CONFORMING FINITE ELEMENT APPROXIMATION
OF THE STOKES PROBLEM

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This paper considers a conforming finite element method for the stationary
Stokes problem in the plane. An easy way of generating continuous and
divergence-free FE-basis functions (e.g. piecewise linear) with small supports is
shown. The approximate solution can be obtained solving a system of linear
algebraic equations.

1. Introduction

The Stokes problem is usually solved by mixed (non-conforming) FE-methods
when the incompressibility condition \( \text{div} \, \mathbf{v} = 0 \) is satisfied only approximately.
Some benefits and also disadvantages of these methods can be found e.g. in
[1, 2, 3, 9, 10]. To fulfill the condition \( \text{div} \, \mathbf{v} = 0 \) exactly, conforming
FE-methods have to be employed (see [2, 4, 6, 10, 11]). Here we present
a conforming method mentioned in [6].

With the help of a stream function and \( C^1 \)-elements in \( \mathbb{R}^2 \), we shall
construct finite element spaces of continuous and divergence-free vector
functions. The method is applicable especially for polygonal domains, since
curved \( C^1 \)-elements are quite complicated [7, 12]. We shall deal with
approximation properties of the above-mentioned FE-spaces and apply them
to the stationary Stokes problem. However, these spaces may also be used for
the Navier–Stokes equations or non-stationary problems.

We denote by \( \Omega \) a bounded plane domain with a Lipschitz boundary \( \partial \Omega \).
The outward unit normal \( n = (n_1, n_2) \) to \( \partial \Omega \) exists almost everywhere (see [8],
p. 88). Let \((\cdot, \cdot)_0\) be the inner product in \( (L^2(\Omega))^d \), \( d \geq 1 \). By \( (H^k(\Omega))^d \), \( k = 0, 1, 2, \ldots \) we mean the Cartesian product of the Sobolev spaces \( H^k(\Omega) \) with the
standard norm \( \| \cdot \|_k \) and seminorm \( \cdot \cdot_k \). Further we define the linear operator
curl: \( H^1(\Omega) \to (L^2(\Omega))^2 \) by

\[
\text{curl} \, s = (\partial_2 s, -\partial_1 s), \quad s \in H^1(\Omega),
\]

[389]
where $\partial_i = \partial/\partial x_i$, and recall that

$$H_0^1(\Omega) = \{ v \in H^1(\Omega) | v = 0 \text{ on } \partial \Omega \}$$

and

$$H_0^2(\Omega) = \left\{ s \in H^2(\Omega) | s = \frac{\partial s}{\partial n} = 0 \text{ on } \partial \Omega \right\}.$$

The homogeneous stationary Stokes problem of the motion of an incompressible viscous fluid in $\Omega$ is classically formulated in the following way:

Given $f \in (L^2(\Omega))^2$ (volumic forces per unit mass) and a constant $\nu > 0$ (dynamic viscosity), find the velocity $u = (u_1, u_2)$ and the pressure $p$ such that

(1) $$- \nu \Delta u + \text{grad } p = f \quad \text{in } \Omega,$$

(2) $$\text{div } u = 0 \quad \text{in } \Omega,$$

(3) $$u = 0 \quad \text{on } \partial \Omega,$$

where $\Delta u = (\Delta u_1, \Delta u_2)$.

We shall be not concerned with the way of finding $p$ (for this see e.g. [1]).

We roughly outline a variational formulation of (1)–(3) to find the velocity $u = (u_1, u_2) \in V$, where

(4) $$V = \{ v \in (H_0^1(\Omega))^2 | \text{div } v = 0 \text{ in } \Omega \}$$

is the space of test functions which satisfy the conditions (2) and (3). Multiplying (1) by an arbitrary function $v \in V$ and integrating over $\Omega$, we arrive at

$$- \nu (\Delta u, v)_0 + (\text{grad } p, v)_0 = (f, v)_0.$$

Now the Green formula yields

(5) $$\sum_{q=1}^{2} (\text{grad } u_q, \text{grad } v_q)_0 = (f, v)_0 \quad \forall v \in V.$$

It follows from the Lax–Milgram lemma that there exists a unique solution to the variational problem (5).

2. The case of simply connected domains

In this section we assume that $\Omega$ is simply connected.

**Theorem 2.1.** The linear mapping

(6) $$\text{curl}: H_0^2(\Omega) \to V$$

is bijective.

**Proof.** For $s \in H_0^2(\Omega)$ evidently $\text{curl } s \in (H_0^1(\Omega))^2$ and $\text{div } \text{curl } s = 0$ in $\Omega$, i.e., $\text{curl } s \in V$ (cf. (4)).
Injectivity. Let \( s \in H_0^2(\Omega) \) be in the kernel of the mapping (6), i.e., \( \text{curl} \ s = 0 \). Since \( \partial_1 s = \partial_2 s = 0 \), the function \( s \) is constant in \( \Omega \), and due to the boundary condition \( s = 0 \) on \( \partial \Omega \), we see that \( s = 0 \) in the whole domain \( \Omega \).

Surjectivity. Let \( v \in V \) be arbitrary. Then by [3], p. 22, there exists the so-called stream function \( s \in H^1(\Omega) \) unique apart from an additive constant (this constant will be chosen later) such that

\[
    v = \text{curl} \ s.
\]

Since \( v \in V \), we find that \( \partial_1 s, \partial_2 s \in H^1(\Omega) \), i.e., \( s \in H^2(\Omega) \). However, \( \partial_1 s = \partial_2 s = 0 \) on \( \partial \Omega \) which implies that

\[
    \frac{\partial s}{\partial t} = \frac{\partial s}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( t = (n_2, -n_1) \) is the unit tangent vector to \( \partial \Omega \). Therefore, \( s \) is constant on \( \partial \Omega \) (as \( \partial \Omega \) is connected). Choosing \( s \) in (7) so that \( s = 0 \) on \( \partial \Omega \), we get that \( s \in H_0^2(\Omega) \).

**Corollary 2.2.** It is

\[
    V = \text{curl} \ H_0^2(\Omega),
\]

where the symbol \( \text{curl} \ H_0^2(\Omega) \) represents the space of the rotations of all functions from \( H_0^2(\Omega) \).

Now, let \( S_h \subset H_0^2(\Omega) \) be an arbitrary finite element space and let us define

\[
    V_h = \text{curl} \ S_h.
\]

From (8) we immediately see that \( V_h \subset V \) (i.e., \( \text{div} \ v_h = 0 \) whenever \( v_h \in V_h \)) and thus \( V_h \) is called the space of divergence-free (solenoidal) finite elements.

**Corollary 2.3.** We have

\[
    \dim V_h = \dim S_h.
\]

If \( \{ s^i \}_{i=1}^m \) is a basis in \( S_h \) and if we set

\[
    v^i = \text{curl} \ s^i, \quad i = 1, \ldots, m,
\]

then \( \{ v^i \}_{i=1}^m \) is a basis in \( V_h \).

The proof follows directly from (9) and Theorem 2.1. Moreover, from (10) we find that

\[
    \text{supp} \ v^i \subseteq \text{supp} \ s^i, \quad i = 1, \ldots, m,
\]

where \( \text{supp} \) denotes a support. Consequently, if the basis \( \{ s^i \}_{i=1}^m \) is generated by the standard \( C^1 \)-elements, then thanks to the definition formula (10), the basis functions \( v^i \) are continuous, exactly divergence-free and by (11) they have small supports (if \( \text{supp} \ s^i \) are small).
Remark 2.4. In [5], Heindel has presented a triangular composed piecewise quadratic \( C^1 \)-element (see fig.) with only 12 degrees of freedom (like the Hsieh–Clough–Tocher element [1]). Hence, the corresponding divergence-free basis functions \( v' = (v'_1, v'_2) \) satisfying (11) are piecewise linear (cf. [2]) as follows from (10). ■

A conforming FE-approximation of the problem (5) will consist in finding \( u_h = (u_{h1}, u_{h2}) \in V_h \subset V \) such that

\[
\sum_{q=1}^{2} \langle \text{grad } u_{hq}, \text{grad } v_{hq} \rangle_0 = \langle f, v_h \rangle_0 \quad \forall v_h \in V_h.
\]

Seeking \( u_h \) in the form

\[
u_h = \sum_{i=1}^{m} c^i v^i,
\]

we obtain from (12) a system of linear algebraic equations

\[
\sum_{q=1}^{2} \sum_{j=1}^{m} \langle \text{grad } v'_q, \text{grad } v'_j \rangle_0 c^i = \langle f, v^i \rangle_0, \quad i = 1, \ldots, m,
\]

for the unknowns \( c^1, \ldots, c^m \). The corresponding matrix is clearly symmetric positive definite and by (11) it can be band.

The next theorem states the convergence of \( u_h \) defined by (12) to the solution \( u \in V \) of the variational problem (5) without any regularity assumptions upon \( u \). However, to derive some rate of convergence, we shall later assume that \( u \) is smooth enough.

**Theorem 2.5.** Let \( \{ S_h \} \) be a system of finite element subspaces of \( H^2_0(\Omega) \) such that the union \( \bigcup_h S_h \) is dense in \( H^2_0(\Omega) \) (with the topology of \( H^2(\Omega) \)). Then

\[
\| u - u_h \|_1 \to 0 \quad \text{as } h \to 0.
\]

**Proof.** By Theorem 2.1 there exists \( z \in H^2_0(\Omega) \) such that

\[
u = \text{curl } z \quad \text{in } \Omega.
\]
Since the bilinear form corresponding to (5) is evidently continuous and $L^2$-elliptic, i.e.,

$$\nu \sum_{q=1}^{2} (\text{grad } v_q, \text{grad } v_q)_0 \geq c \|v\|_1^2 \quad \forall v \in V,$$

we may apply Céa's Lemma (see [1], p. 104). Thus there exists a constant $C > 0$ independent of $V_h$ such that

$$\frac{1}{C} \|u - u_h\|_1 \leq \inf_{v_h \in V_h} \|u - v_h\|_1 = \inf_{s_h \in S_h} \|\text{curl } z - \text{curl } s_h\|_1$$

$$= \inf_{s_h \in S_h} \|\text{grad } (z - s_h)\|_1 \leq \inf_{s \in S} \|z - s\|_2 \to 0 \quad \text{when } h \to 0.$$

**Remark 2.6.** A sufficient condition for the density assumption in Theorem 2.5 can be found in [1], p. 354. Roughly speaking, this condition requires the regularity of a family $\{T_h\}$ of triangulations of a polygonal domain, the existence of a reference $C^1$-element to which all elements are almost-affine equivalent, and the validity of the inclusions

$$P_2(K) \subset P_K \subset H^2(K) \quad \forall K \in T_h,$$

where $P_2(K)$ is the space of quadratic polynomials defined on $K$, and $P_K$ is the space of ansatz-functions of each element $K$ (with appropriate degrees of freedom). The foregoing inclusions are valid e.g. for the Heindel element mentioned in Remark 2.4.

**Remark 2.7.** (The rate of convergence.) Suppose that for some integer $k \geq 1$ and for all $s \in H^k_0(\Omega) \cap H^{k+2}(\Omega)$, we can define an $S_h$-interpolant $\pi_h s \in S_h$ such that

$$\|s - \pi_h s\|_2 \leq c h^k |s|_{k+2},$$

where $c$ is independent of $h$. Then for any $v \in V \cap (H^{k+1}(\Omega))^2$ we may define the $V_h$-interpolant $\Pi_h v \in V_h$ by

$$\Pi_h v = \text{curl } (\pi_h s),$$

where $s$ corresponds to $v$ by Theorem 2.1 and $s \in H^{k+2}(\Omega)$ as $\partial_1 s$, $\partial_2 s \in H^{k+1}(\Omega)$.

Let us suppose that the solution of (5) belongs to $V \cap (H^{k+1}(\Omega))^2$, and let $z \in H^k_0(\Omega) \cap H^{k+2}(\Omega)$ be the corresponding stream function, i.e.,

$$u = \text{curl } z.$$

Then by Céa's Lemma (cf. (13)), (16), (15) and (14), we obtain the following a priori error estimate

$$\frac{1}{C} \|u - u_h\|_1 \leq \inf_{v_h \in V_h} \|u - v_h\|_1 \leq \|u - \Pi_h u\|_1 = \|\text{curl } (z - \pi_h z)\|_1$$

$$\leq \|z - \pi_h z\|_2 \leq c h^k |z|_{k+2} = c h^k |\text{curl } z|_{k+1} = c h^k |u|_{k+1}.$$
Thus the rate of convergence is $k$ and we get the same rate in the $L^2$-norm for the so-called vorticity \( \text{rot } u = \partial_1 u_2 - \partial_2 u_1 \).

3. The case of multiply connected domains

Let \( \Omega \subset \mathbb{R}^2 \) be a multiply connected domain with a Lipschitz boundary, let \( \Omega_1, \ldots, \Omega_r \) (\( 1 \leq r < \infty \)) be all bounded components of the set \( \mathbb{R}^2 - \overline{\Omega} \) and let

\[
\Omega_0 = \Omega \cup \bigcup_{j=1}^{r} \overline{\Omega}_j,
\]

i.e., \( \partial \Omega = \partial \Omega_0 \cup \partial \Omega_1 \cup \ldots \cup \partial \Omega_r \), where \( r \) is the number of holes in \( \Omega \).

First of all we present an analogue of Theorem 2.1.

**Theorem 3.1.** There exist functions \( z^1, \ldots, z^r \in H^2(\Omega) - H^0_0(\Omega) \) such that the mapping

\[
\text{curl: } \mathcal{L}(H^2_0(\Omega) \cup \{z^1, \ldots, z^r\}) \to V,
\]

where \( \mathcal{L} \) denotes the linear span, is bijective.

**Proof.** Let \( z^j \in H^2(\Omega), j = 1, \ldots, r \), be arbitrary functions satisfying

\[
z^j = \delta_{ij} \quad \text{on } \partial \Omega_i, \quad i = 0, \ldots, r, \quad j = 1, \ldots, r,
\]

(\( \delta_{ij} \) is Kronecker's symbol) and

\[
\partial_1 z^j = \partial_2 z^j = 0 \quad \text{on } \partial \Omega, \quad j = 1, \ldots, r.
\]

Note that the distances of the boundaries \( \partial \Omega_j \) are positive because \( \partial \Omega \) is Lipschitz. By Theorem 2.1 we already know that \( \text{curl } H^2_0(\Omega) \subset V \) and due to (19), \( \text{curl } z^j \in V \), too.

Injectivity. According to (18), any \( z^j \) vanishes on \( \partial \Omega_0 \) and thus we may proceed as in Theorem 2.1.

Surjectivity. Let \( v \in V \) be arbitrary. Since \( v = 0 \) on each component \( \partial \Omega_i \), there exists (by [3], p. 22) a stream function \( s \in H^1(\Omega) \) (unique apart from an additive constant) such that

\[
v = \text{curl } s.
\]

As \( \partial_1 s, \partial_2 s \in H^1_0(\Omega) \), we observe again that \( s \in H^2(\Omega) \), \( \partial s / \partial n = 0 \) on \( \partial \Omega \) and that the tangential derivative of \( s \) vanishes on the boundary, i.e., \( \partial s / \partial t = 0 \) on \( \partial \Omega \). This implies that \( s \) equals to a constant \( c_j \) (\( j = 0, 1, \ldots, r \)) on each part \( \partial \Omega_j \).

Let \( s \) in (20) be chosen so that \( c_0 = 0 \), i.e., \( s|_{\partial \Omega_0} = 0 \). Putting

\[
z^0 = s - \sum_{j=1}^{r} c_j z^j,
\]

we find that \( z^0 \in H^2(\Omega) \) and by (18) and (19) it holds that \( z^0 = \partial_1 z^0 = \partial_2 z^0 \) on \( \partial \Omega \). Hence, \( z^0 \in H^2_0(\Omega) \) and the mapping (17) is due to (21) surjective.
COROLLARY 3.2. According to Theorem 3.1, it is

\[ V = \text{curl } Z, \]

where

\[
Z = \mathcal{L} (H^2_0(\Omega) \cup \{z^1, \ldots, z^r\})
\]

\[
= \left\{ z \in H^2(\Omega) \left| \frac{\partial z}{\partial n} = 0 \text{ on } \partial \Omega, z|_{\partial \Omega^s} = 0, \exists c_1, \ldots, c_r \in \mathbb{R}^1; \right. \right.
\]

\[
\left. \left. z|_{\partial \Omega_j} = c_j, j = 1, \ldots, r \right\} \right. \]

We may therefore define the space of divergence-free finite elements as follows

\[ V_h = \text{curl } Z_h, \]

where \( Z_h \) is an arbitrary finite element subspace of \( Z \).

Remark 3.3. Let us set

\[ Z_h = \mathcal{L} (S_h \cup \{z^1, \ldots, z^r\}), \]

where \( z^j \) belong to a fixed finite element space \( X_{h_0} \subset H^2(\Omega) \) and satisfy (18) and (19),

\[ S_h \supset X_{h_0} \cap H^2_0(\Omega), \]

and let the union \( \bigcup_h S_h \) be dense in \( H^2_0(\Omega) \) (with respect to the \( \| \cdot \|_2 \)-norm). Then we may again prove that

\[ \| u - u_h \|_1 \to 0 \quad \text{as } h \to 0. \]

Assuming further (14), we can derive that the rate of convergence is \( k \) when \( u \) is sufficiently smooth. If \( \{s^j\}_{j=1}^r \) is a basis of \( S_h \) then

(22) \[ \{ \text{curl } s^j \}_{j=1}^r \cup \{ \text{curl } z^j \}_{j=1}^r \]

is a basis of \( V_h \). The supports of the basis functions \( \text{curl } z^j \) may have, for instance, a circular shape around any hole \( \Omega_j, j = 1, \ldots, r \). Hence, to save computer memory, we should store only non-zero entries of the Gram matrix corresponding to the basis (22), and then use some iterative method for finding the discrete solution. For \( r \) fixed merely \( O(m) \) memory cells are needed.

References


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