n}$ μ^{s}-a.s. Putting $Q_n = P_{t_1} \vee \ldots \vee P_{t_n}$ we obtain $Q_n \leq Q_{n+1}$ and $\sup_n Q_n = \sup_t P_t$ μ^{s}-a.s. By Lemma 4.3.9, $Q_n(\mu * \delta_{-z_n}) \in \Gamma_{G,0}(G)$ for each $z_n \in G$ with $\mu(D(Q_n) + z_n) = 1$. Application of Corollary 4.3.7 now implies that

$$(\sup_t P_t)(\mu * \delta_{-z}) \in \Gamma_{G,0}(G)$$

for each $z \in G$ with $\mu(D(\sup_t P_t) + z) = 1$. ∎

LEMMA 4.3.10. *Let G be an Abelian metrizable group and $\mu \in M_0^1(G)$. Assume that the measure μ has no G-Gaussian product-factors. Then $\Pi_B(\mu)$ is a commutative subsemigroup of $\mathbb{D}_B(\mu)$.*

Proof. Let $P_1, P_2 \in \Pi_B(\mu)$. Application of Lemma 4.3.4 yields

(a) $I = \sum_{i,j=0}^{1} P_1^{(i)} \wedge P_2^{(j)}$ μ^{s}-a.s.;

(b) $(P_1^{(i)} \wedge P_2^{(j)})(P_1^{(k)} \wedge P_2^{(l)}) = 0$ μ^{s}-a.s. for $(i,j) \neq (k,l)$.

Clearly,

$$P_1^{(0)} \wedge P_2^{(0)} + P_1^{(0)} \wedge P_2^{(1)} \leq P_1, \quad P_1^{(0)} \wedge P_2^{(1)} + P_1^{(1)} \wedge P_2^{(1)} \leq I - P_1,$$
$$P_1^{(0)} \wedge P_2^{(0)} + P_1^{(1)} \wedge P_2^{(0)} \leq P_2, \quad P_1^{(0)} \wedge P_2^{(1)} + P_1^{(1)} \wedge P_2^{(1)} \leq I - P_2.$$

But this implies that

$$P_1 = P_1^{(0)} \wedge P_2^{(0)} + P_1^{(0)} \wedge P_2^{(1)} \ \mu^{\mathrm{s}}\text{-a.s.} \quad \text{and}$$
$$P_2 = P_1^{(0)} \wedge P_2^{(0)} + P_1^{(1)} \wedge P_2^{(0)} \ \mu^{\mathrm{s}}\text{-a.s.} \ ∎$$

THEOREM 4.3.11. *Let G be an Abelian metrizable group and $\mu \in M^1(G)$. Then there exists $P_0 \in \Pi_B(\mu)$ such that*

(i) *$P_0(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for each $x \in G$ with $\mu(D(P_0) + x) = 1$;*

(ii) *for each $x \in G$ with $\mu(D(P_0) + x) = 1$, the measure $(I - P_0)(\mu * \delta_{-x})$ has no G-Gaussian product-factors*;

(iii) *for every $R \in \Pi_B(\mu)$, $RP_0 = P_0R$ μ^{s}-a.s.*;

(iv) *$\{RP_0 : R \in \Pi_B(\mu)\}$ is a commutative subsemigroup of $\mathbb{D}_B(\mu)$.*

Moreover, *the projection P_0 is uniquely determined by conditions* (i) *and* (ii).

Proof. (i) Put

$$\Gamma = \{P \in \Pi_B(\mu) : P(\mu * \delta_{-x_P}) \in \Gamma_{G,0}(G) \text{ for some } x_P \in G \text{ with } \mu(D(P) + x_P) = 1\}.$$

Set $P_0 = \sup\{P : P \in \Gamma\}$. Hence, by Theorem 4.3.8, $P_0(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for each $x \in G$ with $\mu(D(P_0) + x) = 1$.

(ii) is obvious.

(iii) Let $R \in \Pi_B(\mu)$. We define $Q = I - \sum_{i,j=0}^{1} P_0^{(i)} \wedge R^{(j)}$. Application of Lemma 4.3.4 yields

(a) $QR = RQ$ $\mu^{\rm s}$-a.s.;

(b) $R = QR + R \wedge P_0 + R \wedge (I - P_0)$ $\mu^{\rm s}$-a.s.;

(c) $Q(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for each $x \in G$ with $\mu(D(Q) + x) = 1$.

Lemma 4.3.5 now shows that $RQ(\mu*\delta_{-x}) \in \Gamma_{G,0}(G)$ for some $x \in G$ with $\mu(D(P_0)+x) = 1$, which implies that $RQ \le P_0$ and finally that

$$P_0 = P_0(RQ) \qquad \mu^{\rm s}\text{-a.s.} \tag{6}$$

Hence, we conclude from (b) together with (6) that $P_0 R = RP_0$ $\mu^{\rm s}$-a.s.

(iv) follows immediately from (ii) together with Lemma 4.2.10. ∎

THEOREM 4.3.12. *Let G be an Abelian metrizable group and $\mu \in M_0^1(G)$. Assume that there exist $P_1, \ldots, P_n \in \Pi_B(\mu)$ such that*

$$\bigcap_{k=1}^{n} (\operatorname{im} P_k \cup \ker P_k) \cap F = \{0\}$$

for some Borel subgroup F of G with $\mu^{\rm s}(F) = 1$. Then $\mu \in \Gamma_{G,0}(G)$.

Proof. By Theorem 4.3.11 there exists $P_0 \in \Pi_B(\mu)$ such that

(i) $P_0(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for every $x \in G$ with $\mu(D(P_0) + x) = 1$;

(ii) for every $R \in \Pi_B(\mu)$, $RP_0 = P_0 R$ $\mu^{\rm s}$-a.s.;

(iii) $\{RP_0^{(1)} : R \in \Pi_B(\mu)\}$ is a commutative subsemigroup of $\mathbb{D}_B(\mu)$.

Hence, by assumption, for each $(i_1, \ldots, i_n) \in \{0,1\}^n$,

$$P_1^{(i_1)} P_0^{(1)} \ldots P_n^{(i_n)} P_0^{(1)} = 0 \qquad \mu^{\rm s}\text{-a.s.}$$

Since

$$P_0^{(1)} = \sum_{\underline{i}_n \in \{0,1\}^n} P_1^{(i_1)} P_0^{(1)} \ldots P_n^{(i_n)} P_0^{(1)} \qquad \mu^{\rm s}\text{-a.s.},$$

we conclude that $P_0^{(1)} = 0$ $\mu^{\rm s}$-a.s. ∎

COROLLARY 4.3.13. *Let $\mu \in M_0^1(G)$ and $P_1, \ldots, P_n \in \Pi_B(\mu)$. Define*

$$Q = I - \sum_{\underline{i}_n \in \{0,1\}^n} P_1^{(i_1)} \wedge \ldots \wedge P_n^{(i_n)} \qquad \mu^{\rm s}\text{-a.s.}$$

Then

(i) *$Q(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$ for each $x \in G$ with $\mu(D(Q) + x) = 1$;*

(ii) *$QP_k = P_k Q$ $\mu^{\rm s}$-a.s. for $k = 1, \ldots, n$;*

(iii) *$P_k = QP_k + \sum_{\underline{i}_n \in \{0,1\}^n, i_k = 1} P_1^{(i_1)} \wedge \ldots \wedge P_n^{(i_n)}$ $\mu^{\rm s}$-a.s.*

4.4. Product-atoms in Borel decomposability semigroups of probability measures without idempotent factors. We now study the properties of product-atoms in Borel decomposability semigroups of Radon probability measures without idempotent factors.

LEMMA 4.4.1. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $P \in \mathcal{A}_B(\mu)$. Assume that there exists $Q \in \Pi_B(\mu)$ such that $PQ \neq QP$ μ^{s}-a.s. Then for each $x \in G$ with $\mu(D(P)+x) = 1$, $P(\mu * \delta_{-x}) \in \Gamma_{G,0}(G)$.*

Proof. By assumption $P \wedge Q = 0$ μ^{s}-a.s. and $P \wedge (I - Q) = 0$ μ^{s}-a.s. Application of Lemma 4.3.5 now implies that $P(\mu*\delta_{-x}) \in \Gamma_G(G)$ for each $x \in G$ with $\mu(D(P)+x) = 1$. ■

THEOREM 4.4.2. *Let G be an Abelian metrizable group, $\mu \in M_0^1(G)$ and $\mathcal{B}$ be a commutative family of $\mathcal{A}_B(\mu)$. Then $\mathcal{B}$ is at most countable.*

Proof. Put $R = \sup\{P : P \in \mathcal{B}\}$ μ^{s}-a.s. Hence, by Theorem 3.4.10 there exists a sequence (P_n) in $\mathcal{B}$ such that $R = \sup_n P_n$ μ^{s}-a.s.

Let $Q \in (\mathcal{B} \cup \{0\}) \setminus (P_n)$. Since $Q \wedge P_n = 0$ μ^{s}-a.s. we conclude that for each $n \in \mathbb{N}$, $P_n \le I - Q$ μ^{s}-a.s., which implies that $R \le I - Q$ μ^{s}-a.s. Further, $R \le R \vee Q = R + Q \le R$ μ^{s}-a.s. and thus $Q = 0$ μ^{s}-a.s. ■

COROLLARY 4.4.3. *If $\mathcal{A}_B(\mu)$ is commutative then $\mathcal{A}_B(\mu)$ is at most countable.*

THEOREM 4.4.4. *Let G be an Abelian metrizable group and $\mu \in M_0^1(G)$. Then*

(i) *if μ is purely product-atomic then for every $P \in \Pi_B(\mu)$ there exists a commutative at most countable subset $\mathcal{B}_P$ of $\mathcal{A}_B(\mu)$ such that* $\sup\{Q : Q \in \mathcal{B}_B\} = P$ *μ^{s}-a.s.*;

(ii) *if $\Pi_B(\mu)$ is commutative subsemigroup of $\mathbb{D}_B(\mu)$ and there exists a subset $\mathcal{B}$ of $\mathcal{A}_B(\mu)$ such that* $\sup\{P : P \in \mathcal{B}\} = I$ *μ^{s}-a.s. then $\mathcal{B} = \mathcal{A}_B(\mu)$ and μ is purely product-atomic.*

Proof. (i) Let $P \in \Pi_B(\mu)$. We denote by $\mathcal{C}$ the family of all commutative subsets of $\{Q \in \mathcal{A}_B(\mu) : Q \le P\}$. Further, we define the binary relation $\le$ on $\mathcal{C}$ by

$$B_1 \le B_2 \quad \text{iff} \quad B_1 \subset B_2.$$

It is easy to see that $\le$ is a partial ordering in $\mathcal{C}$. By the Kuratowski–Zorn lemma there exists a maximal element B_0 in $\mathcal{C}$. Theorem 4.4.2 implies that B_0 is at most countable. Put $P_0 = \sup\{Q : Q \in B_0\}$. Clearly, $P_0 \le P$. Assume that $P_0 < P$. Since $P - P_0 \neq 0$ and $P - P_0 \in \Pi_B(\mu)$ we conclude that there exists $Q \in \mathcal{A}_B(\mu)$ such that $Q \le P - P_0$. A trivial verification shows that $B_0 \cup \{Q\} \in \mathcal{C}$ and $B_0 < B_0 \cup \{Q\}$. This contradiction proves (i).

(ii) By Theorem 4.4.2 the set $\mathcal{A}_B(\mu)$ is at most countable. Since

$$I \le \sup\{P : P \in \mathcal{B}\} \le \sup\{P : P \in \mathcal{A}_B(\mu)\} \le I$$

we conclude that $\sup\{P : P \in \mathcal{A}_B(\mu)\} = I$ and $\mathcal{B} = \mathcal{A}_B(\mu)$.

Let $Q \in \Pi_B(\mu) \setminus \{0\}$. Assume that $P \le I - Q$ for each $P \in \mathcal{A}_B(\mu)$. But this implies that $I \le I - Q \le I$ and finally that $Q = 0$ μ^{s}-a.s. This contradiction proves (ii). ■

COROLLARY 4.4.5. *Let $\mu \in M_0^1(G)$ be purely product-atomic. Then there exist a sequence $(P_n) \subset \mathcal{A}_B(\mu)$ and an element $x \in G$ such that for each $n \in \mathbb{N}$, $\mu(D(P_n) + n) = 1$ and*

$$\mu = \mathop{\ast}_{n=1}^{\infty} P_n(\mu * \delta_{-x}) * \delta_x.$$

4.5. Product-atomless probability measures without idempotent factors. Let G be an Abelian metrizable group and $\mu \in M^1(G)$. A triangular system $\{P_j^{(n)} : j = 1, \ldots, k_n;\ n = 1, 2, \ldots\}$ of projections in $\Pi_B(\mu) \setminus \{0\}$ is called a *triangular infinitesimal system of projections* corresponding to the measure μ if

(i) $P_j^{(n)} P_i^{(n)} = P_i^{(n)} P_j^{(n)} = 0$ μ^{s}-a.s. for $i, j \in \{1, \ldots, k_n\},\ i \neq j$;

(ii) $\sum_{i=1}^{k_n} P_i^{(n)} = I$ μ^{s}-a.s. for each $n \in \mathbb{N}$;

(iii) $\max\{P_i^{(n+1)} : P_i^{(n+1)} \leq P_j^{(n)}\} = P_j^{(n)}$ for each $n \in \mathbb{N}$ and $j \in \{1, \ldots, k_n\}$;

(iv) for each sequence (j_n) in $\mathbb{N}$ with $j_n \in \{1, \ldots, k_n\}$ one has $P_{j_n}^{(n)} \mu^{\mathrm{s}} \Rightarrow \delta_0$ as $n \to \infty$.

The next two corollaries are immediate from the above definition.

Corollary 4.5.1. *Let x be an element of G with $\mu(D(P_j^{(n)}) + x) = 1$ for all $n \in \mathbb{N}$ and $j = 1, \ldots, k_n$. Then $\{P_j^{(n)}(\mu * \delta_{-x}) : j = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ is a shift infinitesimal triangular system in $M^1(G)$.*

Corollary 4.5.2. *Let F be a proper common subdomain of the triangular infinitesimal system of projections $\{P_j^{(n)}\}$ corresponding to the measure μ. Then*

$$\bigcap_{n=1}^{\infty} \Big\{ \bigcup_{i=1}^{k_n} \operatorname{im} P_i^{(n)} \cap F \Big\} = \bigcup_{\underline{i} \in I} H_{\underline{i}},$$

where $I = \{\underline{i} \in \mathbb{N}^{\infty} : i_n \in \{1, \ldots, k_n\},\ P_{i_{n+1}}^{(n+1)} \leq P_{i_n}^{(n)}\}$ and $H_{\underline{i}} = \bigcap_{n=1}^{\infty} \operatorname{im} P_{i_n}^{(n)} \cap F$. Moreover, if $x_j \in H_{\underline{i}_j}$ for $j = 1, \ldots, m$, $H_{\underline{i}_j} \neq H_{\underline{i}_l}$ for $i_j \neq i_l$ and $\sum_{j=1}^{k_n} x_j = 0$ then $x_1 = \ldots = x_m = 0$.

Lemma 4.5.3. *Let G be an Abelian compact metrizable group and (F_n) be a decreasing sequence of compact subgroups of G. Let $F = \bigcap_{n=1}^{\infty} F_n$. Then $\omega_{F_n} \to \omega_F$.*

Proof. Since (ω_{F_n}) is uniformly tight, there exist an increasing sequence (n_k) of positive integers and a measure $\mu \in M^1(G)$ such that $\omega_{F_{n_k}} \Rightarrow \mu$. A trivial verification shows that $\operatorname{supp}(\mu) \subset F$. Hence $\omega_{F_{n_k}} = \omega_{F_{n_k}} * \omega_F \Rightarrow \mu * \omega_F = \omega_F$, and thus $\mu = \omega_F$. ■

Lemma 4.5.4. *Let G be an Abelian compact metrizable group. Then ω_G has no a triangular infinitesimal system of projections.*

Proof. We assume on the contrary that there exists a triangular infinitesimal system of projections $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ corresponding to ω_G.

It is easy to see that $\{P_i^{(n)}\}$ is a family of continuous projections from G into itself. Application of Lemma 1.5.2 implies that

$$\bigcap_{n=1}^{\infty} \Big(\bigcup_{i=1}^{k_n} \operatorname{im} P_i^{(n)} \Big) \neq \{0\}.$$

Let (j_n) be a sequence of positive integers such that

(a) $j_n \in \{1, \ldots, k_n\}$;

(b) $P_{j_{n+1}}^{(n+1)} \leq P_{j_n}^{(n)}$;

(c) $H = \bigcap_{n=1}^{\infty} \operatorname{im} P_{j_n}^{(n)} \neq \{0\}$;

(d) $P_{j_n}^{(n)} \omega_G \to \delta_0$.

Set $H=\bigcap_{n=1}^{\infty}\operatorname{im}P_{j_n}^{(n)}$ and $H_n=\operatorname{im}P_{j_n}^{(n)}$. Since $P_{j_n}^{(n)}(\omega_G)=\omega_{H_n}$ Lemma 4.5.3 implies that $P_{j_n}^{(n)}\omega_G\Rightarrow\omega_H\neq\delta_0$. This contradicts (d), and the lemma is proved. ■

THEOREM 4.5.5. *Let G be an Abelian metrizable group and $\mu\in M^1(G)$. Suppose that μ has a triangular system of projections $\{P_i^{(n)}:i=1,\ldots,k_n;\ n\in\mathbb{N}\}$. Then $I(\mu)=\{0\}$.*

Proof. Suppose that $I(\mu)\neq\{0\}$. Thus $\mu=\mu*\omega_H$, where $H=I(\mu)$. It is easy to see that $\{P_i^{(n)}|_H\}$ is a triangular system of projections corresponding of ω_H. This contradicts Lemma 4.5.4 and the theorem is proved. ■

LEMMA 4.5.6. *Let G be an Abelian metrizable group and d be some metric in $M_0^1(G)$. Then a measure μ in $M_0^1(G)$ is product-atomless iff for every $P\in\Pi_B(\mu)\setminus\{0\}$,*

$$\inf\{d(Q(\mu^{\mathrm{s}}),\delta_0):0<Q\leq P\}=0.$$

LEMMA 4.5.7. *Let G be an Abelian metrizable group and d be some metric in $M_0^1(G)$. Assume that μ is a product-atomless measure in $M_0^1(G)$. Then for every $\varepsilon>0$ there exist $k\in\mathbb{N}$ and $P_1,\ldots,P_k\in\Pi_B(\mu)\setminus\{0\}$ such that*

(i) $P_iP_j=0$ *μ^{s}-a.s. for* $i\neq j$;
(ii) $\sum_{i=1}^{k}P_i=I$ *μ^{s}-a.s.*;
(iii) $d(P_i(\mu^{\mathrm{s}}),\delta_0)<\varepsilon$ *for* $i=1,\ldots,k$.

Proof. Let $\varepsilon>0$ and let $\mathcal{D}$ be the family of all subsets $\mathcal{P}$ of $\Pi_B(\mu)\setminus\{0\}$ such that

(a) $P_iP_j=0$ μ^{s}-a.s. for $P_i,P_j\in\mathcal{P},\ P_i\neq P_j$;
(b) $d(P(\mu^{\mathrm{s}}),\delta_0)\leq\varepsilon$ for each $P\in\mathcal{P}$.

We define the binary relation $\leq$ on $\mathcal{D}$ by $\mathcal{P}_1\leq\mathcal{P}_2$ iff $\mathcal{P}_1\subset\mathcal{P}_2$. It is easy to see that $\leq$ is a partial ordering in $\mathcal{D}$. By the Kuratowski–Zorn lemma there exists a maximal element $\mathcal{P}_0$ in $\mathcal{D}$. Theorem 3.4.10 now implies that $\mathcal{P}_0$ is at most countable. Put $\mathcal{P}_0=\{P_n:n\in\mathbb{N}\}$. It is easy to verify that $\sup P_n=I$ μ^{s}-a.s. Hence, $\mu^{\mathrm{s}}=\ast_{n=1}^{\infty}P_n(\mu^{\mathrm{s}})$. Since $\ast_{n=k+1}^{\infty}P_n(\mu^{\mathrm{s}})\Rightarrow\delta_0$ as $k\to\infty$, it follows that there exists $m\in\mathbb{N}$ such that $d(\ast_{n=k+1}^{\infty}P_n(\mu^{\mathrm{s}}),\delta_0)<\varepsilon$. Let $Q=\sup\{P_n:n=m+1,m+2,\ldots\}$. It is easy to see that $\{P_1,\ldots,P_m,Q\}$ has the required properties. ■

THEOREM 4.5.8. *Let G be an Abelian metrizable group and $\mu\in M^1(G)$. Then*

(i) *if $I(\mu)=\{0\}$ and μ is product-atomless then μ has a triangular infinitesimal system of projections*;

(ii) *if μ has a triangular infinitesimal system of projections and $\Pi_B(\mu)$ is commutative then $I(\mu)=\{0\}$ and μ is product-atomless.*

Proof. (i) is an immediate consequence of Lemma 4.5.7. (ii) is obvious. ■

COROLLARY 4.5.9. *Let $\mu\in M^1(G)$ have no G-Gaussian product-factors. Then the following conditions are equivalent*:

(i) $I(\mu)=\{0\}$ *and μ is product-atomless*;
(ii) *μ has a triangular infinitesimal system of projections.*

Set

$$m_L=\delta_{2^{-1}}*\mathop{\ast}_{n=0}^{\infty}\frac{1}{2}(\delta_{-2/3^n}+\delta_{2/3^n}).$$

Then m_L is a singular measure on $\mathbb{R}$ with $\operatorname{supp}(m_L) = C$, where C is Cantor's ternary set (see Theorem 6.11 of [12], [13] and [9], p. 178). The measure m_L is said to be the *Cantor–Lebesgue measure* on C.

Let G be an Abelian metrizable group and let ϕ be a homeomorphism from C onto some compact subset K of G. Then the measure $M_L = \phi(m_L)$ is said to be the *Cantor–Lebesgue measure* on K.

LEMMA 4.5.10. *Let G be an Abelian metrizable group, K be a compact subset of G homeomorphic to Cantor's ternary set and let M_L be the Cantor–Lebesgue measure on K. Then there exists a family $\{K_k^{(n)} : k = 1, \ldots, 2^n;\ n \in \mathbb{N}\}$ of compact subsets of K such that*

(i) $\bigcup_{k=1}^{2^n} K_k^{(n)} = K$;
(ii) $K_k^{(n)} \cap K_j^{(n)} = \emptyset$ *if* $k \neq j$;
(iii) $M_L(K_k^{(n)}) = 2^{-n}$ *for* $k = 1, \ldots, 2^{-n}$;
(iv) $K_k^{(n)} = K_{2k-1}^{(n+1)} \cup K_{2k}^{(n+1)}$.

LEMMA 4.5.11. *Let K be an independent subset of $\mathbb{R}$ homeomorphic to Cantor's ternary set and let M_L be the Cantor–Lebesgue measure on K. Then the measure $e(M_L)$ is product-atomless.*

Proof. By Lemma 4.5.10, there exists a family $\{K_k^{(n)} : k = 1, \ldots, 2^n;\ n \in \mathbb{N}\}$ of compact subsets of C such that

(i) $\bigcup_{k=1}^{2^n} K_k^{(n)} = K$;
(ii) $K_k^{(n)} \cap K_j^{(n)} = \emptyset$ if $k \neq j$;
(iii) $M_L(K_k^{(n)}) = 2^{-n}$ for $k = 1, \ldots, 2^{-n}$;
(iv) $K_k^{(n)} = K_{2k-1}^{(n+1)} \cup K_{2k}^{(n+1)}$.

We denote by $F_k^{(n)}$ the subgroup of $\mathbb{R}$ algebraically generated by $K_k^{(n)}$ for $k = 1, \ldots, 2^n$ and $n \in \mathbb{N}$. Let F be the subgroup of $\mathbb{R}$ algebraically generated by K. It is easy to see that $F = \bigoplus_{k=1}^{2^n} F_k^{(n)}$. Since $F_k^{(n)}$ are σ-compact for all $k = 1, \ldots, 2^n$ and $n \in \mathbb{N}$, Lemma 1.4.1 implies that for every $i = 1, \ldots, 2^n$ and $n \in \mathbb{N}$ there exists a Borel-measurable projection $P_i^{(n)}$ from F into itself with $\operatorname{im} P_i^{(n)} = F_i^{(n)}$ and $\ker P_i^{(n)} = \bigoplus_{k=1, k\neq i}^{2^n} F_k^{(n)}$. A trivial verification shows that $\{P_i^{(n)} : i = 1, \ldots, 2^n;\ n \in \mathbb{N}\}$ is a triangular infinitesimal system of projections corresponding to $e(M_L)$. Moreover, by Theorem 5.1.7 below, $\Pi_B(e(M_L))$ is commutative. Application of Theorem 4.5.8(ii) now ends the proof of the lemma. ∎

LEMMA 4.5.12. *Let K be a p-independent subset of $\mathbb{Z}_p^\infty$ for some prime p, homeomorphic to Cantor's ternary set, and let M_L be the Cantor-Lebesgue measure on K. Then the measure $e(M_L)$ is product-atomless and $I(e(M_L)) = \{0\}$.*

The proof is similar to that of Lemma 4.5.11 and will be omitted.

THEOREM 4.5.13. *Let G be an Abelian metrizable group, K be a subset of G homeomorphic to Cantor's ternary set and let M_L be the Cantor–Lebesgue measure on K. Suppose that either K is independent, or K is p-independent for some prime p. Then the measure $e(M_L)$ is product-atomless.*

P r o o f. We first assume that K is independent. Let K_1 be an independent subset of $\mathbb{R}$ homeomorphic to Cantor's ternary set. We will denote by F the subgroup of G algebraically generated by K and by F_1 the subgroup of $\mathbb{R}$ generated algebraically by K_1. Since K and K_1 are homeomorphic Theorem 1.4.6 implies that there exists a Borel isomorphism ϕ from F onto F_1 such that the restriction of ϕ to K is a homeomorphism from K onto K_1. Further, by Lemma 4.5.11, the measure $e(\phi(M_L))$ is product-atomless. A trivial verification shows that $e(M_L)$ is product-atomless. This completes the proof of the first part of the theorem.

Let K be p-independent for some prime p. The proof of this part of the theorem is similar to that of the first part and will be omitted. ■

THEOREM 4.5.14. *Let G be an Abelian metrizable complete nondiscrete group. Then there exists a nondegenerate product-atomless measure μ in $M_0^1(G)$.*

P r o o f. Theorem 1.4.5 implies that there exists a subset K of G homeomorphic to Cantor's ternary set such that either K is independent, or K is p-independent for some prime p. Now, application of Theorem 4.5.13 ends the proof. ■

4.6. Canonical product-decomposition of probability measures. We are now ready to prove an analogue of the classical Khinchin theorem (see [14]).

THEOREM 4.6.1. *Let G be an Abelian metrizable group and $\mu \in M_0^1(G)$. Then there exist measures $\mu_0, \mu_1, \mu_2 \in M_0^1(G)$, $x_0 \in G$ and Borel subgroups G_0, G_1, G_2 of G such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;
(ii) μ_0 *is a Gaussian measure in the sense of Gnedenko*;
(iii) μ_1 *is a product-atomless measure without G-Gaussian product-factors*;
(iv) μ_2 *is a purely product-atomic measure without G-Gaussian product-factors*;
(v) $\mu_i(G_i) = 1$ *for* $i = 0, 1, 2$;
(vi) $G_0 \cap G_1 = \{0\}$ *and* $(G_0 + G_1) \cap G_2 = \{0\}$.

Moreover, the measures μ_0, μ_1 and μ_2 are uniquely determined up to degenerate convolution factors.

P r o o f. Theorem 4.3.11 implies that there exist $P_0 \in \Pi_B(\mu)$ and $y \in G$ such that

(a) $\mu(D(P_0) + y) = 1$;
(b) $P_0(\mu * \delta_{-y}) \in \Gamma_{G,0}(G)$;
(c) the measure $(I - P_0)(\mu * \delta_{-y})$ has no G-Gaussian product-factors;
(d) $\Pi_B((I - P_0)(\mu * \delta_{-y}))$ is commutative.

Set $\mu_0 = P_0(\mu * \delta_{-y})$, $G_0 = \operatorname{im} P_0$, $\nu = (I - P_0)(\mu * \delta_{-y})$ and $F = \ker P_0$. Putting

$$Q = \sup \mathcal{A}_B(\nu),$$

we deduce from Theorem 4.4.4(ii) that the measure $Q(\nu * \delta_{-z})$ is purely product-atomic for every $z \in F$ with $\nu\{D(Q) + z\} = 1$. Let $\mu_1 = (I - Q)(\nu * \delta_{-z})$, $\mu_2 = Q(\nu * \delta_{-z})$, $G_1 = F \cap \ker P$ and $G_2 = F \cap \operatorname{im} P$. A trivial verification shows that μ_1 is product-atomless. ■

A convolution decomposition of the measure $\mu \in M^1(G)$ of the form

$$\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x,$$

where $\mu_0, \mu_1, \mu_2 \in M^1(G)$ and $x \in G$, is called a *canonical product-decomposition* of μ if the following conditions hold:

(i) μ_0 is a Gaussian measure in the sense of Gnedenko;
(ii) μ_1 is a product-atomless measure without G-Gaussian product-factors;
(iii) μ_2 is a purely product-atomic measure without G-Gaussian product-factors;
(iv) there exist Borel subgroups G_0, G_1, G_2 of G such that $G_0 \cap G_1 = \{0\}$, $(G_0 + G_1) \cap G_2 = \{0\}$ and $\mu_i(G_i) = 1$ for $i = 0, 1, 2$.

COROLLARY 4.6.2. *Let G be an Abelian metrizable group and $\mu \in M_0^1(G)$. Then μ has a canonical product-decomposition uniquely determined up to degenerate convolution factors.*

Proof. This follows at once from Theorem 4.6.1. ■

COROLLARY 4.6.3. *Let $\mu \in M^1(G)$. Then there exist measures $\mu_0, \mu_1, \mu_2 \in M^1(G)$, $x_0 \in G$ and Borel subgroups G_0, G_1, G_2 of G such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;
(ii) $\pi_{I(\mu)}(\mu_0)$ *is a Gaussian measure in the sense of Gnedenko*;
(iii) $\pi_{I(\mu)}(\mu_1)$ *is a product-atomless measure without G-Gaussian product-factors*;
(iv) $\pi_{I(\mu)}(\mu_2)$ *is a purely product-atomic measure without G-Gaussian product-factors*;
(v) $\mu_i(G_i) = 1$ *for* $i = 0, 1, 2$;
(vi) $G_0 \cap G_1 \subset I(\mu)$ *and* $(G_0 + G_1) \cap G_2 \subset I(\mu)$.

Moreover, there exists a sequence $(\lambda_n) \subset M^1(G)$ such that $\mu_2 = \ast_{n=1}^{\infty} \lambda_n$, and for every $n \in \mathbb{N}$, the measure $\pi_{I(\mu)}(\lambda_n)$ is product-indecomposable.

Proof. This follows from Theorem 2.4.9 together with Theorem 4.6.1. ■

V. Product-decomposability of probability measures on locally convex metrizable spaces

5.1. Strong product-decomposability of probability measures on metrizable linear spaces. The present section contains basic definitions and theorems on strong product-decomposability of probability measures on metrizable linear spaces. Since many results of this section are analogous to those of Chapters III and IV we omit the proofs.

LEMMA 5.1.1. *Let E be a metrizable linear space and G be a σ-compact subgroup of E. Then* $\text{lin}(G)$ *is a σ-compact subspace of E.*

LEMMA 5.1.2. *Let E be a metrizable linear space. Then $M^1(E) = M_0^1(E)$.*

LEMMA 5.1.3. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then there exists a σ-compact subspace F of E such that $\mu(E) = 1$.*

Proof. For every $n \in \mathbb{N}$ there exists a compact C_n of E with $\mu(E \setminus C_n) \le 1/n$. Without loss of generality we may assume that $C_n \subset C_{n+1}$. Since

$$\mathrm{lin}(C_n) = \bigcup_{m=1}^{\infty} \Big(\bigcup_{k=1}^{\infty} \underbrace{[-k,k]C_n + \ldots + [-k,k]C_n}_{m\,\text{times}} \Big)$$

we conclude that $\mathrm{lin}(C_n)$ is σ-compact and $\mathrm{lin}(C_n) \subseteq \mathrm{lin}(C_{n+1})$. Putting $F = \bigcup_{n=1}^{\infty} \mathrm{lin}(C_n)$ one deduces that F is a σ-compact linear space with $\mu(F) = 1$. ■

LEMMA 5.1.4. *If E is a metrizable linear space then $I(E) = I_0(E)$.*

Let E be a metrizable space and $\mu \in M^1(E)$. An algebraic linear operator A in E is called a *linear measurable operator in G* if

(i) $D_l(A) \in \mathrm{Bo}(E)$;
(ii) A is Borel-measurable;
(iii) $\mu(D_l(A) + x_0) = 1$ for certain $x_0 \in E$.

We denote by $\mathcal{L}in(E;\mu)$ the set of all linear measurable operators A in E.

COROLLARY 5.1.5. *$\mathcal{L}in(E;\mu) \subset \mathrm{Add}(E;\mu)$.*

Let E be a metrizable space and $\mu \in M^1(E)$. A Borel subspace F of E is said to be a *linear subdomain* of $A \in \mathcal{L}(E;\mu)$ if $F \subset D_l(A)$ and $\mu^{\mathrm{s}}(F) = 1$.

COROLLARY 5.1.6. *If G is a σ-compact subdomain of $A \in \mathcal{L}in(E;\mu)$ then $\mathrm{lin}(G)$ is a σ-compact linear subdomain of A.*

COROLLARY 5.1.7. *The relation "$A = B$ μ^{s}-a.s." for $A, B \in \mathcal{L}in(E;\mu)$ is an equivalence relation.*

Remark. $\mathcal{L}in(E;\mu)$ denotes the set of all measurable operators in E. At the same time the elements of $\mathcal{L}in(E;\mu)$ are understood as equivalence classes of the relation "equality μ^{s}-a.s.", but this should not lead to any confusion.

THEOREM 5.1.8. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then $\mathcal{L}in(E;\mu)$ is a metrizable linear space with the metric d. Moreover, $\mathcal{L}in(E;\mu)$ is a closed subgroup of $\mathrm{Add}(E;\mu)$.*

Let E be a metrizable linear space and $\mu \in M^1(E)$. A linear operator $P \in \mathcal{L}in(E;\mu)$ is called a *linear measurable projection* if there exists a linear subdomain F of P such that $P|_F$ is an algebraic linear projection.

Let E be a metrizable linear space and $\mu \in M^1(E)$. A Borel subspace F of E is said to be

(i) a *proper linear subdomain* of a projection $P \in \mathcal{L}(E;\mu)$ if F is an invariant linear subdomain of P with $Px = P^2x$ for each $x \in F$;

(ii) a *proper linear common subdomain* of a family of projections $\mathcal{P}$ in $\mathcal{L}in(G;\mu)$ if F is a proper linear subdomain of P for each $P \in \mathcal{P}$.

THEOREM 5.1.9. *Let E be a metrizable linear space, $\mu \in M^1(E)$ and $P \in \mathcal{L}in(E;\mu)$. Then the following statements are equivalent*:

(i) *P is a linear measurable projection*;
(ii) $P^2 \in \mathcal{L}in(E;\mu)$ *and* $P = P^2$ μ^{s}*-a.s.*;
(iii) *there exists a σ-compact proper linear subdomain H of P such that*
 (a) $Px = P^2x$ *for each* $x \in H$;
 (b) $H \cap \operatorname{im} P$, $H \cap \ker P$ *are σ-compact.*

The proof is similar to that of Theorem 3.1.15 and will be omitted.

Let E be a metrizable linear space and $\mu \in M^1(E)$. We define

$$\mathbb{D}_{SB}(\mu) = \mathbb{D}_B(\mu) \cap \mathcal{L}in(E;\mu), \qquad \mathbb{S}_{SB}(\mu) = \mathbb{S}_B(\mu) \cap \mathcal{L}in(E;\mu).$$

Corollary 5.1.10. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then*

(i) $\mathbb{D}_{SB}(\mu)$ *with the metric d is a right metrizable semigroup under multiplication of operators*;
(ii) $\mathbb{D}_{SB}(\mu)$ *is closed in* $\mathcal{L}in(E;\mu)$;
(iii) $\mathbb{S}_{SB}(\mu)$ *is a closed subsemigroup of* $\mathbb{D}_{SB}(\mu)$.

The semigroup $\mathbb{D}_{SB}(\mu)$ is called the *strong Borel decomposability semigroup* of the measure μ, and the subsemigroup $\mathbb{S}_{SB}(\mu)$ is called the *strong Borel symmetry semigroup* of μ.

We denote by $\varPi_{SB}(\mu)$ the subset of $\mathbb{D}_{SB}(\mu)$ consisting of all linear projections.

Theorem 5.1.11. *$\varPi_{SB}(\mu)$ is a complete sublattice of $\varPi_B(\mu)$.*

The proof is similar to that of Theorem 3.4.11.

Let E be a metrizable linear space. A measure μ in $M^1(E)$ is said to be *strongly product-decomposable* if there exist two nondegenerate measures μ_1, μ_2, two Borel subspace E_1, E_2 of E and an element x in E such that

(i) $\mu = \mu_1 * \mu_2 * \delta_x$;
(ii) $\mu_i(E_i) = 1$ for $i = 1, 2$;
(iii) $E_1 \cap E_2 = \{0\}$.

In the opposite case, μ is said to be *strongly product-indecomposable*.

Corollary 5.1.12. *Every strongly product-decomposable measure $\mu \in M^1(E)$ is product-decomposable.*

The following example shows that there exists a product-decomposable measure which is strongly product-indecomposable.

Example 5.1.1. Let γ be a Gaussian measure on $\mathbb{R}^2$ with $\mathrm{supp}(\gamma) = \{(x, \pi x) : x \in \mathbb{R}\}$ and let ν be a probability measure on $\mathbb{R}^2$ with $\mathrm{supp}(\nu) = \mathbb{Z}^2$. Let $\mu = \gamma * \nu$. It is easy to see that μ is product-decomposable and strongly product-indecomposable.

Corollary 5.1.13. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then the following statements are equivalent*:

(i) *μ is strongly product-decomposable*;
(ii) $\varPi_{SB}(\mu) \neq \{0, I\}$.

A nondegenerate measure ν in $M^1(E)$ is said to be a *strong product-factor* of a measure μ in $M^1(E)$ if there exist $P \in \Pi_{SB}(\mu)$ and $x \in E$ with $\mu(D_l(P)+x) = 1$ such that $\nu = P(\mu * \delta_{-x})$.

A linear projection $P \in \Pi_{SB}(\mu) \setminus \{0\}$ is said to be a *strong product-atom* if

$$\{Q \in \Pi_{SB}(\mu) : 0 \le Q \le P\} = \{0, I\}.$$

We denote by $\mathcal{A}_{SB}(\mu)$ the set of all strong product-atoms from $\Pi_{SB}(\mu)$.

A measure μ in $M^1(E)$ is said to be

(i) *strongly product-atomic* if for every $P \in \Pi_{SB}(\mu) \setminus \{0\}$ there exists $Q \in \mathcal{A}_{SB}(\mu)$ such that $0 < Q \le P$;

(ii) *strongly product-atomless* if it has no strong product-atoms.

LEMMA 5.1.14. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then*

(i) $\mathbb{D}(\mu)$ *is a subsemigroup of* $\mathbb{D}_{SB}(\mu)$;

(ii) $\mathcal{A}(\mu) \subset \mathcal{A}_{SB}(\mu)$.

Let E be a metrizable linear space and $\mu \in M^1(E)$. A triangular system $\{P_j^{(n)} : j = 1,\dots,k_n;\ n = 1,2,\dots\}$ of linear projections in $\Pi_{SB}(\mu) \setminus \{0\}$ is called a *triangular infinitesimal system of linear projections* corresponding to the measure μ if

(i) $P_j^{(n)}P_i^{(n)} = P_i^{(n)}P_j^{(n)} = 0$ μ^{s}-a.s. for $i,j \in \{1,\dots,k_n\}$, $i \ne j$;

(ii) $\sum_{i=1}^{k_n} P_i^{(n)} = I$ μ^{s}-a.s. for each $n \in \mathbb{N}$;

(iii) $\max\{P_i^{(n+1)} : P_i^{(n+1)} \le P_j^{(n)}\} = P_j^{(n)}$ for each $n \in \mathbb{N}$ and $j \in \{1,\dots,k_n\}$;

(iv) for each sequence (j_n) in $\mathbb{N}$ with $j_n \in \{1,\dots,k_n\}$ one has $P_{j_n}^{(n)}\mu^{\mathrm{s}} \Rightarrow \delta_0$ as $n \to \infty$.

COROLLARY 5.1.15. *Let E be a metrizable linear space and let $\mu \in M^1(E)$ be nondegenerate. Let $\{P_j^{(n)} : j = 1,\dots,k_n;\ n = 1,2,\dots\}$ be a triangular infinitesimal system of linear projections corresponding to the measure μ. Then $\dim E = \infty$.*

THEOREM 5.1.16. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then*

(i) *if μ is strongly product-atomless then μ has a triangular infinitesimal system of linear projections*;

(ii) *if μ has a triangular infinitesimal system of linear projections and $\Pi_{SB}(\mu)$ is commutative then μ is strongly product-atomless.*

The proof is similar to that of Theorem 4.5.8.

A measure μ in $M^1(E)$ is said to be a *strong Gaussian measure in the sense of Gnedenko* (*SG-Gaussian*) if there exist $\nu \in M^1(E)$ and $P \in \Pi_B(\mu \otimes \nu)$ such that the condition

$$(E \times \{0\} \cup \{0\} \times E) \cap ((\operatorname{im} P) \cup (\ker P)) \cap F = \{(0,0)\}$$

holds for some Borel subspace F of $E \times E$ with the property $(\mu \otimes \nu)^{\mathrm{s}}(F) = 1$.

The class of all *SG*-Gaussian measures on E will be denoted by $\Gamma_{SG}(E)$.

COROLLARY 5.1.17. $\Gamma_{SG}(E) \subset \Gamma_G(E)$.

THEOREM 5.1.18. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then there exist measures $\mu_0, \mu_1, \mu_2 \in M^1(E)$, $x_0 \in E$ and Borel subspaces E_0, E_1, E_2 of E such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;

(ii) μ_0 *is a strong Gaussian measure in the sense of Gnedenko*;

(iii) μ_1 *is a strongly product-atomless measure without SG-Gaussian strong product-factors*;

(iv) μ_2 *is a purely strongly product-atomic measure without SG-Gaussian strong product-factors*;

(v) $\mu_i(E_i) = 1$ *for* $i = 0, 1, 2$;

(vi) $E_0 \cap E_1 = \{0\}$ *and* $(E_0 + E_1) \cap E_2 = \{0\}$.

Moreover, the measures μ_0, μ_1 and μ_2 are uniquely determined up to degenerate convolution factors.

The proof is similar to that of Theorem 4.6.1 and will be omitted.

A convolution decomposition of the measure $\mu \in M^1(E)$ of the form

$$\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x,$$

where $\mu_0, \mu_1, \mu_2 \in M^1(E)$ and $x \in E$, is called a *canonical strong product-decomposition* of μ if

(i) μ_0 is a strong Gaussian measure in the sense of Gnedenko;

(ii) μ_1 is a strong product-atomless measure without strong G-Gaussian product-factors;

(iii) μ_2 is a strong purely product-atomic measure without strong G-Gaussian product-factors;

(iv) there exist Borel subspaces E_0, E_1, E_2 of E such that $E_0 \cap E_1 = \{0\}$, $(E_0 + E_1) \cap E_2 = \{0\}$ and $\mu_i(E_i) = 1$ for $i = 0, 1, 2$.

COROLLARY 5.1.19. *Let E be a metrizable linear space and $\mu \in M^1(E)$. Then μ has a canonical strong product-decomposition uniquely determined up to degenerate convolution factors.*

5.2. Infinitely divisible probability measures on locally convex metrizable spaces. Let E be a locally convex metrizable space. A measure $\mu \in M^1(E)$ is said to be *Gaussian* if for every $x' \in E'$, the measure $x'(\mu)$ is a Gaussian measure on $\mathbb{R}$.

The class of all Gaussian measures in $M^1(E)$ will be denoted by $\Gamma(E)$.

We now state some lemmata.

LEMMA 5.2.1. *Let E be a locally convex metrizable space and $\mu \in I(E)$. Then μ has a unique canonical representation $\mu = \varrho * \widetilde{e}(M)$, where ϱ is a symmetric Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with a Lévy measure M. In particular, M is uniquely determined.*

P r o o f. See Theorem 1.9 of [5]. ∎

LEMMA 5.2.2. *Let E be a locally convex metrizable space and $\mu \in M^1(E)$. Suppose that there exists a triangular uniformly infinitesimal system $\{\mu_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ in $M^1(E)$ such that*

$$\mathop{\ast}_{i=1}^{k_n} \mu_i^{(n)} * \delta_{x_n} \Rightarrow \mu$$

for some sequence (x_n) in E. Then $\mu \in I(E)$.

Proof. This follows at once from Theorem 1.9 of [5].

THEOREM 5.2.3. *Let E be a locally convex metrizable space, $\mu \in I(E)$ and $P \in \Pi_B(\mu)$. Suppose that $\mu = \varrho * \widetilde{e}(M)$ is the canonical representation of μ, where ϱ is a symmetric Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with the Lévy measure M. Then*

(i) $P \in \Pi_B(\varrho)$;

(ii) $P \in \Pi_B(\widetilde{e}(M))$ *and* $M = PM + (I-P)M$;

(iii) *for every $x \in E$ with $\widetilde{e}(M)(D(P)+x) = 1$ there exist $y, z \in E$ such that $P(\widetilde{e}(M) * \delta_{-x}) = \widetilde{e}(P(M)) * \delta_{-y}$ and $(I-P)(\widetilde{e}(M) * \delta_{-x}) = \widetilde{e}((I-P)(M)) * \delta_{-z}$.*

Proof. Let x_0 be some element of E with $\mu(D(P)+x_0) = 1$. Putting $\nu = \mu * \delta_{-x_0}$ we obtain $\nu = P\nu * (I-P)\nu$. Let F be a σ-compact proper subdomain of P such that $\operatorname{im} P \cap F$ and $\ker P \cap F$ are σ-compact. Moreover, $P(\nu)(\operatorname{im} P \cap F) = (I-P)(\ker P \cap F) = 1$. Since Lemma 3.1.20 implies that $P(\nu), (I-P)(\nu) \in I(E)$ we conclude that $P(\nu) = \varrho_1 * \widetilde{e}(M_1)$ and $(I-P)(\nu) = \varrho_2 * \widetilde{e}(M_2)$, where ϱ_1, ϱ_2 are symmetric Gaussian measures on G, $\widetilde{e}(M_1)$ and $\widetilde{e}(M_2)$ are generalized Poisson measures on E. The equality

$$\varrho * \widetilde{e}(M) = \varrho_1 * \widetilde{e}(M_1) * \varrho_2 * \widetilde{e}(M_2)$$

implies that $\varrho = \varrho_1 * \varrho_2$ and $\widetilde{e}(M) = \widetilde{e}(M_1) * \widetilde{e}(M_2)$. Combining Lemma 2.3.3 with Lemma 2.3.12 we deduce that

$$\varrho_1(\operatorname{im} P \cap F) = \widetilde{e}(M_1)(\operatorname{im} P \cap F) = \varrho_2(\ker P \cap F) = \widetilde{e}(M_2)(\ker P \cap F) = 1,$$
$$M_1(E \setminus \operatorname{im} P \cap F) = M_2(E \setminus \ker P \cap F) = 0.$$

Then one easily checks that $P \in \Pi_B(\varrho) \cap \Pi_B(\widetilde{e}(M))$, $M_1 = PM$ and $M_2 = (I-P)M$. ∎

THEOREM 5.2.4. *Let E be a metrizable locally convex space and let $\mu = \widetilde{e}(M)$ be a generalized Poisson measure on E. Then $\Pi_B(\mu) = \Pi_B(\mu^{\mathrm{s}})$.*

Proof. Let $P \in \Pi_B(E)$ and let F be a proper σ-compact subdomain of P such that $\operatorname{im} P \cap F$ and $\ker P \cap F$ are σ-compact. Hence,

$$M(E \setminus ((\operatorname{im} P \cap F) \cup (\ker P \cap F))) \le (M + \overline{M})(E \setminus ((\operatorname{im} P \cap F) \cup (\ker P \cap F))) = 0.$$

Writing $M_1 = M|_{\operatorname{im} P \cap F}$ and $M_2 = M|_{\ker P \cap F}$ we conclude that $M = M_1 + M_2$, $M_1 + \overline{M}_1 = PM + P\overline{M}$ and $M_2 + \overline{M}_2 = (I-P)M + (I-P)\overline{M}$. Theorem 5.2.3 now implies that $(\widetilde{e}(M_1))^{\mathrm{s}} = P(\widetilde{e}(M+\overline{M}))$ and $(\widetilde{e}(M_2))^{\mathrm{s}} = (I-P)(\widetilde{e}(M+\overline{M}))$. Since $(\widetilde{e}(M_1))^{\mathrm{s}}(\operatorname{im} P \cap F) = (\widetilde{e}(M_2))^{\mathrm{s}}(\ker P \cap F) = 1$ we deduce that there exist $x, y \in E$ such that

$$\widetilde{e}(M_1)(\operatorname{im} P \cap F + x) = \widetilde{e}(M_2)(\ker P \cap F + y) = 1.$$

An easy computation shows that $P \in \Pi_B(\mu)$. ∎

THEOREM 5.2.5. *Let E be a metrizable locally convex space and let $\mu = \widetilde{e}(M)$ be a generalized Poisson measure on E. Then $\Pi_B(\mu)$ is commutative.*

Proof. Let F be a σ-compact proper common subdomain of projections P and Q such that $\operatorname{im} P \cap F$, $\ker P \cap F$, $\operatorname{im} Q \cap F$, $\ker Q \cap F$ are σ-compact. Theorem 5.2.3 now implies that $M(E \setminus ((\operatorname{im} P \cap F) \cup (\ker P \cap F))) = M(E \setminus ((\operatorname{im} Q \cap F) \cup (\ker Q \cap F))) = 0$. Hence,

$$M(E \setminus ((\operatorname{im} P \cap \operatorname{im} Q \cap F) \cup (\operatorname{im} P \cap \ker Q \cap F) \cup (\ker P \cap \operatorname{im} Q \cap F) \cup (\ker P \cap \ker Q \cap F))) = 0.$$

Writing $N = M + \overline{M}$, $N_1 = N|_{\operatorname{im} P \cap \operatorname{im} Q \cap F}$, $N_2 = N|_{\operatorname{im} P \cap \operatorname{im} Q \cap F}$, $N_3 = N|_{\operatorname{im} P \cap \operatorname{im} Q \cap F}$, $N_4 = N|_{\operatorname{im} P \cap \operatorname{im} Q \cap F}$, we conclude that $\mu^{\mathrm{s}} = \widetilde{e}(N)$, $PN = N_1 + N_2$, $(I-P)N = N_3 + N_4$, $QN = N_1 + N_3$ and $(I-Q)N = N_2 + N_4$. But this implies that

$$P\mu^{\mathrm{s}} = \widetilde{e}(PN) = \widetilde{e}(N_1) * \widetilde{e}(N_2), \quad (I-P)\mu^{\mathrm{s}} = \widetilde{e}((I-P)N) = \widetilde{e}(N_3) * \widetilde{e}(N_4),$$
$$Q\mu^{\mathrm{s}} = \widetilde{e}(QN) = \widetilde{e}(N_1) * \widetilde{e}(N_3), \quad (I-Q)\mu^{\mathrm{s}} = \widetilde{e}((I-Q)N) = \widetilde{e}(N_2) * \widetilde{e}(N_4).$$

Now, by Lemma 2.3.3,

$$\begin{aligned}\widetilde{e}(N_1)(\operatorname{im} P \cap F) &= \widetilde{e}(N_2)(\operatorname{im} P \cap F) = \widetilde{e}(N_3)(\ker P \cap F) = \widetilde{e}(N_4)(\ker P \cap F)\\ &= \widetilde{e}(N_1)(\operatorname{im} Q \cap F) = \widetilde{e}(N_2)(\ker Q \cap F)\\ &= \widetilde{e}(N_3)(\operatorname{im} Q \cap F) = \widetilde{e}(N_4)(\ker Q \cap F) = 1.\end{aligned}$$

Since

$$\begin{aligned}\widetilde{e}(N_1)(\operatorname{im} P \cap \operatorname{im} Q \cap F) &= \widetilde{e}(N_2)(\operatorname{im} P \cap \ker Q \cap F) = \widetilde{e}(N_3)(\ker P \cap \operatorname{im} Q \cap F)\\ &= \widetilde{e}(N_4)(\ker P \cap \ker Q \cap F) = 1\end{aligned}$$

we conclude that

$$\mu^{\mathrm{s}}(\operatorname{im} P \cap \operatorname{im} Q \cap F + \operatorname{im} P \cap \ker Q \cap F + \ker P \cap \operatorname{im} Q \cap F + \ker P \cap \ker Q \cap F) = 1.$$

We set $H = \operatorname{im} P \cap \operatorname{im} Q \cap F \oplus \operatorname{im} P \cap \ker Q \cap F \oplus \ker P \cap \operatorname{im} Q \cap F \oplus \ker P \cap \ker Q \cap F$. It is easy to see H is a σ-compact proper common subdomain of P and Q, and $PQx = QPx$ for all $x \in H$. ■

5.3. Gaussian measures on locally convex metrizable spaces

THEOREM 5.3.1. *Let E be a locally convex metrizable space. Then*

$$\Gamma(E) = \Gamma_B(E) = \Gamma_P(E) = \Gamma_G(E) = \Gamma_{SG}(E).$$

Proof. It suffices to show that $\Gamma_G(E) \subset \Gamma(E)$. Let $\varrho \in \Gamma_G(E)$. By assumption there exist $P, Q \in \Pi_B(\varrho)$ and a common subdomain G of projections P and Q such that

$$(\operatorname{im} P \cup \ker P) \cap (\operatorname{im} Q \cup \ker Q) \cap G = \{0\}.$$

We set $\nu = |\varrho|^2$. Hence, by Theorem 3.4.3, $(PQ)_{\underline{ij}_n} \nu \Rightarrow \delta_0$ as $n \to \infty$ for $i, j = 0, 1$. As application of Lemma 3.4.4 we find that the triangular system $\{(PQ)_{\underline{k}_n} \nu : \underline{k}_n \in J_n;\ n \in \mathbb{N}\}$ is uniformly infinitesimal. Thus, $\nu \in I(E)$. Let $\nu = \gamma * \widetilde{e}(M)$ be the canonical representation of ν, where γ is a symmetric Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with the Lévy measure M. Hence, Theorem 5.2.3 together with Theorem 5.2.5 implies that $P, Q \in \Pi_P(\widetilde{e}(M))$ and $PQ = QP$ $(\widetilde{e}(M))^{\mathrm{s}}$-a.s. Thus

$(\widetilde{e}(M))^{\rm s} = \delta_0$ and finally $\nu \in \Gamma(E)$. Application of the Cramér theorem then shows $\varrho \in \Gamma(E)$. ■

Let (ϱ_n) be a sequence of symmetric Gaussian measures on $\mathbb{R}$ with $\int x^2 \varrho_n(dx) = 1$. We define $\varrho_0 = \bigotimes_{n=1}^{\infty} \varrho_n$. Clearly, ϱ_0 is a Gaussian symmetric measure on $\mathbb{R}^\infty$.

THEOREM 5.3.2. *Let $\varrho \in M^1(\mathbb{R}^\infty)$. Then ϱ is a Gaussian measure iff $\varrho = A\varrho_0 * \delta_x$, where $x \in \mathbb{R}^\infty$ and A is a linear continuous injective mapping from $\mathbb{R}^\infty$ into itself such that $A(\mathbb{R}^\infty)$ is closed.*

The proof is prepared by two lemmata.

LEMMA 5.3.3. *Let B be a symmetric nonnegative definite bilinear form on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$. Then*

1. *There exists a Hamel basis (e'_n) of $\mathbb{R}_0^\infty$ such that*
   (i) *$B(e'_n, e'_m) = 0$ for $n \neq m$;*
   (ii) *$B(e'_n, e'_n) \in \{0, 1\}$;*
   (iii) *$B(x', y') = \sum_{n=1}^{\infty} B(e'_n, e'_n)\langle e_n, x'\rangle\langle e_n, y'\rangle$,*

where (e_n) is the Schauder basis of $\mathbb{R}^\infty$ such that $\langle e_k, e'_n\rangle = \delta_{k,n}$.

2. *If (f'_n) is an arbitrary Hamel basis of $\mathbb{R}_0^\infty$ with corresponding Schauder basis (f_n) of $\mathbb{R}^\infty$ then for all $x', y' \in \mathbb{R}_0^\infty$,*

$$B(x', y') = B_0(Cx', Cy'),$$

where $B_0(\cdot,\cdot) = \sum_{n=1}^{\infty} B(e'_n, e'_n)\langle f_n, \cdot\rangle\langle f_n, \cdot\rangle$ and C is a topological automorphism of $\mathbb{R}_0^\infty$ defined by $Ce'_n = f'_n$ for $n \in \mathbb{N}$.

Proof. 1. Let $E = \{x' \in \mathbb{R}_0^\infty : B(x', x') = 0\}$. Then one easily checks that E is a linear subspace of $\mathbb{R}_0^\infty$ and $B(x', y') = 0$ for all $x' \in E$, $y' \in \mathbb{R}_0^\infty$. Let F be an algebraically complementary subspace to E. Hence, $B(x', x') > 0$ for all $x' \in F \setminus \{0\}$. By the Schmidt orthogonalization there exists a Hamel basis (f'_n) of F such that $B(f'_n, f'_k) = \delta_{n,k}$. Let (g'_n) be a Hamel basis of E. It is easy to see that $(f'_n) \cup (g'_n)$ has the required properties.

2. This is obvious. ■

COROLLARY 5.3.4. *If B_1 is a symmetric nonnegative bilinear form on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that*

$$B_1(x', x') \le B(x', x') \quad \text{for all } x' \in \mathbb{R}_0^\infty,$$

then there exists a sequence $(b_n) \subset [0, 1]$ such that

$$B(x', y') = \sum_{n=1}^{\infty} b_n B(e'_n, e'_n)\langle e'_n, x'\rangle\langle e'_n, y'\rangle \quad \text{for all } x', y' \in \mathbb{R}_0^\infty.$$

LEMMA 5.3.5. *For any complex-valued function ϕ on $\mathbb{R}_0^\infty$ the following statements are equivalent*:

(i) *there exists a Gaussian measure ϱ on $\mathbb{R}^\infty$ with $\widehat{\varrho} = \phi$;*

(ii) *there exist $x_0 \in \mathbb{R}^\infty$ and a symmetric nonnegative bilinear form Q on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that*

$$\phi(x') = \exp\left\{i\langle x_0, x'\rangle - \tfrac{1}{2}Q(x', x')\right\}$$

for all $x' \in \mathbb{R}_0^\infty$.

Proof. See Theorem 2.1.5 of [25]. ■

Proof of Theorem 5.3.2. Let ϱ be a Gaussian measure on $\mathbb{R}^\infty$. By Lemma 5.3.5 there exist $x_0 \in \mathbb{R}^\infty$ and a symmetric nonnegative bilinear form Q on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that

$$\phi(x') = \exp\left\{i\langle x_0, x'\rangle - \tfrac{1}{2}Q(x', x')\right\}$$

for all $x' \in \mathbb{R}_0^\infty$.

Let (e_n') be a Hamel basis of $\mathbb{R}_0^\infty$ with corresponding Schauder basis (e_n) of $\mathbb{R}^\infty$ such that

(i) $Q(e_n', e_m') = 0$ for $n \neq m$;
(ii) $Q(e_n', e_n') \in \{0, 1\}$;
(iii) $Q(x', y') = \sum_{n=1}^\infty Q(e_n', e_n')\langle e_n', x'\rangle\langle e_n', y'\rangle$ for all $x', y' \in \mathbb{R}_0^\infty$.

Also, let C' be the topological automorphism of $\mathbb{R}_0^\infty$ defined by $C'e_n' = \varepsilon_n$ for $n \in \mathbb{N}$. Hence, by Lemma 5.3.5,

$$\widehat{\gamma}(x') = \exp\left\{-\tfrac{1}{2}Q_0(C'x', C'x')\right\} = \widehat{\gamma}_0(C'x') \quad \text{for all } x' \in \mathbb{R}_0^\infty,$$

where $\gamma = \varrho * \delta_{-x_0}$, $Q_0(x', y') = \sum_{n=1}^\infty Q(e_n', e_n')\langle \varepsilon_n, x'\rangle\langle \varepsilon_n, y'\rangle$ for all $x', y' \in \mathbb{R}_0^\infty$ and γ_0 is the Gaussian measure on $\mathbb{R}^\infty$ such that

$$\widehat{\gamma}_0(x') = \exp\left\{-\tfrac{1}{2}Q_0(x', x')\right\} \quad \text{for all } x' \in \mathbb{R}_0^\infty.$$

This implies that $\gamma = C\gamma_0$, where C is the adjoint mapping to C'.

Clearly, there exists an injective linear continuous mapping B from $\mathbb{R}^\infty$ into itself such that $B(\mathbb{R}^\infty)$ is closed and $B(\varrho_0) = \gamma_0$. Thus $\varrho = CB(\varrho_0) * \delta_{x_0}$. ■

We now state the following fundamental theorem on Gaussian measures on locally convex metrizable spaces:

THEOREM 5.3.6. *Let E be a metrizable locally convex space and $\varrho \in M^1(E)$. Then the following statements are equivalent*:

(i) *there exist a σ-compact subspace V of $\mathbb{R}^\infty$ with $\varrho_0(V) = 1$, an injective linear Borel-measurable mapping S from V into E such that $S(V)$ is separable and $x \in E$ such that $\varrho = S(\varrho_0) * \delta_x$.*

(ii) *ϱ is a Gaussian measure on E.*

Proof. The implication (i)⇒(ii) is obvious.

(ii)⇒(i). By Lemma 5.1.3 there exists a σ-compact subspace E_0 of E such that $\varrho(E_0) = 1$. Without loss of generality we may assume that ϱ is symmetric. Since E_0 is a metrizable locally convex separable space by Lemma 1.8.8 there exists an injective linear continuous mapping T from E_0 into $\mathbb{R}^\infty$. Let $F = T(E_0)$. Hence, F is σ-compact. It is easy to check that $T(\varrho)$ is a Gaussian symmetric measure on $\mathbb{R}^\infty$ and $T(\varrho)(F) = 1$. By Theorem 5.3.2 there exists an injective linear continuous mapping from $\mathbb{R}^\infty$ into itself such that $A(\mathbb{R}^\infty)$ is closed and $T(\varrho) = A(\varrho_0)$. Write $V = A^{-1}(F)$. Thus V is σ-compact and $\varrho_0(V) = 1$. Putting $S = T^{-1}(A|_V)$ we obtain $\varrho = S\varrho_0$. ■

COROLLARY 5.3.7. *Let E be a metrizable locally convex space and μ be a Gaussian measure on E. Then there exists a sequence $(P_n) \subset \Pi_B(\mu)$ such that*

(i) *for every $n \in \mathbb{N}$, $D(P_n)$ is a Borel subspace of E and P_n is linear*;
(ii) $\dim(\operatorname{im} P_n) = 1$ *for all* $n \in \mathbb{N}$;
(iii) $P_n P_m = P_m P_n = 0$ μ^{s}-*a.s.*;
(iv) $\sup_n \sum_{k=1}^n P_k = I$ μ^{s}-*a.s.*;
(v) $\mu = \ast_{n=1}^{\infty} (\mu * \delta_{-x}) * \delta_x$ *for some $x \in E$ with $\mu(D(P_n) + x) = 1$ for all $n \in \mathbb{N}$.*

COROLLARY 5.3.8. *Let E be a Fréchet space and $\mu \in M^1(E)$. Then μ is a Gaussian measure iff there exists a sequence (x_n) in E such that $\mu = \mathcal{L}(\sum_{n=1}^{\infty} \gamma_n x_n + x)$, where (γ_n) is a sequence of real independent*, *symmetric Gaussian random variables and x is an element of E. Moreover*, *without loss of generality we may assume that for every $n \in \mathbb{N}$ there exists a Borel subspace V_n of E such that*

(i) $x_n \notin V_n$;
(ii) $\mathcal{L}(\sum_{k=1, k \neq n}^{\infty} \gamma_k x_k)(V_n) = 1$.

LEMMA 5.3.9. *Let ϱ be a nondegenerate Gaussian measure on $\mathbb{R}$ and let G be a Borel subgroup of $\mathbb{R}$ with $\varrho(G) = 1$. Then $G = \mathbb{R}$.*

THEOREM 5.3.10. *Let E be a locally convex metrizable space and $\varrho \in G(E)$. Then ϱ is purely product-atomic.*

This follows at once from Corollary 5.3.8 together with Lemma 5.3.9.

THEOREM 5.3.11. *Let E be a metrizable locally convex space and μ_1, μ_2 be Gaussian measures on E. Then there exist Gaussian measures ν_1, ν_2, γ on E, $x_1, x_2 \in E$ and Borel subspaces E_1 and E_2 of E such that*

(i) $\mu_i = \nu_i * \gamma * \delta_{x_i}$ *for* $i = 1, 2$;
(ii) $\nu_i(E_i) = 1$ *for* $i = 1, 2$;
(iii) $E_1 \cap E_2 = \{0\}$.

Proof. Set $\mu = \mu_1 * \mu_2$. By Theorem 5.3.6, there exist a σ-compact subspace V of E, an element x of E with $\mu * \delta_{-x}(V) = 1$ and an injective linear continuous map T from V into $\mathbb{R}^\infty$ such that $T(\mu * \delta_{-x}) = \varrho_0$. Clearly, there exists $y \in E$ such that $\mu_1 * \delta_{-x+y}(V) = 1$ and $\mu_2 * \delta_{-x-y}(V) = 1$. Putting $\varrho_1 = T(\mu_1 * \delta_{-x+y})$ and $\varrho_2 = T(\mu_2 * \delta_{-x-y})$ we obtain $\varrho_0 = \varrho_1 * \varrho_2$. Corollary 5.3.4 implies that there exist sequences $(c_n^{(1)}), (c_n^{(2)}) \subset [0, 1]$ such that

(a) $c_n^{(1)} + c_n^{(2)} = 1$;
(b) $\widehat{\varrho}_i(y') = \exp\{-\frac{1}{2} \sum_{n=1}^{\infty} c_n^{(i)} (y'_n)^2\}$ for $y' = (y'_n) \in \mathbb{R}_0^\infty$.

Let $c_n = c_n^{(1)} \wedge c_n^{(2)}$ and $d_n^{(i)} = c_n^{(i)} - c_n$ for $i = 1, 2$. Lemma 5.3.6 now shows that there exist Gaussian measures λ_1, λ_2 and η on $\mathbb{R}^\infty$ such that for all $y' \in \mathbb{R}_0^\infty$,

$$\widehat{\lambda}_i(y') = \exp\Big\{-\frac{1}{2} \sum_{n=1}^{\infty} d_n^{(i)} (y'_n)^2\Big\} \quad \text{for } i = 1, 2 \quad \text{and} \quad \widehat{\eta}(y') = \exp\Big\{-\frac{1}{2} \sum_{n=1}^{\infty} c_n (y'_n)^2\Big\}.$$

Set $F_i = \overline{\mathrm{lin}\{\varepsilon_n : d_n^{(i)} \neq 0\}}$ for $i = 1, 2$. Hence, F_1 and F_2 are closed subspaces of $\mathbb{R}^\infty$, $F_1 \cap F_2 = \{0\}$ and $\lambda_i(F_i) = 1$ for $i = 1, 2$. Clearly, $\varrho_i = \lambda_i * \eta$ for $i = 1, 2$. Putting $E_i = T^{-1}(F_i \cap T(V)), \nu_i = T^{-1}(\lambda_i)$ for $i = 1, 2$ and $\gamma = T^{-1}(\eta)$ one obtains $E_1 \cap E_2 = \{0\}$ and $\nu_i(E_i) = 1$ for $i = 1, 2$. Moreover, $\mu_1 * \delta_{-x+y} = \nu_1 * \eta$ and $\mu_2 * \delta_{-x-y} = \nu_2 * \eta$. ∎

5.4. Product-atomless probability measures on locally convex metrizable spaces

THEOREM 5.4.1. *Let E be a locally convex metrizable space and $\mu \in M^1(E)$. Suppose that μ has a triangular infinitesimal system of projections $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ and let G be a proper common subdomain of $\{P_i^{(n)}\}$. Then*

(i) *$\mu = \widetilde{e}(M)$ for some Lévy measure M;*

(ii) *$M(E \setminus \bigcup_{\underline{i} \in I} H_{\underline{i}}) = 0$ and $M(H_{\underline{i}}) = 0$ for all $\underline{i} \in I$, where $I = \{\underline{i} \in \mathbb{N}^\infty : \underline{i}(n) = i_n \in \{1, \ldots, k_n\},\ P_{i_{n+1}}^{(n+1)} \le P_{i_n}^{(n)}\}$ and $H_{\underline{i}} = \bigcap_{n=1}^\infty \mathrm{im}\, P_{i_n}^{(n)} \cap G$;*

(iii) *if μ is continuous then M is infinite;*

(iv) *if μ has a nondegenerate discrete part then M is finite and $\mu_{\mathrm{d}} = e^{-M(E)}\delta_0, \mu_{\mathrm{c}} = (1 - e^{-M(E)}) \sum_{n=1}^\infty M^{*n}/n!$;*

(v) *$\varPi_B(\mu)$ is commutative.*

Proof. Without loss of generality we may assume that E is a Fréchet space.

(i) Let x be an element of E such that $\mu(G + x) = 1$. Hence

$$\mu = P_1^{(n)}(\mu * \delta_{-x}) * \ldots * P_{k_n}^{(n)}(\mu * \delta_{-x}) * \delta_x \quad \text{for all } n \in \mathbb{N}.$$

Lemma 5.2.2 now implies that $\mu \in I(E)$. Let $\mu = \varrho * \widetilde{e}(M)$ be the canonical representation of μ, where ϱ is a symmetric Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with the Lévy measure M. Further, by Theorem 5.2.3 together with Theorem 5.3.10, $\varrho = \delta_0$.

(ii) Theorem 5.2.3 yields for all $n \in \mathbb{N}$,

$$M\Big(E \setminus \bigcup_{i=1}^{k_n} \mathrm{im}\, P_i^{(n)} \cap G\Big) = 0.$$

Let $\underline{i} \in I$. Since $H_{\underline{i}} = \bigcap_{n=1}^\infty \mathrm{im}\, P_{i_n}^{(n)} \cap G$, $\mathrm{im}\, P_{i_{n+1}}^{(n+1)} \cap G \subset \mathrm{im}\, P_{i_n}^{(n)} \cap G$ and $P_{i_n}^{(n)}(\mu^{\mathrm{s}}) \Rightarrow \delta_0$ we conclude that $M(H_{\underline{i}}) = 0$.

(iii) and (iv) are obvious.

(v) follows from Lemma 4.3.10. ∎

COROLLARY 5.4.2. *Let E be a locally convex metrizable space and $\mu \in M^1(E)$. Then μ is product-atomless iff it has a triangular infinitesimal system of projections.*

This follows at once from Corollary 4.5.9.

THEOREM 5.4.3. *Let E be a locally convex metrizable space and $\mu \in M^1(E)$. Suppose that μ has a triangular infinitesimal system of linear projections $\{P_i^{(n)} : i = 1, \ldots, k_n;$ $n \in \mathbb{N}\}$ and let F be a proper common linear subdomain of $\{P_i^{(n)}\}$. Then*

(i) $\mu = \widetilde{e}(M)$ *for some Lévy measure* M;

(ii) $M(E \setminus \bigcup_{\underline{i} \in I} H_{\underline{i}}) = 0$ *and* $M(H_{\underline{i}}) = 0$ *for all* $\underline{i} \in I$, *where* $I = \{\underline{i} \in \mathbb{N}^\infty : \underline{i}(n) = i_n \in \{1, \ldots, k_n\},\ P^{(n+1)}_{i_{n+1}} \le P^{(n)}_{i_n}\}$ *and* $H_{\underline{i}} = \bigcap_{n=1}^\infty \operatorname{im} P^{(n)}_{i_n} \cap G$;

(iii) *if* μ *is continuous then* M *is infinite*;

(iv) *if* μ *has a nondegenerate discrete part then* M *is finite and* $\mu_{\mathrm{d}} = e^{-M(E)}\delta_0, \mu_{\mathrm{c}} = (1 - e^{-M(E)}) \sum_{n=1}^\infty M^{*n}/n!$;

(v) $\Pi_B(\mu)$ *is commutative.*

The proof is similar to that of Theorem 5.4.1 and will be omitted.

5.5. Canonical product-decomposition and canonical strong product-decomposition of probability measures on locally convex metrizable spaces. The following theorem is a modified version of Theorem 4.6.1.

THEOREM 5.5.1. *Let* E *be a locally convex metrizable space and* $\mu \in M^1(E)$. *Then there exist measures* $\mu_0, \mu_1, \mu_2 \in M^1(E)$, $x_0 \in E$ *and Borel subgroups* G_0, G_1, G_2 *of* E *such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;

(ii) μ_0 *is a symmetric Gaussian measure*;

(iii) $\mu_1 = \widetilde{e}(M)$ *is a product-atomless generalized Poisson measure with some Lévy measure* M;

(iv) μ_2 *is a purely product-atomic measure without Gaussian product factors*;

(v) $\mu_i(G_i) = 1$ *for* $i = 0, 1, 2$;

(vi) $G_1 \cap G_2 = \{0\}$ *and* $G_0 \cap (G_1 + G_2) = \{0\}$.

Moreover, the measures μ_0 *and* M *are uniquely determined and* μ_2 *is uniquely determined up to degenerate convolution factors.*

Proof. This follows from Theorem 4.6.1 together with Theorem 5.4.1. ∎

THEOREM 5.5.2. *Let* E *be a locally convex metrizable space and* $\mu \in M^1(E)$. *Then there exist measures* $\mu_0, \mu_1, \mu_2 \in M^1(E)$, $x_0 \in E$ *and Borel subspaces* E_0, E_1, E_2 *of* E *such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;

(ii) μ_0 *is a symmetric Gaussian measure*;

(iii) $\mu_1 = \widetilde{e}(M)$ *is a strongly product-atomless generalized Poisson measure with some Lévy measure* M;

(iv) μ_2 *is a purely strongly product-atomic measure without strong Gaussian product-factors*;

(v) $\mu_i(E_i) = 1$ *for* $i = 0, 1, 2$;

(vi) $E_1 \cap E_2 = \{0\}$ *and* $E_0 \cap (E_1 + E_2) = \{0\}$.

Moreover, the measures μ_0 *and* M *are uniquely determined and* μ_2 *is uniquely determined up to degenerate convolution factors.*

Proof. This follows from Theorem 5.1.18 together with Theorem 5.4.3. ∎

Remark. Example 5.1.1 shows that "product-decomposability of Radon probability measures" and "strong product-decomposability of Radon probability measures" are distinct notions.

THEOREM 5.5.3. *Let E be a finite-dimensional locally convex metrizable space and $\mu \in M^1(E)$. Then there exist $\gamma, \mu_1, \ldots, \mu_k \in M^1(E)$, $x_0 \in E$ and Borel subspaces $E_0, E_1, \ldots, E_k$ of E such that*

(i) $\mu = \gamma * \mu_1 * \ldots * \mu_k * \delta_x$;
(ii) *γ is a symmetric Gaussian measure*;
(iii) *$\mu_1, \ldots, \mu_k$ are strongly product-indecomposable measures without Gaussian strong product-factors*;
(iv) *$\gamma(E_0) = 1$ and $\mu_i(E_i) = 1$ for $i = 1, \ldots, k$*;
(v) $E_0 \cap E_1 = \{0\}$, $(E_0 + E_1) \cap E_2 = \{0\}$, $\ldots$, $(E_0 + \ldots E_{k-1}) \cap E_k = \{0\}$.

The measure γ is uniquely determined and the measures $\mu_1, \ldots, \mu_k$ are uniquely determined up to degenerate convolution factors.

COROLLARY 5.5.4. *Let E be a finite-dimensional locally convex metrizable space and $\mu \in M^1(E)$ be full. Then there exist $P_0 \in \Pi(\mu)$, $P_1, \ldots, P_k \in \mathcal{A}(\mu)$ such that*

(i) *$P_0 + \ldots + P_k = I$ and $P_i P_j = 0$ for $i \neq j$*;
(ii) *$P(\mu)$ is a Gaussian measure and $P_1(\mu), \ldots, P_k(\mu)$ have no Gaussian strong product-factors.*

VI. Product decomposability of probability measures on LCA metrizable groups

6.1. Initial results on probability measures

LEMMA 6.1.1. *Let G be a LCA metrizable group and F be a Borel subgroup of G. Then either*

(i) *F is an open-closed subgroup of G, or*
(ii) $\omega_G(F) = 0$.

LEMMA 6.1.2. *Let G be a LCA metrizable group and $\mu \in M^1(G)$. Then either*

(i) *$\bigcap \mathcal{G}_\mu = \{\mathrm{supp}_\mathrm{g}(\mu)\}$ and $\mathrm{supp}_\mathrm{g}(\mu)$ is open-closed, or*
(ii) *there exists $H \in \mathcal{G}_\mu$ such that $\omega_G(H) = 0$.*

Proof. Suppose that $\omega_F(H \cap \mathrm{supp}_\mathrm{g}(\mu)) > 0$ for every $H \in \mathcal{G}_\mu$. But Lemma 6.1.1 implies that for every $H \in \mathcal{G}_\mu$, $H \cap \mathrm{supp}_\mathrm{g}(\mu)$ is an open-closed subgroup of $\mathrm{supp}_\mathrm{g}(\mu)$ and $\mu^\mathrm{s}(H \cap \mathrm{supp}_\mathrm{g}(\mu)) = 1$. Hence, $\mathrm{supp}_\mathrm{g}(\mu) \subset H$. ∎

COROLLARY 6.1.3. *Let G be a LCA metrizable group and $\mu \in M^1(G)$ be a measure such that $\omega_G(F) > 0$ for every $F \in \mathcal{G}_\mu$. Then*

(i) *$\mathrm{supp}_\mathrm{g}(\mu)$ is open-closed and $\bigcap \mathcal{G}_\mu = \{\mathrm{supp}_\mathrm{g}(\mu)\}$*;
(ii) *for each $A \in \mathbb{D}_B(\mu)$, $A(\mathrm{supp}_\mathrm{g}(\mu)) \subset \mathrm{supp}_\mathrm{g}(\mu)$ and $A|_{\mathrm{supp}_\mathrm{g}(\mu)}$ is continuous.*

Proof. This follows at once from Lemma 3.2.10 together with Lemma 6.1.2. ■

LEMMA 6.1.4. *Let G be a LCA metrizable group and $\mu \in M^1(G)$. Suppose that for some $n \in \mathbb{N}$, $(\mu^{\rm s})^{*n}$ has the nondegenerate absolutely continuous part with respect to ω_G. Then* $\operatorname{supp}_{\rm g}(\mu)$ *is open-closed and* $\bigcap \mathcal{G}_\mu = \{\operatorname{supp}_{\rm g}(\mu)\}$.

Proof. Let ν be the absolutely continuous part of $(\mu^{\rm s})^{*n}$ and $H \in \mathcal{G}_\mu$. Since $\nu(\operatorname{supp}_{\rm g}(\mu) \cap H) = 1$ we conclude that $\omega_G(\operatorname{supp}_{\rm g}(\mu) \cap H) > 0$, which implies that $\operatorname{supp}_{\rm g}(\mu) \cap H$ is an open-closed subgroup of $\operatorname{supp}_{\rm g}(\mu)$ and thus $\operatorname{supp}_{\rm g}(\mu) \subset H$. ■

LEMMA 6.1.5. *Let G be a LCA metrizable group and μ be a generalized Poisson measure on G with $I(\mu) = \{0\}$. Suppose that there exist Lévy measures M_1, M_2 on G such that $\mu = \widetilde{e}(M_1) = \widetilde{e}(M_2)$. Then $M_1 + \overline{M}_1 = M_2 + \overline{M}_2$.*

Proof. Set $N_i = M_i + \overline{M}_i$ for $i = 1, 2$. Hence $\mu^{\rm s} = \widetilde{e}(N_1) = \widetilde{e}(N_2)$ and $\widehat{\mu^{\rm s}}(y) = \exp\{-2\int(1-\Re\langle x, y\rangle)N_i(dx)\}$ for $i = 1, 2$ and all $y \in G'$. Thus there exists a one-parameter convolution semigroup $(\nu_t)_{t\ge 0}$ with $\nu_t \Rightarrow \delta_0$ as $t \to 0$ such that

$$\widehat{\nu_t}(y) = \exp\Big\{ -2t\int(1 - \Re\langle x, y\rangle)\, N_i(dx)\Big\}$$

for $i = 1, 2$ and all $y \in G'$. Hence, by Theorem 4.10.1 of [16], $M_1 + \overline{M}_1 = M_2 + \overline{M}_2$. ■

THEOREM 6.1.6. *Let G be a LCA metrizable group, $\mu \in I(G)$ and $P \in \Pi_B(\mu)$. Suppose that $\mu = \varrho * \widetilde{e}(M)$ is the canonical representation of μ, where ϱ is a symmetric P-Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with a Lévy measure M. Then*

(i) $P \in \Pi_B(\varrho)$;

(ii) $P \in \Pi_B(\widetilde{e}(M))$ *and* $M = PM + (I - P)M$;

(iii) *for every $x \in G$ with $\widetilde{e}(M)(D(P) + x) = 1$ there exist $y, z \in G$ such that $P(\widetilde{e}(M) * \delta_{-x}) = \widetilde{e}(P(M)) * \delta_{-y}$ and $(I - P)(\widetilde{e}(M) * \delta_{-x}) = \widetilde{e}((I - P)(M)) * \delta_{-z}$.*

Proof. Let x_0 be some element of G with $\mu(D(P) + x_0) = 1$. Putting $\nu = \mu * \delta_{-x_0}$ we obtain $\nu = P\nu * (I-P)\nu$. Let F be a σ-compact proper subdomain of P such that $\operatorname{im} P \cap F$ and $\ker P \cap F$ are σ-compact. Moreover, $P(\nu)(\operatorname{im} P \cap F) = (I-P)(\nu)(\ker P \cap F) = 1$. Since Lemma 3.1.20 implies that $P(\nu), (I - P)(\nu) \in I(G)$ we conclude that $P(\nu) = \varrho_1 * \widetilde{e}(M_1)$ and $(I-P)(\nu) = \varrho_2 * \widetilde{e}(M_2)$, where ϱ_1, ϱ_2 are symmetric P-Gaussian measures on G, $\widetilde{e}(M_1)$ and $\widetilde{e}(M_2)$ are generalized Poisson measures on G. Since $\varrho * \widetilde{e}(M) = \varrho_1 * \widetilde{e}(M_1) * \varrho_2 * \widetilde{e}(M_2)$ we conclude that $\varrho = \varrho_1 * \varrho_2$ and $\widetilde{e}(M) = \widetilde{e}(M_1) * \widetilde{e}(M_2)$. Lemma 2.3.3, together with Lemma 2.3.12, implies that

$$\varrho_1(\operatorname{im} P \cap F) = \widetilde{e}(M_1)(\operatorname{im} P \cap F) = \varrho_2(\ker P \cap F) = \widetilde{e}(M_2)(\ker P \cap F) = 1$$

and

$$M_1(G \setminus \operatorname{im} P \cap F) = M_2(G \setminus \ker P \cap F) = 0.$$

It is easy to verify that $P \in \Pi_B(\varrho)$ and $P \in \Pi_B(\widetilde{e}(M))$. Further, Lemma 6.1.5 implies $M + \overline{M} = (M_1 + \overline{M}_1) + (M_2 + \overline{M}_2)$ and thus

$$\begin{aligned}M(G\setminus(\operatorname{im}P\cup\ker P)\cap F)&\\ &\le (M+\overline{M})(G\setminus(\operatorname{im}P\cup\ker P)\cap F)\\ &=((M_1+\overline{M_1})+(M_2+\overline{M_2}))(G\setminus(\operatorname{im}P\cup\ker P)\cap F)\\ &\le (M_1+\overline{M_1})(G\setminus\operatorname{im}P\cap F)+(M_2+\overline{M_2})(G\setminus\ker P\cap F)=0.\ \blacksquare\end{aligned}$$

THEOREM 6.1.7. *Let G be a LCA metrizable group and let $\mu=\widetilde{e}(M)$ be a generalized Poisson measure on G. Then $\Pi_B(\mu)=\Pi_B(\mu^{\mathrm{s}})$.*

Proof. Let $P\in\Pi_B(G)$ and let F be a σ-compact proper subdomain of P such that $\operatorname{im}P\cap F$ and $\ker P\cap F$ are σ-compact. Hence,

$$M(G\setminus((\operatorname{im}P\cap F)\cup(\ker P\cap F)\le (M+\overline{M})(G\setminus((\operatorname{im}P\cap F)\cup(\ker P\cap F)=0.$$

Writing $M_1=M|_{\operatorname{im}P\cap F}$ and $M_2=M|_{\ker P\cap F}$ we conclude that $M=M_1+M_2, M_1+\overline{M}_1=PM+P\overline{M}$ and $M_2+\overline{M}_2=(I-P)M+(I-P)\overline{M}$. Theorem 6.1.6 now implies that $(\widetilde{e}(M_1))^{\mathrm{s}}=P(\widetilde{e}(M+\overline{M}))$ and $(\widetilde{e}(M_2))^{\mathrm{s}}=(I-P)(\widetilde{e}(M+\overline{M}))$. Since

$$(\widetilde{e}(M_1))^{\mathrm{s}}(\operatorname{im}P\cap F)=(\widetilde{e}(M_2))^{\mathrm{s}}(\ker P\cap F)=1$$

we see that there exist $x,y\in G$ such that

$$\widetilde{e}(M_1)(\operatorname{im}P\cap F+x)=\widetilde{e}(M_2)(\ker P\cap F+y)=1.$$

An easy computation shows that $P\in\Pi_B(\mu)$. ∎

THEOREM 6.1.8. *Let G be a LCA metrizable group and let $\mu=\widetilde{e}(M)$ be a generalized Poisson measure on G. Then $\Pi_B(\mu)$ is commutative.*

Proof. Let F be a σ-compact proper common subdomain of projections P and Q such that $\operatorname{im}P\cap F$, $\ker P\cap F$, $\operatorname{im}Q\cap F$, $\ker Q\cap F$ are σ-compact. Theorem 6.1.6 now implies that $M(G\setminus((\operatorname{im}P\cap F)\cup(\ker P\cap F)))=M(G\setminus((\operatorname{im}Q\cap F)\cup(\ker Q\cap F)))=0$. Hence,

$$\begin{aligned}M(G\setminus((\operatorname{im}P\cap\operatorname{im}Q\cap F)&\cup(\operatorname{im}P\cap\ker Q\cap F)\\ &\cup(\ker P\cap\operatorname{im}Q\cap F)\cup(\ker P\cap\ker Q\cap F)))=0.\end{aligned}$$

Writing $N=M+\overline{M}$, $N_1=N|_{\operatorname{im}P\cap\operatorname{im}Q\cap F}$, $N_2=N|_{\operatorname{im}P\cap\ker Q\cap F}$, $N_3=N|_{\ker P\cap\operatorname{im}Q\cap F}$, $N_4=N|_{\ker P\cap\ker Q\cap F}$ we conclude that $\mu^{\mathrm{s}}=\widetilde{e}(N)$, $PN=N_1+N_2$, $(I-P)N=N_3+N_4$, $QN=N_1+N_3$, $(I-Q)N=N_2+N_4$. But this implies that

$$\begin{aligned}P\mu^{\mathrm{s}}&=\widetilde{e}(PN)=\widetilde{e}(N_1)*\widetilde{e}(N_2), &\quad (I-P)\mu^{\mathrm{s}}&=\widetilde{e}((I-P)N)=\widetilde{e}(N_3)*\widetilde{e}(N_4),\\ Q\mu^{\mathrm{s}}&=\widetilde{e}(QN)=\widetilde{e}(N_1)*\widetilde{e}(N_3), &\quad (I-Q)\mu^{\mathrm{s}}&=\widetilde{e}((I-Q)N)=\widetilde{e}(N_2)*\widetilde{e}(N_4).\end{aligned}$$

Now, by Lemma 2.3.3,

$$\begin{aligned}\widetilde{e}(N_1)(\operatorname{im}P\cap F)&=\widetilde{e}(N_2)(\operatorname{im}P\cap F)=\widetilde{e}(N_3)(\ker P\cap F)\\ &=\widetilde{e}(N_4)(\ker P\cap F)=\widetilde{e}(N_1)(\operatorname{im}Q\cap F)\\ &=\widetilde{e}(N_2)(\ker Q\cap F)=\widetilde{e}(N_3)(\operatorname{im}Q\cap F)=\widetilde{e}(N_4)(\ker Q\cap F)=1.\end{aligned}$$

Since

$$\begin{aligned}\widetilde{e}(N_1)(\operatorname{im}P\cap\operatorname{im}Q\cap F)&=\widetilde{e}(N_2)(\operatorname{im}P\cap\ker Q\cap F)=\widetilde{e}(N_3)(\ker P\cap\operatorname{im}Q\cap F)\\ &=\widetilde{e}(N_4)(\ker P\cap\ker Q\cap F)=1\end{aligned}$$

we conclude that

$$\mu^{\mathrm{s}}(\operatorname{im} P \cap \operatorname{im} Q \cap F + \operatorname{im} P \cap \ker Q \cap F + \ker P \cap \operatorname{im} Q \cap F + \ker P \cap \ker Q \cap F) = 1.$$

Put $H = (\operatorname{im} P \cap \operatorname{im} Q \cap F) \oplus (\operatorname{im} P \cap \ker Q \cap F) \oplus (\ker P \cap \operatorname{im} Q \cap F) \oplus (\ker P \cap \ker Q \cap F)$. It is easy to see H is a σ-compact proper common subdomain of P and Q and $PQx = QPx$ for all $x \in H$. ∎

6.2. Gaussian measures

THEOREM 6.2.1. *Let G be a LCA metrizable group and $\mu \in \Gamma_{G,0}(G)$. Then $\mu \in \Gamma_P(G)$.*

Proof. Let $\mu \in \Gamma_{G,0}(E)$. By assumption there exist $P, Q \in \Pi_B(\varrho)$, a common subdomain F of P and Q, and $x \in G$ such that $\mu(F + x) = 1$ and $(\operatorname{im} P \cup \ker) \cap (\operatorname{im} Q \cup \ker Q) \cap F = \{0\}$. Let $i, j \in \{0, 1\}$. Theorem 3.4.3 implies that

$$PQ_{\underline{ij}_n}(\mu * \delta_{-x}) * \delta_{x_n^{(i,j)}} \Rightarrow \delta_0 \quad \text{as } n \to \infty$$

for some sequence $(x_n^{(i,j)})$ in G. Application of Lemma 3.4.6 yields that the triangular system $\{(PQ)_{\underline{k}_n}(\mu * \delta_{-x}) : \underline{k}_n \in J_n;\ n \in \mathbb{N}\}$ is shift uniformly infinitesimal. Thus, $\mu \in I_0(G)$. Let $\mu = \gamma * \widetilde{e}(M)$ be the canonical representation of μ, where γ is a symmetric G-Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with a Lévy measure M. Hence, by Theorem 6.2.3 below, $P, Q \in \Pi_B(\widetilde{e}(M))$, $M = P(M) + (I - P)(M)$ and $M = Q(M) + (I - Q)(M)$. Since $\widetilde{e}(M)(F) = 1$ we conclude that $M(G \setminus F) = 1$, which implies that

$$M(G \setminus (\operatorname{im} P \cup \ker P) \cap F) = M(G \setminus (\operatorname{im} Q \cup \ker Q) \cap F) = 0,$$

and thus

$$\begin{aligned} 0 &= M((G \setminus (\operatorname{im} P \cup \ker P) \cap F) \cup (G \setminus (\operatorname{im} Q \cup \ker Q) \cap F)) \\ &= M(G \setminus (\operatorname{im} P \cup \ker P) \cap (\operatorname{im} Q \cup \ker Q) \cap F) = M(G \setminus \{0\}). \end{aligned}$$

Finally, $M = \delta_0$. But this implies that $\mu \in \Gamma_P(E)$. ∎

LEMMA 6.2.2. *Let B_0 be a continuous biadditive symmetric nonnegative form on $(\mathbb{R}^n \times \mathbb{Z}_0^\infty) \times (\mathbb{R}^n \times \mathbb{Z}_0^\infty)$. Then there exists exactly one bilinear symmetric nonnegative form B on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that B_0 is the restriction of B to $(\mathbb{R}^n \times \mathbb{Z}_0^\infty) \times (\mathbb{R}^n \times \mathbb{Z}_0^\infty)$.*

Proof. If $\underline{x} = (x_k) \in \mathbb{R}^n \times \mathbb{Z}_0^\infty$ then $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $x_m \in \mathbb{Z}$ for all $m > n$. Set $a_{ij} = B_0(\varepsilon_i, \varepsilon_j)$ for $i, j \in \mathbb{N}$. Hence, (a_{ij}) is an infinite real matrix such that

$$B_0(\underline{x}, \underline{y}) = \sum_{i,j=1}^{\infty} a_{ij} x_i y_j \quad \text{for all } \underline{x}, \underline{y} \in \mathbb{R}^n \times \mathbb{Z}_0^\infty.$$

Putting

$$B(\underline{x}, \underline{y}) = \sum_{i,j=1}^{\infty} a_{ij} x_i y_j \quad \text{for all } \underline{x}, \underline{y} \in \mathbb{R}_0^\infty$$

we obtain a bilinear symmetric nonnegative definite form on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that B_0 is the restriction of B to $(\mathbb{R}^n \times \mathbb{Z}_0^\infty) \times (\mathbb{R}^n \times \mathbb{Z}_0^\infty)$.

Let C be a bilinear symmetric nonnegative definite form on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that

$$B(\underline{x}, \underline{y}) = C(\underline{x}, \underline{y}) \quad \text{for all } \underline{x}, \underline{y} \in \mathbb{R}^n \times \mathbb{Z}_0^\infty.$$

Since for $\underline{x}, \underline{y} \in \mathbb{Q}_0^\infty$ there exists $k \in \mathbb{N}$ with $k\underline{x}, k\underline{y} \in \mathbb{Z}_0^\infty$ we conclude that

$$B(\underline{x}, \underline{y}) = k^{-2}B(k\underline{x}, k\underline{y}) = k^{-2}C(k\underline{x}, k\underline{y}) = C(\underline{x}, \underline{y}).$$

The continuity of B and C yields $B(\underline{x}, \underline{y}) = C(\underline{x}, \underline{y})$ for all $\underline{x}, \underline{y} \in \mathbb{R}_0^\infty$. ∎

THEOREM 6.2.3. *Let G be a LCA metrizable group and G_0 be the component of G. Then there exist a closed subgroup H of $\mathbb{R}^\infty$ and an additive continuous mapping T from H onto G_0 such that for every symmetric measure μ on G there exists exactly one symmetric Gaussian measure γ on $\mathbb{R}^\infty$ such that*

(i) $T\gamma = \mu$;
(ii) $\gamma(H) = 1$.

Proof. Let μ be a symmetric P-Gaussian measure on G. By Corollary 5.2.12 of [11], $\mu(G_0) = 1$. Since G_0 is connected, Theorem 2.9.14 of [10] together with Theorem 2.2.6 of [18] implies that G_0 is topologically isomorphic to $\mathbb{R}^n \times K$, where $n \in \mathbb{N}$ and K is some closed subgroup of $\mathbb{T}^\infty$. Hence, without loss of generality we may assume that $G_0 = \mathbb{R}^n \times \mathbb{T}^\infty$. Application of Theorem 5.2.9 of [11] now yields $\widehat{\mu}(\underline{x}) = \exp\{-B_0(\underline{x}, \underline{x})\}$ for all $\underline{x} \in \mathbb{R}^n \times \mathbb{Z}_0^\infty$, where B_0 is some symmetric, biadditive, nonnegative definite form on $(\mathbb{R}^n \times \mathbb{Z}_0^\infty) \times (\mathbb{R}^n \times \mathbb{Z}_0^\infty)$.

Let T be the mapping from $\mathbb{R}_0^\infty$ onto $\mathbb{R}^n \times \mathbb{T}_0^\infty$ defined by $T\underline{x} = \underline{y}$ for $\underline{x} = (x_n) \in \mathbb{R}_0^\infty$, where $y_j = x_j$ for $1 \le j \le n$ and $y_j = \exp(2\pi i x_j)$ for all $j > n$. It is easy to see that T is additive and continuous.

Let γ be a P-Gaussian measure on $\mathbb{R}^\infty$ such that $\widehat{\gamma}(\underline{x}) = \exp\{-B(\underline{x}, \underline{x})\}$ for all $\underline{x} \in \mathbb{R}_0^\infty$, where B is the symmetric bilinear nonnegative definite form on $\mathbb{R}_0^\infty \times \mathbb{R}_0^\infty$ such that B_0 is the restriction of B to $(\mathbb{R}^n \times \mathbb{Z}_0^\infty) \times (\mathbb{R}^n \times \mathbb{Z}_0^\infty)$. A trivial verification shows that $T\gamma = \mu$. ∎

COROLLARY 6.2.4. *Let μ_1 and μ_2 be symmetric P-Gaussian measures on G. Suppose that there exist Borel subgroups G_1 and G_2 of G such that $G_1 \cap G_2 = \{0\}$ and $\mu_i(G_i) = 1$ for $i = 1, 2$. Then there exist Borel subgroups H_1 and H_2 of $\mathbb{R}^\infty$ such that $H_1 \cap H_2 = \{0\}$ and $\gamma_i(H_i) = 1$ for $i = 1, 2$, where γ_1 and γ_2 are symmetric Gaussian measures on $\mathbb{R}^\infty$ such that $T\gamma_i = \mu_i$ for $i = 1, 2$.*

Proof. By Theorem 6.2.3 there exist symmetric Gaussian measures γ_1 and γ_2 on $\mathbb{R}^\infty$ such that $T\gamma_i = \mu_i$ for $i = 1, 2$. Application of Theorem 5.3.11 shows that there exist Borel subgroups H_1 and H_2 of $\mathbb{R}^\infty$ such that $H_1 \cap H_2 = \{0\}$ and $\gamma_i(H_i) = 1$ for $i = 1, 2$. ∎

COROLLARY 6.2.5. *Let μ be a symmetric P-Gaussian measures on G and γ be a symmetric Gaussian measure on $\mathbb{R}^\infty$ with $T\gamma = \mu$. Then for every $P \in \Pi_B(\mu)$ there exists exactly one $Q \in \Pi_B(\gamma)$ such that $TQ\gamma = P\mu$ and $T(I - Q)\gamma = (I - P)\mu$.*

COROLLARY 6.2.6. *Let $\mu \in \Gamma_P(G)$. Then μ is purely product-atomic.*

This follows at once from Theorem 5.3.11 together with Corollary 6.2.5.

Remark. Some similar results appeared in [6].

Lemma 6.2.7. *Let P be an additive operator from $\mathbb{T}^2$ into $\mathbb{T}^2$ such that $P \notin \{0, I\}$. Then the following conditions are equivalent*:

(i) *P is an additive projection*;

(ii) *the matrix of the adjoint operator of P in the basis $\{(1,0),(0,1)\}$ of $\mathbb{Z}^2$ has the form $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$, where $a, b, c \in \mathbb{Z}$ and $bc = a(1-a)$.*

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the matrix of the adjoint operator of P. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \quad \text{and} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

A trivial verification shows that $a + d = 1$. ∎

Lemma 6.2.8. *Let ϱ be a Gaussian measure on $\mathbb{R}^2$ with*

$$\widehat{\varrho}(t_1, t_2) = \exp\left\{-\tfrac{1}{2}(s_1 t_1^2 + s_2 t_2^2)\right\}, \quad (t_1, t_2) \in \mathbb{R}^2,$$

where $s_1, s_2 \in \mathbb{R}_+$, and let P_i, $i = 1, 2$, be the projection from $\mathbb{R}^2$ into itself defined by $P_i(x_1, x_2) = \left(\frac{1}{2}(1+(-1)^{i-1})x_1, \frac{1}{2}(1+(-1)^i)x_2\right)$. Let P be a projection from $\mathbb{R}^2$ into itself. Then the following conditions are equivalent:

(i) $P \in \Pi_B(\varrho) \setminus \{0, I, P_1, P_2\}$;

(ii) (a) *the matrix of P in the basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$ has the form $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$, where $0 < a < 1$, $b, c \in \mathbb{R} \setminus \{0\}$, $bc = a(1-a)$*;

(b) $s_1 b = s_2 c$.

Proof. Let γ be a Gaussian measure on $\mathbb{R}^2$ and P be a projection from $\mathbb{R}^2$ into itself. An easy computation shows that $P \in D(\gamma)$ iff $S = P'SP + (I-P)'S(I-P)$, where S is the covariance operator of γ. The rest of the proof is clear. ∎

Theorem 6.2.9. $\Gamma_P(\mathbb{T}) = \{\delta_x : x \in G\}$.

Proof. This follows from Corollary 6.2.5 and Lemmas 6.2.7 and 6.2.8. ∎

6.3. Product-atomless probability measures

Lemma 6.3.1. *Let G be a LCA metrizable group and $\mu \in M_0^1(G)$. Suppose that μ has a triangular infinitesimal system of projections $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$. Then $\mu = \widetilde{e}(M)$ for some Lévy measure M.*

Proof. Let x be an element of G such that $\mu(G_0 + x) = 1$. Hence $\mu = P_1^{(n)}(\mu * \delta_{-x}) * \ldots * P_{k_n}^{(n)}(\mu * \delta_{-x}) * \delta_x$ for all $n \in \mathbb{N}$. Theorem 4.5.2 of [16] now implies that $\mu \in I(E)$. Let $\mu = \varrho * \widetilde{e}(M)$ be the canonical representation of μ, where ϱ is a symmetric P-Gaussian measure on G and $\widetilde{e}(M)$ is a generalized Poisson measure with a Lévy measure M. Further, by Theorem 6.1.6 together with Corollary 6.2.6, $\varrho = \delta_0$. ∎

Theorem 6.3.2. *Let μ be a probability measure either on $\mathbb{R}$ or on $\mathbb{T}$, such that $(\mu^{\mathrm{s}})^{*n}$ has a nondegenerate absolutely continuous part for some $n \in \mathbb{N}$. Then μ is product-indecomposable.*

This follows at once from Lemma 4.1.2 together with Lemma 6.1.4.

THEOREM 6.3.3. *Let G be an Abelian metrizable group and let $\mu \in M_0^1(G)$. Suppose that* $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is a discrete subgroup. Then μ is a purely product-atomic measure without G-Gaussian product-factors.*

Proof. Let $H = \mathrm{supp}_{\mathrm{g}}(\mu)$. Then H is at most countable. In particular, H is a LCA group. It is easy to see that $\bigcap \mathcal{G}_\mu = \{H\}$. Since $\{0\}$ is the component of H we conclude that μ is a measure without G-Gaussian product factors.

Assume that μ has a triangular infinitesimal system of projections $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$. Lemma 6.3.1 now implies that $\mu = e(M) * \delta_x$ for some finite Lévy measure M and some $x \in G$. Let G_0 be a proper common subdomain of $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$. Further, by Theorem 6.1.6, for all $n \in \mathbb{N}$,

$$M\Big(E \setminus \bigcup_{i=1}^{k_n} \mathrm{im}\, P_i^{(n)} \cap G_0\Big) = 0.$$

Let $\underline{i} \in I$. Since $H_{\underline{i}} = \bigcap_{n=1}^{\infty} \mathrm{im}\, P_{i_n}^{(n)} \cap G_0$, $\mathrm{im}\, P_{i_n+1}^{(n+1)} \cap G_0 \subset \mathrm{im}\, P_{i_n}^{(n)} \cap G_0$ and $P_{i_n}^{(n)}(\mu^{\mathrm{s}}) \Rightarrow \delta_0$ we conclude that $M(H_{\underline{i}}) = 0$.

An easy computation shows that the set $\{\underline{i} \in I : H_{\underline{i}} \neq \{0\}\}$ is at most countable. Hence, $M = 0$ and μ is degenerate. This contradicts the assumption and the lemma is proved. ∎

THEOREM 6.3.4. *Let G be a LCA metrizable group and $\mu \in M_0^1(E)$ be product-atomless. Suppose that $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ is a triangular infinitesimal system of projections corresponding to the measure μ. Let G_0 be a proper common subdomain of $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$. Then*

(i) $\mu = \widetilde{e}(M)$ *for some Lévy measure* M;

(ii) $M(E \setminus \bigcup_{\underline{i} \in I} H_{\underline{i}}) = 0$; $M(H_{\underline{i}}) = 0$ *for all* $i \in I$, *where* $I = \{\underline{i} \in \mathbb{N}^\infty : \underline{i}(n) = i_n \in \{1, \ldots, k_n\},\ P_{i_{n+1}}^{(n+1)} \leq P_{i_n}^{(n)}\}$ *and* $H_{\underline{i}} = \bigcap_{n=1}^{\infty} \mathrm{im}\, P_{i_n}^{(n)} \cap G_0$;

(iii) *if μ is continuous then M is unbounded*;

(iv) *if μ has a nondegenerate discrete part then M is finite and* $\mu_{\mathrm{d}} = e^{-M(E)}\delta_0, \mu_{\mathrm{c}} = (1 - e^{-M(E)}) \sum_{n=1}^{\infty} M^{*n}/n!$;

(v) $\varPi_B(\mu)$ *is commutative.*

Proof. (i) follows at once from Lemma 6.3.1.

(ii) Theorem 6.1.6 implies for all $n \in \mathbb{N}$,

$$M\Big(E \setminus \bigcup_{i=1}^{k_n} \mathrm{im}\, P_i^{(n)} \cap G_0\Big) = 0.$$

Let $\underline{i} \in I$. Since $H_{\underline{i}} = \bigcap_{n=1}^{\infty} \mathrm{im}\, P_{i_n}^{(n)} \cap G_0$, $\mathrm{im}\, P_{i_{n+1}}^{(n+1)} \cap G_0 \subset \mathrm{im}\, P_{i_n}^{(n)} \cap G_0$ and $P_{i_n}^{(n)}(\mu^{\mathrm{s}}) \Rightarrow \delta_0$ we conclude that $M(H_{\underline{i}}) = 0$.

(iii) and (iv) are obvious.

(v) follows from Lemma 4.3.10. ∎

THEOREM 6.3.5. *Let G be a LCA metrizable group and $\mu \in M_0^1(E)$ be product-atomless. Suppose that either*

(i) $\mathrm{supp}_{\mathrm{g}}(\mu)$ *has a nontrivial component or*
(ii) $\mathrm{supp}_{\mathrm{g}}(\mu)$ *is a compactly generated nondiscrete group.*

Then there exists $H \in \mathcal{G}_\mu$ *such that* $\omega_G(H) = 0$.

Proof. We assume on the contrary that for every $F \in \mathcal{G}_\mu$, $\omega_G(F) > 0$. Let $\{P_i^{(n)} : i = 1, \ldots, k_n;\ n \in \mathbb{N}\}$ be a triangular infinitesimal system of projections of μ. Corollary 6.1.3 now implies that $P_i^{(n)}(\mathrm{supp}_{\mathrm{g}}(\mu)) \subset \mathrm{supp}_{\mathrm{g}}(\mu)$ and $P_i^{(n)}|_{\mathrm{supp}_{\mathrm{g}}(\mu)}$ is continuous. Moreover, by Theorem 1.5.1 there exists a sequence (m_n) such that $m_n \in \{1, \ldots, k_n\}$, $P_{m_{n+1}}^{(n+1)} \le P_{m_n}^{(n)}$ and $\bigcap_{n=1}^\infty \mathrm{im}\, P_{m_n}^{(n)}|_{\mathrm{supp}_{\mathrm{g}}(\mu)} \ne \{0\}$. It is easy to see that

$$P_{m_n}^{(n)}(\mu^{\mathrm{s}}) \Rightarrow \delta_0. \tag{1}$$

Put $P = \inf_n P_{m_n}^{(n)}$. Theorem 3.4.7 implies that $P(\mathrm{supp}_{\mathrm{g}}(\mu)) \subset \mathrm{supp}_{\mathrm{g}}(\mu)$, $\mathrm{im}\, P \cap \mathrm{supp}_{\mathrm{g}}(\mu) = \bigcap_{n=1}^\infty \mathrm{im}\, P_{m_n}^{(n)}|_{\mathrm{supp}_{\mathrm{g}}(\mu)} \ne \{0\}$ and

$$P_{m_n}^{(n)} \to P \quad \text{in } \mu^{\mathrm{s}}. \tag{2}$$

From (1) and (2) we conclude that $\mu^{\mathrm{s}} = (I - P)\mu^{\mathrm{s}}$. Hence, $\mu^{\mathrm{s}}(\ker P) = 1$. Since P is continuous we obtain $\mathrm{supp}_{\mathrm{g}}(\mu) \subset \ker P$ and $\mathrm{supp}_{\mathrm{g}}(\mu) \setminus \ker P \ne \emptyset$. This contradiction proves the theorem. ∎

COROLLARY 6.3.6. *Let* $\mu \in M^1(\mathbb{R}^n)$ *be product-atomless. Then there exists* $H \in \mathcal{G}_\mu$ *such that* $\omega_{\mathbb{R}^n}(H) = 0$.

6.4. Canonical product-decomposition of probability measures. The following theorem is a modified version of Theorem 4.6.1.

THEOREM 6.4.1. *Let* E *be a LCA metrizable group and* $\mu \in M_0^1(G)$. *Then there exist measures* $\mu_0, \mu_1, \mu_2 \in M_0^1(G)$, *an element* x_0 *in* G *and Borel subgroups* G_0, G_1, G_2 *of* G *such that*

(i) $\mu = \mu_0 * \mu_1 * \mu_2 * \delta_x$;
(ii) μ_0 *is a symmetric* P*-Gaussian measure* ;
(iii) μ_1 *is a purely product-atomic measure without* P*-Gaussian product factors*;
(iv) $\mu_2 = \widetilde{e}(M)$ *is a product-atomless generalized Poisson measure for some Lévy measure* M;
(v) $\mu_i(G_i) = 1$ *for* $i = 0, 1, 2$;
(vi) $G_0 \cap G_1 = \{0\}$ *and* $(G_0 + G_1) \cap G_2 = \{0\}$.

Moreover, the measure μ_0 *is uniquely determined and the measures* μ_1 *and* μ_2 *are uniquely determined up to degenerate convolution factors.*

Proof. This follows from Theorem 4.1.6 together with Theorem 6.3.4. ∎

References

[1] S. Banach, *Théorie des opérations linéaires*, Monograf. Mat. 1, Warszawa, 1932.

[2] J. F. Berglund, H. D. Junghenn and P. Milnes, *Compact Right Topological Semigroups and Generalizations of Almost Periodicity*, Lecture Notes in Math. 663, Springer, Berlin, 1984.

[3] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[4] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[5] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmaße auf Badrikianschen Räumen*, Z. Warsch. Verw. Gebiete 34 (1976), 285–311.

[6] G. M. Fel'dman, *Arithmetic of Probability Distributions, and Characterization Problems on Abelian Groups*, Transl. Math. Monographs 116, Amer. Math. Soc., Providence, R.I., 1993.

[7] L. Fuchs, *Infinite Abelian Groups I*, Academic Press, New York, 1970.

[8] B. V. Gnedenko, *On a theorem of S. N. Bernshteĭn*, Izv. Akad. Nauk SSSR Ser. Mat. 12 (1948), 97–100 (in Russian).

[9] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Springer, New York, 1979.

[10] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer, Berlin, 1963.

[11] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer, Berlin, 1977.

[12] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. 38 (1935), 48–88.

[13] R. Kershner and A. Wintner, *On symmetric Bernoulli convolutions*, Amer. J. Math. 57 (1935), 541–548.

[14] A. I. Khinchin, *Contributions à l'arithmétique des lois de distribution*, Bull. Mosk. Univ. Sect. A 1 (1) (1937), 6–17.

[15] G. Köthe, *Topological Linear Spaces I*, Springer, Berlin, 1969.

[16] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.

[17] W. Rudin, *Independent perfect sets in groups*, Michigan Math. J. 5 (1958), 159–161.

[18] — , *Fourier Analysis on Groups*, Interscience, New York, 1962.

[19] H. H. Schaefer, *Topological Vector Spaces*, Springer, New York, 1971.

[20] L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Tata Inst. fund. Res. Stud. math. 6, Oxford Univ. Press, Bombay, 1973.

[21] A. Tortrat, *Convolutions dénombrables équitendues dans un groupe topologique X*, in: Les Probabilités sur les Structures Algébriques, Colloques Internationaux du CNRS, Clermont-Ferrand 1969, Éditions du Centre National de la Recherche Scientifique, Paris, 1970, 327–343.

[22] — , *Structure des lois indéfiniment divisibles* ($\mu \in I = I(X)$) *dans un espace vectoriel topologique* (*séparé*) X, in: Symposium on Probability Methods in Analysis, Lecture Notes in Math. 31, Springer, 1967, 299–328.

[23] K. Urbanik, *Lévy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), 119–148.

[24] — , *Operator semigroups associated with probability measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), 75–76.

[25] N. Vakhania, *Probability Distributions in Linear Spaces*, Mecniereba, Tbilisi, 1971 (in Russian).

[26] N. Vakhania, S. A. Chobanjan and V. I. Tarieladze, *Probability Distributions on Banach Spaces*, Reidel, Dordrecht, 1987.

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