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Internal and forcing models for the impredicative theory of classes

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Introduction*

The purpose of this work is the study of some metamathematical problems of the impredicative theory of classes. This theory has a complicated history, but may be attributed, in principle, to A. Morse [10]. An axiomatization using standard methods appeared in [4]; the same theory appears in [11]. In both of these works the theory is developed.

The axioms used here are essentially due to Tarski. Almost the same axioms were used in [1]. This system of axioms is formulated in primitive language (i.e. using only $\in$ as a non-logical constant). This is an advantage over the other standard axiom systems.

The paper can be divided into five parts. The first part, which consists of Chapters I–III, develops the theory itself. Chapter I gives the axioms of the theory and introduces the elementary concepts used. Chapter II is devoted to a generalization of the principle of definitions by recursion, which is very useful for further developments. Chapter III studies well-ordered classes, especially those theorems that are peculiar to this theory.

The second part, consisting of Chapter IV, studies general metamathematical notions as applied to the theory. In particular, the notions of model and satisfaction (generalized from [11]), and absoluteness.

The third part, which comprises Chapters V and VI, discusses some internal models for the theory. In Chapter V, the universe of constructible classes is constructed. Other methods for developing the constructible classes have been obtained by Tharp [17], Solovay (private communication), and Marek [7]. However, I believe that the construction presented here is simpler. My method gives the relative consistency of strong versions of the axiom of choice. A proof of this has not appeared in print before this paper, but seems to have been obtained by Tharp and Solovay. Chapter VI contains a discussion of relative constructibility (where a new axiom for simplifying the metamathematics of the theory is proposed) and ordinal definability.

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The fourth part includes Chapter VII. The complexity of the axiom system is studied in it. In particular, a proof of the non-finite axiomatizability of the theory is given. Besides this, it is proved that theorems restricted to sets cannot be obtained by a finite extension of Zermelo–Fraenkel (ZF). This Chapter uses the papers [9] and [5]

The last part, which contains Chapters VIII and IX, discusses forcing models. A proof of the independence of different forms of the axiom of choice is given in IX. This proof uses ordinal definability as obtained in Chapter VI.

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I. Axiom system and elementary consequences

1. Axioms. The theory will be formulated in first order logic with identity, with just one non-logical constant $\in$ as a primitive symbol. As variables we use small or capital italic letters with or without subindices. We also use the description operator $\mathscr{U}$. The notions of formula and term are defined recursively, as usual. We shall use Greek letters φ, ψ, $\theta, \ldots$ as metalinguistic variables that refer to formulas. If φ is a formula and x a variable, $\mathscr{U}x\varphi$ is a term. $\mathscr{U}x\varphi$ is interpreted as the x that satisfies φ if there is one and only one such x, or as the empty class otherwise. We shall use $\tau, \sigma, \ldots$ for terms.

In some metamathematical contexts we shall assume, for simplicity, that $\vee$, $\neg$, $\exists$, $=$, and $\in$ are the only primitive symbols, and that the variables are only $v_0, v_1, v_2, \ldots$ All other symbols will be considered as abbreviations. The axioms may easily be formulated with just these symbols.

In the theory of classes that we shall study we talk about two kinds of objects: classes and sets. However, we admit only one sort of variables and express 'x is a set' by '$\exists u(x \in u)$'.

The axioms of the theory, due to Tarski, are the following:

(I) Axiom schema of class determination: For every formula φ in which A is not free, the following is an axiom (φ may contain any free variables except A):

$$\exists A \forall x\big(x \in A \leftrightarrow \varphi \wedge \exists u(x \in u)\big).$$

(II) Axiom of extensionality:

$$\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B.$$

(III) Axiom of the empty set:

$$\forall x(x \notin a) \rightarrow \exists u(a \in u).$$

(IV) Axiom of unions:

$$\big(\forall x(x \in a \leftrightarrow \exists y(x \in y \in b)) \wedge \exists u(b \in u)\big) \to \exists u(a \in u).$$

(V) Axiom of the power set:

$$\big(\forall x(x \in a \leftrightarrow \forall y(y \in x \to y \in b)) \wedge \exists u(b \in u)\big) \to \exists u(a \in u).$$

(VI) Axiom of replacement:

$$\Big(\forall x\Big(x \in a \to \exists y\big(y \in b \wedge \forall z(z \in a \to (y \in z \leftrightarrow z = x))\big)\Big) \wedge \exists u(b \in u)\Big) \to \exists u(a \in u).$$

(VII) Axiom of infinity:

$$\exists x \exists a \exists u\big(x \in a \in u \wedge \forall t(t \in a \to \exists y(t \in y \in a))\big).$$

(VIII) Axiom of regularity or foundations:

$$\exists x(x \in A) \to \exists x(x \in A \wedge \forall y(y \in x \to y \notin A)).$$

This is an infinite system of axioms, since there is one Axiom $(\mathrm{I})_\varphi$ for each formula φ. As we shall see in [14], it is not possible to replace the system by a finite system. $(\mathrm{I})_\varphi$ asserts the existence of the class of all sets that satisfy φ.

Axiom (III) says that the empty class is a set. Axioms (IV), (V), and (VI) have the form:

$$\big(\psi(a, b) \wedge \exists u(b \in u)\big) \to \exists u(b \in u),$$

where $\psi(a, b)$ is a formula with only a and b free. In (IV), $\psi(a, b)$ says that a is the union of b, and hence (IV) says that the union of a set is a set. (V) expresses the fact that the power-class of a set is a set. The interpretation of Axiom (VI) is somewhat more complicated. In order to give an idea of its content, let us define the operation F_a by

$$F_a(y) = \mathscr{U} z(z \in a \wedge y \in z) \quad \text{for every } y \in b.$$

Then (VI) asserts that if b is a set and a contains exactly the sets $F_a(y)$ for $y \in b$, then a is also set.

Axioms (I)–(VI) imply that there is an infinite class. Axioms (VII) and (VIII) assure us that there is an infinite set. (VIII) is the usual axiom of regularity.

Later on, we shall introduce several forms of the axiom of choice.

This axiom system has several advantages over the usual one for this theory (see [11], p. 7). In the first place, it uses $\in$ as the only non-logical symbol, and the expression of the axioms in primitive symbols is not unduly long. Also, the unusual Axiom (VI), has the advantage that it implies the axiom of pairing (and hence IV′ of [11], p. 7), besides the usual axiom of replacement (V′ of [11], p. 7). A sketch of the proof of these implications appears in [1].

Since these axioms are not very familiar, I shall give them using symbols defined in the next section.

The theory with axioms (I)–(VIII) we shall call MT. As we shall see below, MT is definitely stronger than the Zermelo–Fraenkel theory (ZF, see [6]) and the Von Neumann–Bernays–Gödel theory (NBG, see [3]). It is not very much stronger, however, since ZF plus 'there is an inaccesible cardinal' is stronger than MT.

We use the same types of definitions as in [3], p. 12. Namely:

1. *Special classes* A introduced by terms τ without free variables. These definitions are of the form:

$$A = \tau.$$

2. *n-ary notions* $\mathscr{B}$ introduced by the stipulation

$$\mathscr{B}(X_1, \ldots, X_n) \leftrightarrow \varphi,$$

where φ is a formula with at most $X_1, \ldots, X_n$ free.

Unary notions are called *collections*. For collections $\mathscr{B}, \mathscr{C}, \mathscr{C}_1$ and $\mathscr{C}_2$ we write:

$\mathscr{B} \simeq \mathscr{C}$ iff $\forall X\big(\mathscr{B}(X) \leftrightarrow \mathscr{C}(X)\big)$,
$\mathscr{B} \mathrel{\tilde{\subseteq}} \mathscr{C}$ iff $\forall X\big(\mathscr{B}(X) \rightarrow \mathscr{C}(X)\big)$,
$\mathscr{B} \simeq \mathscr{C}_1 \mathbin{\tilde{\cup}} \mathscr{C}_2$ iff $\forall X\big(\mathscr{B}(X) \leftrightarrow \mathscr{C}_1(X) \vee \mathscr{C}_2(X)\big)$,
$\mathscr{B} \simeq \mathscr{C}_1 \mathbin{\tilde{\cap}} \mathscr{C}_2$ iff $\forall X\big(\mathscr{B}(X) \leftrightarrow \mathscr{C}_1(X) \wedge \mathscr{C}_2(X)\big)$,
$\mathscr{B} \simeq \tilde{\sim}\, \mathscr{C}$ iff $\forall X\big(\mathscr{B}(X) \leftrightarrow \neg \mathscr{C}(X)\big)$.

If $\mathscr{D}$ is a binary nation we write:

$\mathscr{B} \simeq \tilde{\bigcup} \{\mathscr{D}_Y \colon \mathscr{C}(Y)\}$ iff $\forall X\Big(\mathscr{B}(X) \leftrightarrow \exists Y\big(\mathscr{C}(Y) \wedge \mathscr{D}(Y, X)\big)\Big)$.
$\overline{\mathscr{B}}(X_1, \ldots, X_n)$ is an abbreviation for $\mathscr{B}(X_1) \wedge \ldots \wedge \mathscr{B}(X_n)$.

3. An *n-ary operation* F is introduced by a term τ. Thus:

$$F(X_1, \ldots, X_n) = \tau,$$

where τ contains at most $X_1, \ldots, X_n$ free.

For two n-ary operations F and G we shall write

$$F \simeq G \quad \text{iff} \quad \forall X_1, \ldots, \forall X_n\big(F(X_1, \ldots, X_n) = G(X_1, \ldots X_n)\big).$$

$F \circ G$ is the operation defined by

$$F \circ G(X) = F\big(G(X)\big).$$

If F is a one-one operation, F^{-1} will designate the inverse operation. $\mathscr{R}_F(X)$ is the collection defined by $\exists Y\big(X = F(Y)\big)$, and $\mathscr{R}_{F,\mathscr{C}}(X)$, when $\mathscr{C}$ is a collection, will stand for the collection defined by $\exists Y\big(\mathscr{C}(Y) \wedge X = F(Y)\big)$.

4. In some cases we shall introduce *special variables* a through the stipulation that formulas of the form

$$\forall a \varphi(a) \quad \text{mean} \quad \forall X\big(\mathscr{B}(X) \rightarrow \varphi(X)\big),$$

and those the form

$$\exists\, \alpha\varphi(\alpha) \quad \text{mean} \quad \exists X\,(\mathscr{B}(X) \wedge \varphi(X)),$$

where $\mathscr{B}$ is a collection previously defined.

A special class is an object of our theory. Similarly, the result of applying an operation to classes $X_1, \ldots, X_n$ is a class and thus an object of MT.

It is clear that all symbols defined through methods 1–4 can be eliminated.

2. Definitions and elementary consequences. I shall give in this section notations for the main concepts used in this paper. Their definitions are the usual ones. Thus, instead of giving these definitions explicitly, I shall explain their meanings in words (in most cases).

Because of Axioms (I) and (II), we can introduce the following definition schema:

$$\{x\colon \varphi\} = \mathscr{U} A\big(\forall x\big(x \in A \leftrightarrow \varphi \wedge \exists\, u\,(x \in u)\big)\big),$$

where φ is any formula and A is a variable not occurring free in φ. We clearly have

$$x \in \{x\colon \varphi\} \leftrightarrow \varphi \wedge \exists\, u(x \in u).$$

The *universal class* is denoted by $V (= \{x\colon x = x\})$. It is evident that

$$x \in V \leftrightarrow \exists\, u(x \in u).$$

Thus, from now on, we express 'x is a set' by '$x \in V$'.

The following are the main notations:

For arbitrary classes A, B, C, we have:

$A \subseteq B$, $B \supseteq A$: A is a subclass of B,
$A \subset B$, $B \supset A$: A is a proper subclass of B,
0: the empty class,
$\{A\}$, $\{A, B\}$, $\{A, B, C\}, \ldots$: singletons, pairs, triples, ...,
$\mathscr{S} A$: the power class of A,
$\bigcup A$: the union of A,
$A \cup B$: A union B,
$\bigcap A$: the intersection of A,
$A \cap B$: A intersected with B,
$A \sim B$: the set-theoretical difference of A and B,
$\sim B$: the complement of B,
$\langle a, b\rangle$: the ordered pair of a and b,
$A \times B$: the cartesian product of A and B,
$A \subseteq V \times V$: A is a relation.

For relations R, S, we have:

xRy: $\langle y, x\rangle \in R$,

ID: the identity relation,

EL: the elementhood relation,

IN: the inclusion relation,

$R \circ S$: the composition of R and S,

R^{-1}: the inverse of R,

$\mathscr{D}R$: the domain of R $(= \{x: \exists y(\langle y, x\rangle \in R)\})$,

$R \restriction B$: R restricted in its domain to B,

$R^{*}A$: the image of A under R,

$O_R(x) = (R^{-1*}\{x\}) \sim \{x\}$,

$R_x = R \restriction O_R(x)$,

R reflexive antisymmetric, transitive, connected with their usual meanings,

R is well-founded $\leftrightarrow \forall X\big(X \subseteq \mathscr{D}R \cup \mathscr{D}R^{-1} \wedge X \neq 0 \rightarrow \exists y\big(y \in X \wedge X \cap \cap O_R(y) = 0\big)\big)$,

R is a well-ordering with its usual meaning,

$R \circ R^{-1} \subseteq \mathrm{ID}$: R is a function.

For functions F, we have:

$F`x$: the value of x under F,

${}^{B}A$: the class of functions with domain B and range included in A,

${}^{B}A(F) \leftrightarrow F$ is a function $\wedge\ \mathscr{D}F = B \wedge \mathscr{D}F^{-1} \subseteq A$,

$\Pi_x\langle\tau\colon \varphi\rangle$: the generalized cartesian product of the classes $\tau(x)$ such that $\varphi(x)$,

$A \approx_F B$: F is a one-to-one function from A onto B,

$A \approx B$: A is equinumerous with B (i.e. $\exists F(A \approx_F B)$).

Axioms (III), (IV) and (V) can now be formulated as follows:

(III) $0 \in V$.

(IV) $b \in V \rightarrow \bigcup b \in V$.

(V) $b \in V \rightarrow \mathscr{P}b \in V$.

In [1], the proof of the following facts is indicated:

(1) $\{A\} \in V$.

(2) $B \in V \wedge A \subseteq B \rightarrow A \in V$.

(3) $\{A, B\} \in V$.

(4) $A, B \in V \rightarrow A \cup B \in V$.

(5) F is a function $\wedge\ A \in V \rightarrow F^{*}A \in V$.

Thus, the usual form of the axiom of replacement or substitution is obtained.

We now define the function that corresponds to an operation F by

$$\langle F(x)\colon \varphi\rangle = \{\langle F(x), x\rangle\colon \varphi\}.$$

Given (3), Axiom (VI) is equivalent to the following particular case of (5). Let F_a be the operation defined in the previous section, i.e.

$$F_a(y) = \mathcal{U}x(y \in x \in a) \quad \text{for } y \in b.$$

Let $\bar{F}_a$ be the corresponding function, i.e.

$$\bar{F}_a = \langle F_a(y)\colon\ y \in b\rangle.$$

Then, (VI) is equivalent to

$$a = \bar{F}_a^* b \wedge b \in V \rightarrow a \in V.$$

Thus, (VI) can be replaced by the conjuction of (3) and (5). I now introduce some special definitions:

DEFINITION 2.1 (*The ordered pair for arbitrary classes*):

$$[A, B] = A \times \{0\} \cup B \times \{1\}.$$

DEFINITION SCHEMA 2.2. Let τ be a term and φ a formula. Then:

(i) $_{X_1,\ldots,X_n}\{\tau\colon\ \varphi\} = \{y\colon\ \exists X_1, \ldots, \exists X_n(y = \tau \wedge \varphi)\}$.

(ii) $\bigcup_{X_1,\ldots,X_n}\{\tau\colon\ \varphi\} = \{y\colon\ \exists X_1, \ldots, \exists X_n(y \in \tau \wedge \varphi)\}$.

(iii) $\bigcap_{X_1,\ldots,X_n}\{\tau\colon\ \varphi\} = \{y\colon\ \forall X_1, \ldots, \forall X_n(\varphi \rightarrow y \in \tau)\}$.

If all free variables of τ are among $X_1, \ldots, X_n$, we omit these variables before the braces.

From these definitions we see that in MT the union and intersection of a collection of classes defined by a formula φ ($\bigcup\{X\colon\ \varphi\}$ and $\bigcap\{X\colon\ \varphi\}$, respectively) is a class.

Thus, we can introduce the *transitive closure* of any class A or relation R by

DEFINITION 2.3.

(i) $\mathrm{TC}(A) = \bigcap\{X\colon\ A \cup X \subseteq X\}$.

(ii) $R^\infty = \bigcap\{X\colon\ R \cup X \circ X \subseteq X\}$.

From now on we use the following stipulation of variables: lower-case letters stand for sets. Capital letters denote arbitrary classes. Thus,

$$\forall x\varphi \quad \text{means} \quad \forall x(x \in V \rightarrow \varphi),$$

and

$$\exists x\varphi \quad \text{means} \quad \exists x(x \in V \wedge \varphi).$$

A very useful device in this theory is the possibility of representing any unary operation F restricted to a class by a relation. Thus:

THEOREM SCHEMA 2.4. *Let τ be a term and φ a formula. Then*

$$\exists R\big(R \subseteq V \times V \wedge \forall x(\varphi \rightarrow \tau = R^*\{x\})\big).$$

Proof. Define

$$R = \bigcup\{\tau \times \{x\}:\ \varphi\}.$$

DEFINITION SCHEMA 2.5.

$$_x[\tau:\ \varphi] = \bigcup\{\tau \times \{x\}:\ \varphi\}.$$

This device gives us a way of treating operations restricted to classes as objects of the theory. The collection of classes defined by

$$\exists x\big(x \in B \wedge X = F(x)\big),$$

where F is a unary operation, can be represented by the relation $[F(x):\ x \in B]$, in the sense that any class X of this collection is the image of $\{x\}$ for a certain $x \in B$. We shall call collections of this kind *superclasses*.

We have the following simple consequence, similar to the axiom of replacement:

THEOREM SCHEMA 2.6. *Let F be an operation and $\mathscr{C}$ a superclass. Then $\mathscr{R}_{F,\mathscr{C}}$ is a superclass.*

Proof. Let $\mathscr{C}(X) \leftrightarrow \exists x\big(x \in B \wedge G(x) = X\big)$. Then

$$\mathscr{R}_{F,\mathscr{C}}(X) \leftrightarrow \exists x\Big(x \in B \wedge X = F\big(G(x)\big)\Big).$$

II. Principles of definitions by recursion

3. Monotone operations. The purpose of this chapter is to generalize the principles of definitions by recursion. In this section we shall prove, as an auxiliary lemma, a property of monotone operations. This theorem is essentially contained in [16].

Let F be a unary operation. By 'F is monotone' we abbreviate

$$\forall A \forall B\big(A \subseteq B \rightarrow F(A) \subseteq F(B)\big).$$

It is easy to see that the condition 'F is monotone' is equivalent to each of the following:

(1) For every formula φ (with arbitrary free variables):

$$\bigcup\{F(X):\ \varphi\} \subseteq F\big(\bigcup\{X:\ \varphi\}\big).$$

(2) For every formula φ (with arbitrary free variables):

$$F\big(\bigcap\{X:\ \varphi\}\big) \subseteq \bigcap\{F(X):\ \varphi\}.$$

THEOREM SCHEMA 3.1. *Let F be a monotone unary operation. Then*

$$\exists C \exists D\big(F(C) = C \wedge F(D) = D \wedge C = \bigcap\{X: F(X) = X\} \wedge D = \bigcup\{X:\ F(X) = X\}\big).$$

Proof. Let

$$C = \bigcap \{X: F(X) \subseteq X\}.$$

Suppose that X is such that $F(X) \subseteq X$. Then $C \subseteq X$ and thus

$$F(C) \subseteq F(X) \subseteq X.$$

That is, $F(C) \subseteq C$. But then $F(F(C)) \subseteq F(C)$. Therefore $C \subseteq F(C)$ and $C = F(C)$.

For D take $D = \bigcup \{X: X \subseteq F(X)\}$.

4. Recursion principles. The following theorem is easy to prove:

THEOREM 4.1 (Induction principle for well-founded relations).

(i) R *is well-founded* $\wedge \forall x \big(x \in \mathscr{D}R \cup \mathscr{D}R^{-1} \wedge O_R(x) \subseteq A \to x \in A\big) \to \mathscr{D}R \cup \mathscr{D}R^{-1} \subseteq A$.

(ii) $\mathscr{S}A \subseteq A \to A = V$.

(iii) R *is well-founded* $\to R^{\infty}$ *is well-founded.*

(ii) is the induction principle for the $\in$-relation. It is equivalent to the axiom of foundation.

THEOREM 4.2 (Definitions by recursion).

(i) R *is well-founded* $\wedge \forall x \big(O_R(x) \in V\big) \wedge {}^V V(H) \to \exists ! F \big({}^{\mathscr{D}R \cup \mathscr{D}R^{-1}} V(F) \wedge \forall x \big(x \in \mathscr{D}R \cup \mathscr{D}R^{-1} \to F`x = H`(F \restriction O_R(x))\big)$;

(ii) ${}^V V(H) \to \exists ! F \big({}^V V(F) \wedge \forall x (F`x = H`(F \restriction x))\big)$.

(ii) is a special form of (i) using the $\in$-relation. Instead of proving this theorem, we shall prove a stronger version.

THEOREM SCHEMA 4.3. *Let H be a binary operation, and R a well-founded relation. Then there is a unique operation F such that*

$$F(x) = H\big(x, [F(y): y \in O_R(x)]\big)$$

for all $x \in \mathscr{D}R \cup \mathscr{D}R^{-1}$.

Proof. The uniqueness is proved by induction. To prove the existence we shall find a relation S such that

$$S = \big[H\big(x, S \restriction O_R(x)\big): x \in \mathscr{D}R \cup \mathscr{D}R^{-1}\big].$$

We shall then define $F(x) = S^{*}\{x\}$.

We first assume that R is well-founded and transitive. Define the following monotone operation:

$$G(T) = \Big[\bigcup \big\{H(x, Y): Y \subseteq T \restriction O_R(x) \wedge Y = \big[H\big(y, Y \restriction O_R(y)\big): y \in O_R(x)\big]\big\}: x \in \mathscr{D}R \cup \mathscr{D}R^{-1}\Big].$$

It is clear that

$$T \subseteq T' \to G(T) \subseteq G(T').$$

Thus, by 3.2, there is an S such that $G(S) = S$. We have to show that for this S

$$(+) \qquad S^*\{x\} = H(x, S \restriction O_R(x)) \quad \text{for all } x \in \mathfrak{D}R \cup \mathfrak{D}R^{-1}.$$

In order to show this, we shall prove something a little stronger, namely:

$$(++) \qquad Y \subseteq S \restriction O_R(x) \wedge Y = [H(y, Y \restriction O_R(y)); y \in O_R(x)] \to Y = S \restriction O_R(x).$$

Since $G(S) = S$, it is clear that $(++)$ implies $(+)$. We shall prove $(++)$ by induction (4.1(i)). Assume it is true for $y \in O_R(x)$ and suppose

$$Y \subseteq S \restriction O_R(x) \wedge Y = [H(y, Y \restriction O_R(y)): y \in O_R(x)].$$

Let $y \in O_R(x)$. Since R is transitive, $O_R(y) \subseteq O_R(x)$. Then

$$Y \restriction O_R(y) \subseteq S \restriction O_R(y) \wedge Y \restriction O_R(y) = [H(u, Y \restriction O_R(u)); u \in O_R(y)].$$

Hence, by the induction hypothesis,

$$Y \restriction O_R(y) = S \restriction O_R(y) \quad \text{for all } y \in O_R(x).$$

Thus, since

$$Y = [H(y, Y \restriction O_R(y)): y \in O_R(x)],$$

we have

$$Y = [H(y, S \restriction O_R(y)): y \in O_R(x)].$$

But, again from the induction hypothesis,

$$S^*\{y\} = H(y, S \restriction O_R(y)) \quad \text{for all } y \in O_R(x).$$

Therefore

$$Y = [S^*\{y\}: y \in O_R(x)] = S \restriction O_R(x).$$

Suppose now that R is only well-founded. Then R^∞ is well-founded and transitive.

Define

$$\bar{H}(x, T) = H(x, T \restriction O_R(x)).$$

We have just proved that there is an S such that

$$S = [\bar{H}(x, S \restriction O_{R^\infty}(x)): x \in \mathfrak{D}R^\infty \cup \mathfrak{D}R^{\infty-1}].$$

Then

$$S = [H(x, S \restriction O_R(x)): x \in \mathfrak{D}R \cup \mathfrak{D}R^{-1}].$$

The extension of the proof to well-founded relations that are not transitive was obtained by Karl-Heinz Diener. The proof of 4.3 given here is based on a proof of 4.2 by Reinhardt and Tarski.

5. The rank function. The rank function, denoted by ϱ, is well known. We shall only indicate its main properties. In MT, it is simpler to introduce ϱ directly by recursion, and deduce the properties of ordinals from ϱ. We shall indicate how this is done. The main ideas for this development are due to Tarski.

DEFINITION 5.1 (By recursion using 4.2 (ii)).

$$^{V}V(\varrho) \wedge \varrho`x = \varrho^* x \cup \bigcup \varrho^* x.$$

We can extend ϱ to an operation defined for all classes by:

$$\varrho(X) = \varrho^*(X) \cup \bigcup \varrho^*(X).$$

We shall use the same symbol ϱ for this operation.

The main properties we need are the following:

(1) $\mathscr{D}\varrho^{-1}$ is well-ordered by IN and EL$\cup$ID and these relations restricted to $\mathscr{D}\varrho^{-1}$ coincide.

(2) $\varrho X = \mathrm{TC}(\varrho X)$, that is, ϱX is transitive. In particular, $\mathscr{D}\varrho^{-1} = \varrho V$ is transitive.

(3) $\varrho X = \varrho^* \mathrm{TC}(X) = \varrho^* \varrho X$.

This means that the rank of a class is a class of ranks, namely the class of the ranks smaller than the given rank.

(4) $(X \in V \leftrightarrow \varrho^{-1*} \varrho X \in V) \wedge (X \in V \leftrightarrow \varrho X \in V)$.

(5) $X \notin V \rightarrow \varrho X = \varrho V = \mathscr{D}\varrho^{-1}$.

We take the class of ordinals, OR, to be $\mathscr{D}\varrho^{-1}$. It is easy to show that this definition coincides with the usual one. We use small Greek letters $\alpha, \beta, \gamma, \ldots$ for ordinals.

It is well known that it is possible to decompose V into sets $V`\alpha$ for each $\alpha \in \mathrm{OR}$.

We first define, for any $x \in V$,

DEFINITION 5.2.

$$V`x = \varrho^{-1*} \varrho x.$$

That is, $V`x$ is the set of sets of rank less than the rank of x. Thus we have

$$V`x = \{y\colon \varrho y \in \varrho x\} \in V.$$

It is also clear that

$$V`x = V`\varrho x \quad \text{since} \quad \varrho`\varrho x = \varrho x.$$

Thus

$$V`x = V\alpha \quad \text{for } \alpha = \varrho`x \in \mathrm{OR}.$$

Therefore we obtain:

$$V`0 = 0,$$

$$V`(\alpha \cup \{\alpha\}) = \mathscr{P}\, V`\alpha,$$

$$\text{if } \alpha = \bigcup \alpha, \text{ then } V`\alpha = \bigcup \{V`_\beta\colon \beta \in \alpha\},$$

$$V = \bigcup \{V`\alpha\colon \alpha \in \mathrm{OR}\}.$$

It is also true that IN well-orders $\{V`\alpha\colon \alpha \in \mathrm{OR}\}$ and that IN is the same as EL$\cup$ID when restricted to this class.

The next concept, that of a distinguished subset of a class is useful for some purposes (D. Scott).

DEFINITION 5.3.

$$dsX = X \cap \bigcap (V^* \mathscr{P} X \sim \{0\}).$$

$ds\,X$ is the set of all elements of X of minimal rank.

The following are the main properties of ds:

(1) $ds\,X \in V$,

(2) $ds\,X \in \mathscr{P}X$,

(3) $ds0 = 0$,

(4) $X \neq 0 \rightarrow ds\,X \neq 0$,

(5) $y \in \mathscr{P}X \sim \{0\} \rightarrow \varrho(ds\,X) \subseteq \varrho y$,

(6) $y \in ds\,X \wedge z \in X \rightarrow \varrho y \subseteq \varrho z$.

The proofs of these facts are simple.

III. Well-ordering relations

6. Well-ordering relations. R is a *well-ordering* if R is a well-founded, connected, transitive, reflexive, and antisymmetric relation.

Since every well-ordering is a well-founded relation, the principle of induction 4.1 is also valid for well-orderings.

It is also easy to prove the following induction principle.

THEOREM SCHEMA 6.1. *Let φ be a formula such that for every well-ordering R we have*

$$\forall x \big(x \in \mathscr{D}R \rightarrow \varphi(R_x)\big) \rightarrow \varphi(R).$$

Then

$$\forall R \big(R \text{ is a well-ordering } \rightarrow \varphi(R)\big).$$

In the rest of this section we shall give some properties of well-orderings which hold without the assumption that $O_R(x) \in V$ for every $x \in \mathscr{D}R$. R, S, T will denote well-orderings. We shall write $x \leqslant_R y$ instead of xRy;

$x <_R y$ means $x \leqslant_R y$ and $x \neq y$. The symbol $\cong$ is used for isomorphism between well-orderings; $\underset{R}{\preceq}$ is an isomorphic embedding; $R \prec S$ means $R \preceq S$ and $R \not\cong S$; and $\overset{R}{V}B$ is the R-least upper bound of B, if it exists.

Most of the proofs of the theorems for this section are similar to the corresponding proofs in ZF; thus, in most cases, we shall not give them. However, some of these proofs are not valid in NBG.

THEOREM 6.2.

(i) $A \subseteq \mathscr{D}R \wedge \forall x \forall y (x \in A \wedge y \leqslant_R x \to y \in A) \to \exists x \big(x \in \mathscr{D}R \wedge A = O_R(x)\big)$.
(ii) $\forall x (x \in \mathscr{D}R \to R \not\cong R_x)$.
(iii) $\forall x \forall y (x, y \in \mathscr{D}R \wedge x \neq y \to R_x \not\cong R_y)$.

THEOREM 6.3.

$$R \cong S \vee \exists x (x \in \mathscr{D}R \wedge R_x \cong S) \vee \exists y (y \in \mathscr{D}S \wedge R \cong S_y).$$

COROLLARY 6.4. (i) $R \preceq S \leftrightarrow R \cong S \vee \exists y (y \in \mathscr{D}S \wedge R \cong S_y)$.
(ii) $R \preceq S \vee S \preceq R$.

If $[F(x)\colon x \in A]$ represents a superclass of well-orderings and $A \subseteq \mathscr{D}R$, where R is a well-ordering, we can define, in the usual way, the ordered sum of $[F(x)\colon x \in A]$, denoted by:

$$\sum_R [F(x)\colon x \in A].$$

In a similar way, we obtain the ordered sum of two well-orderings: $R_0 + R_1$.

The following definition and theorem are due to R. Van Wesep.

DEFINITION SCHEMA 6.5. Let F be an operation such that, for all $x \in A$, $F(x)$ is a well-ordering (i.e. $[F(x)\colon x \in A]$ represents a superclass of well-orderings). For $x, y \in A$, let xEy iff $F(x) \cong F(y)$. Let $B = \{ds(E^*\{x\})\colon x \in A\}$, and for each $y \in B$, let $G(y)$ be the relation determined by:

$$\mathscr{D}\big(G(y)\big) = \big\{f\colon \mathscr{D}f = y \wedge \forall u \forall z \big(u, z \in y \to f`u \in \mathscr{D}\big(F(u)\big) \wedge \wedge f`z \in \mathscr{D}\big(F(z)\big) \wedge F(u)_{f`u} \cong F(z)_{f`z}\big)\big\},$$

and

$$f\big(G(y)\big)g \leftrightarrow \exists z \big(z \in y \wedge f`z F(z) g`z\big).$$

Suppose now that R is a relation such that $\mathscr{D}R = B$ and $xRy \leftrightarrow \exists z \exists u \big(u \in x \wedge z \in y \wedge F(u) \preceq F(z)\big)$. Let

$$S = \sum_R [G(y)\colon y \in B].$$

Define:

$$\sup[F(x)\colon x \in A] = \begin{cases} S_u, & \text{where } u \text{ is the } S\text{-least } y \text{ such that } F(x) \preceq S_y \text{ for all } x \in A, \text{ if there is such an } y; \\ S, & \text{if } F(x) \preceq S \text{ for every } x \in A, \text{ and there is no } y \text{ such that } F(x) \preceq S_y \text{ for every } x \in A; \\ V, & \text{otherwise.} \end{cases}$$

THEOREM SCHEMA 6.6. *Let F be an operation such that, for every $x \in A$, $F(x)$ is a well-ordering and let $T = \sup[F(x)\colon x \in A]$. Then, we have:*

(i) *T is a well-ordering,*

(ii) $\forall x\big(x \in A \to F(x) \preceq T\big)$,

(iii) $\forall y\big(y \in \mathscr{D}T \to \exists x\big(x \in A \wedge T_y \preceq F(x)\big)\big)$.

Proof. We shall use the notation used in Definition 6.5. It is not difficult to verify the following facts:

(1) $G(y)$ is a well-ordering for every $y \in B$;

(2) if $x \in y \in B$, then $F(x) \cong G(y)$;

(3) R is a well-ordering;

(4) S is a well-ordering;

(5) for every $y \in B$, $G(y) \preceq S$.

From (1)–(5), and Definition. 6.5, the conclusion follows immediately.

Since all well-orderings are well-founded relations, Theorem 4.3, the principle of definitions by recursion, applies to them. From 4.3 we obtain the following principle:

THEOREM SCHEMA 6.7. *Let H be a unary operation that satisfies the following condition:*

$$\big(R, S \textit{ are well-orderings } \wedge R \cong S \wedge \mathscr{D}T = \mathscr{D}R \wedge \mathscr{D}T' = \mathscr{D}S \wedge$$
$$\wedge \forall x \forall y\,(x \in \mathscr{D}R \wedge y \in \mathscr{D}S \wedge R_x \cong S_y \to T^*\{x\} = T'^*\{y\})\big) \to H(T) = H(T').$$

Then there exists a unique operation F such that for every well-ordering R we have

$$F(R) = H\big([F(R_y)\colon y \in \mathscr{D}R]\big).$$

Proof. The uniqueness is proved by induction.

Proof of the existence:

From 4.3 we infer that for each well-ordering R there is an operation F_R such that, for $x \in \mathscr{D}R$,

$$F_R(x) = H\big([F_R(y)\colon y <_R x]\big).$$

Let us define for $S \prec R$

$$\overline{F}_R(S) = F_R(x), \quad \text{where } R_x \cong S.$$

We then define

$$F(R) = \overline{F}_{R+1}(R).$$

We have to show that F satisfies the conclusion of the theorem. First we show by induction that:

$$\text{If } R \prec S, T, \text{ then } \overline{F}_S(R) = \overline{F}_T(R).$$

We have

$$\begin{aligned} \overline{F}_S(R) &= F_S(x), \quad \text{with } R \cong S_X, \\ &= H\big([F_S(y) : y <_S x]\big) \\ &= H\big([\overline{F}_S(S_y) : y <_S x]\big). \end{aligned}$$

But since $S_y \prec R$ for $y <_S x$, we have by the induction hypothesis

$$\overline{F}_S(S_y) = \overline{F}_T(S_y).$$

If $T_z \cong S_y$, we have

$$\overline{F}_T(S_y) = \overline{F}_T(T_z).$$

Let us suppose that $T_{x'} \cong S_x \cong R$.
By the property of H, we get:

$$\begin{aligned} \overline{F}_S(R) &= H\big([\overline{F}_T(T_z) : z <_T x']\big) \\ &= H\big([F_T(z) : z <_T x']\big) \\ &= F_T(x') = \overline{F}_T(R). \end{aligned}$$

Thus, we conclude that $F(R) = \overline{F}_S(R)$ for any $S \succ R$.
Therefore:

$$\begin{aligned} F(R) &= \overline{F}_{R+1}(R) = F_{R+1}(\langle 0, 1\rangle) \\ &= H\big([F_{R+1}(y) : y <_{R+1} \langle 0, 1\rangle]\big) \\ &= H\Big([\overline{F}_{R+1}\big((R+1)_y\big) : y <_{R+1} \langle 0, 1\rangle]\Big) \\ &= H\Big(\big[F\big((R+1)_y\big) : y <_{R+1} \langle 0, 1\rangle\big]\Big). \end{aligned}$$

Let $R_y \cong (R+1)_{y'}$; then

$$F(R) = H\big([F(R_{y'}) : y' \in \mathscr{D}R]\big)$$

by the condition on H.

7. Ordinals and well-order types. As we saw in 5, the class $\mathscr{D}\varrho^{-1}$ will be our class of ordinals OR. The theory of ordinals can be developed as usual and we shall assume it to be known.

DEFINITION 7.1.

$$\mathrm{EN} = \{\langle f, r\rangle: r \text{ is a well-ordering } \wedge$$
$$\wedge \exists \alpha \big(f \in {}^{\alpha}\mathscr{D}r \wedge f^{-1} \in {}^{\mathscr{D}r}\alpha \wedge \forall\xi\forall\eta\,(\xi, \eta \in \alpha \to (\xi \subseteq \eta \leftrightarrow f`\xi \leqslant_r f`\eta))\big)\}.$$

EN is a function such that, for each well-ordering $r \in V$, $\mathrm{EN}`r$ is the isomorphism of an ordinal with $r$. $\mathscr{D}(\mathrm{EN}`r)$ is the ordinal isomorphic to r.

THEOREM 7.2. *R is a well-ordering $\wedge R \notin V \to \mathscr{D}R \approx$ OR.*

Proof. Let $A`\alpha = \mathscr{D}R \cap (V`\alpha+1 \sim V`\alpha)$ for each $\alpha \in \mathrm{OR}$. Then, $\mathscr{D}R = \bigcup\{A`\alpha:\ \alpha \in \mathrm{OR}\}$, and $A`\alpha \cap A`\beta = 0$ for $\alpha \neq \beta$. Take $R`\alpha = R \restriction A`\alpha$. It is clear that $R`\alpha$ is a well-ordering, $R`\alpha \in V$, and $\mathscr{D}R`\alpha = A`\alpha$. Let $\beta`\alpha = \mathscr{D}(\mathrm{EN}`R`\alpha)$. Then

$$\mathscr{D}R = \bigcup\{A`\alpha:\ \alpha \in \mathrm{OR}\} \approx [\beta`\alpha:\ \alpha \in \mathrm{OR}] \approx \mathrm{OR}.$$

The following theorem is due to W. Marek and P. Zbierski [8].

THEOREM 7.3. *T is a well-ordering $\wedge T \subseteq \mathrm{OR}\times\mathrm{OR} \to \exists G\big({}^{\mathscr{D}G}\mathscr{D}T(G) \wedge$ $\wedge (\mathscr{D}G \in \mathrm{OR} \vee \mathscr{D}G = \mathrm{OR}) \wedge \forall\gamma\big(\gamma \in \mathscr{D}T \to \exists\beta(\beta \in \mathscr{D}G \wedge \gamma \leqslant_T G`\beta)\big) \wedge \forall\alpha\forall\beta$ $(\alpha,\beta \in \mathscr{D}G \wedge \alpha \subset \beta \to G`\alpha <_T G`\beta)\big)$.*

The last clauses say that G is cofinal in T and that G is strictly increasing.

Proof. (1) If T is a set, the proof is clear.

(2) Suppose T is a proper class. We may assume that $0 \notin \mathscr{D}T$.

Define by recursion on OR the function G as follows: for $\alpha \in \mathrm{OR}$ let

$$A_\alpha = \{\gamma:\ \forall\beta(\beta \in \alpha \to G`\beta \in \gamma \wedge G`\beta <_T \gamma)\}.$$

Define

$$G`\alpha = \begin{cases} \bigcap A_\alpha, & \text{if } A_\alpha \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

There are two cases:

(a) Suppose, for a certain α, $G`\alpha = 0$. Then $A_\alpha = 0$ and hence $\forall\gamma\big(\overset{T}{V}\{G`\beta:\ \beta \in \alpha\} <_T \gamma \to \gamma \subseteq \bigcup\{G`\beta:\ \beta \in \alpha\}\big)$. Thus

$$T \restriction \{\gamma:\ \overset{T}{V}\{G`\beta:\ \beta \in \alpha\} <_T \gamma\} \in V$$

and hence case (1) is applicable.

(b) On the other hand, if $G`\alpha \neq 0$ for all α, then ${}^{\mathrm{OR}}\mathscr{D}T(G)$ and G is cofinal in T.

From the preceding two theorems we infer that any well-ordering is cofinal with an ordinal or OR:

COROLLARY 7.4.

$$T \text{ is a well-ordering } \rightarrow \exists G\big({}^{\mathscr{D}G}\mathscr{D}T(G) \wedge (\mathscr{D}G \in \mathrm{OR} \vee \mathscr{D}G = \mathrm{OR}) \wedge \wedge \forall x\big(x \in \mathscr{D}T \rightarrow \exists \alpha(\alpha \in \mathscr{D}G \wedge x \leqslant_T G`\alpha) \wedge \wedge \forall \alpha \forall \beta(\alpha, \beta \in \mathscr{D}G \wedge \alpha \subset \beta \rightarrow G`\alpha <_T G.\beta)\big)\big).$$

Thus, for any well-ordering T, we can define:

$$\mathrm{cf}(T) = \bigcap\big\{\Gamma : \Gamma \in \mathrm{OR} \vee \Gamma = \mathrm{OR} \wedge \exists G\big({}^{\Gamma}\mathscr{D}T(G) \wedge \wedge \forall \alpha \forall \beta(\alpha, \beta \in \Gamma \wedge \alpha \subset \beta \rightarrow G`\alpha <_T G`\beta) \wedge \forall x\big(x \in T \rightarrow \exists \alpha(\alpha \in \mathscr{D}G \wedge \wedge x \leqslant_T G`\alpha)\big)\big)\big\}.$$

It is possible to develop the theory of strictly increasing and continuous functions for arbitrary well-orderings R.

DEFINITION 7.5. Let R be a well-ordering, ${}^{\mathscr{D}R}\mathscr{D}R(F)$. Then

(i) F is *strictly increasing* $\leftrightarrow \forall x \forall y(x, y \in \mathscr{D}R \wedge x <_R y \rightarrow F`x <_R F`y)$;

(ii) F is *continuous* $\leftrightarrow \forall X(X \subseteq \mathscr{D}R \wedge \overset{R}{V}X \in \mathscr{D}R \rightarrow F\overset{R}{V}X = \overset{R}{V}F^*X)$;

(iii) F is *normal* $\leftrightarrow F$ is strictly increasing and continuous.

Using the principle of recursion, we can define the iteration of a function, thus:

DEFINITION 7.6. Let R be a well-ordering, with $\mathrm{cf}(R) \supset \omega$, and ${}^{\mathscr{D}R}\mathscr{D}R(F)$. Then

$$F^{(0)} = \mathrm{ID} \restriction \mathscr{D}F,$$

$$F^{(n+1)} = F \circ F^{(n)},$$

$$F^{(\omega)}`x = \overset{R}{V}\{F^{(n)}x : n \in \omega\} \quad \text{for all } x \in \mathscr{D}R.$$

With this definition we can prove

THEOREM 7.7. *R is a well-ordering* $\wedge\, \mathrm{cf}(R) \supset \omega \wedge {}^{\mathscr{D}R}\mathscr{D}R(F) \wedge F$ *is normal* $\rightarrow \forall x\big(x \in \mathscr{D}R \rightarrow \exists y(x \leqslant_R y \wedge F`y = y)\big)$.

Proof. Take $y = F^{(\omega)}`x$.

In order to be able to generalize 7.7 to operations defined on well-orderings, we need to add a further axiom to our system. We add an operation symbol TW and the following axiom:

(IX) $\forall R \forall S\big(R, S \text{ are well-orderings } \rightarrow \big(R \cong \mathrm{TW}(R) \wedge \big(R \cong S \rightarrow \mathrm{TW}(R) = \mathrm{TW}(S)\big)\big)\big)$.

We shall say, for any operation F, that F is a definition of types of well-orderings if F satisfies the properties of TW in (IX). In Chapter V we shall prove that if MT is consistent, then MT $+$(IX) $=$ MTT is also consistent. It is not known, however, if is consistent to assume that is impossible to define an operation that defines w.o. types. Another unknown question is whether MTT is a conservative extension of MT.

In the rest of this chapter we work in MTT.

DEFINITION SCHEMA 7.8. (1) R is a *w.o. type* $\leftrightarrow \mathcal{T}(R) \leftrightarrow R = \mathrm{TW}(R)$.

(2) Let F be an operation such that for every w.o. type R, $F(R)$ is a w.o. type (we shall call F a *type operation*).

(i) F is *increasing* $\leftrightarrow \forall R \forall S\big(R, S$ w.o. types $\wedge R \preceq S \to F(R) \preceq F(S)\big)$;

(ii) F is *strictly increasing* $\leftrightarrow \forall R \forall S\big(R, S$ w.o. types $\wedge R \prec S \to F(R) \prec F(S)\big)$;

(iii) F is *continuous* $\leftrightarrow$ for every operation G and every A such that $G(x)$ is a type for every $x \in A$, we have:

$$F\big(\sup[G(x):\ x \in A]\big) = \mathrm{TW}\big(\sup\big[F\big(G(x)\big):\ x \in A\big]\big);$$

(iv) F is *normal* $\leftrightarrow F$ is strictly increasing and continuous.

(v) If $\mathscr{C}$ is a collection of w.o. types, we define $\inf\{R:\ \mathscr{C}(R)\} = \inf \mathscr{C} =$ least type R such that $\mathscr{C}(R)$. In particular for superclasses of types we write $\inf[G(x):\ x \in A]$. When the collection is finite, i.e., $\mathscr{C}(R_0), \ldots, \mathscr{C}(R_n)$, we will write $\inf[R_0, \ldots, R_n]$.

It is clear that, for any collection $\mathscr{C}$ of types, we have

$$\mathscr{C}(\inf \mathscr{C}).$$

We now generalize the usual theorems for normal functions to the case of normal operations.

THEOREM SCHEMA 7.9. *Let F be a type operation. Then the following conditions are equivalent*:

(i) *F is increasing.*

(ii) $\forall R \forall S\big(R, S$ *are w.o. types* $\to$
$$F(\sup[R, S]) = \mathrm{TW}\big(\sup[F(R),\ F(S)]\big)\big).$$

(iii) $\forall R \forall S\big(R, S$ *are w.o. types* $\to F(\inf[R, S]) = \inf[F(R), F(S)]\big)$.

(iv) *For any operation G and class $A \neq 0$ such that $G(x)$ is a type for every $x \in A$ we have*

$$F\big(\inf[G(x):\ x \in A]\big) = \inf\big[F\big(G(x)\big):\ x \in A\big].$$

Proof: It is easy to show that (ii), (iii), and (iv) imply (i). As an example we will show that (i) implies (iv). Let us suppose then that F is increasing. We clearly have

$$F\big(\inf[G(x):\ x \in A]\big) \preceq \inf\big[F\big(G(x)\big):\ x \in A\big].$$

On the other hand,

$$\inf[G(x):\ x \in A] = G(y) \quad \text{for some } y \in A.$$

Hence

$$F(\inf[G(x):\ x \in A]) = F(G(y)),$$

and

$$\inf[F(G(x)):\ x \in A] \preceq F(G(y)).$$

Therefore

$$F(\inf[G(x):\ x \in A]) \cong \inf[F(G(x)):\ x \in A].$$

But, since both sides are w.o. types, they must be equal.

THEOREM SCHEMA 7.10. *Let F be a type operation. Then*

$$F \textit{ is strictly increasing} \to \forall R\,(R \textit{ w.o. types} \to R \preceq F(R)).$$

Proof. Similar to the corresponding proof for functions.

THEOREM SCHEMA 7.11. *Let $\mathscr{C}$ be a collection of types. Then*

$$\forall R\,(R \textit{ w.o. type} \to \exists S\,(S \succeq R \wedge \mathscr{C}(S)))$$

iff there is no superclass $\mathscr{B}$ such that $\mathscr{B} \simeq \mathscr{C}$. (We shall say in this case: '$\mathscr{C}$ is not a superclass'.)

Proof. (1) Suppose there is such a superclass $\mathscr{B}$. By 6.4, sup $\mathscr{B}$ is a w.o. and

$$\neg\exists S\,(S \succeq \mathrm{TW}(\sup\mathscr{B}+1) \wedge \mathscr{C}(S)).$$

(2) If, on the other hand, there is an R such that $S \preceq R$ for all S with $\mathscr{C}(S)$, we define, for $x \in \mathscr{D}(R+1)$, $G(x) = \mathrm{TW}(R+1_x)$ iff $\mathscr{C}(\mathrm{TW}(R+1_x))$. Hence $\mathscr{C}$ is the superclass $[G(x):\ x \in A]$, where

$$A = \{x:\ \mathscr{C}(\mathrm{TW}(R+1_x))\}.$$

THEOREM SCHEMA 7.12. *Let $\mathscr{C}$ be a collection of types that is not a superclass. Then:*

There is a unique type operation F such that F is strictly increasing and $\mathscr{C} \simeq \mathscr{R}_{F,\mathscr{T}}$.

Proof. We shall first prove the existence of F using the recursion principle 6.5.

Define

$$F(R) = \inf\{S:\ \mathscr{C}(S) \wedge \forall x\,(x \in \mathscr{D}R \to S \neq F(\mathrm{TW}(R_x)))\}.$$

It is easy to show that F is strictly increasing. We shall show now that $\mathscr{C} \simeq \mathscr{R}_{F,\mathscr{T}}$, i.e.,

$$\forall R\,(\mathscr{C}(R) \leftrightarrow \exists S\,(S \text{ w.o. type} \wedge R = F(S))).$$

It is clear from the definition of F that $\mathscr{R}_{F,\mathscr{T}} \sqsubseteq \mathscr{C}$. Let us suppose then that R satisfies

$$\mathscr{C}(R) \quad \text{and} \quad \sim \mathscr{R}_{F,\mathscr{T}}(R).$$

Since F is strictly increasing, the inverse operation, F^{-1}, exists, and since $\mathscr{T}$ is not a superclass $\mathscr{R}_F$ is not a superclass, because $\mathscr{T} \simeq \mathscr{R}_{F^{-1},R_F}$.

Hence, by 7.11, there is a type S such that $\mathscr{R}_{F,\mathscr{T}}(S)$ and $S \succ R$.

Let $T = \inf\{S\colon \mathscr{R}_{F,\mathscr{T}}(S) \wedge R \prec S\}$.

Then we have $\mathscr{R}_{F,\mathscr{T}}(T)$ and $R \prec T$.

Thus $T = F(S) = \inf\{S'\colon \mathscr{C}(S') \wedge \forall x\big(x \in \mathscr{D}S \rightarrow S' \neq F(\mathrm{TW}(S_x))\big)\}$. Let us suppose now that S' is a type with

$$S' = F\big(\mathrm{TW}(S_x)\big) \quad \text{for a certain } x \in \mathscr{D}S.$$

Then $\mathscr{R}_{F,\mathscr{T}}(S') \wedge S' \prec T$.

If $R \prec S'$, we would have $T \preceq S'$; hence $S' \preceq R$. But $S' \neq R$, because $\mathscr{R}_{F,\mathscr{T}}(S') \wedge \sim \mathscr{R}_{F,\mathscr{T}}(R)$. Then, since both are types, $S' \prec R$.

Thus we have for every $x \in \mathscr{D}S$, $F\big(\mathrm{TW}(S_x)\big) \prec R$.

Therefore $T \preceq R$. But this contradicts the assumption that $R \prec T$.

Thus we have proved $\mathscr{C} \sqsubseteq \mathscr{R}_{F,\mathscr{T}}$ and hence $\mathscr{C} \simeq \mathscr{R}_{F,\mathscr{T}}$. We now prove the uniqueness: Suppose F and G have the desired properties. Then F^{-1} and G^{-1} are also strictly increasing and so are $G^{-1} \circ F$ and $F^{-1} \circ G$.

Hence $R \preceq G^{-1}\big(F(R)\big)$ and $R \preceq F^{-1}\big(G(R)\big)$, i.e.,

$$G(R) \preceq F(R) \quad \text{and} \quad F(R) \preceq G(R).$$

But, since $F(R)$, $G(R)$ are types, $F(R) = G(R)$.

THEOREM SCHEMA 7.13. *Let F be a type operation. Then F is continuous $\wedge\, \mathscr{C}$ is superclass $\wedge\, \mathscr{C} \sqsubseteq \mathscr{R}_{F,\mathscr{T}} \wedge \mathscr{C} \neq 0 \rightarrow \mathscr{R}_{F,\mathscr{T}}\big(\mathrm{TW}(\sup\mathscr{C})\big)$.*

Proof. Let F be continuous, $\mathscr{C} \neq 0$, $\mathscr{C}$ a superclass $\mathscr{C} \sqsubseteq \mathscr{R}_{F,\mathscr{T}}$.

Let $\mathscr{B}(R) \leftrightarrow \mathscr{C}\big(F(R)\big) \wedge \forall S\big(S$ w.o. type $\wedge\, F(S) = F(R) \rightarrow R \preceq S\big)$. It is clear that $\mathscr{B} \neq 0 \wedge \mathscr{B} \subseteq \mathscr{T}$.

On the other hand, F restricted to $\mathscr{B}$ is one-one and $\mathscr{R}_{F,\mathscr{B}} \simeq \mathscr{C}$. Therefore $\mathscr{B}$ is a superclass.

From the continuity of F we deduce:

$$\mathrm{TW}(\sup\mathscr{C}) = \mathrm{TW}\big(\sup\{F(R)\colon \mathscr{B}(R)\}\big) = F\big(\mathrm{TW}(\sup\{R\colon \mathscr{B}(R)\})\big).$$

THEOREM SCHEMA 7.14. *Let F be a type operation. Then $\big(F$ is normal $\leftrightarrow F$ is strictly increasing, and for every superclass $\mathscr{C} \subseteq \mathscr{R}_{F,\mathscr{T}}$ with $\mathscr{C} \neq 0$ we have $\mathscr{R}_F\big(\mathrm{TW}(\sup\mathscr{C})\big)\big)$.*

Proof. From left to right the proof is obtained from the preceding theorem.

Suppose now that F is strictly increasing and $\mathscr{R}_F$ is closed under suprema of superclasses. We have to show that if $\mathscr{C}$ is a superclass of w.o. types and $\mathscr{C} \neq 0$, then $F(\mathrm{TW}\sup\mathscr{C}) \preceq \mathrm{TW}(\sup\mathscr{R}_{F,\mathscr{C}})$. The other inequality is clear because F is increasing.

We know that

$$\mathrm{TW}(\sup\mathscr{R}_{F,\mathscr{C}}) = F(R) \quad \text{for some } R.$$

Thus it will be enough to show that

$$\sup\mathscr{C} \preceq R.$$

Let us suppose that $R \prec \sup\mathscr{C}$.

Then

$$R \prec S \quad \text{for some } S \text{ with } \mathscr{C}(S).$$

Hence

$$F(R) \prec F(S) \preceq \mathrm{TW}(\sup\mathscr{R}_{F,\mathscr{C}}) = F(R).$$

But this is a contradiction.

Therefore we have the following corollary.

Corollary schema 7.15. *Let $\mathscr{C}$ be a collection of w.o. types that is not a superclass and is such that for every non-empty superclass $\mathscr{B} \mathrel{\breve{\subseteq}} \mathscr{C}$ we have $\mathscr{C}(\mathrm{TW}(\sup\mathscr{B}))$. Then there is a unique normal type operation F such that*

$$\mathscr{C} \simeq \mathscr{R}_{F,\mathscr{T}}.$$

Definition schema 7.16. Let F be a type operation. Then, for each type R, define by recursion on ordinals

$$\begin{aligned} H(R, 0) &= R, \\ H(R, \alpha+1) &= F(H(R, \alpha)), \\ \alpha = \bigcup \alpha \neq 0 \to H(R, \alpha) &= \mathrm{TW}(\sup[F(R, \beta)\colon \beta \in \alpha]). \end{aligned}$$

Then put $F^{(\alpha)}(R) = H(R, \alpha)$.

It is clear that $F^{(\alpha+1)}(R) = F(F^{(\alpha)}(R))$.

Theorem schema 7.17. *Let F be a normal type operation. Then*

$$\forall R(R \text{ w.o. type} \to \exists S(S \succeq R \wedge S = F(S))).$$

Proof. Take $S = F^{(\omega)}(R)$.

IV. Models and satisfaction

8. Satisfaction. We shall define satisfaction for formulas in structures whose objects are classes, i.e., sets or proper classes. Thus our structures will be represented by superclasses.

A *structure* $\mathfrak{A}$ is a pair $[A, R]$, where A is a class and R a relation with $\mathscr{D}R \subseteq A$. The elements of the structure $\mathfrak{A}$, as a model, are the classes $R^*\{x\}$ for $x \in A$. Thus we shall consider the relation R as representing the corresponding superclass and write it $[R^*\{x\}: x \in A]$.

For any unary operation F, $\mathfrak{A} = [A, [F(x): x \in A]]$ is a structure. In most cases we shall write structures in this manner. A is the class of codes for the "elements" of the structure, i.e., given $x \in A$, it is possible to obtain uniquely the "element" of $\mathfrak{A}$, $F(x)$. From $[F(x): x \in A]$ we cannot recover A, because we might have

$$[F(x): x \in A] = [F(x): x \in B],$$

with $A \sim B \neq 0$. This happens when $F(x) = 0$ for $x \in A \sim B$. Thus, if we do not include A in the definition of a structure, we cannot handle well the empty set as "element" of $\mathfrak{A}$.

In order to define satisfaction we shall generalize the procedure which appears in [11], Chapter I. Our object-language has the symbols $\in$, $=$, $\neg$, $\vee$, and $\exists$, and variables v_i, for $i \in \omega$. We represent expressions by sequences of ordinals. The number i stands for the variable v_i, and ω, $\omega+1$, $\omega+2$, $\omega+3$, and $\omega+4$ for the other symbols. With the expression φ we correlate a sequence designated by $\ulcorner\varphi\urcorner$, and $\mathrm{Fv}\varphi$ is the set of numbers that correspond to the free variables of φ. The set of formulas is denoted by Fm.

We assume a deductive system, and write, for a set of sentences, $\Sigma \cup \{\varphi\}$, $\Sigma \vdash \varphi$ for φ is *derivable from* Σ. A *theory* is a set of sentences closed under deduction in a given language. We shall also need the concepts of *extension*, *conservative extension*, and *extension by definitions* of a theory. For details see [14], 4.1–4.6. MT is the set of all sentences that are derivable from axioms (I)–(VIII) in a language with $\in$ as the only non-logical symbols. MTT contains $\in$ and TW as non-logical symbols and has (I)–(IX) as axioms.

If $\mathfrak{A} = [A, [F(x): x \in A]]$ is a model, an assignment for the formula φ has to be, intuitively, an operation

$$H: Fv(\varphi) \to \{F(x): x \in A\}.$$

We would like to define satisfaction $\models$ in such a way that the following is true:

$\mathfrak{A} \models v_i \in v_j[H]$ iff $H(i) \in H(j)$,
$\mathfrak{A} \models v_i = v_j[H]$ iff $H(i) = H(j)$,
$\mathfrak{A} \models \neg\varphi[H]$ iff not $\mathfrak{A} \models \varphi[H]$,
$\mathfrak{A} \models (\varphi \vee \psi)[H]$ iff $\mathfrak{A} \models \varphi[H \upharpoonright \mathrm{Fv}\varphi]$ or $\mathfrak{A} \models \psi[H \upharpoonright \mathrm{Fv}\psi]$,
$\mathfrak{A} \models \exists v_i\varphi[H]$ iff there is an assignment G for φ such that $G \upharpoonright \mathrm{Fv}(\exists v_i\varphi) = H$ and $\mathfrak{A} \models \varphi[G]$.

In order to do this, we should construct a class $\mathrm{St}_{\mathfrak{A}}$ such that

$$\langle H, \ulcorner\varphi\urcorner\rangle \in \mathrm{St}_{\mathfrak{A}} \quad \text{iff} \quad \mathfrak{A} \models \varphi[H].$$

In our case there is one serious difficulty. Assignments H cannot be functions because, in general, $H(x) \notin V$. In order to overcome this difficulty we use the following definition:

DEFINITION 8.1. G is an *assignment for the formula* φ *in* $\mathfrak{A} = [A, [F(x)\colon x \in A]]$ iff $G \subseteq V \times V$, $\mathscr{D}G$ is a finite subset of ω, and

$$\forall i \big(i \in \mathrm{Fv}\varphi \to \exists x(x \in A \wedge G^*\{i\} = F(x))\big).$$

The next difficulty is the necessity of talking about collections of assignments. For this, we shall define a class $C_{\mathfrak{A}}$ and an operation $H_{\mathfrak{A}}$ such that $[H_{\mathfrak{A}}(x)\colon x \in C_{\mathfrak{A}}]$ represents the superclass of all assignments in $\mathfrak{A}$.

DEFINITION 8.2. Let $\mathfrak{A} = [A, [F(x)\colon x \in A]]$. Then

(i) $C_{\mathfrak{A}} = \bigcup\{{}^cA\colon c \subseteq \omega \wedge c \text{ is finite}\}$.

(ii) If $g \in C_{\mathfrak{A}}$, we define,

$$H_{\mathfrak{A}}(g) = [F(g`i)\colon i \in \mathscr{D}g],$$

i.e., $\mathscr{D}H_{\mathfrak{A}}(g) \subseteq \mathscr{D}g$, and $H_{\mathfrak{A}}(g)(i) = F(g`i)$ for $i \in \mathscr{D}g$.

Thus we are able to construct a class $\mathrm{St}_{\mathfrak{A}}$ such that

$$\langle g, \ulcorner\varphi\urcorner\rangle \in \mathrm{St}_{\mathfrak{A}} \quad \text{iff} \quad \mathfrak{A} \models \varphi[H_{\mathfrak{A}}(g)].$$

In order to do this, we have to define an operation $K_{\mathfrak{A}}$ such that, for every formula φ, $K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) \subseteq {}^{\mathrm{Fv}\varphi}A$ represents the collection of assignments which satisfy φ. $\mathrm{St}_{\mathfrak{A}}$ will then be the relation $[K_{\mathfrak{A}}(\ulcorner\varphi\urcorner)\colon \ulcorner\varphi\urcorner \in \mathrm{Fm}]$, i.e.,

$$\langle g, \ulcorner\varphi\urcorner\rangle \in \mathrm{St}_{\mathfrak{A}} \quad \text{iff} \quad g \in K_{\mathfrak{A}}(\ulcorner\varphi\urcorner).$$

The definition of $K_{\mathfrak{A}}$ is done by recursion (4.3) over the relation R defined by

$$\ulcorner\varphi\urcorner R \ulcorner\psi\urcorner \quad \text{iff} \quad \ulcorner\varphi\urcorner \text{ is a subformula of } \ulcorner\psi\urcorner.$$

This relation is clearly well-founded.

DEFINITION 8.3. Let $\mathfrak{A} = [A, [F(x)\colon x \in A]]$. Then

(i) $\ulcorner\varphi\urcorner = \ulcorner v_i \in v_j\urcorner \to K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) = \{g\colon g \in {}^{\mathrm{Fv}\varphi}A \wedge H_{\mathfrak{A}}(g)(i) \in H_{\mathfrak{A}}(g)(j)\}$;

(ii) $\ulcorner\varphi\urcorner = \ulcorner v_i = v_j\urcorner \to K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) = \{g\colon g \in {}^{\mathrm{Fv}\varphi}A \wedge H_{\mathfrak{A}}(g)(i) = H_{\mathfrak{A}}(g)(j)\}$;

(iii) $\ulcorner\varphi\urcorner = \ulcorner\neg\psi\urcorner \to K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) = {}^{\mathrm{Fv}\varphi}A \sim K_{\mathfrak{A}}(\ulcorner\psi\urcorner)$;

(iv) $\ulcorner\varphi\urcorner = \ulcorner\psi_1 \vee \psi_2\urcorner \to K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) = \{g\colon g \in {}^{\mathrm{Fv}\varphi}A \wedge (g \restriction \mathrm{Fv}\,\psi_1 \in K_{\mathfrak{A}}(\ulcorner\psi_1\urcorner) \vee$
$\vee\, g \restriction \mathrm{Fv}\,\psi_2 \in K_{\mathfrak{A}}(\ulcorner\psi_2\urcorner))\}$;

(v) $\ulcorner\varphi\urcorner = \ulcorner\exists v_i\psi\urcorner \wedge i \notin \mathrm{Fv}\,\psi \to K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) = K_{\mathfrak{A}}(\ulcorner\psi\urcorner)$;

(vi) $\ulcorner\varphi\urcorner = \ulcorner\exists v_i\psi\urcorner \wedge i \in \mathrm{Fv}\,\psi \to K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) = \{g\colon g \in {}^{\mathrm{Fv}\varphi}A \wedge \exists x(x \in A \wedge g \cup$
$\cup \{\langle x, i\rangle\} \in K_{\mathfrak{A}}(\ulcorner\psi\urcorner)\})$.

DEFINITION 8.4. Let $\mathfrak{A} = [A, [F(x)\colon x \in A]]$ and $\ulcorner\varphi\urcorner \in \mathrm{Fm}$.

(i) Let G be an assignment for φ in $\mathfrak{A}$. Then

$$\mathfrak{A} \models \varphi[G] \quad \text{iff} \quad \exists g\big(g \in K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) \wedge \forall i\,(i \in \mathrm{Fv}\varphi \to G^*\{i\} = H_{\mathfrak{A}}(g)(i))\big).$$

(ii) Let $X_0, \ldots, X_{n-1}$ be classes and $\mathrm{Fv}\varphi \subseteq n$. Then

$$\mathfrak{A} \models \varphi[X_0, \ldots, X_{n-1}] \quad \text{iff} \quad \exists g\big(g \in K_{\mathfrak{A}}(\ulcorner\varphi\urcorner) \wedge \wedge \forall i(i \in \mathrm{Fv}\varphi \to H_{\mathfrak{A}}(g)(i) = X_i)\big).$$

(iii) Let G be an assignment for $\exists v_0\varphi$ with $0 \notin \mathscr{D}G$, and X a class. Then

$$\mathfrak{A} \models \varphi[X, G] \quad \text{iff} \quad \mathfrak{A} \models \varphi[X \times \{0\} \cup G].$$

(iv) Let St be the operation that for each structure $\mathfrak{A}$ gives

$$\mathrm{St}_{\mathfrak{A}} = [K_{\mathfrak{A}}(\ulcorner\varphi\urcorner)\colon \ulcorner\varphi\urcorner \in \mathrm{Fm}].$$

(v) Let B be a class, $\mathfrak{B} = [B, [x\colon x \in B]]$, and $g \in {}^{\mathrm{Fv}\varphi}B$. Then

$$B \models \varphi[g] \quad \text{if} \quad \mathfrak{B} \models \varphi[H_{\mathfrak{B}}(g)].$$

(vi) Let φ be a sentence and Σ a set of sentences. Then

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi[0];$$

$$\mathfrak{A} \models \Sigma \quad \text{iff} \quad \mathfrak{A} \models \varphi \text{ for every } \varphi \in \Sigma.$$

It is easy to prove (see [11]) that $V \models \varphi$, where φ is any axiom of Zermelo–Fraenkel. That is, in MT it is possible to prove that there is a transitive, standard model of ZF, namely V, and hence the consistency of ZF. Since the consistency of ZF can be expressed as an arithmetical statement, this means that there is a sentence of arithmetic which is provable in MT but not in ZF.

We shall use $\mathfrak{A}$, at the same time, for the structure $[A, [F(x)\colon x \in A]]$ and the corresponding collection; thus

$$\mathfrak{A}(X) \leftrightarrow \exists x\,(x \in A \wedge X = F(x)).$$

We say that a collection $\mathscr{B}$ is transitive if

$$\forall X\big(\mathscr{B}(X) \to \forall y\,(y \in X \to \mathscr{B}(y))\big).$$

The next definition introduces the relativization of notions and operations to a given collection $\mathscr{B}$.

DEFINITION SCHEMA 8.5. Let $\mathscr{B}$ be a collection such that $\mathscr{B}(0)$, $\mathscr{C}$ is an n-ary notion, and G an n-ary operation. We define the relativization of $\mathscr{C}$ and G to $\mathscr{B}$, $\mathscr{C}^{\mathscr{B}}$ and $G^{\mathscr{B}}$ by recursion on the definition of $\mathscr{C}$ or G:

(i) $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow X_i \in X_j$. Then $\mathscr{C} \simeq \mathscr{C}^{\mathscr{B}}$.

(ii) $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow X_i = X_j$. Then $\mathscr{C} \simeq \mathscr{C}^{\mathscr{B}}$.

(iii) $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow \neg \mathscr{D}(X_1, \ldots, X_n)$. Then
$$\mathscr{C}^{\mathscr{B}}(X_1, \ldots, X_n) \leftrightarrow \neg \mathscr{D}^{\mathscr{B}}(X_1, \ldots, X_n).$$

(iv) $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow \mathscr{D}_1(X_{i_1}, \ldots, X_{i_n}) \vee \mathscr{D}_2(X_{j_1}, \ldots, X_{j_p})$. Then
$$\mathscr{C}^{\mathscr{B}}(X_1, \ldots, X_n) \leftrightarrow \mathscr{D}_1^{\mathscr{B}}(X_{i_1}, \ldots, X_{i_n}) \vee \mathscr{D}_2^{\mathscr{B}}(X_{j_1}, \ldots, X_{j_p}).$$

(v) $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow \exists X \mathscr{D}(X, X_1, \ldots, X_n)$. Then
$$\mathscr{C}^{\mathscr{B}}(X_1, \ldots, X_n) \leftrightarrow \exists X(\mathscr{B}(X) \wedge \mathscr{D}^{\mathscr{B}}(X, X_1, \ldots, X_n)).$$

(vi) $G(X_1, \ldots, X_n) = \mathscr{U} X \mathscr{D}(X, X_1, \ldots, X_n)$. Then
$$G^{\mathscr{B}}(X_1, \ldots, X_n) = \mathscr{U} X\big(\mathscr{B}(X) \wedge \mathscr{D}^{\mathscr{B}}(X, X_1, \ldots, X_n)\big).$$

The following theorem is easy to prove.

THEOREM SCHEMA 8.6. *Let $\mathfrak{A}$ be a structure such that $\mathfrak{A}(0)$. Suppose $\mathscr{C}$ is an n-ary notion and G an n-ary operation and that we have $\overline{\mathfrak{A}}(X, X_1, \ldots, X_n)$; then*

(i) $\mathfrak{A} \models \mathscr{C}(v_1, \ldots, v_n)[X_1, \ldots, X_{\mathfrak{A}}]$ *iff* $\mathscr{C}^{\mathfrak{A}}(X_1, \ldots, X_n)$;

(ii) $\mathfrak{A} \models v_0 = G(v_1, \ldots, v_n)[X, X_1, \ldots, X_n]$ *iff* $X = G^{\mathfrak{A}}(X_1, \ldots, X_n)$.

Finally, we define another notion related to models.

DEFINITION 8.7. Let $\mathfrak{A}, \mathfrak{B}$ be structures; then $\mathfrak{A} \prec \mathfrak{B}$ iff $\mathfrak{A} \sqsubseteq \mathfrak{B}$ and for every formula φ and every assignment G for φ in $\mathfrak{A}$ we have $\mathfrak{A} \models \varphi[G]$ iff $\mathfrak{B} \models \varphi[G]$ ($\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$).

9. Absoluteness. This study of absoluteness is based on [14], pp. 265–270.

We introduce the following definition:

DEFINITION SCHEMA 9.1. Let $\mathscr{B}$ be a collection such that $\mathscr{B}(0)$, $\mathscr{C}$ is an n-ary notion and G an n-ary operation. We say that

(i) $\mathscr{C}$ is *$\mathscr{B}$-absolute* iff for every $X_1, \ldots, X_n$ such that $\overline{\mathscr{B}}(X_1, \ldots, X_n)$ we have $\mathscr{C}(X_1, \ldots, X_n)$ iff $\mathscr{C}^{\mathscr{B}}(X_1, \ldots, X_n)$.

(ii) G is *$\mathscr{B}$-absolute* iff for every $X_1, \ldots, X_n$ with $\overline{\mathscr{B}}(X_1, \ldots, X_n)$ we have $G(X_1, \ldots, X_n) = G^{\mathscr{B}}(X_1, \ldots, X_n)$.

(iii) $\mathscr{C}$ is *X_1-$\mathscr{B}$-complete* iff $\big(\overline{\mathscr{B}}(X_2, \ldots, X_n) \wedge \mathscr{C}(X_1, X_2, \ldots, X_n) \to \mathscr{B}(X_1)\big)$.

We shall sometimes apply (with some abuse of language) $\mathscr{B}$-absoluteness and completeness to a formula φ or a term τ when those concepts hold for the corresponding notion or operation.

From now on we assume that $\mathscr{B}$ is a transitive collection such that $\mathscr{B}(0)$; $\mathscr{C}, \mathscr{D}$ are notions, and G, H operations.

THEOREM SCHEMA 9.2.

(i) *If $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow X_i \in X_j$ (or $X_i = X_j$), then $\mathscr{C}$ is $\mathscr{B}$-absolute.*

(ii) *If* $\mathscr{C}$, $\mathscr{D}$ *are* $\mathscr{B}$*-absolute, then* $\sim\mathscr{C}$, $\mathscr{C}\,\bar{\cup}\,\mathscr{D}$, $\mathscr{C}\,\bar{\cap}\,\mathscr{D}$ *are* $\mathscr{B}$*-absolute.*

(iii) *If* $\mathscr{C}$ *is* $\mathscr{B}$*-absolute and* X_1*-*$\mathscr{B}$*-complete and* $\mathscr{D}(X_2, \ldots, X_n) \leftrightarrow \exists X_1 \mathscr{C}(X_1, \ldots, X_n)$*, then* $\mathscr{D}$ *is* $\mathscr{B}$*-absolute.*

(iv) *If* $\mathscr{D}$ *is* $\mathscr{B}$*-absolute and* X_1*-*$\mathscr{B}$*-complete and* $\mathscr{C}$ *is* $\mathscr{B}$*-absolute, then* $\exists X_1\big(\mathscr{D}(X_1, Y_2, \ldots, Y_m) \wedge \mathscr{C}(X_1, \ldots, X_n)\big)$ *and* $\forall X_1\big(\mathscr{D}(X_1, Y_2, \ldots, Y_n) \rightarrow \mathscr{C}(X_1, \ldots, X_n)\big)$ *are* $\mathscr{B}$*-absolute.*

In particular, $\exists X_1\big(X_1 \in Y \wedge \mathscr{C}(X_1, \ldots, X_n)\big)$ *and* $\forall X_1\big(X_1 \in Y \rightarrow \mathscr{C}(X_1, \ldots, X_n)\big)$ *are* $\mathscr{B}$*-absolute.*

(v) *If* $\mathscr{D}_1$, $\mathscr{D}_2$ *are* $\mathscr{B}$*-absolute and* $\mathscr{C}(X_1, \ldots, X_n) \leftrightarrow \exists X \mathscr{D}_1(X, X_1, \ldots, X_n) \leftrightarrow \forall X \mathscr{D}_2(X, X_1, \ldots, X_n)$*, then* $\mathscr{C}$ *is* $\mathscr{B}$*-absolute.*

(vi) G *is* $\mathscr{B}$*-absolute* $\leftrightarrow X = G(X_1, \ldots, X_n)$ *is* $\mathscr{B}$*-absolute.*

(vii) *If* $\mathscr{C}$*,* G*, and* H *are* $\mathscr{B}$*-absolute, then* $\mathscr{C}\big(G(X_1, \ldots, X_n), Y_2, \ldots, Y_m\big)$ *and* $H\big(G(X_1, \ldots, X_n), Y_2, \ldots, Y_m\big)$ *are* $\mathscr{B}$*-absolute.*

(viii) *If* $\mathscr{C}$ *is* $\mathscr{B}$*-absolute and for all* $X_1, \ldots, X_n$ $\overline{\mathscr{B}}(X_1, \ldots, X_n) \wedge \wedge \exists! X \mathscr{C}(X, X_1, \ldots, X_n) \rightarrow \exists! X\big(\mathscr{B}(X) \wedge \mathscr{C}(X, X_1, \ldots, X_n)\big)$*, then* $G(X_1, \ldots, X_n) = \mathscr{U} X \mathscr{C}(X, X_1, \ldots, X_n)$ *is* $\mathscr{B}$*-absolute.*

The proof of this theorem consists of trivial verifications. To prove (vii) we use (v), and for (viii) we use (vi).

DEFINITION SCHEMA 9.3. $\mathscr{B}$ is *set-closed* iff

$$\forall X\big(X \in V \wedge \mathscr{B}(X) \rightarrow \exists Y\big(\mathscr{B}(Y) \wedge X \in Y\big)\big).$$

It is clear that $\mathscr{B}$ is set-closed iff $\exists Y(X \in Y)$ is $\mathscr{B}$-absolute.

If Σ is a set of sentences, we will write $\Sigma^{\mathscr{B}}$ iff we have for every $\varphi \in \Sigma$, $\varphi^{\mathscr{B}}$.

THEOREM SCHEMA 9.4. *If* $\mathscr{B}$ *is set-closed and* $\mathrm{MT}^{\mathscr{B}}$*, then:*

(i) *If* $\mathscr{C}$ *is* $\mathscr{B}$*-absolute and* X_1*-*$\mathscr{B}$*-complete, then*

$$F(X_2, \ldots, X_n) = \{y\colon \mathscr{C}(y, X_2, \ldots, X_n)\} \text{ is } \mathscr{B}\text{-absolute.}$$

(ii) *If* $\mathscr{C}$ *is* $\mathscr{B}$*-absolute,* X*-*$\mathscr{B}$*-complete and* G *is* $\mathscr{B}$*-absolute, then*

$$F(X_1, \ldots, X_n) = \bigcup \{G(X)\colon \mathscr{C}(X, X_1, \ldots, X_n)\} \text{ is } \mathscr{B}\text{-absolute.}$$

Proof. We have

(1) $\{y\colon \mathscr{C}(y, X_2, \ldots, X_n)\} = \mathscr{U} A\big(\forall y\big(y \in A \rightarrow \mathscr{C}(y, \ldots, X_n) \wedge \exists u(y \in u)\big) \wedge \forall y\big(\mathscr{C}(y, \ldots, X_n) \wedge \exists u(y \in u) \rightarrow y \in A\big)\big)$.

By $\mathrm{MT}^{\mathscr{B}}$, the existence and uniqueness of A is assured. By 9.2, absoluteness is obtained.

(2) $\bigcup \{G(X)\colon \mathscr{C}(X, X_1, \ldots, X_n)\} = \big\{y\colon \exists X\big(y \in G(X) \wedge \mathscr{C}(X, X_1, \ldots, X_n)\big)\big\}$.

THEOREM SCHEMA 9.5. *If* $\mathrm{MT}^{\mathscr{B}}$*,* H *is* $\mathscr{B}$*-absolute, and* $\mathscr{B}(R)$*, then*

$$F(x) = H\big([F(y)\colon y <_R x]\big)$$

is $\mathscr{B}$*-absolute.*

Proof. By induction on R using 9.4.

10. Models of MT. The first theorem of this section gives sufficient conditions for a structure to be a model of MT.

THEOREM SCHEMA 10.1. *Let $\mathfrak{A}$ be a transitive collection. Suppose that*

(i) $\mathfrak{A}$ *is set-closed*;

(ii) *If φ is any formula with $n+1$ free variables and $\bar{\mathfrak{A}}(X_1, \ldots, X_n)$, then*

$$\mathfrak{A}\big(\{x\colon \varphi(X_1, \ldots, X_n)\}^{\mathfrak{A}}\big);$$

(iii) $\mathfrak{A}(\omega)$.

Then, if $\mathfrak{A}$ satisfies these conditions, we have $\mathrm{MT}^{\mathfrak{A}}$.

Proof. The proof consists of easy verifications.

(1) Extensionality and regularity hold because $\mathfrak{A}$ is transitive.

(2) Infinity, because $\mathfrak{A}(\omega)$ and (i).

(3) The axiom of class determination holds because of (ii).

(4) The other axioms follow from (ii) and (i). For instance, we shall prove $(\mathrm{VI})^{\mathfrak{A}}$. Let $\mathfrak{A}(A)$ and $\mathfrak{A}(B)$, and

$$\Big[\forall x\Big(x \in A \to \exists y\big(y \in B \wedge \forall z(z \in A \to (y \in z \leftrightarrow y \in x))\big)\Big)\Big]^{\mathfrak{A}} \wedge \big(\exists u(B \in u)\big)^{\mathfrak{A}}.$$

Since $\mathfrak{A}$ is transitive, we have

$$\forall x\Big[x \in A \to \exists y\big(y \in B \wedge \forall z(z \in A \to (y \in z \leftrightarrow z \in x))\big)\Big] \wedge \exists u(B \in u).$$

Applying (VI), we obtain $A \in V$. Hence, by (i), A is a set in $\mathfrak{A}$.

The following theorem is a form of a reflection principle adequate for $\mathrm{MTT}\big(\mathrm{MT} + (\mathrm{IX})\big)$. Its proof is an adaptation of the proof of Theorem 2.1 of [11], Chapter II.

THEOREM SCHEMA 10.2 (MTT). *Let $\mathfrak{A}$ be a collection satisfying*:

(i) $\mathfrak{A} \simeq \bigcup\{\mathfrak{A}_R\colon \mathscr{T}(R)\}$.

(ii) *For every type R, $\mathfrak{A}_R$ is a transitive structure such that* $\mathfrak{A}_R(0)$.

(iii) $\mathscr{T}(R, S) \wedge R \preceq S \to \mathfrak{A}_R \sqsubseteq \mathfrak{A}_S$.

(iv) $\mathscr{T}(R) \wedge R$ *has no last element* $\to \mathfrak{A}_R \simeq \bigcup\{\mathfrak{A}_S\colon S \prec R \wedge \mathscr{T}(S)\}$.

Under these conditions, $\mathfrak{A}$ also satisfies:

For every formula φ, there is a normal type operation F_φ such that if $R = F_\varphi(R)$ and φ has n free variables we have, for every $X_1, \ldots, X_n$ with $\bar{\mathfrak{A}}_R(X_1, \ldots, X_n)$,

$$\varphi^{\mathfrak{A}}(X_1, \ldots, X_n) \quad \textit{iff} \quad \varphi^{\mathfrak{A}_R}(X_1, \ldots, X_n).$$

Proof. We prove this by recursion on formulas.

(1) If φ is atomic, we take $F_\varphi(R) = R$ for every type R.

(2) If φ is $\neg\psi$ we take $F_\varphi \simeq F_\psi$.

(3) If φ is $\psi_1 \vee \psi_2$ we take $F_\varphi \simeq F_{\psi_1} \circ F_{\psi_2}$.

(4) Let φ be $\exists v_i \psi$.

Let $\mathfrak{A}_R = [A_{R'}[F_R(x):\ x \in A_R]]$. For $g \in {}^{\mathrm{Fv}\varphi}A_R$ define:

$$\begin{aligned} G_\varphi(g, R) = \inf\{R' :\ & R' \text{ is type } \wedge R' \succeq R \wedge \\ & \wedge \exists X\big(\mathfrak{A}(X) \wedge \psi^{\mathfrak{A}}(X, H_{\mathfrak{A}_R}(g)(i_0) \ldots H_{\mathfrak{A}_R}(g)(i_{n-1}))\big) \\ & \rightarrow \exists X\big(\mathfrak{A}_{R'}(X) \wedge \psi^{\mathfrak{A}}(X, H_{\mathfrak{A}_R}(g)(i_0), \ldots, H_{\mathfrak{A}_R}(g)(i_{n-1}))\big)\}, \end{aligned}$$

where $\{i_0, \ldots, i_{n-1}\} = \mathrm{Fv}\varphi$.

Let $J_\varphi(R) = \mathrm{TW}(\sup[G_\varphi(g, R):\ g \in {}^{\mathrm{Fv}\varphi}A_R])$. Define $H_\varphi(R)$ by recursion as follows:

$H_\varphi(0) = J_\varphi(0)$,

$H_\varphi(R+1) = \sup[H_\varphi(\mathrm{TW}(R)), \mathrm{TW}(J_\varphi(R+1)+1)]$,

$H_\varphi(R) = \mathrm{TW}(\sup[H_\varphi(S):\ S \prec R])$ if R has no last element.

Let $F_\varphi \simeq F_\psi \circ H_\varphi$. It is now easy to show that F_φ has the required properties.

Note that this theorem implies that, for every φ and for every R, there is an $R' \succeq R$ such that if $\overline{\mathfrak{A}}_{R'}(X_1, \ldots, X_n)$ we have

$$\varphi^{\mathfrak{A}}(X_1, \ldots, X_n) \quad \text{iff} \quad \varphi^{\mathfrak{A}_{R'}}(X_1, \ldots, X_n).$$

V. The axiom of constructibility

11. The axiom of constructibility. We shall define simultaneously by recursion applying 4.3 and 6.5 an operation TW that defines w.o. types and a structure L_S for each w.o. S (at least up to a certain w.o. R).

L_S consists of the classes constructible at stage S. Because of the impredicativeness of MT, we need to go through a sufficient number of well-orderings, S, of the universe in our construction. It is necessary, however, to choose one well-ordering S from each collection of isomorphic well-orderings. In case S is a set we have the ordinals available for this purpose. For proper classes we do not have this; we need to define an operation TW that defines w.o. types. First, a preliminary definition.

DEFINITION 11.1. Assume $\langle \ulcorner\varphi_i\urcorner :\ i \in \omega\rangle$ is a sequence of all formulas. Let $\mathfrak{A} = [A, [F(x):\ x \in A]]$ be a structure. Then:

(i) $\bar{C}_{\mathfrak{A}} = [{}^{\mathrm{Fv}(\exists v_0 \varphi_m)}A:\ m \in \omega]$.

(ii) Let $D_{\mathfrak{A}}$ be the operation defined by

$$D_{\mathfrak{A}}(\langle g, m\rangle) = \{y:\ \mathfrak{A}(y) \wedge \mathfrak{A} \models \varphi_m[y, H_{\mathfrak{A}}(g)]\} \quad \text{for all } \langle g, m\rangle \in \bar{C}_{\mathfrak{A}}.$$

(iii) $D(\mathfrak{A}) = [\bar{C}_{\mathfrak{A}}, [D_{\mathfrak{A}}(y):\ y \in \bar{C}_{\mathfrak{A}}]]$.

(iv) $\overline{\mathfrak{A}} = \{x:\ \mathfrak{A}(x)\}$.

We proceed now to define by recursion (using Theorem 6.5) L_R and $\mathrm{TW}'(R)$ for w.o. R.

DEFINITION 11.2. We define by recursion for R well-orderings a well-ordering $\mathrm{TW}'(R)$, a class B_R, an operation G_R, and a structure L_R:

(i) If $R \in V$, then $\mathrm{TW}'(R) = \mathrm{IN} \upharpoonright \mathscr{D}(EN`R)$.

(ii) If $R \notin V, R \cong S+1$ for a certain S, and $\mathrm{TW}'(S) \cong S$, then $\mathrm{TW}'(R) = \mathrm{TW}'(S)+1$.

(iii) $B_0 = \{0\}$, $G_0(0) = 0, L_0 = [B_0, [G_0(x)\colon x \in B_0]]$.

(iv) If $R \cong S+1$ for a certain w.o. S, then $B_R = [\{0\}, \bar{C}_{L_S}]$, $G_R(\langle 0, 0\rangle) = \mathrm{TW}'(S)$ and $G_R(\langle y, 1\rangle) = D_{L_S}(y)$ for $y \in \bar{C}_{L_S}$; $L_R = [B_R, [G_R(x)\colon x \in B_R]]$.

(v) If $R \not\cong S+1$ for all w.o. S, define $B_R = [B_{R_x}\colon x \in \mathscr{D}R]$, and $G_R(\langle y, x\rangle) = G_{R_x}(y)$ for $\langle y, x\rangle \in B_R$; $L_R = [B_R, [G_R(x)\colon x \in B_R]]$.

(vi) Let $R \notin V$, $R \not\cong S+1$ for all w.o. S, and assume that TW′ is defined for all $S \prec R$ with $\mathrm{TW}'(S) \cong S$, and L_S is defined for all $S \preceq R$.

1st S t a g e. Define for $g \in^{\mathrm{Fv}(\exists v_0 \varphi_m)} B_T$ and $T \prec R$

$$A(T, m, g) = \{y\colon L_R \models \varphi_m[y, H_{L_T}(g)]\}.$$

For $x \in \bar{L}_R$, let

$$\mu(x) = \inf\{T\colon T = \mathrm{TW}'(T) \wedge L_T(x)\}.$$

Define

$$B(T, m, g) = \sup[\mu(y)\colon y \in A(T, m, g)];$$

and

$$C(T) = \sup[B(T, m, g)\colon m \in \omega \wedge g \in {}^{\mathrm{Fv}(\exists v_0 \varphi_m)}B_T].$$

Now, if there is a T such that $C(T) \cong R$, define

$$\mathrm{TW}'(R) = C(T_0),$$

where $T_0 = \inf\{T\colon C(T) \cong R\}$.

Otherwise, we proceed to the second stage.

2nd S t a g e. For $g \in^{\mathrm{Fv}(\exists v_0 \varphi_m)} B_T$ and $T \prec R$, define

$$J(T, m, g) = \inf\{T'\colon T' = \mathrm{TW}'(T') \wedge T' \prec R \wedge T' \succeq T \wedge \exists X(L_R(X) \wedge \wedge L_R \models \varphi_m[X, H_{L_T}(g)]) \rightarrow \exists X(L_{T'}(X) \wedge L_R \models \varphi_m[X, H_{L_T}(g)])\}.$$

Define also

$$\mathbf{L}(T) = \sup[J(T, m, g)\colon m \in \omega \wedge g \in {}^{\mathrm{Fv}(\exists v_0 \varphi_m)}B_T],$$

if $T \prec R$; and $\mathbf{L}(T) = T$, if $T \cong R$. It is clear that for $T \preceq R$, $\mathbf{L}(T) \preceq R$.

Define for each $T \preceq R$ by recursion on ω

$$M(T, 0) = T,$$
$$M(T, m+1) = \mathbf{L}(M(T, m)).$$

Let $N(T) = \sup[M(T, m)\colon m \in \omega]$. We have $N(T) \preceq R$ for each $T \preceq R$.

It is easy to show that $L_{N(T)} \prec L_R$ (i.e. $L_{N(T)}$ is an elementary substructure of L_R, Definition 8.7) for all $T \preceq R$.

There are two cases to consider:

(1) There is a type $T \prec R$ such that

$$N(T) \cong R.$$

Then define $\mathrm{TW}'(R) = N(T_0)$, where

$$T_0 = \inf\{T\colon T = \mathrm{TW}'(T) \wedge N(T) \cong R\}.$$

(2) Otherwise, define $\mathrm{TW}'(R) = 0$.

(vii) If for a certain $S \prec R$, $\mathrm{TW}'(S) \overset{\sim}{\neq} S$, define

$$\mathrm{TW}'(R) = 0 \quad \text{and} \quad L_R = 0 = B_R = G_R.$$

(viii) Define $\mathscr{T}'(R)$ iff $R = \mathrm{TW}'(R) \wedge R$ w.o., and

$$\mathscr{L} = \bigcup \{L_R\colon \mathscr{T}'(R)\}.$$

If we call $\mathscr{V}$ the collection defined by

$$\mathscr{V}(X) \leftrightarrow X = X,$$

the axiom of constructibility is given by the formula $\mathscr{V} \simeq \mathscr{L}$, which is equivalent to:

$$\forall X \exists R \big(\mathscr{T}'(R) \wedge L_R(X)\big).$$

We then have the following theorem:

THEOREM 11.3. (i) $\mathrm{MT}^{\mathscr{L}}$,

(ii) $(\mathscr{V} \simeq \mathscr{L})^{\mathscr{L}}$.

Proof. (i) We shall use 10.1.

It is clear that $\mathscr{L}$ is transitive and that $\mathscr{L}(\omega)$. Thus we need to show only

(1) $\forall x \big(\mathscr{L}(x) \to (x \in \mathscr{V})^{\mathscr{L}}\big)$,

(2) $\forall X_0, \ldots, \forall X_{n-1} \big(\overline{\mathscr{L}}(X_0, \ldots, X_{n-1}) \to \mathscr{L}\big(\{x\colon \varphi(x, X_0, \ldots, X_{n-1})\}^{\mathscr{L}}\big)\big)$.

Proof of (1). Suppose $\mathscr{L}(x)$. Then there is a T such that $\mathscr{L}(T)$ and $L_T(x)$. Then

$$L_{T+1}(\{y\colon y = x\}) \quad \text{and} \quad L_{T+1} \sqsubseteq \mathscr{L}.$$

Hence $\mathscr{L}(\{x\})$ and (1) is proved.

Proof of (2). There are two cases:

(a) $\mathscr{L} = L_S$ for a certain S. Then $\mathrm{TW}'(S) = 0$ and S is the first such.

Let $\varphi = \varphi_n$ be a formula with $m+1$ free variables and let $L_S(X_0), \ldots$ $\ldots, L_S(X_{m-1})$. Then there is a type $T \prec S$ such that

$$L_T(X_0), \ldots, L_T(X_{m-1}).$$

Let

$$Q = \{x\colon \varphi(x, X_0, \ldots, X_{m-1})\}^{L_S}$$
$$= \{x\colon x \in L_S \wedge L_S \models \varphi[x, X_0, \ldots, X_{m-1}]\}.$$

By the 1st stage, $Q \subseteq L_{C(T)}$, with $C(T) \prec S$. By the 2nd stage, $C(T) \preceq N(C(T))$ and $L_{N(C(T))} \prec L_S$. Hence $L_{N(C(T))}(Q)$. But, since S has no last element, $\mathrm{TW}'(N(C(T))+1) \prec S$. Hence $L_S(Q)$.

(b) $\mathscr{L} = \bigcup \{L_{\mathrm{TW}'(R)}\colon R$ a well-ordering$\}$. We use a similar argument to that used in (a), employing the reflection principle 10.2, since TW′, in this case, is an operation that defines w.o. types for all well-orderings R.

(ii) is obtained immediately because $\mathrm{TW}'(R)$ and L_R are defined by recursion, and we have $\mathscr{L}(\mathrm{TW}'(R))$ for every well-ordering R such that $R \cong \mathrm{TW}'(R)$ (11.2(iv)). Thus, by absoluteness (9.5), we get (ii).

We see that with this procedure we obtain the relative consistency of $\mathscr{V} \simeq \mathscr{L}$. We call $\mathrm{MT}+\mathscr{V} \simeq \mathscr{L}$, MT$L$.

THEOREM 11.4. (i) $\mathrm{MTT}+\mathscr{V} \simeq \mathscr{L}$ *is an extension by definitions of* MTL (i.e. *in* MTL *it is possible to define an operation that defines w.o. types*).

(ii) $\mathrm{MT}+\neg\exists S$ (S *is a w.o.* $\wedge\ L_S \models \mathrm{MT}$) $\vdash$ TW′ *is an operation that defines w.o. types.*

Proof. (i) Suppose that there is a w.o. S in L such that for all w.o. T with $T = \mathrm{TW}'(T)$ we have $T \prec S$. Then $\mathscr{L}$ can be represented in $\mathscr{L}$ itself by the superclass $[L_{TW'(S_x)}\colon x \in \mathscr{D}S]$. This is a contradiction. Therefore there is no such S. Thus we define TW for well-orderings R in $\mathscr{L}$ by

$$\mathrm{TW}(R) = \inf\{T\colon T = \mathrm{TW}'(T) \cong R\}.$$

(ii) is immediate from 11.2.

There are some open questions concerning the existence of an operation that defines w.o. types:

(1) Is it possible to prove in MT the existence of an operation that defines w.o. types?

(2) If it is not possible to define in MT such an operation, is the addition of an operation TW that defines w.o. types a conservative extension of MT?

We shall now turn to the axiom of choice. We consider the following two strong versions of this axiom:

(A) Let T be an extension of MT. Then $\mathrm{T}+\mathrm{WO}_{\mathscr{C}}$ is the theory obtained from T by adding a new binary predicate $\mathscr{C}$ and the following axiom schema:

$\mathrm{WO}_{\mathscr{C}}$: $\mathscr{C}$ is reflexive, antisymmetric, transitive, and connected; for every n and every n-ary notion $\mathscr{A}$ we have

$$\forall X_1, \ldots, \forall X_{n-1}\exists X_0\big(\mathscr{A}(X_0, \ldots, X_{n-1}) \wedge \forall Y\big(\mathscr{A}(Y, X_1, \ldots, X_{n-1}) \to \mathscr{C}(X_0, Y)\big)\big);$$

and

$$\forall X \exists R \big(R \subseteq V \times V \wedge \forall Y(\mathscr{C}(Y, X) \to \exists x(Y = R^* \{x\}))\big).$$

Thus, according to $\mathrm{WO}_{\mathscr{C}}$, $\mathscr{C}$ is a well-ordering of all classes in which every proper segment can be coded by a relation (see [8]).

(B) Let T be an extension of MTT. Then $T + \mathrm{WO}_Q$ is the theory obtained from T by adding a new unary operation symbol Q and the following axiom:

$$\mathrm{WO}_Q\colon\ \forall X \exists R \big(R = \mathrm{TW}(R) \wedge X = Q(R)\big).$$

It is easy to see that $\mathrm{MTT} + \mathrm{WO}_{\mathscr{C}}$ is an extension by definitions of $\mathrm{MT} + \mathrm{WO}_{\mathscr{C}}$. Just take for each w.o. R

$$\mathrm{TW}(R) = \text{the } \mathscr{C}\text{-first w.o. } S \text{ such that } S \cong R.$$

It is also easy to prove that $\mathrm{MTT} + \mathrm{WO}_{\mathscr{C}} + \mathrm{WO}_Q$ is an extension by definition of $\mathrm{MT} + \mathrm{WO}_{\mathscr{C}}$ and $\mathrm{MTT} + \mathrm{WO}_Q$.

The following theorem has an easy proof:

THEOREM 11.5. $\mathrm{MTT} + \mathscr{V} \simeq \mathscr{L} + \mathrm{WO}_Q$ *(or* $\mathrm{MT}L + \mathrm{WO}_{\mathscr{C}}$*) is an extension by definitions of* $\mathrm{MT}L$ *(i.e. in* $\mathrm{MT}L$ *it is possible to define* TW *and* Q *(or* $\mathscr{C}$*)).*

Obviously, both $\mathrm{MTT} + \mathrm{WO}_Q$ and $\mathrm{MT} + \mathrm{WO}_{\mathscr{C}}$ imply that V can be well-ordered and hence, by 7.3, that $V \approx \mathrm{OR}$. Thus

$$\mathrm{MT}L \vdash V \approx \mathrm{OR}.$$

$V \approx \mathrm{OR}$ is Axiom E of Gödel (see [3]), i.e., the global axiom of choice. It is clear that MT+E implies the local axiom of choice. (For other forms of the axiom of choice see [8].)

Other, slightly weaker, forms of $\mathrm{WO}_{\mathscr{C}}$ and WO_Q that we shall use are the following:

(C) Let T be an extension of MT. Then $\mathrm{T} + \mathrm{WO}_{\mathscr{C}}^+$ is the theory obtained from T by adding a new ternary predicate $\mathscr{C}^+$ and the axiom $\mathrm{WO}_{\mathscr{C}}^+$, which is the axiom obtained by replacing in $\mathrm{WO}_{\mathscr{C}}$ $\mathscr{C}(Y, X)$ by $\mathscr{C}^+(Z, Y, X)$ and prefixing $\exists Z$.

(D) Let T be an extension of MT. Then $\mathrm{T} + \mathrm{WO}_Q^+$ is the theory obtained form T by adding a new binary operation symbol Q^+ and the axiom

$$\mathrm{WO}_Q^+\colon\ \exists Y \forall X \exists R \big(R = \mathrm{TW}(R) \wedge X = Q^+(R, Y)\big).$$

It is clear that $\mathrm{MTT} + \mathrm{WO}_Q + \mathrm{WO}_Q^+$ is a conservative extension of $\mathrm{MTT} + \mathrm{WO}_Q^+$, and $\mathrm{MT} + \mathrm{WO}_{\mathscr{C}} + \mathrm{WO}_{\mathscr{C}^+}$ is also a conservative extension of $\mathrm{MT} + \mathrm{WO}_{\mathscr{C}^+}$. Hence $\mathrm{MTT} + \mathrm{WO}_Q^+ \vdash \mathrm{E}$.

It is also possible to prove that $\mathrm{MT}L$ implies $V = L$, i.e., every set can be constructed from an ordinal (see [3], [7]) and thus the generalized continuum hypothesis. These results, however, will appear elsewhere.

VI. Ordinal definability

12. Existence of hierarchies of all classes and relative constructibility. In [12], Myhill and Scott proved that it was possible in ZF to define a collection (a class for ZF) that contains all definable sets. The same problem, i.e., to define a collection that contains all definable classes, is open for MT. In the metalanguage it is of course clear what constitutes a definable class, i.e., a special class in our terminology (see Section 1). A special class is a class A such that $A = \tau$ for a term τ without free variables.

The first metatheorem of [12] is also valid for MT. In order to formulate it, we need some terminology:

We shall say that a binary notion $\mathscr{C}$ is a definable well-ordering of the collection $\mathscr{B}$ iff the following two conditions are satisfied:

(i) $\mathscr{C}$ is connected, transitive, antisymmetric and reflexive.

(ii) For every collection $\mathscr{A} \mathrel{\overline{\subseteq}} \mathscr{B}$ have

$$\exists X\big(\mathscr{A}(X) \wedge \forall Y\big(\mathscr{A}(Y) \rightarrow \mathscr{C}(X, Y)\big)\big).$$

METATHEOREM 12.1. *Every collection $\mathscr{B}$ that contains all definable (special) classes includes every collection with a definable well-ordering.*

The proof is exactly the same as for the first metatheorem of [12]. Notice that for 12.1 we need a weaker concept of well-ordering than that which was required in Axiom $\mathrm{WO}_{\mathscr{C}}$. For possible further applications, we may notice that 12.1 remains true for any extension of MT.

Myhill and Scott proved that there is a collection in ZF that contains all definable sets. The proof, however, makes essential use of the hierarchy of sets $\{V^{\cdot}\alpha\colon \alpha \in \mathrm{OR}\}$. Since we have to obtain a collection that contains all definable classes, we would need a corresponding hierarchy of classes. The rank operation is not useful for obtaining such a hierarchy, because for every proper class X we have $\varrho X = \mathrm{OR}$.

We have for MTT a principle of reflection (10.2). In order to use this principle to obtain the collection which contains all definable classes, we need to add a new axiom.

Let T be an extension of MTT. The theory T+(X) is obtained from T by adding a new unary operation W and the following axiom:

$$\text{(X)}\ \ \forall R\big(\mathscr{T}(R) \rightarrow W(R) \text{ is a structure}\big) \wedge \mathscr{V} \simeq \overline{\bigcup}\{W(R)\colon \mathscr{T}(R)\}.$$

We shall call MTT+(X), MTH.

It is clear that an operation with this property of W can be defined in MTL, namely

$$W(R) = L_R \quad \text{for every type } R.$$

Thus MTH $+\mathscr{V} \simeq \mathscr{L}$ is an extension by definitions of MTL. Therefore MTH is consistent if MT is consistent.

We now pass to the discussion of relative constructibility.

DEFINITION 12.2. Let $\mathscr{A} = [F(x)\colon x \in A]$ be a superclass with $A \neq 0$.

(i) Define the structure $L_R[\mathscr{A}]$ and $\mathrm{TW}'_{\mathscr{A}}(R)$ for R w.o. in the same way as L_R and TW′ (11.2) but replacing clause (iii) by

(iii′) $$L_0[\mathscr{A}] = [A, [F(x)\colon x \in A]].$$

(ii) $\mathscr{L}[\mathscr{A}] \simeq \bigcup\{L_R[\mathscr{A}]\colon R \text{ a w.o. and } R = \mathrm{TW}'_{\mathscr{A}}(R)\}$.

(iii) Let A be a class. Then $L_R[A]$, $\mathrm{TW}'_A(R)$, and $L[A]$ are $L_R[\mathscr{A}]$, $\mathrm{TW}'_{\mathscr{A}}[R]$, and $\mathscr{L}[\mathscr{A}]$, where $\mathscr{A} = [x\colon x \in A]$.

(iv) (Definition schema). Let TW be an operation that defines w.o. types and $\mathscr{T}$ the corresponding collection of types. Define $L_{\mathscr{T},R}[\mathscr{A}]$ and $\mathscr{L}_{\mathscr{T}}[\mathscr{A}]$ as in 11.2, but using the types already available in $\mathscr{T}$, and hence omitting clauses (i), (ii), and (vi). Also, change (iii) to (iii′) as in (i) above.

We call the theory MT $+\mathscr{V} \simeq \mathscr{L}[\mathscr{A}]$, MT$L[\mathscr{A}]$; and

$$\mathrm{MT} + \mathscr{V} \simeq \mathscr{L}_{\mathscr{T}}[\mathscr{A}],\ \mathrm{MT}L_{\mathscr{T}}[\mathscr{A}].$$

In the same way as 11.3 and 11.4, we can prove:

THEOREM 12.3 (i) $\mathrm{MT}^{\mathscr{L}[\mathscr{A}]}$ *and* $\mathrm{MT}^{\mathscr{L}_{\mathscr{T}}[\mathscr{A}]}$.

(ii) $(\mathscr{V} \simeq \mathscr{L}[\mathscr{A}])^{\mathscr{L}[\mathscr{A}]}$ *and* $(\mathscr{V} \simeq \mathscr{L}_{\mathscr{T}}[\mathscr{A}])^{\mathscr{L}_{\mathscr{T}}[\mathscr{A}]}$.

(iii) MTT $+\mathscr{V} \simeq \mathscr{L}[\mathscr{A}]$ *is an extension by definitions of* MT$L[\mathscr{A}]$.

We shall be interested in the special cases MT$L[V]$ and MT$L_{\mathscr{T}}[V]$, which we shall call MTV and MTV$_{\mathscr{T}}$, respectively.

$$\mathrm{MTH} + \mathscr{V} \simeq \mathscr{L}[V] \quad \text{and} \quad \mathrm{MTH} + \mathscr{V} \simeq \mathscr{L}_{\mathscr{T}}[V]$$

are extensions by definition of MTV and MTV$_{\mathscr{T}}$, respectively. Thus we may use (X) (and (IX)) in MTV.

MTV is a very natural extension of MT, since it preserves all sentences relativized to sets. To be more precise, we shall introduce the following definition. We shall call L_{MT} the language of MT, the language which has $\in$ and V as non-logical symbols. The primitive language of MT, with $\in$ as the only primitive, is L_{ZF}.

DEFINITION 12.4. Let φ be a formula of L_{MT}. Then

(i) φ^V is the formula obtained form φ by replacing $\exists v_i \psi$ by $\exists v_i(v_i \in V \wedge \psi)$, and $\forall v_i \psi$ by

$$\exists v_i(v_i \in V \to \psi).$$

(ii) φ is predicative iff φ is ψ^V for some formula ψ (i.e. all quantifiers in φ are relativized to sets).

(iii) Let $\mathbf{T}$ be a theory whose language contains L_{MT}; we define:

$$\mathrm{T}^* = \{\varphi\colon \varphi \text{ is a sentence of } L_{\mathrm{ZF}} \wedge \mathrm{T} \vdash \varphi^V\}.$$

We can prove the following theorem:

THEOREM 12.5.

$$\mathrm{MT}^* = \mathrm{MTT}^* = \mathrm{MTH}^* = \mathrm{MTV}^*_{\mathscr{T}} = \mathrm{MTV}^*.$$

Proof. Since $\mathrm{MTT} + \mathscr{V} \simeq \mathscr{L}[V]$ and $\mathrm{MTH} + \mathscr{V} \simeq \mathscr{L}[V]$ are extensions by definition, and hence conservative extensions of MTV, we have

$$\mathrm{MT}^* \subseteq \mathrm{MTT}^* \subseteq \mathrm{MTH}^* \subseteq \mathrm{MTV}^*_{\mathscr{T}} \subseteq \mathrm{MTV}^*.$$

Thus we only have to prove that

$$\mathrm{MTV}^* \subseteq \mathrm{MT}^*.$$

Suppose then that there is a sentence ψ of L_{ZF} such that $\mathrm{MTV} \vdash \psi^V$ and not $\mathrm{MT} \vdash \psi^V$. Hence $\mathrm{MT} \cup \{\neg \psi^V\}$ is consistent. Let $\mathscr{B}$ be a collection such that $\mathrm{MT}^{\mathscr{B}}$ and $(\neg\psi^V)^{\mathscr{B}}$. We construct $(\mathscr{L}[V])^{\mathscr{B}}$. Then MTV is true in $(\mathscr{L}[V])^{\mathscr{B}}$ and $(\mathscr{L}[V])^{\mathscr{B}}$ has the same sets as $\mathscr{B}$. Hence $\neg\psi^V$ is true in $(\mathscr{L}[V])^{\mathscr{B}}$. But this contradicts $\mathrm{MTV} \vdash \psi^V$.

Theorem 12.5 implies that MTV is a good extension of MT because it does not disturb the most important objects, namely the sets. Everything that we do in this paper can be done in MTV. Therefore MTV seems to be an appropiate extension of MT, which serves both for the theory and the metatheory. The only drawback of MTV is the complicated form of the axiom $\mathscr{V} \simeq \mathscr{L}[V]$.

An interesting property of MTV is the following: $\mathrm{MTV} + \mathrm{WO}^+_{\mathscr{C}}$ is an extension by definitions of $\mathrm{MTV} + \mathrm{E}$.

Since $\mathrm{MTV} + \mathrm{WO}_{\mathscr{C}} + \mathrm{WO}^+_{\mathscr{C}}$ is a conservative extension of $\mathrm{MTV} + \mathrm{WO}^+_{\mathscr{C}}$, we have

(+) $\mathrm{MTV} + \mathrm{WO}_{\mathscr{C}}$ is a conservative extension of $\mathrm{MTV} + \mathrm{E}$.

Thus

THEOREM 12.6.

$$(\mathrm{MT} + \mathrm{WO}_{\mathscr{C}})^* = (\mathrm{MTV} + \mathrm{WO}_{\mathscr{C}})^* = (\mathrm{MTV} + \mathrm{E})^* = (\mathrm{MT} + \mathrm{E})^*.$$

The first and third equalities are proved as in Theorem 12.5. The second one is obtained from (+).

13. Ordinal definability. We are now ready to define in MTH a collection that contains all definable classes. This is possible in MTH because of the following reflection principle, which is an immediate consequence of 10.2. In the whole of this section we work in MTH. We could also work in MTV, which has the same primitive symbols as MT.

THEOREM 13.1. *For every term τ with n free variables, for every type R, there is a type $R' \succeq R$ such that if $\overline{W(R')}(X_1, \ldots, X_n)$, then*
$$\tau^{(R')}(X_1, \ldots, X_n) = \tau(X_1, \ldots, X_n).$$

We can now define the collection of classes definable by formulas with well-ordering types as parametres.

DEFINITION 13.2. Let $W(R) = [A_R, [F_R(x): x \in A_R]]$ for every type R. Define:

$$\mathscr{OD}(X) \leftrightarrow \exists n \exists \ulcorner\tau\urcorner \exists R \exists g \big(n \in \omega \wedge \ulcorner\tau\urcorner \text{ is a term } \wedge$$
$$\wedge \operatorname{Fv}(\tau) = n \sim \{0\} \wedge \mathscr{T}(R) \wedge g \in {}^n A_R \wedge H_{W(R)}(g)(0) = X \wedge$$
$$\wedge \forall i \big(i \in n \sim \{0\} \to \mathscr{T}\big(H_{W(R)}(g)(i)\big)\big) \wedge W(R) \models v_0 = \tau[H_{W(R)}(g)]\big).$$

By 13.1, $\mathscr{OD}$ contains all classes that are definable in MTH with types as parameters, i.e., those X for which there is a term τ with n free variables and types $R_1, \ldots, R_n$ such that
$$X = \tau(R_1, \ldots, R_n).$$

DEFINITION 13.3. (1) Let X be a class and R a type. Then

(i) $C_{W(R)}(X) = \big\{g: \exists n \big(n \in \omega \wedge g \in {}^n A_R \wedge H_{W(R)}(g)(0) = X \wedge \forall i \big(i \in n - \{0\} \to \mathscr{T}\big(H_{W(R)}(g)(i)\big)\big)\big)\big\}$.

(ii) $T(R, X)$ is the ordering of $C_{W(R)}(x)$ given by $gT(R, X)f$ iff $\mathscr{D}g \subset \mathscr{D}f \vee \mathscr{D}g = \mathscr{D}f \wedge$ if i_0 is the least $i \in \mathscr{D}g$ such that $H_{W(R)}(g)(i) \neq H_{W(R)}(f)(i)$, we have $H_{W(R)}(g)(i_0) \prec H_{W(R)}(f)(i_0)$.

(iii) For $g \in C_{W(R)}(X)$, define $T_{R,X}(g) = \operatorname{TW}\big(T(R, X)_g\big)$.

(2) Let X be a class, and $\langle \tau_i: i \in \omega \rangle$ a sequence of all terms τ such that $\operatorname{Fv}\tau = n \sim \{0\}$ for a certain $n \in \omega$. Then:

(i) $J(X) = \inf\big\{R: \mathscr{T}(R) \wedge \exists n \exists \ulcorner\tau\urcorner \exists g \big(n \in \omega \wedge \ulcorner\tau\urcorner$ is a term $\wedge \operatorname{Fv}\tau = n \sim \{0\} \wedge g \in C_{W(R)}(X) \wedge \mathscr{D}g = n \wedge W(R) \models \ulcorner v_0 = \tau\urcorner [H_{W(R)}(g)]\big)\big\}$;

(ii) $l(X) = \inf\big\{i: \exists g \big(g \in C_{W(J(X))}(X) \wedge \mathscr{D}g = \operatorname{Fv}\tau_i \cup \{0\} \wedge W\big(J(X)\big) \models v_0 = \tau_i [H_{W(J(X))}(g)]\big)\big\}$;

(iii) $K(X) = \inf\big\{R: \exists g \big(g \in C_{W(J(X))}(X) \wedge \mathscr{D}g = \operatorname{Fv}\tau_{l(X)} \cup \{0\} \wedge W\big(J(X)\big) \models v_0 = \tau_{l(X)} [H_{W(J(X))}(g)] \wedge R = T_{W(J(X)),X}(g)\big)\big\}$;

(iv) $\mathscr{C}(X, Y) \leftrightarrow \mathscr{OD}(X, Y) \wedge \big(J(X) \prec J(Y) \vee \big(J(X) = J(Y) \wedge l(X) \subset l(Y)\big) \vee \big(J(X) = J(Y) \wedge l(X) = l(Y) \wedge K(X) \prec K(Y)\big)\big)$.

It is easy to see that $\mathscr{C}$ is a definable well-ordering of $\mathscr{OD}$ with the property that

$$(+) \qquad \forall X \exists R \forall Y \big(\mathscr{C}(Y, X) \leftrightarrow \exists y (R^* \{y\} = Y)\big).$$

THEOREM 13.4. (i) $\mathscr{T} \subseteq \mathscr{OD}$.

(ii) *If τ is a term with n free variables and $\overline{\mathscr{OD}}(X_1, \ldots, X_n)$, then $\overline{\mathscr{OD}}\big(\tau(X_1, \ldots, X_n)\big)$.*

Proof. (i) is clear.

(ii) We can see that the collection $\mathscr{B}$ defined by

$$\mathscr{B}(X)\leftrightarrow\exists X_1,\ldots,\exists X_n\big(X=\tau(X_1,\ldots,X_n)\wedge\overline{\mathscr{O}\mathscr{D}}(X_1,\ldots,X_n)\big)$$

has a definable well-ordering induced by $\mathscr{C}$. Hence, by 12.1

$$B \mathrel{\bar{\subseteq}} \mathscr{O}\mathscr{D}.$$

To be able to generalize (ii) to other situations we need another type of proof.

Let $X=\tau(X_1,\ldots,X_n)$ for τ a term.

Suppose $X_i=\tau_i^{W(R_i)}(S_0,\ldots,S_{n_i})$.

Choose $R\succeq R_i, S_0,\ldots,S_{n_i}$ such that $\overline{W(R)}(R_i,S_j)$, for every $i\leq n$, $j\leq n_i$.

Let $R'\succeq R$ be such that

$$X=\tau\big(\tau_i^{W(R_i)}(S_0,\ldots,S_{n_i})\ldots\big)=\tau^{W(R')}(\ldots).$$

Therefore $\mathscr{O}\mathscr{D}(X)$.

As in ZF, $\mathscr{O}\mathscr{D}$ might not be a model of MT. Hence we define the collection $\mathscr{H}\mathscr{O}\mathscr{D}$ by

DEFINITION 13.5.

$$\mathscr{H}\mathscr{O}\mathscr{D}(X)\leftrightarrow\mathscr{O}\mathscr{D}(X)\wedge\mathrm{TC}(X)\mathrel{\bar{\subseteq}}\mathscr{O}\mathscr{D}.$$

THEOREM 13.6.

$$\mathscr{H}\mathscr{O}\mathscr{D}(X)\leftrightarrow X\mathrel{\bar{\subseteq}}\mathscr{H}\mathscr{O}\mathscr{D}\wedge\mathscr{O}\mathscr{D}(X).$$

Using 10.1 we can easily prove:

THEOREM 13.7.

$$\mathrm{MT}^{\mathscr{H}\mathscr{O}\mathscr{D}}.$$

Now, $\mathscr{C}$ restricted to $\mathscr{H}\mathscr{O}\mathscr{D}$ is a definable well-ordering of $\mathscr{H}\mathscr{O}\mathscr{D}$ that satisfies $(+)$ for $\mathscr{H}\mathscr{O}\mathscr{D}$. By 7.5 it is possible to assume

$$(*)\qquad \mathscr{T}(R)\to R\subseteq \mathrm{OR}\times\mathrm{OR}.$$

This is so because it is possible to define an operation F such that for each R w.o. $R\notin V$, $\mathscr{D}R\approx_{F(R)}\mathrm{OR}$. Thus, if $\mathscr{T}$ does not satisfy $(*)$, we can define a new $\mathscr{T}'$ and TW$'$ by

$\mathrm{TW}'(R)=$ the ordering of OR, S, that is isomorphic to $\mathrm{TW}(R)$ by $F(\mathrm{TW}(R))$.

Since $\mathrm{OR}\mathrel{\bar{\subseteq}}\mathscr{H}\mathscr{O}\mathscr{D}$ and hence $\mathrm{OR}\times\mathrm{OR}\mathrel{\bar{\subseteq}}\mathscr{H}\mathscr{O}\mathscr{D}$, we have

$$\mathscr{T}'\mathrel{\bar{\subseteq}}\mathscr{H}\mathscr{O}\mathscr{D}.$$

We can define the relations that code the initial segments of $\mathscr{C}$ using $\mathscr{T}$. Thus these relations are all in $\mathscr{H}\mathscr{O}\mathscr{D}$. Hence we have:

THEOREM 13.8.

$$(\mathrm{WO}_{\mathscr{C}})^{\mathscr{H}\mathscr{O}\mathscr{D}}.$$

Therefore the axiom of choice E and the local axiom of choice are true in $\mathscr{HOD}$.

For many applications in proofs of independence, we need to have a model $\mathscr{N}$ of MT in which all classes are definable from types, determined by a given operation TW which defines w.o. types, and elements of a given class A. That is, for every X with $\mathscr{N}(X)$, there are types $R_1, \ldots, R_n$; $x_1, \ldots, x_m \in A$; and a term τ with at most $v_0, \ldots, v_{n+m}$ free, such that

$$X = \tau(R_1, \ldots, R_n, x_1, \ldots, x_m).$$

Let us see how to proceed in order to obtain such an $\mathscr{N}$, assuming only MT. This procedure will be used to prove the independence of different forms of the axiom of choice in Chapter IX.

We first construct $\mathscr{L}_{\mathscr{T}}[V]$ as in 12.2, where $\mathscr{T}$ is the collection of types given by TW. It is clear that $\mathscr{L}_{\mathscr{T}}[V]$ satisfies the reflection principle 13.1, taking $W(R) = L_{\mathscr{T},R}[V]$. Thus it is possible to give the following definition:

DEFINITION SCHEMA 13.9. Let TW be an operation that defines w.o. types, and $\mathscr{T}$—the corresponding collection of types. Let

$$L_{\mathscr{T},R}[V] = [B_R, [G_R(x)\colon x \in B_R]]$$

for every type R. Then:

(i) $\mathscr{OD}_{\mathscr{T},A}(X) \leftrightarrow \exists n \exists \ulcorner\tau\urcorner \exists R \exists g \Big\{n \in \omega \wedge \ulcorner\tau\urcorner$ is a term $\wedge$
$\wedge \operatorname{Fv}\tau = n \sim \{0\} \wedge \mathscr{T}(R) \wedge g \in {}^{n}B_R \wedge H_{L_{\mathscr{T},R}[V]}(g)(0) = X \wedge$
$\wedge \forall i \Big(i \in n \sim \{0\} \to \Big(\mathscr{T}\big(H_{L_{\mathscr{T},R}[V]}(g)(i)\big) \vee$
$\vee\ H_{L_{\mathscr{T},R}[V]}(g)(i) \in A\Big)\Big) \wedge L_{\mathscr{T},R}[V] \models v_0 = \tau[H_{L_{\mathscr{T},R}[V]}(g)]\Big)\Big\}.$

(ii) $\mathscr{HOD}_{\mathscr{T},A}(X) \leftrightarrow \operatorname{TC}(X) \subseteq \mathscr{HOD}_{\mathscr{T},A} \wedge \mathscr{OD}(X)$.

If $\mathscr{HOD}_{\mathscr{T},A}(X)$, then X is definable in MT form types in $\mathscr{T}$ and elements of A. By a similar proof to that of 13.7, we can prove

$$\mathrm{MT}^{\mathscr{HOD}_{\mathscr{T},A}}.$$

However, it is not true, in general, that the axioms of choice are true in $\mathscr{HOD}_{\mathscr{T},A}$.

Let us see now how close we can get to a collection that contains all definable classes in MT itself.

We first consider the theory MTV of 12. Now in MTV we have for any formula φ,

$$\mathrm{MTV} \vdash \varphi^{V} \quad \text{iff} \quad \mathrm{MT} \vdash \varphi^{V}.$$

Hence, for any term τ in which the quantifiers $\forall$, $\exists$ are restricted to sets, we have in MT for $\mathscr{L}[V](X_1, \ldots, X_n)$

$$\tau(X_1, \ldots, X_n) = \tau^{\mathscr{L}[V]}(X_1, \ldots, X_n).$$

Thus, if we repeat the procedure of 13.9, we obtain a class $\mathcal{OD}_{\mathcal{T},V}$, which contains all classes definable with formulas restricted to sets, using as parameters the types $\mathcal{T}$ defined in the construction of $\mathcal{L}[V]$.

The corresponding collection $\mathcal{HOD}_{\mathcal{T},V}$ is a model of

$$\mathrm{MT}+\mathrm{WO}_{\mathscr{C}}.$$

VII. Complexity of the axiom system

14. Non-finite axiomatizability and non-axiomatizability with sentences of bounded unrestricted quantifier depth. Given the results of the last chapters, it is easy to show that MT is not finitely axiomatizable. We first add one axiom to MT, namely $\mathscr{V} \simeq \mathscr{L}$ or $\mathscr{V} \simeq \mathscr{L}[V]$. Hence, if MT were finitely axiomatizable, MTV would also be such. But the reflection principle is valid for MTV. Suppose φ were the axiom of MTV. Then we would have in MTV

$$L_R[V] \models \varphi \quad \text{for some type } R.$$

Hence we would be able to prove the consistency of MTV in MTV.

Using Montague's [9] and Kreisel–Levy's [5] methods, it is posible to prove something stronger. Let us consider the theory given by the following axioms:

(I) Extensionality:

$$\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B.$$

(II) Axiom schema of class determination. For every formula φ in which A is not free we have the following axiom:

$$\exists A \forall x\big(x \in A \leftrightarrow \varphi \wedge \exists U(x \in U)\big).$$

(III) Empty set:

$$\forall x(x \notin a) \rightarrow \exists U(a \in U).$$

(IV) Pairs:

$$\forall x(x \in a \leftrightarrow x = b \vee x = c) \rightarrow \exists U(a \in U).$$

Let us call this theory general class theory, GC. It is clear that GC is a subtheory of MT. In GC we have the schema of class construction, arbitrary unions and intersections of collections, relations. We shall use for these the same symbols as in MT.

We also have in GC the possibility of representing an arbitrary operation restricted to a class by a relation. Thus we have superclasses $[F(x) \colon x \in B]$. Theorem 2.4 is valid in GC.

Since there is a model of GC constituted by all classes whose elements are sets hereditarily of cardinality less than or equal to two, we cannot use the usual (von Neumann) natural numbers. We use, instead, Zermelo's numbers:

$$Z = \bigcap\{X\colon\ 0 \in X \wedge \forall x(x \in X \to \{x\} \in X)\}.$$

With this definition, arithmetic is interpretable in GC, and hence we can interpret the syntax of the language in GC.

It is not true, in general, in GC that $x \in V$ implies $x \times \{i\} \in V$. However, $x \in V$ implies $\{\langle x, i\rangle\} \in V$. In order to be able to work with relations, for each function r we define:

$$\bar{r} = \bigcup\{r`u \times \{u\}\colon\ u \in \mathscr{D}r\}.$$

DEFINITION 14.1. Define by recursion, for each $n \in \omega$, the subset F_n of the formulas of L_{MT}:

$$F_0 = \bigcap\{X\colon\ X \text{ contains the atomic formulas and is closed under } \vee,\ \neg,\ \exists v_j(v_j \in V \wedge \ldots)\},$$

$$F_{m+1} = \bigcap\{X\colon\ \{\exists v_{i_0} \ldots \exists v_{i_{p-1}}\varphi\colon\ \varphi \in F_m \wedge p \in \omega \wedge i \in {}^p\omega\} \subseteq X \text{ and } X \text{ is closed under } \vee,\ \neg, \text{ and } \exists v_j(v_j \in V \wedge \ldots)\}.$$

Define $\mathrm{Dp}(\varphi) = \bigcap\{n\colon\ \varphi \in F_n\}$ for each formula φ of L_{MT}. $\mathrm{Dp}(\varphi)$ is read depth of φ.

We can give a definition of satisfaction for formulas of depth n. This is possible because the principle of definitions by recursion 4.3 is valid in GC.

DEFINITION 14.2. (i) Define by recursion on the relation of subformula for a formula φ of L_{MT} with $\mathrm{Dp}(\varphi) = 0$, and a relation R with $\mathscr{D}R \subseteq \mathrm{Fv}\varphi$, $K_0(\ulcorner\varphi\urcorner, R)$, thus:

(a) $\ulcorner\varphi\urcorner = \ulcorner v_i \in v_j\urcorner \to K_0(\ulcorner\varphi\urcorner, R) = \{r\colon\ r \text{ is a function} \wedge \mathscr{D}R \cap \mathscr{D}r = 0 \wedge \mathscr{D}r \subseteq \mathrm{Fv}\varphi \wedge (R \cup \bar{r})^*\{i\} \in (R \cup \bar{r})^*\{j\}\}$.

(b) $\ulcorner\varphi\urcorner = \ulcorner v_i = v_j\urcorner \to K_0(\ulcorner\varphi\urcorner, R) = \{r\colon\ r \text{ is a function} \wedge \mathscr{D}R \cap \mathscr{D}r = 0 \wedge \mathscr{D}r \subseteq \mathrm{Fv}\varphi \wedge (R \cup \bar{r})^*\{i\} = (R \cup \bar{r})^*\{j\}\}$.

(c) $\ulcorner\varphi\urcorner = \neg\psi \to K_0(\ulcorner\varphi\urcorner, R) = \{r\colon\ r \text{ is a function} \wedge \mathscr{D}R \cap \mathscr{D}r = 0 \wedge \mathscr{D}r \subseteq \mathrm{Fv}\varphi \wedge r \notin K_0(\ulcorner\psi\urcorner, R)\}$.

(d) $\ulcorner\varphi\urcorner = \ulcorner\psi_1 \vee \psi_2\urcorner \to K_0(\ulcorner\varphi\urcorner, R) = \{r\colon r \text{ is a function } \wedge \mathscr{D}R \cap \mathscr{D}r = 0 \wedge \wedge \mathscr{D}r \subseteq \mathrm{Fv}\varphi \wedge (r{\restriction}\mathrm{Fv}\psi_1 \in K_0(\psi_1, R{\restriction}\mathrm{Fv}\psi_1) \vee r{\restriction}\mathrm{Fv}\psi_2 \in K_0(\psi_2, R{\restriction}\mathrm{Fv}\psi_2))\}$.

(e) $\ulcorner\varphi\urcorner = \ulcorner\exists v_i(v_i \in V \wedge \psi)\urcorner \wedge i \notin \mathrm{Fv}\psi \to K_0(\ulcorner\varphi\urcorner, R) = K_0(\ulcorner\psi\urcorner, R)$.

(f) $\ulcorner\varphi\urcorner = \ulcorner\exists v_i(v_i \in V \wedge \psi)\urcorner \wedge i \in \mathrm{Fv}\psi \to K_0(\ulcorner\varphi\urcorner, R) = \{r\colon\ r \text{ is a function} \wedge \mathscr{D}R \cap \mathscr{D}r = 0 \wedge \mathscr{D}r \subseteq \mathrm{Fv}\varphi \wedge \exists s(s \subseteq V \times V \wedge \mathscr{D}s \subseteq \{i\} \wedge r \cup s \in K_0(\ulcorner\psi\urcorner, R))\}$.

(ii) Suppose that the operation K_n is defined for formulas of depth n. Let φ be a formula of $\mathrm{Dp}(\varphi) = n+1$ and R a relation with $\mathscr{D}R \subseteq \mathrm{Fv}\varphi$. Define $K_{n+1}(\ulcorner\varphi\urcorner, R)$ for formulas φ of depth $n+1$ by recursion:

(a) $\mathrm{Dp}(\ulcorner\varphi\urcorner) = n \to K_{n+1}(\ulcorner\varphi\urcorner, R) = K_n(\ulcorner\varphi\urcorner, R)$;

(b) $\ulcorner\varphi\urcorner = \ulcorner\exists v_{i_0} \ldots \exists v_{i_{p-1}} \psi\urcorner \wedge \mathrm{Dp}(\psi) = n \to K_{n+1}(\ulcorner\varphi\urcorner, R) = \bigcup\{K_n(\ulcorner\psi\urcorner, S): \mathscr{D}S \subseteq \mathscr{D}R \cup \{i_0 \ldots i_{p-1}\} \wedge S{\restriction}\mathrm{Fv}\varphi = R\}$;

(c)–(f) similar to clauses (c)–(f) in (i) with K_0 replaced by K_{n+1}.

(iii) For φ of depth n we define

$$R\,\mathrm{sat}_n \ulcorner\varphi\urcorner \leftrightarrow R \subseteq V \times V \wedge \mathscr{D}R \subseteq \mathrm{Fv}\varphi \wedge 0 \in K_n(\ulcorner\varphi\urcorner, R).$$

(iv) For a sentence φ of depth n we define

$$\mathrm{Tr}_n(\ulcorner\varphi\urcorner) \leftrightarrow 0\,\mathrm{sat}_n \ulcorner\varphi\urcorner.$$

We now apply Theorem 6 of [5] with GC as U, and obtain:

THEOREM 14.3. *No consistent extension of* GC *with the same symbols is axiomatizable with sentences of bounded depth.*

In particular, this is true for MT.

Now let (In) be the general schema of transfinite induction, i.e.,

$$(\mathrm{In}_\varphi)\Big[\varphi(0) \wedge \forall\alpha\big(\varphi(\alpha) \to \varphi(\alpha+1)\big) \wedge \Big(\alpha = \bigcup\alpha \wedge \forall\beta\big(\beta \in \alpha \to \varphi(\beta)\big) \to \varphi(\alpha)\Big)\Big] \to \forall\alpha\big(\alpha \in \mathrm{OR} \to \varphi(\alpha)\big).$$

Take NBG $+ \{\mathrm{In}_\varphi$: φ a formula$\}$ and call it NBGI. Now, it is possible to prove, applying Theorems 8 and 4 of [5], that MT cannot be obtained by extending NBGI with a set of axioms of bounded depth. As an example of the type of reasoning involved I shall prove, for simplicity, the following

THEOREM 14.4. *If* MT *is consistent, then* MT *is not a finite extension of* NBGI (*i.e., obtained by adding a finite number of axioms to* NBGI).

Proof. If MT were a finite extension of NBGI, MTV would also be such an extension. Let MTV $=$ NBGI $+\varphi$. By the reflection principle, there is an $R \succ \mathrm{OR}$, R without a last element, such that

$$L_R[V] \models \varphi.$$

But it is easy to prove that, for any R with no last element,

$$L_R[V] \models \mathrm{NBGI}.$$

Hence

$$L_R[V] \models \mathrm{NBGI} + \varphi,$$

i.e.,

$$L_R[V] \models \mathrm{MTV}.$$

This is impossible, since then the consistency of MTV would be provable in MTV itself.

As was stated above, by using Theorems 8 and 4 of [5] we can also obtain:

THEOREM 14.5. *If* MT *is consistent, then* MT *is not obtained from* NBGI *by adding a set of sentences of bounded depth.*

15. Complexity of axioms for the predicative sentences. In this section we shall study the relation between ZF and the set of predicative sentences valid in MT. The language of ZF (L_{ZF}) is the same as the primitive language of MT. V is not a symbol of L_{ZF}.

As in 13, let $\mathrm{MT}^* = \{\psi\colon \psi$ is a sentence of $L_{\mathrm{ZF}} \wedge \mathrm{MT} \vdash \psi^V\}$. Hence MT^* is the theory obtained from the predicative sentences provable in MT. It is clear that $\mathrm{ZF} \subseteq \mathrm{MT}^*$. Since the consistency of ZF can be expressed by a numerical sentence, and hence by a sentence of ZF, say Con ZF, and we proved it in MT, $\mathrm{Con\,ZF} \in \mathrm{MT}^* \sim \mathrm{ZF}$. Thus MT^* is strictly larger than ZF. We shall prove in this section, using Theorem 10 of [5], that MT^* is much larger than ZF. We first have to introduce a new notion of depth for formulas of ZF.

Definition 15.1. Let φ be a formula of L_{ZF}. Then we define $\mathrm{Dep}\,(\varphi)$ similarly to $\mathrm{Dp}(\varphi)$ (Definition 14.1), replacing everywhere

$$\exists v_i(v_i \in V \wedge \ldots) \quad \text{by} \quad \exists v_i(v_i \in v_j \wedge \ldots).$$

In Theorem 10 of [5] we will take $S_2 = K = \mathrm{ZF}$ and $S_1 = \mathrm{MT}$. The interpretation of a formula ψ of ZF in MT is given by ψ^V. We have $S_1|K = \mathrm{MT}|\mathrm{ZF} = \mathrm{MT}^*$ and $S_2|K = S_2 = \mathrm{ZF}$.

Theorem 10 of [5] as applied to our case says:

Theorem 15.2. *Let A be a set of sentences of L_{ZF} closed under conjunction and let $\sigma(x)$ and $T(x)$ be formulas of L_{MT} and L_{ZF}, respectively, such that:*

(i) *For each sentence φ of L_{ZF}*, $\mathrm{MT} \vdash \sigma(\ulcorner\varphi\urcorner) \leftrightarrow \varphi^V$.

(ii) $\mathrm{MT} \vdash \sigma(\ulcorner\varphi \to \psi\urcorner) \to \big(\sigma(\ulcorner\varphi\urcorner) \to \sigma(\ulcorner\psi\urcorner)\big)$.

(iii) $\mathrm{MT} \vdash \mathrm{Pr}(\ulcorner\varphi\urcorner)^V \to \sigma(\ulcorner\varphi\urcorner)$, *where* $\mathrm{Pr}(\ulcorner\varphi\urcorner)$ *is the formula of* ZF *that asserts* $\mathrm{ZF} \vdash \varphi$.

(iv) $\mathrm{MT} \vdash \forall\ulcorner\varphi\urcorner\big(\ulcorner\varphi\urcorner \in A \to \big(T(\ulcorner\varphi\urcorner)^V \to \sigma(\ulcorner\varphi\urcorner)\big)\big)$.

Then no consistent extension of MT^* *can be obtained from* ZF *by the addition of any subset of A.*

Now let us take A to be a set of formulas of ZF of bounded depth (in the sense of Dep). We can define truth for them in ZF similarly to 14.2. Thus, this is $T(\ulcorner\varphi\urcorner)$ for $\ulcorner\varphi\urcorner \in A$. Now, for $\varphi \in L_{\mathrm{ZF}}$, we define in MT

$$\sigma(\ulcorner\varphi\urcorner) \leftrightarrow V \models \varphi.$$

It is now easy to show (i), (ii), (iii), (iv). (iii) comes from the fact that all sentences provable in ZF are valid in V. Thus we can prove:

Theorem 15.3. *No consistent extension of* MT^* *with the same symbols can be obtained from* ZF *by adding a set of sentences of L_{ZF} of bounded depth (in the sense of* Dep).

Let us now consider NBGI. Take, as above,

$$\mathrm{NBGI}^* = \{\psi\colon \psi \text{ is a sentence of } L_{\mathrm{ZF}} \wedge \mathrm{NBGI} \vdash \psi^V\}.$$

We already know (12.6) that $\mathrm{MT}^* = \mathrm{MTV}^*$. Thus, taking $K = \mathrm{ZF}$, $S_2 = \mathrm{NBGI}$ and $S_1 = \mathrm{MTV}$ and defining the interpretation of a formula of ZF, ψ, in MTV and NBGI to be ψ^V, and the interpretation of a sentence of L_{NBGI} to be $\varphi^{\mathscr{B}}$, where $\mathscr{B}$ is $L_{\mathrm{TW(OR}+\omega)}[V]$, we obtain:

THEOREM 15.4. *No consistent extension of* MT^* *with the same symbols can be obtained from* NBGI^* *by adding a set of sentences of* L_{ZF} *of bounded depth.*

In this case we define for φ a sentence in L_{NBGI}

$$\sigma(\varphi) \leftrightarrow L_{\mathrm{TW(OR}+\omega)}[V] \models \varphi.$$

The rest of the proof is similar.

It is also easy to obtain a comparable result for NBGI^* and ZF. Therefore we have

$$\mathrm{ZF} \subset \mathrm{NBGI}^* \subset \mathrm{MT}^*$$

and none of the theories is an extension of bounded depth of the preceding one.

VIII. Forcing models

16. Forcing. In this chapter we shall develop the theory of forcing following [1], which is suggested in [13]. Most of the theorems of this section are proved in [1]; hence we shall omit some of the proofs and sketch others. In parentheses I shall indicate the numbers of the corresponding definitions or theorems in [1].

In the whole of this chapter and the next one, we work countable transitive structures of the form $\mathfrak{A} = [M, [x\colon x \in M]]$, where M is a countable transitive set. We identify the model $\mathfrak{A}$ with M. We work with models M of MT+E(MTE) and MTL.

As is well known, in order to do forcing with transitive models M, we must assume the consistency of MT with the existence of a well-founded model of MT. However, we can obtain our proofs assuming only the consistency of MT, by working in MTV (which is consistent if MT is consistent). In MTV we can use the procedure described in [13], p. 366: add a new constant N with axioms that say that N is a well-founded model of MT. We can prove that this theory is a conservative extension of MTV, because the reflection principle 10.2 is valid for MTV.

We start with the definition of notions of forcing and generic classes.

DEFINITION 16.1 (2.1). (i) A partially ordered system $[C, \leqslant_c]$ with largest element 1_c is called a *notion of forcing*. The elements of C are called *conditions*. p, q, r, s, t will be used for conditions. If $p, q \in C$ and $p \leqslant_c q$, we say that p is an *extension* of q. If $p, q \in C$ and there is an $r \in C$ such that $r \leqslant_c p, q$, we say that p and q are *compatible*. $p \wedge q$ is the greatest lower bound of p and q if it exists.

(ii) A subclass D of C is *C-dense* if for every $p \in C$ there is an extension of p in D.

(iii) A subclass D of C is a *C-section* if every extension of a condition in D is in D.

DEFINITION 16.2 (2.2). Let M be a structure, $[C, \leqslant_c] \in M$, a notion of forcing in M.

(i) A subclass G of C is *C-generic over M* if the following conditions hold:

(G1) $1_c \in G$.

(G2) $p \in G \wedge q \geqslant p \rightarrow q \in G$.

(G3) $\forall p \forall q\big(p, q \in G \rightarrow \exists r(r \in G \wedge r \leqslant p, q)\big)$.

(G4) For all C-dense classes $D \in M$, $G \cap D \neq 0$.

(ii) G is *strongly C-generic over M* iff G is C-generic and

(G5) for all ordinals $\beta \in M$ there is a set $C' \in V^M$ such that $C' \subseteq C$, and for all sequences of dense sections $[D_\alpha\colon\ \alpha \in \beta]$, there is a $q \in G$ satisfying

$$\forall \alpha\big(\alpha \in \beta \rightarrow \exists p\big(p \in C' \cap G \wedge (p \wedge q) \text{ exists} \wedge (p \wedge q) \in D_\alpha\big)\big).$$

It is clear that if C is a set in M, then every C-generic set G is strongly C-generic.

LEMMA 16.3 (2.3). *Let M be a countable set, $[C, \leqslant]$ a notion of forcing and $p \in C$. Then there is a class G which is C-generic over M and such that $p \in G$.*

DEFINITION 16.4 (2.4). Let D be a subclass of C in M, $p \in C$. Then D is *C-dense below p* if every extension of p has an extension in D.

LEMMA 16.4 (2.5), (2.6). (i) *Let G be C-generic over M, $p \in G$ and let D be C-dense in M below p. Then $G \cap D \neq 0$.*

(ii) *Let D be a section in M. If $D \cap G \neq 0$ for every G which is C-generic over M, then D is dense.*

We now define the model $M[G]$ for any model M, and G, C-generic over M.

DEFINITION 16.5 (III). Suppose that $[C, \leqslant]$ is a notion of forcing in a countable transitive model M of MT and that G is C-generic over M. Define:

(i) $a \in_G b \leftrightarrow \exists p(p \in G \wedge \langle a, p\rangle \in b)$.

(ii) $\bar{b} = K_G(b) = \{K_G(a)\colon\ a \in_G b\}$.

(iii) $M[G] = \{K_G(a)\colon\ a \in M\}$.

(iv) $\hat{b} = \{\langle \hat{a}, 1_c\rangle\colon\ a \in b\}$.

(ii) is a definition by recursion on $\in_G$ that is obviously well-funded. (iv) is by recursion on $\in$ and it can be done in M, since M is a model of TM.

It is easy to see that

$$K_G(\hat{b}) = b.$$

Hence

$$M \subseteq M[G].$$

Also, if $G' = \{\langle \hat{p}, p\rangle : p \in C\}$, then $G' \in M$ and $K_G(G') = G$. Hence $G \in M[G]$.

We now introduce the concept of forcing. From now on, we take M to be a countable transitive model of MT.

DEFINITION 16.6 (4.6). Let $\varphi, \varphi_1, \varphi_2$ be formulas with free variables among $v_1, \ldots, v_n$, C a notion of forcing, $p \in C$. We assume that r, q, s, s', s'' range over C.

(i) $p \Vdash X_1 = X_2$ iff $\forall x (\forall q \leqslant p)(\exists r \geqslant q)[(\exists s \geqslant r)(\langle x, s\rangle \in X_1 \cup X_2) \rightarrow (\exists s' \geqslant r)(\exists s'' \geqslant r)\exists y \exists y' (\langle y, s'\rangle \in X_1 \wedge \langle y', s''\rangle \in X_2 \wedge r \Vdash x = y \wedge r \Vdash x = y')]$;

(ii) $p \Vdash X_1 \in X_2$ iff $(\forall q \leqslant p)(\exists r \leqslant q)(\exists s \geqslant r)\exists x(\langle x, s\rangle \in X_2 \wedge r \Vdash x = X_1)$;

(iii) $p \Vdash \neg\varphi(X_1, \ldots, X_n)$ iff $(\forall q \leqslant p)\neg(q \Vdash \varphi(X_1, \ldots, X_n))$;

(iv) $p \Vdash (\varphi_1 \wedge \varphi_2)(X_1, \ldots, X_n)$ iff $p \Vdash \varphi_1(X_1, \ldots, X_n) \wedge p \Vdash \varphi_2(X_1, \ldots, X_n)$;

(v) $p \Vdash \forall v_0 \varphi(X_1, \ldots, X_n)$ iff $\forall X(p \Vdash \varphi(X, X_1, \ldots, X_n))$.

Part (i) of the definition is done by recursion on $\varrho X_1 \cup \varrho X_2$. The rest is done by recursion on formulas. This definition is essentially 4.6 of [1] but part (a) of 4.6 contains a small error that has been corrected here.

If M is a model of MT, then 16.6 can be formulated inside M. When there is a possibility of confusion we shall write $p \Vdash_M \varphi$ if we think of the definition in M. An immediate corollary of the definition is the following:

THEOREM 16.7 (4.3). *If φ is a formula with free variables $v_1, \ldots, v_n$ and $[C, \leqslant]$ is a notion of forcing in M, then there is a formula ψ_φ with free variables $v_0, \ldots, v_{n+1}$ such that for every $p \in C$ and every $X_1, \ldots, X_n \in M$,*

$$p \Vdash_M \varphi(X_1, \ldots, X_n) \quad \textit{iff} \quad M \Vdash \psi_\varphi[p, X_1, \ldots, X_n, C, \leqslant].$$

We shall sketch the proof by recursion of the following theorem:

THEOREM 16.8 (4.2), (4.4). *Let φ be a formula with free variables among $v_1, \ldots, v_n$, $[C, \leqslant]$ a notion of forcing in M and $p \in C$. Then for every $X_1, \ldots, X_n \in M$ we have:*

(i) *$p \Vdash_M \varphi(X_1, \ldots, X_n)$ iff for every generic G with $p \in G$, $M[G] \models \varphi[\bar{X}_1, \ldots, \bar{X}_n]$.*

(ii) *$M[G] \models \varphi[\bar{X}_1, \ldots, \bar{X}_n]$ iff there is a $p \in G$, such that $p \Vdash_M \varphi(X_1, \ldots, X_n)$.*

(iii) *If $p \Vdash_M \varphi(X_1, \ldots, X_n)$ and $q \leqslant p$, then $q \Vdash_M \varphi(X_1, \ldots, X_n)$.*

Proof. (iii) is an immediate corollary of (i). Also, (ii) from right to left is a corollary of (i). Hence it will be enough to prove simultaneously by induction on formula, (i) and

(ii′) If $M[G] \models \varphi[\bar{X}_1, \ldots, \bar{X}_n]$, then there is a $p \in G$ such that $p \Vdash_M \varphi(X_1, \ldots, X_n)$.

We shall do here only the case where φ is $v_1 = v_2$, where some small corrections have to be made to the proof in [1]. The rest of the proof is on pages 7 and 8 of [1].

Hence let us suppose that φ is $v_1 = v_2$. The proof for this case is done by induction on $\varrho X_1 \cup \varrho X_2$. Thus suppose (i), (ii′) are true for sets of rank less than $\varrho X_1 \cup \varrho X_2$. We have

$$M[G] \models v_1 = v_2[\bar{X}_1, \bar{X}_2] \text{ iff } \bar{X}_1 = \bar{X}_2,$$
$$\text{iff } (\forall x \in M)[x \in_G X_1 \cup X_2 \to (\exists y \in M)(\exists y' \in M)$$
$$(y \in_G X_1 \wedge y' \in_G X_2 \wedge \bar{y} = \bar{y}' = \bar{x})].$$

By the induction hypotheses on (ii′), this is equivalent to:

$$(*) \quad (\forall x \in M)[(\exists s \in G)(\langle x, s\rangle \in X_1 \cup X_2) \to (\exists r \in G)(\exists s' \geqslant r)(\exists s'' \geqslant r)(\exists y \in M)(\exists y' \in M)(\langle y, s'\rangle \in X_1 \wedge \langle y, s''\rangle \in X_2 \wedge r \Vdash x = y = y')].$$

Proof of (i). (1) Suppose, in the first place, that $p \Vdash X_1 = X_2$, G is generic with $p \in G$, $x \in M$, and $(\exists s \in G)(\langle x, s\rangle \in X_1 \cup X_2)$. Let $q \leqslant p, s$ with $q \in G$. Define in M:

$$D = \{r\colon r \leqslant q \wedge [(\exists s \geqslant r)(\langle x, s\rangle \in X_1 \cup X_2) \to (\exists s' \geqslant r)(\exists s'' \geqslant r)$$
$$\exists y \exists y'(\langle y, s'\rangle \in X_1 \wedge \langle y', s''\rangle \in X_2 \wedge r \Vdash y = y' = x)]\}.$$

Since $p \Vdash X_1 = X_2$, D is C-dense in M below q.

Using 16.4, we get an $r \in D \cap G$. Then $r \leqslant q \leqslant s$.

Hence $(\exists s \geqslant r)(\langle x, s\rangle \in X_1 \cup X_2)$. Then

$$(\exists s' \geqslant r)(\exists s'' \geqslant r)(\exists y \in M)(\exists y' \in M)(\langle y, s'\rangle \in X_1 \wedge \langle y', s''\rangle \in X_2 \wedge r \Vdash y = y' = x).$$

This proves $(*)$ for G.

(2) Suppose now that, for every generic G with $p \in G$, $(*)$ is satisfied, and let $q \leqslant p$.

Case 1. $\neg(\exists s \geqslant q)(\langle x, s\rangle \in X_1 \cup X_2)$. In this case we take $r = q$ in the definition of $p \Vdash X_1 = X_2$.

Case 2. $(\exists s \geqslant q)(\langle x, s\rangle \in X_1 \cup X_2)$. Let G be a generic class with $q \in G$ (its existence is assured by 16.3). Since $q \leqslant s, p$, we have $s, p \in G$. Therefore

$$(\exists s \in G)(\langle x, s\rangle \in X_1 \cup X_2).$$

Then by $(*)$:

$$(\exists r \in G)(\exists s' \geqslant r)(\exists s'' \geqslant r)(\exists y \in M)(\exists y' \in M)(\langle y, s'\rangle \in X_1 \wedge \langle y', s''\rangle \in X_2 \wedge r \Vdash y = y' = x).$$

It is clear that we can take $r \leqslant q$, and the assertion is proved.

Proof of (ii'). Suppose G is generic and satisfies:

$$M[G] \models v_1 = v_2[\bar{X}_1, \bar{X}_2].$$

Then G satisfies (*). Define in M

$$\begin{aligned} D = \{p:\ \exists x(\forall r \leqslant p)[(\exists s \geqslant r(\langle x, s\rangle \in X_1 \cup X_2) \wedge \\ \wedge(\forall s' \geqslant r)(\forall s'' \geqslant r)\forall y \forall y'(r \Vdash y = y' = x \\ \to (\langle y, s'\rangle \notin X_1 \vee \langle y', s''\rangle \notin X_2)] \vee p \Vdash X_1 = X_2)\}. \end{aligned}$$

We infer that D is C-dense, because if no $q \leqslant p$ satisfies the first disjunct then $p \Vdash X_1 = X_2$. Thus, let $p \in D \cap G$.

Suppose

$$\begin{aligned} (+) \quad & (\exists x \in M)(\forall r \leqslant p)[(\exists s \geqslant r)(\langle x, s\rangle \in X_1 \cup X_2) \wedge \\ & \wedge (\forall s' \geqslant r)(\forall s'' \geqslant r)(\forall y \in M)(\forall y' \in M)(r \Vdash y = y' = x \\ & \to \langle y, s'\rangle \notin X_1 \vee \langle y', s''\rangle \notin X_2)]. \end{aligned}$$

Then

$$(++) \qquad (\exists x \in M)(\exists s \geqslant p)(\langle x, s\rangle \in X_1 \cup X_2).$$

Suppose now that

$$\begin{aligned} (\exists r \in G)(\exists s' \in G)(\exists s'' \in G)(\exists y \in M)(\exists y' \in M)(\langle y, s'\rangle \in X_1 \wedge \langle y', s''\rangle \\ \in X_2 \wedge r \models y = y' = x). \end{aligned}$$

Then we can take $r \leqslant p, s'$, contradicting $(+)$. Hence we have

$$\begin{aligned} (\forall r \in G)(\forall s' \in G)(\forall s'' \in G)(\forall y \in M)(\forall y' \in M)(r \Vdash y \\ = y' = x \to \langle y, s'\rangle \notin X_1 \vee \langle y', s''\rangle \notin X_2), \end{aligned}$$

i.e.,

$$(\forall y \in M)(\forall y' \in M)[(\exists r \in G)(r \Vdash y = y' = x) \to y \notin_G X_1 \vee y' \notin_G X_2],$$

and by (i)

$$(\forall y \in M)(\forall y' \in M)(\bar{y} = \bar{y}' = \bar{x} \to y \notin_G X_1 \vee y' \notin_G X_2).$$

Therefore $\bar{x} \notin \bar{X}_1 \vee \bar{x} \notin \bar{X}_2$. But, by $(++)$, $\bar{x} \in \bar{X}_1 \vee \bar{x} \in \bar{X}_2$. Hence $\bar{X}_1 \neq \bar{X}_2$, contradicting (*). Thus p has to satisfy the second disjunct of the definition of D, i.e., $p \Vdash X_1 = X_2$.

It is now possible to prove the following theorem.

THEOREM 16.9 (5.1). *Let M be a countable transitive model of* MTE, *$[C, \leqslant]$ a notion of forcing in M, G a subclass of C which is strongly C-generic over M. Then $M[G]$ is a countable transitive model of* MTE, *with the same ordinals as M, which includes M and contains G; it is the smallest such model.*

It is clear that $M[G]$ is countable and transitive and hence satisfies the axioms of extensionality, regularity, and empty set. The proof of the rest of the theorem is obtained from the following lemmas, whose proof I shall only sketch.

LEMMA 16.10 (5.2). *For all $a \in M[G]$, if there is a $\bar{b} \in M[G]$ with $a \in \bar{b}$, then $a \in \overline{V^M}$. Furthermore, there is a $c \in V^M$ such that $\bar{c} = a$.*

COROLLARY 16.11 (5.3).

$$V^{M[G]} = \overline{V^M}.$$

LEMMA 16.12 (5.4). *Let φ be a formula with free variables $v_0, \ldots, v_n$ and $\bar{X}_1, \ldots, \bar{X}_n \in M[G]$. Then $B = \{a\colon a \in M[G] \wedge a \in \overline{V^M} \wedge M[G] \models \varphi[a, \bar{X}_1, \ldots, \bar{X}_n]\} \in M[G]$.*

Proof. We prove that $A = \{\langle b, p\rangle\colon b \in V^M \wedge p \Vdash \varphi(b, X_1, \ldots, X_n)\}$ is such that $A \in M$ and $\bar{A} = B$.

LEMMA 16.13 (5.5).

$$a \in V^{M[G]} \rightarrow \exists b \exists C' \big(C' \subseteq C \wedge b, C' \in V^M \wedge \mathscr{P}a \cap M[G] \subseteq K_G^*(\mathscr{P}(\mathscr{D}b^{-1} \times C')) \cap M\big).$$

Proof. Let $b \in V^M$ with $\bar{b} = a$ and let $g \in {}^{\beta}\mathscr{D}b^{-1}$ be an enumeration of b in V^M.

We use strong genericity over sequences of dense sections of length β given, for each A with $\bar{A} \subseteq a$, by

$$D_{a,A} = \{p\colon p \Vdash g\text{`}a \in A \vee p \Vdash g\text{`}a \notin A\}.$$

We obtain $C', q_A \in G$ for each $[D_{\alpha,A}\colon \alpha \in \beta]$, and set

$$c = \{\langle d, p\rangle\colon \langle d, p\rangle \in \mathscr{D}b^{-1} \times C' \wedge (p \wedge q_A \Vdash d \in A)\}.$$

Then $\bar{c} = \bar{A}$ and $c \in \mathscr{P}(\mathscr{D}b^{-1} \times C') \cap M$.

LEMMA 16.14 (5.5).

$$a \in V^{M[G]} \wedge B \in \mathscr{P}a \cap M[G] \rightarrow B \in V^{M[G]}.$$

Lemma 16.13 is the only place where we use the existence of a C' that works for all sequences of a given length. 16.13 and 16.14 are the essential lemmas for proving the power set axiom.

LEMMA 16.15 (5.6).

$$A \in M[G] \wedge \exists a(a \in V^M \wedge A \subseteq K_G^* a) \rightarrow A \in V^{M[G]}.$$

Proof. We take $d = a \times \{1\}$. Then $d \in V^M$; hence $\bar{d} \in V^{M[G]}$ and $A \subseteq \bar{d}$.

From 16.15, 16.13 we immediately derive

LEMMA 16.16 (5.7).

$$a \in V^{M[G]} \rightarrow \bigcup a \in V^{M[G]}.$$

LEMMA 16.17 (5.8).

$$a \in V^{M[G]} \to \mathscr{P} a \cap M[G] \in V^{M[G]}.$$

The next lemma proves our version of the axiom of replacement, which is slightly different from that of [1].

LEMMA 16.18 (5.9).

$$\bar{A} \in M[G] \wedge \bar{b} \in V^{M[G]} \wedge \forall x \big(x \in \bar{A} \to \exists y \big(y \in \bar{b} \wedge y \in x \wedge \wedge x = \mathscr{U} z (y \in z \wedge z \in \bar{A})\big)\big) \to \bar{A} \in V^{M[G]}.$$

Proof. We may assume that $b \in V^M$. Let $g \in {}^{\beta}\mathscr{D} b^{-1}$ be an enumeration of $\mathscr{D} b^{-1}$ in V^M and let φ be $v_3 \in v_1 \wedge v_2 \in v_3$. Define:

$$D_a = \{p\colon \exists x \big(p \Vdash (\exists v_3 \varphi \to \varphi)(A, g^{\cdot} a, x)\big)\}.$$

Then $[D_a\colon a \in \beta]$ is a sequence of C-dense sections in M, by 16.4. Hence, by strong genericity, we find the appropriate $q \in G$ and $C' \in V^M$.

Let

$$A_{c,p} = \{x\colon p \wedge q \Vdash (\exists v_3 \varphi \to \varphi)(A, c, x)\}$$

and

$$f \in \Pi \langle A_{c,p}\colon \langle c, p\rangle \in (\mathscr{D} b^{-1} \times C') \wedge A_{c,p} \neq 0\rangle.$$

Then

$$f^*(\mathscr{D} b^{-1} \wedge C') \in V^M.$$

We shall prove

$$\bar{A} \subseteq K_G^*\big(f^*(\mathscr{D} b^{-1} \times C')\big).$$

If $d \in \bar{A}$, then $d = \mathscr{U} z(\bar{c} \in z \wedge z \in \bar{A})$ for a certain $c \in \mathscr{D} b^{-1}$; hence there is a p such that $A_{c,p} \neq 0$.

Let $x = f^{\cdot}\langle c, p\rangle$. Then $p \wedge q \Vdash (\exists v_3 \varphi \to \varphi)(A, c, x)$. Hence $M[G] \models (\exists v_3 \varphi \to \varphi)[\bar{A}, \bar{c}, \bar{x}]$. But $d \in \bar{A} \wedge \bar{c} \in d$: thus

$$M[G] \models \exists v_3 \varphi[\bar{A}, \bar{c}].$$

Hence

$$\bar{x} \in \bar{A} \wedge \bar{c} \in \bar{X},$$

i.e.,

$$\bar{x} = d,$$

because

$$x \in f^*(\mathscr{D} b^{-1} \times C'), \qquad d \in K_G^*\big(f^*(\mathscr{D} b^{-1} \times C')\big).$$

We then use 16.15 to obtain $\bar{A} \in V^{M[G]}$.

LEMMA 16.19 (5.10). *If N is a model of* MT *which includes M and* *contains G, then there is an operation K defined in N such that, for all $X \in M$.*

$$K(G, X) = K_G(X).$$

Hence there is a function in N whose restriction to V^M is $K_G \upharpoonright V^M$, Moreover, $M[G] \subseteq N$.

Proof. Define in N, by recursion on ϱY, for each $A \in N$:

$$K(A, Y) = \{K(A, x) \colon (\exists p \in A)(\langle x, p\rangle \in Y)\}.$$

It is easy to prove by recursion, using the transitivity of M, that for $X \in M$

$$K(G, X) = K_G(X).$$

The rest of the theorem follows.

LEMMA 16.20 (5.11).

$$\mathrm{OR}^M = \mathrm{OR}^{M[G]}.$$

LEMMA 16.21 (5.12).

$$M[G] \models \mathrm{E}.$$

With this lemma we complete the proof of Theorem 16.9.

We recall some properties of $\mathscr{V} \simeq \mathscr{L}$ (Theorem 11.4). In MTL it is possible to define the following operations and collections:

(1) TW′ is defined by recursion. TW is an operation derived form TW′ that defines w.o. types.

(2) $\mathscr{T}$ is the collection of types, and we have:

$$\mathscr{T}(R) \leftrightarrow R = \mathrm{TW}'(R).$$

(3) Q is a unary operation such that

$$\mathrm{MT}L \vdash \forall X \exists R \big(\mathscr{T}(R) \wedge X = Q(R)\big).$$

DEFINITION 16.22. Let N be a model of MT. We say that $N \models \mathrm{WO}^+$ iff the following condition is satisfied:

It is possible to define in N a unary operation $\overline{\mathrm{TW}}$ and a binary operation Q^+ such that $\overline{\mathrm{TW}}$ defines w.o. types in N, and

$$N \models \exists Y \forall X \exists R \big(R = \overline{\mathrm{TW}}(R) \wedge Q^+(Y, R) = X\big).$$

THEOREM 16.23. *If $M \models \mathrm{MT}L$, then $M[G] \models \mathrm{WO}^+$ and $\mathscr{T}^M = \mathscr{T}^{M[G]}$.*

Proof. It is clear, since $M \subseteq M[G]$, that all types in M are in $M[G]$. We prove by recursion that the types in M are well-orderings in $M[G]$.

Since TW′ is defined by recursion from an absolute operation, for a well-ordering R in $M[G]$ that belongs to M, we have

$$\mathrm{TW}^M(R) = \mathrm{TW}^{M[G]}(R).$$

Therefore

$$\mathscr{T}^M \subseteq \mathscr{T}^{M[G]}.$$

Let Q be an operation in M such that

$$M \models \forall X \exists R(\mathscr{T}(R) \wedge X = Q(R)).$$

By 16.19, there is a binary operation K such that, for $X \in M$,

$$K(G, X) = K_G(X).$$

Hence, for R with $\mathscr{T}^M(R)$, let

$$H(G, R) = K(G, Q(R)).$$

Then itis clear that

$$M[G] \models \exists Y \forall X \exists R(\mathscr{T}(R) \wedge X = H(Y, R)),$$

i.e.,

$$M[G] \models \mathrm{WO}^+.$$

We shall now prove that TW defines types for all well-orderings of $M[G]$.

Suppose there is a well-ordering $S \in M[G]$ such that $S \succ R$ for all types R. Take S to be the least such well-ordering.

Define in $M[G]$:

$$A = \{x\colon x \notin H(G, \mathrm{TW}(S_x))\}.$$

Then

$$A = H(G, \mathrm{TW}(S_y)) \quad \text{for a certain } y \in \mathscr{D}S.$$

Hence $y \in A \leftrightarrow y \notin A$, a contradiction. Therefore we have proved WO^+ in $M[G]$ and $\mathscr{T}^M = \mathscr{T}^{M[G]}$.

17. Products of notions of forcing and coherent notions. This section is based on Shoenfield [13], Section 8, and [1], Section IV. We start with products of notions of forcing. We assume that M is a countable transitive model of MT.

DEFINITION 17.1. Let $[C_1, \leqslant_1]$, $[C_2, \leqslant_2]$ be notions of forcing. Then

$$[C_1, \leqslant_1] \otimes [C_2, \leqslant_2] = [C, \leqslant]$$
$$\leftrightarrow C = C_1 \times C_2 \wedge (\langle p_1, q_1\rangle \leqslant \langle p_2, q_2\rangle \leftrightarrow p_1 \leqslant_1 p_2 \wedge q_1 \leqslant_2 q_2).$$

When speaking of notions of forcing, we shall write $C_1 \times C_2$ instead of $[C_1, \leqslant_1] \otimes [C_2, \leqslant_2]$.

THEOREM 17.2 (Product theorem). *Let $[C_1, \leqslant_1]$, $[C_2 \leqslant_2]$ be notions of forcing in M. Suppose that every $C_1 \times C_2$-generic class over M is strongly* *generic. Then, if G_1 is C_1-generic over M and G_2 is C_2-generic over $M[G_1]$, then $G_1 \times G_2$ is strongly $C_1 \times C_2$-generic over M and $M[G_1 \times G_2] = M[G_1][G_2]$. Moreover, every set which is strongly $C_1 \times C_2$-generic over M is obtained in this way, every C_1-generic class over M is strongly C_1-generic, and every C_2-generic class over $M[G_1]$ is strongly C_2-generic.*

Proof. It is easy to verify (G1), (G2), and (G3) for $G_1 \times G_2$. Let D be a $C_1 \times C_2$-dense class in M. We must show that

$$(G_1 \times G_2) \cap D \neq 0.$$

Let

$$D_2 = \{p_2\colon (\exists p_1 \in G_1)(\langle p_1, p_2 \rangle \in D)\}.$$

It will be enough to show that $D_2 \cap G_2 \neq 0$, i.e., since $D_2 \in M[G_1]$, that D_2 is C_2-dense.

Let $q_2 \in C_2$; we must find $p_2 \leqslant q_2$ and $p_1 \in G_1$ such that $\langle p_1, p_2 \rangle \in D$. Thus, if we define

$$D_1 = \{p_1\colon (\exists p_2 \leqslant_2 q_2)(\langle p_1, p_2 \rangle \in D)\},$$

we must show that $D_1 \cap G_1 \neq 0$, i.e., since $D_1 \in M$, that D_1 is C_1-dense.

Let $q_1 \in C_1$ and choose $\langle p_1, p_2 \rangle \leqslant \langle q_1, q_2 \rangle$ such that $\langle p_1, p_2 \rangle \in D$. Then $p_1 \leqslant q_1$ and $p_1 \in D_1$.

Therefore we have proved that $G_1 \times G_2$ is $C_1 \times C_2$-generic over M, and hence strongly $C_1 \times C_2$-generic.

We shall now prove that every (strongly) generic subclass of $C_1 \times C_2$ is obtained as a product of strongly generic subclasses of C_1, C_2.

Let G be (strongly) generic over M and let G_1, G_2 be the projections of G on C_1, C_2, respectively.

Clearly, $G \subseteq G_1 \times G_2$. On the other hand, let $\langle p_1, p_2 \rangle \in G_1 \times G_2$. Then for some $q_1, q_2, \langle p_1, q_2 \rangle, \langle q_1, p_2 \rangle \in G$. Hence, we have a common extension $\langle r_1, r_2 \rangle$. Since $\langle r_1, r_2 \rangle \leqslant \langle p_1, p_2 \rangle$, we have $\langle p_1, p_2 \rangle \in G$. Thus $G_1 \times G_2 = G$.

We have to prove now that G_1 is strongly C_1-generic over M and G_2 is strongly C_2-generic over $M[G_1]$. The verification of (G1), (G2), (G3) for G_1, G_2 is easy. We shall verify (G5) leaving a similar verification of (G4), to the reader (see [13], p. 367).

Let β be given and let $C' \subseteq C_1 \times C_2$ satisfy (G5) for G and β. Let $C' = C_1' \times C_2'$, where C_1', C_2' are the respective projections. We shall prove that C_1', C_2' satisfy (G5) for G_1, G_2, respectively, and β.

Let $[D_{\alpha 1}\colon \alpha \in \beta]$ be a sequence of C_1-dense sections in M. Then $[D_{\alpha 1} \times C_2\colon \alpha \in \beta]$ is a sequence of $C_1 \times C_2$-dense sections in M. Thus, there is a $q = \langle q_1, q_2 \rangle \in G$ which satisfies (G5) for G and this sequence. Hence, q_1 and C_1' satisfy (G5) for G_1 and $[D_{\alpha 1}\colon \alpha \in \beta]$.

We shall prove (G5) for G_2. Let $[D_{\alpha 2}\colon \alpha \in \beta]$ be a sequence of C_2-

dense sections in $M[G_1]$. Let

$$R = [D_{\alpha 2}\colon \alpha \in \beta].$$

It is clearly possible, from the definition of K_{G_1}, to find $R' \in M$ such that, for all $\alpha \in \beta$,

$$K_{G_1}(R'(\hat{\alpha})) = R^*\{\alpha\}.$$

Let φ be the formula that says 'v_0 is a dense section'.

We show that

$$D_\alpha = \{\langle p_1, p_2\rangle\colon p_1 \models (\varphi \rightarrow v_1 \in v_0)(R'(\hat{\alpha}), \hat{p}_2)\}$$

is a $C_1 \times C_2$-dense section. It is easy to see that it is a section. We show that it is dense. Let $\langle q_1, q_2\rangle$ be given, and G', C_1-generic over M with $q_1 \in G'$. If $K_{G'}(R'(\hat{\alpha}))$ is C_2-dense, choose $p_2 \leqslant q_2$ so that $p_2 \in K_{G'}(R'(\hat{\alpha}))$; otherwise let $p_2 = q_2$. In either case $M[G'] \models (\varphi \rightarrow v_1 \in v_0)[\overline{R'\{\hat{\alpha}\}}, \dot{p}_2]$. Hence a certain $p_1 \in G'$ forces $(\varphi \rightarrow v_1 \in v_0)(R'(\hat{\alpha}), \hat{p}_2)$ with $p_1 \leqslant q_1$. Then

$$\langle p_1, p_2\rangle \leqslant \langle q_1, q_2\rangle \quad \text{and} \quad \langle p_1, p_2\rangle \in D_\alpha.$$

Therefore $[D_\alpha\colon \alpha \in \beta]$ is a sequence of $C_1 \times C_2$-dense sections in M. Thus, by (G5) for G, there is a $q \in G$ such that

$$(\forall \alpha \in \beta)(\exists p \in C' \cap G)(p \wedge q \in D_\alpha).$$

Let $p = \langle p_1, p_2\rangle$, $q = \langle q_1, q_2\rangle$. Then

$$p \wedge q = \langle p_1 \wedge q_1, p_2 \wedge q_2\rangle.$$

We have $p_1 \wedge q_1 \in G_1$. Since $p \wedge q \in D_\alpha$,

$$p_1 \wedge q_1 \Vdash (\varphi \rightarrow v_1 \in v_0)(R'(\hat{\alpha}), \widehat{p_2 \wedge q_2}).$$

Hence

$$M[G_1] \models (\varphi \rightarrow v_1 \in v_0)[R^*\{\alpha\}, p_2 \wedge q_2].$$

But

$$M[G_1] \models \varphi[R^*\{\alpha\}];$$

thus

$$p_2 \wedge q_2 \in R^*\{\alpha\} = D_{\alpha,2}.$$

Also

$$p_2 \in C_2' \cap G_2,$$

therefore q_2 satisfies (G5) for $D_{\alpha 2}$ and G_2.

We shall write $M[G_1, G_2]$ for $M[G_1], [G_2]$.

COROLLARY 17.3. *Let C_1 and C_2 be notions of forcing in M. Suppose that every $C_1 \times C_2$-generic class over M is strongly generic. Let G_1 be C_1-*

generic over M, and G_2, C_2-generic over $M[G_1]$. Then G_1 is strongly C_1-generic over $M[G_2]$, G_2 is strongly C_2-generic over $M[G_1]$ (and hence over M), and

$$M[G_2, G_1] = M[G_1, G_2].$$

The following type of infinite product (the weak power) will be the only one considered:

Definition 17.4. Let $[C, \leqslant]$ be a notion of forcing and let b a set. Then $[C, \leqslant]^b = [C^b, \leqslant^b]$, where:

(i) $C^b = \{p: p \in {}^bC \wedge \{i: i \in b \wedge p`i \neq 1_c\}$ is finite$\}$,

(ii) $p \leqslant^b q \leftrightarrow \forall i(i \in b \to p`i \leqslant q`i)$.

It is easy to see that if $[C, \leqslant]$, $b \in M$, then $[C, \leqslant]^b$ is a notion of forcing *in M*.

It is clear that if C is a notion of forcing in M and $F \in M$ a one-one function with domain C, then F^*C is also a notion of forcing with the obvious ordering. Also, for any strongly C-generic class G, F^*G is also strongly C-generic and

$$M[G] = M[F^*G].$$

Hence, if $b = d \cup c$ and $d \cap c = 0$ with $d, c \in M$, then C^b is isomorphic to $C^d \times C^c$ with the corresponding orderings. In particular, if $i \in b$ and $d = b \sim \{i\}$, then C^b is naturally isomorphic to $C^d \times C$. In either case, we can apply the product theorem.

We shall now introduce coherent notions of forcing. The number of the corresponding theorem or definition in [1] will appear in parentheses. Since we are not interested in cardinality problems in this paper, some proofs may be somewhat simpler.

Definition 17.5 (6.2).

(i) Let $[C, \leqslant]$ be a notion of forcing. Then $[C, \leqslant]$ is *coherent* iff:

(i′) $C = \bigcup\{C_\xi: \xi \in \mathrm{OR}\}$, where $C_\xi \in V$ and

$$\forall\xi\forall\eta(\xi, \eta \in \mathrm{OR} \wedge \xi \subseteq \eta \to C_\xi \subseteq C_\eta).$$

(i″) For every $\xi \in \mathrm{OR}$, there is a subclass C^ξ of C such that if $p \in C_\xi$ and $q \in C^\xi$, $p \wedge q$ exists and the mapping F_ξ, defined by $F_\xi`\langle q, r\rangle = q \wedge r$ for every $\langle q, r\rangle \in C_\xi \times C^\xi$, is an isomorphism of $[C_\xi \times C^\xi, \leqslant']$ onto $[C, \leqslant]$, where $\langle p, q\rangle \leqslant' \langle r, s\rangle$ iff $p \leqslant r$ and $q \leqslant s$.

(i‴) If p is compatible with q, the $p \wedge q$ exists.

(ii) Let $[C, \leqslant]$ be a coherent notion of forcing. Define for $\xi \in \mathrm{OR}$ and $p \in C$, $(p)_\xi$ and $(p)^\xi$ by the condition

$$F_\xi`\langle (p)_\xi, (p)^\xi\rangle = p.$$

Lemma 17.6 (6.4). *If $[C, \leqslant]$ is a coherent notion of forcing in M and G is C-generic over M, then $G \cap C_\xi = G_\xi$ is C_ξ-generic over M for every $\xi \in \mathrm{OR}^M$.*

DEFINITION 17.7 (6.5). (i) $|a| = \bigcap\{\xi : \xi \in \mathrm{OR} \wedge \xi \approx a\}$ ($|a|$ is the *cardinal* of a), we shall use m, n for cardinals, i.e., initial ordinals.

(ii) C is *m-closed* iff for every linearly ordered set $b \subseteq C$ such that $|b| \subset m$, there is a $p \in C$ such that $p < q$ for every $q \in b$.

(iii) If m is a cardinal, m^+ is the *successor cardinal* of m.

(iv) C *satisfies the m-chain condition* iff, for every subclass E of incompatible elements of C, we have $|E| \subset m$.

THEOREM 17.8 (6.10). *Let $[C, \leqslant]$ be a coherent notion of forcing in M, m an increasing sequence of cardinals in M cofinal with OR^M. Suppose, furthermore, that for every $\xi \in \mathrm{OR}^M$, C^ξ is m_ξ^+-closed and C_ξ satisfies the m_ξ^+-chain condition.*

Then, every C-generic class over M is strongly C-generic over M.

The proof is obtained, as that of 6.10 in [1], from 6.7 and 6.8 of [1]. Notice that it is possible to eliminate the set H in these theorems for the proof of 17.8 (and 6.10).

THEOREM 17.9. *Let $[C, \leqslant]$ be a coherent notion of forcing in M that is a proper class in M, and let G be strongly C-generic over M. Then:*

If $x \in V^{M[G]}$, then there is a $\xi \in \mathrm{OR}^M$ such that $x \in V^{M[G_\xi]}$ i.e.,

$$V^{M[G]} = \bigcup\{V^{M[G_\xi]} : \xi \in \mathrm{OR}^M\}.$$

Proof. It is clear that

$$\bigcup\{V^{M[G_\xi]} : \xi \in \mathrm{OR}\} \subseteq V^{M[G]}.$$

Let $x \in V^{M[G]}$. By 16.10, there is a $y \in V^M$ such that

$$\bar{y} = x.$$

For $p \in \mathrm{TC}(y)$, define

$$f'p = \bigcap\{\xi : \xi \in \mathrm{OR}^M \wedge p \in C_\xi\}.$$

We have $f \in M$ and hence

$$f^*\big(C \cap \mathrm{TC}(y)\big) \subseteq \eta \quad \text{for some } \eta \in \mathrm{OR}^M.$$

It is clear that $\bar{y} \in M[G_\eta]$ and also that $\bar{y}$ is a set in $M[G_\eta]$.

IX. Independence of axioms of choice

18. Automorphisms of notions of forcing. Sections 18 and 19 are based on [13], Section 9. Shoenfield, however, worked in ZF.

DEFINITION 18.1. F is an *automorphism* of the notion of forcing $[C, \leqslant]$ iff F is an isomorphism of $[C, \leqslant]$ onto itself.

We have the following obvious lemma:

LEMMA 18.2. *Let F be an automorphism of C which is in M. Then, for every strongly C-generic over M, G, F^*G is strongly C-generic over M and $M[G] = M[F^*G]$.*

DEFINITION 18.3. Let $\mathscr{A}$ be a collection of automorphism of C such that $\mathscr{A} \subseteq M$, and let $A \in M$. Then:

(i) A is $\mathscr{A}$-invariant $\leftrightarrow \forall F\big(\mathscr{A}(F) \to K_G(A) = K_{F*G}(A)\big)$;

(ii) C is $\mathscr{A}$-homogeneous $\leftrightarrow \forall p \forall q (p, q \in C \to \exists F \ \big(\mathscr{A}(F) \wedge (F^{-1}(p)$ and q are compatible$)\big)$;

(iii) C is homogeneous $\leftrightarrow C$ is $\mathscr{A}$-homogeneous, where $\mathscr{A}$ is the collection of all automorphism in M.

We have the following theorem:

THEOREM 18.4 (9.2 of [13]). *Let $\mathscr{A}$ be a collection of automorphisms of C such that $\mathscr{A} \subseteq M$; suppose that C is $\mathscr{A}$-homogeneous in M, that every generic subclass of C is strongly generic and that $X_1, \ldots, X_n$ are $\mathscr{A}$-invariant elements of M. Let φ be a formula with n free variables. Then*

$$M[G] \models \varphi[\bar{X}_1, \ldots, \bar{X}_n] \quad \textit{iff} \quad 1_c \Vdash \varphi(X_1, \ldots, X_n).$$

Proof. Since $1_c \in G$, $1_c \Vdash \varphi(X_1, \ldots, X_n)$ implies that

$$M[G] \models \varphi[\bar{X}_1, \ldots, \bar{X}_n].$$

Suppose now that $M[G] \models \varphi[\bar{X}_1, \ldots, \bar{X}_n]$ but $\neg(1_c \Vdash \varphi(X_1, \ldots, X_n))$. By the former, a certain $p \Vdash \varphi(X_1, \ldots, X_n)$; by the latter, a certain $q \Vdash \neg\varphi(X_1, \ldots, X_n)$. Choose F, with $\mathscr{A}(F)$ such that $F^{-1}\dot{p}$ and q are compatible. Let G' be a strongly generic class containing $F^{-1}\dot{p}$ and q; there are such G', by 16.3 and the hypotheses. By 18.2, F^*G' is strongly generic. Since $p \in F^*G'$, we have

$$M[F^*G'] \models \varphi[K_{F*G'}(X_1), \ldots, K_{F*G'}(X_n)]$$

and, since $q \in G'$,

$$M[G'] \models \neg\varphi[K_{G'}(X_1), \ldots, K_{G'}(X_n)].$$

But, by 18.2, $M[F^*G'] = M[G']$, and since $X_1, \ldots, X_n$ are $\mathscr{A}$-invariant,

$$K_{F*G'}(X_i) = K_{G'}(X_i) \quad \text{for } i = 1, \ldots, n.$$

Thus we get a contradiction.

It is clear that every element of M of the form $\hat{X}$ for a certain $X \in M$ is $\mathscr{A}$-invariant for any $\mathscr{A}$, because $K_G(\hat{X}) = X$ for every $X \in M$ and every generic G.

Hence, for all ordinals $\alpha \in M[G]$, $\hat{\alpha}$ is $\mathscr{A}$-invariant. The same is true for well-order types $R \in M[G]$ if $M \models \mathrm{MT}L$.

THEOREM 18.5 (9.3 of [13]). *Let $\mathscr{A}$ be a collection of automorphisms of C such that $\mathscr{A} \subseteq M$; suppose that C is $\mathscr{A}$-homogeneous, G a strongly C-generic class over M, $X_1, \ldots, X_n$ $\mathscr{A}$-invariant elements of M. Let τ be a term* *with n free variables $v_1, \ldots, v_n$ and let*

$$M[G] \models v_0 = \tau[\bar{X}, \bar{X}_1, \ldots, \bar{X}_n].$$

Then

$$\bar{X} \cap M \in M.$$

Moreover, if $\bar{X}$ is a set in $M[G]$, $\bar{X} \cap M$ is a set in M.

Proof. We have

$$\bar{y} \in \bar{X} \leftrightarrow M[G] \models v_0 \in \tau[\bar{y}, \bar{X}_1, \ldots, \bar{X}_n].$$

For $y \in M$, $y = \bar{\hat{y}}$. Hence

$$y \in \bar{X} \leftrightarrow M[G] \models v_0 \in \tau[\bar{\hat{y}}, \bar{X}_1, \ldots, \bar{X}_n].$$

By 18.4

$$y \in \bar{X} \cap M \leftrightarrow 1 \Vdash \hat{y} \in \tau(X_1, \ldots, X_n).$$

Thus $\bar{X} \cap M \in M$. If $\bar{X}$ is a set in $M[G]$, then

$$\varrho\bar{X} \in \mathrm{OR}^{M[G]} \quad \text{and} \quad \varrho(\bar{X} \cap M) \subseteq \varrho\bar{X}.$$

Since ϱ is absolute,

$$\bar{X} \cap M \text{ is a set } M.$$

Therefore, in particular, we have

COROLLARY 18.6. *If X is definable in $M[G]$ from elements of M and C is homogeneous, then $X \cap M \in M$.*

19. Independence of the local axiom of choice. In this section we shall construct a model N of MT in which the local axiom of choice, AC, is not valid. AC says that every set is equinumerous to an ordinal. It will be true in N that there is no mapping from the ordinals onto $\mathscr{S}(\omega)$. The proof if obtained by adapting the one found in [13] for ZF.

We consider the following notion of forcing:

DEFINITION 19.1.

(i) $C = \{p\colon \mathscr{D}p \subseteq \omega \wedge \mathscr{D}p$ is finite $\wedge\ \mathscr{D}p^{-1} \subseteq \{0,1\}\}$;

(ii) $p \leqslant q \leftrightarrow p \supseteq q$.

Notice that C is a set in M; thus every C-generic class is strongly generic.

LEMMA 19.2. *If G is C-generic over M, then $G \notin M$.*

Proof. We can see that for any $f \in {}^{\omega}2 \cap M$,

$$D_f = \{p\colon p \in C \wedge p \nsubseteq f\}$$

is C-dense in M. Thus $G \cap D_f \neq 0$ and hence $\bigcup G \neq f$ for every $f \in {}^{\omega}2 \cap M$.

Consider now C^{ω}. If $C \in M$, then $C^{\omega} \in M$. Let G be a C^{ω}-generic set over M. Let

$$G_i = \{p`i\colon p \in G\} \quad \text{and} \quad H = \{G_i\colon i \in \omega\}.$$

Suppose that $M \models \mathrm{MT}L$. Obtain $M[G]$. We have by 16.23 $M[G] \models \mathrm{WO}^+$ and $\mathscr{T}^M = \mathscr{T}^{M[G]}$. Construct the submodel N of $M[G]$ given by

$$N = (\mathscr{HOD}_{\mathscr{T},H})^{M[G]}.$$

(For the construction of $\mathscr{HOD}_{\mathscr{T},H}$, see 13.9.) We have

THEOREM 19.3. *$N \models \neg\mathrm{AC}$, in particular,*

$$N \models \neg\exists f(\mathscr{D}f \in \mathrm{OR} \wedge \mathscr{D}f^{-1} = \mathscr{S}(\omega)).$$

Proof. Suppose that there is an f such that $\mathscr{D}f \in \mathrm{OR}$ and for all $a \in N$ with $a \subseteq \omega$, $a = f`\alpha$ for some $\alpha \in \mathrm{OR}$. Then for a certain n, f is definable from order types and $G_0, \ldots, G_{n-1}$ in $M[G]$.

Hence every element of $\mathscr{S}(\omega) \cap N$ is definable from order types and $G_0, \ldots, G_{n-1}$.

We now regard C^ω as the product $C' \times C''$, where $C' = {}^nC$ and $C'' = C^{\omega \sim n}$. Let G', G'' be the projections of G on C', C'' respectively. By 17.2, G' is C'-generic over M, G'' is C''-generic over $M' = M[G']$, and $M'[G''] = M[G]$. Applying 17.2 again, G_n is C-generic over M'. Hence, if

$$a = \left(\bigcup G_n\right)^* \{i\},$$

then $a \subseteq \omega$, and $a \notin M'$. Since $a \cap M' = a$, $a \cap M' \notin M'$.

On the other hand, C'' is homogeneous over M' [because: If $p, q \in C''$, let π be a permutation of $\omega \sim n$ which moves the $i \in \omega$ such that $p(i) \neq 1$ (finite in number) to j such that $q(j) = 1$. Then $\bar{\pi}$ defined by

$$\left(\bar{\pi}(p)\right)`i = p`\pi`i$$

is the required automorphism].

Also, a is definable from elements of M'; hence

$$a \cap M' \in M'.$$

But this is a contradiction.

By using methods similar to these, but in a more complicated way, and applying them several times, it is possible to extend Solovay's result [15] to MT.

20. Independence of the global axiom of choice. The independence of E from NBG+AC was proved by Easton [2]. We shall give a simple proof that E is independent of MT+AC.

Let M be a model of MT such that $M \models \mathrm{MT}L$. Let

$$\langle m_\xi \colon \xi \in \mathrm{OR}^M \rangle$$

be the increasing sequence of all regular cardinals in M. Define in M:

DEFINITION 20.1.

(i) $C = \{p \colon \exists \xi(\mathscr{D}p \subseteq m_\xi \wedge \mathscr{D}p^{-1}) \subseteq 2 \wedge \forall \eta(|\mathscr{D}p \cap m_\eta| < m_\eta)\}$.

(ii) $(p)_\xi = p \upharpoonright m_\xi$.
(iii) $(p)^\xi = p \sim p \upharpoonright m_\xi$.
(iv) $(p)^\xi_\eta = (p)_\eta \sim (p)_\xi$.
(v) $C_\xi = \{(p)_\xi\colon p \in C\}$.
(vi) $C^\xi = \{(p)^\xi\colon p \in C\}$.
(vii) $C^\xi_\eta = \{(p)^\xi_\eta\colon p \in C\}$.
(viii) $p \leqslant q \leftrightarrow p \supseteq q$.

It is clear that C is a coherent notion of forcing and that C^ξ is m_ξ^+-closed. In a purely combinatorial way we can prove that C_ξ satisfies the m_ξ^+-chain condition (see [1], 8.2 or [13], 10.2). Hence every generic subclass of C is strongly generic (17.8).

Since C is coherent, we have that $C_\xi \times C^\xi$ is isomorphic with C.

It is also easy to prove that if $\eta \supset \xi$, C_η is isomorphic to $C_\xi \times C^\xi_\eta$.

Now let G be C-generic over M. Let $H = \{G_\xi\colon \xi \in \mathrm{OR}\}$, where $G_\xi = C_\xi \cap G$.

We know that G_ξ is C_ξ-generic over M by 17.6. Construct $M[G]$. Then $M[G]$ is a model of WO^+ and E in which $\mathscr{T}^M = \mathscr{T}^{M[G]}$ (16.23).

LEMMA 20.2. *Let $G^\xi_{\xi+1}$ be the projection of $G_{\xi+1}$ on $C^\xi_{\xi+1}$. Then $G^\xi_{\xi+1}$ is $C^\xi_{\xi+1}$-generic over $M[G_\xi]$ and $G^\xi_{\xi+1} \notin M[G_\xi]$.*

Proof. The first part of the conclusion is obtained from 17.2. The second part is proved similarly to 19.1.

Let

$$N = (\mathscr{H}O\mathscr{D}_{\mathscr{T}^M, H \cup V^M})^{M[G]}.$$

It is possible to define N, since H, $V^M \in M[G]$. It is clear that $N \models \mathrm{MT}$. We shall prove

THEOREM 20.3. (i) $V^{M[G]} = V^N$,

(ii) $N \models \mathrm{AC}$.

Proof. (ii) is obtained immediately from (i) since $M[G] \models \mathrm{AC}$.

Proof of (i). It is clear that $V^N \subseteq V^{M[G]}$. Let $x \in V^{M[G]}$. By 17.9, $x \in V^{M[G_\xi]}$ for a certain ξ. But then x is definable from elements of V^M and G_ξ. Hence $x \in V^N$.

THEOREM 20.4.

$$N \models \neg \mathrm{E}.$$

In particular, $N \models \neg\exists F\big(F$ *is a function* $\wedge\ \mathscr{D}F = \mathrm{OR} \wedge \mathscr{D}F^{-1} = \mathscr{S}(\mathrm{OR})\big)$.

Proof. Suppose there is an F such that $\mathscr{D}F = \mathrm{OR}$ and $\mathscr{D}F^{-1} = \mathscr{S}(\mathrm{OR})$ in N. Then there is a ξ such that F is definable from elements of M and G_ξ (for $\eta \subset \xi$, G_η is definable in terms of G_ξ and ordinals). Hence every subset of OR that is in N is definable from elements of M and G_ξ.

Consider

$$a = \left(\bigcup G^{\xi}_{\xi+1}\right)^{*}\{1\} \subseteq \mathrm{OR}^{M}.$$

By 20.2, $a \notin M[G_{\xi}]$. But $a \subseteq \mathrm{OR}^{M} \subseteq M[G_{\xi}]$; thus,

$$a \cap M[G_{\xi}] = a \notin M[G_{\xi}].$$

On the other hand, a is definable from elements of $M[G_{\xi}]$ in $M[G_{\xi}, G^{\xi}] = M[G]$, where G^{ξ} is the projection of G on C^{ξ}. Now C^{ξ} is homogeneous (because: let $p, q \in C^{\xi}$; then taking $m_{\eta} \supseteq \mathscr{D}p \cup \mathscr{D}q$ and $\pi \in {}^{m_{\eta}}(m_{\eta+1} \sim m_{\eta})$ we obtain $\bar{\pi}$ by

$$\bar{\pi}\,{}^{\cdot}p = p \circ \pi^{-1},$$

and it is clear that $\bar{\pi}$ is an automorphism of C^{ξ} in M and $\bar{\pi}\,{}^{\cdot}p \cap q = 0$). Therefore, by 18.6,

$$a \in M[G_{\xi}],$$

and we obtain a contradiction.

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