ELEMENTARY AND ANALYTIC THEORY OF NUMBERS
BANACH CENTER PUBLICATIONS, VOLUME 17
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

AUTOMORPHIC FORMS

R. W. BRUGGEMAN

Mathematics Department, Rijksuniversiteit, Utrecht, Netherlands

The main purpose of this paper is to discuss some relations satisfied by Fourier coefficients of automorphic forms. The most important one of those relations is the sum formula of Kuznetsov(1), relating Fourier coefficients of automorphic forms and Kloosterman sums. This sum formula is the subject of part II. Two other relations are discussed in part III. Part I serves to introduce the concept of automorphic form, and to state results which are needed later on.

These notes are an expanded version of the lectures I have given in September 1982 at the Banach Center, Warsaw. I am grateful for the invitation to participate in the semester on Number Theory.

I thank Jeannette Guilliamse of the Mathematics Department of Utrecht University for the typing of these notes.

I. AUTOMORPHIC FORMS AND AUTOMORPHIC MODELS

I.0

For those who are new to this subject we discuss first the upper half plane and the modular group, and give examples of automorphic forms.

In Part II we shall want to consider spaces of automorphic forms in which a certain Lie algebra acts; for that purpose the language of "automorphic models" is convenient. Here we introduce this language and state results needed in Part II.

(1) The formula in question in a slightly different form was found independently by the author of this article, cf. [1] and [2], (*editor's remark*).

I.1. Introduction

1.1. The upper half plane

The upper half plane $\mathfrak{h} = \{z \in \boldsymbol{C} \mid \operatorname{Im} z > 0\}$ is a model for the hyperbolic non-euclidean planar geometry. The non-euclidean lines are:

(i) the intersections with $\mathfrak{h}$ of the circles with center on the real axis,

(ii) the vertical lines in $\mathfrak{h}$.

The group G of orientation-preserving non-euclidean motions consists of the transformations

$$z \mapsto g \cdot z = \frac{az+b}{cz+d} \quad \text{with} \quad g = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\boldsymbol{R})/\{\pm I\}.$$

($\mathrm{Sl}_2(\boldsymbol{R})$ consists of the real 2×2 matrices with determinant one.) We identify G and $\mathrm{Sl}_2(\boldsymbol{R})/\{\pm I\}$.

On $\mathfrak{h}$ one has the G-invariant measure $d\mu(z) = y^{-2}\, dx\, dy$; here and later on $z \in \mathfrak{h}$ is written as $z = x+iy$, x, $y \in \boldsymbol{R}$.

Remark that G also acts in $\boldsymbol{R} \cup \{\infty\}$, the "boundary" of $\mathfrak{h}$.

1.2. The modular group

1.2.1. Let us first consider $P = \boldsymbol{R}^2$, as a model of the euclidean planar geometry. Its group of orientation-preserving motions is

$$\mathrm{SO}_2(\boldsymbol{R}) \cdot T \quad \text{(semi-direct product)},$$

with $\mathrm{SO}_2(\boldsymbol{R})$ the group of rotations around the origin and T the group of translations.

In $T \cong \boldsymbol{R}^2$ we may consider the discrete subgroup $\Delta \cong \boldsymbol{Z}^2$, generated by $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+1)$. Each point $p \in P$ may be moved into the region D (see Fig. 1) by some element of Δ. A Δ-invariant function on P is known as soon as we know its values on D.

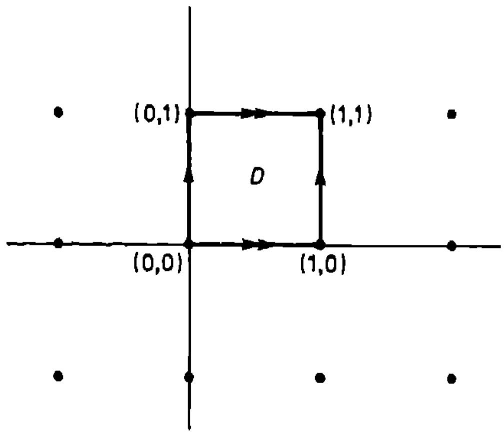

Fig. 1

The quotient space $\Delta\backslash P$ is a torus; it is compact. One may view it as D with its boundaries glued together.

Δ is just one easy example of a discrete subgroup of $SO_2(\boldsymbol{R})\cdot T$; there are many more, some not contained in T.

1.2.2. In the hyperbolic case one may also look for discrete subgroups Γ of G and consider the quotient space $\Gamma\backslash\mathfrak{h}$. There are a lot of possibilities. Most things in these lectures hold for rather general Γ (it should be discrete in G, and $\Gamma\backslash\mathfrak{h}$ should have finite volume with respect to $d\mu(z)$). In order not to complicate the exposition we work with one example, the full modular group: $\Gamma = Sl_2(\boldsymbol{Z})/\{\pm I\}$, consisting of the transformations $z \mapsto \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \boldsymbol{Z}$, $ad-bc = 1$. For number theory this group and some of its subgroups are the most interesting ones.

1.2.3. Each $z \in \mathfrak{h}$ may be moved into

$$F = \{z \in \mathfrak{h} \mid |\operatorname{Re} z| \leqslant \tfrac{1}{2}, |z| \geqslant 1\}$$

by an element of Γ.

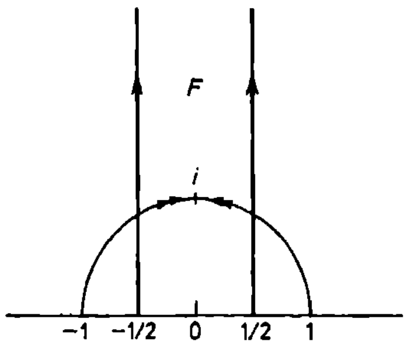

Fig. 2

By glueing the boundaries of F in the way indicated in Figure 2, one obtains the quotient space $\Gamma\backslash\mathfrak{h}$. This space has finite volume with respect to the measure coming from $d\mu(z)$, but it is not compact. If one looks at it through differential-geometrical spectacles one sees a spherical surface, with an infinitely long tentacle. (The metric $(ds)^2 = y^{-2}(dx)^2 + y^{-2}(dy)^2$ on $\mathfrak{h}$ gives the non-euclidean distances; it gives also a metric on the quotient.)

Through algebraic-geometrical spectacles one does not see the tentacle. One views $\Gamma\backslash\mathfrak{h}$ as the set of complex points of an affine curve, which is completed to a projective curve $X(\Gamma)$ by adding one point. The genus of $X(\Gamma)$ is zero for the modular group. The set of points of $X(\Gamma)$ may be viewed as

$$\Gamma\backslash\mathfrak{h}^* \quad \text{with} \quad \mathfrak{h}^* = \mathfrak{h} \cup \boldsymbol{Q} \cup \{\infty\}.$$

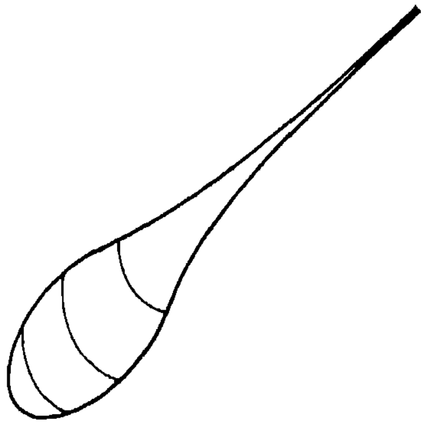

Fig. 3

The elements of $\boldsymbol{Q} \cup \{\infty\}$ are called parabolic points, they are elements of $\boldsymbol{R} \cup \{\infty\}$ which are fixed by some non-trivial elements of Γ with trace 2. The subgroup $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \; n \in \boldsymbol{Z} \right\}$ of Γ fixes ∞. The point of $X(\Gamma)$ corresponding to $\boldsymbol{Q} \cup \{\infty\}$ is called the *cusp*. (More general groups Γ may have more than one cusp; if there is no cusp at all, then $\Gamma \backslash \mathfrak{h}$ itself is compact.)

1.3. Invariant functions

1.3.1. In the example of 1.2.1 the functions on $\Delta \backslash P$ correspond to the Δ-invariant functions on P. Fourier expansion expresses them as infinite sums of multiples of $(x, y) \mapsto e^{2\pi i(nx+my)}$ with n, $m \in \boldsymbol{Z}$. Those exponential functions are the eigenfunctions of the euclidean Laplace operator $-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. So they give the spectral decomposition of $L^2(\Delta \backslash P)$ for the Laplace operator.

1.3.2. The non-euclidean Laplace operator in $\mathfrak{h}$ is

$$L = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It is invariant under the transformations of G.

One may try to expand "all" Γ-invariant functions on $\mathfrak{h}$ in eigenfunctions of L. As $\Gamma \backslash \mathfrak{h}$ is not compact, this is bound to be a more difficult task than in the situation of 1.3.1.

Some eigenfunctions of L on $\Gamma \backslash \mathfrak{h}$ are easily found:

1.3.2.1. The constant functions.

1.3.2.2. The Eisenstein series:

$$e_s(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im}(\gamma \cdot z))^{1/2+s};$$

this series converges uniformly on compact sets if $\operatorname{Re} s > \frac{1}{2}$. ($z \mapsto y^{1/2+s}$ is clearly a Γ_∞-invariant eigenfunction of L. As L is G-invariant, it follows that also e_s is an eigenfunction with the same eigenvalue.) Actually, e_s is an Epstein-zeta function:

$$2\zeta(1+2s)\,e_s(z) = \zeta(1+2s) \sum_{\substack{c,d\in \mathbf{Z}\\ (c,d)=1}} (y/|cz+d|^2)^{1/2+s}$$
$$= y^{1/2+s} \sum_{n,m\in \mathbf{Z}}{}' |nz+m|^{-1-2s}$$
$$= y^{1/2+s} \sum_{n,m\in \mathbf{Z}}{}' \left((y^2+x^2)\cdot n^2+2x\cdot mn+m^2\right)^{-1/2-s}$$

($\sum'$ means: $(n, m) \neq (0, 0)$.) So one probably is not surprised at $s \mapsto e_s(z)$ possessing a meromorphic continuation, giving more Γ-invariant eigenfunctions of L.

1.3.2.3. We have not exhausted the supply of eigenfunctions on $\Gamma\backslash\mathfrak{h}$. In the modular case this is not too difficult to see, if one accepts the fact that $L^2(\Gamma\backslash\mathfrak{h})$ is "spanned" by eigenfunctions of L:

Define on F:

$$\varphi(z) = \begin{cases} \sin 2\pi x & \text{if } 3 < y \leqslant 4, \\ 0 & \text{otherwise;} \end{cases}$$

extend φ to a Γ-invariant function on $\mathfrak{h}$. Clearly $\varphi \in L^2(\Gamma\backslash\mathfrak{h})$, but φ is orthogonal to all functions mentioned in 1.3.2.1 and 1.3.2.2. So there should be other eigenfunctions of L of which φ is a linear combination. We need even infinitely many, for φ is not continuous, and each eigenfunction of L is smooth. (L is an elliptic operator, hence all its eigenfunctions are real analytic.)

None of these Γ-invariant eigenfunctions of L is explicitly known; compare [12] for some numerical results.

1.3.3. DEFINITION. $f\colon \mathfrak{h} \to \mathbf{C}$ is a *real analytic modular form* if

(i) $Lf = \lambda f$ for some $\lambda \in \mathbf{C}$,

(ii) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,

(iii) $|f(z)| \ll y^a$ for $y \to \infty$, uniformly in x, for some $a \in \mathbf{R}$.

The last condition ensures that we exclude functions which grow too fast at the cusp.

The first one who has studied these functions systematically is Maass, [22].

1.4. Holomorphic modular forms

$\Gamma\backslash\mathfrak{h}$ parametrizes the isomorphy classes of elliptic curves over C. Study of elliptic functions soon leads to the holomorphic Eisenstein series:

$$G_k(z) = \sum_{m,n\in \mathbf{Z}}{}' (mz+n)^{-k}, \qquad k \geqslant 4,\ k \text{ even}.$$

(By $\sum'$ is denoted the sum over all (m, n) except $(0, 0)$.)

1.4.1. DEFINITION. f: $\mathfrak{h} \to C$ is a *holomorphic modular form* of weight k if

(i) f is holomorphic,

(ii) $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

(iii) $f(z) = O(1)$ for $y \to \infty$.

Here we only consider even weights. The linear space of holomorphic modular forms of weight k is denoted M_k. Remark that $G_k \in M_k$.

1.4.2. Each $f \in M_k$ satisfies $f(z+1) = f(z)$. This implies that f has a Fourier series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

By condition (iii) there are no terms with $n < 0$.

One may show that

$$E_k(z) := \frac{1}{2\zeta(k)} G_k(z) = 1 - (2k/B_k) \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}$$

with B_k the kth Bernoulli number, and

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Multiplication gives bilinear maps $M_k \times M_l \to M_{k+l}$. As $\dim M_k = 1$ for $4 \leqslant k \leqslant 10$ and $k = 14$, we obtain e.g. $E_{10} = E_4 E_6$. This implies relations for σ_3, σ_5 and σ_9.

Further $\dim M_{12} = 2$. One may choose α and $\beta \in C$ in such a way that $\Delta = \alpha E_4^3 + \beta E_6^2$ satisfies

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$$

with $\tau(1) = 1$. The numbers $\tau(n)$ are all integral and satisfy nice relations amounting to

$$\sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_{p \text{ prime}} (1 - \tau(p) p^{-s} + p^{11-2s})^{-1}$$

for $\operatorname{Re} s$ large.

1.4.3. A modular form like Δ for which the Fourier coefficient of order zero vanishes is called a *cusp form*. $M_k \supset S_k := \{$holomorphic cusp forms of weight $k\}$.

If $f \in S_k$ then $z \mapsto y^{k/2} |f(z)|$ is a bounded, Γ-invariant function. From this follows that the Fourier coefficients satisfy

$$|a_n| \ll n^{k/2} \quad \text{for} \quad n \to \infty.$$

Deligne [4] has proved that

$$|a_n| \ll n^{(k-1)/2+\varepsilon} \quad \text{for } n \to \infty, \quad \text{for each } \varepsilon > 0.$$

1.5. Why study modular forms?

1.5.1. The holomorphic modular forms are very important for number theory. In 1.4 this has barely been shown.

1.5.2. In 1.3 is mentioned the spectral decomposition of L in $L^2(\Gamma\backslash\mathfrak{h})$. Even when studying questions arising from holomorphic forms, one may have to deal with functions on $\Gamma\backslash\mathfrak{h}$. For instance, if $f \in M_k$ then $z \mapsto y^k |f(z)|^2$ is a function on $\Gamma\backslash\mathfrak{h}$. For example, in [10] Good, for more general Γ, uses the spectral decomposition to prove an estimate for the Fourier coefficients of holomorphic cusp forms.

1.5.3. The discrete spectrum of L in $L^2(\Gamma\backslash\mathfrak{h})$ is interesting in itself. It is connected to the geometric structure of $X(\Gamma)$. Here the central theorem is Selberg's trace formula, see e.g. [7], [34].

1.5.4. For Δ as in 1.4

$$L(s) = \int_0^\infty \Delta(iy)\, y^{s-1}\, dy$$

converges for all $s \in \mathbf{C}$. For $\mathrm{Re}\, s$ large:

$$L(s) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \tau(n)\, n^{-s}.$$

The equality $\Delta(-1/z) = z^{12}\Delta(z)$ implies a functional equation for L.

Hecke, [11], established in this way a correspondence between holomorphic automorphic forms and certain Dirichlet series with analytic continuation and functional equation. Maass, [22], tried to extend this correspondence to a wider class of Dirichlet series and arrived at real analytic forms.

Jacquet–Langlands, [13], show that the correspondence mentioned above is in fact a correspondence between certain L-series and certain irreducible representations of adele groups. From this point of view real analytic modular forms are as natural as holomorphic ones.

1.5.5. Petersson, [25], has given an expression for the Fourier coefficients of certain holomorphic cusp forms in terms of Kloosterman sums and Bessel functions. The sum formula in part II may be seen as a generalization of Petersson's formula. This sum formula has turned out to be useful if one wants to estimate sums of Kloosterman sums.

1.6. Remarks

1.6.1. Holomorphic modular forms have been studied for a long time. Of many possible references we mention [21], [33], [19].

For real analytic modular forms one might turn to [23], [31], [32], [14], [34], [35].

1.6.2. One may define real analytic modular forms of arbitrary real weight, [23], [31], [32]. The holomorphic forms may be considered as special cases of real analytic ones.

One understands better what the weight of a modular form really is, if one considers not functions on $\mathfrak{h}$, but on G. See I.2.

1.6.3. In studying modular forms one may study forms of a fixed weight or consider all weights simultaneously. The latter approach leads to the study of representations of the Lie algebra of G in spaces of modular forms.

In the former approach some complications are avoided; this approach is suitable for e.g. the trace formula of Selberg. On the other hand the Jacquet–Langlands theory is highly representational.

These notes are moderately representational. In the second part we shall see that all even weights crop up, even if one tries to derive the sum formula in weight zero only.

1.6.4. In general one says "automorphic form" instead of "modular form". In "modular form" the discrete subgroup Γ is understood to be the modular group or a congruence subgroup of it. Although these lectures discuss only the full modular group, we shall call the functions we study "automorphic forms" from now on.

I.2. Automorphic forms on the group

We define automorphic forms as functions on the group, and show that the functions discussed in I.1 all arise in this way.

The discrete group Γ is $\mathrm{Sl}_2(\boldsymbol{Z})/\{\pm I\}$; with minor modifications everything goes through for more general discrete subgroups of G.

2.1. Subgroups of G and notations

Let $k(\theta) = \pm\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $a(y) = \pm\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$ for $y > 0$ and $n(x) = \pm\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. For $z \in \mathfrak{h}$ we put $p(z) = n(x)a(y)$.

$K = \{k(\theta) |\ 0 \leqslant \theta < \pi\}$ is a maximal compact subgroup of $G = \mathrm{Sl}_2(\boldsymbol{R})/\{\pm I\}$.

$N = \{n(x) | \ x \in \boldsymbol{R}\}$, $A = \{a(y) | \ y > 0\}$. The group A normalizes N: $a(y)n(x)a(y)^{-1} = n(xy)$. $P = NA$ is a subgroup of G; the map $z \mapsto p(z)$ gives an isomorphism of real analytic varieties $\mathfrak{h} \to P$.

By $g \mapsto g \cdot i$, we get a map $G \to \mathfrak{h}$, which gives an identification $G/K \to \mathfrak{h}$. A section of $g \mapsto g \cdot i$ is given by $z \mapsto p(z)$.

Topologically $G \cong N \times A \times K$, the Iwasawa decomposition. By $g = n(x)a(y)k(\theta)$ we get coordinates (x, y, θ) on G. The second order differential operator

$$\omega = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+y\,\frac{\partial^2}{\partial x\,\partial\theta},$$

the Casimir operator, is invariant under left and right translations. Moreover, ω generates the ring of all left- and right-invariant differential operators.

(A differential operator D is right invariant if $D(R_g f) = R_g(Df)$ for all $g \in G$, $f \in C^\infty(G)$; $R_g f(x) = f(xg)$. Left-invariance is defined similarly.)

$$\Gamma_\infty = \Gamma \cap N = \{n(k) | \ k \in \boldsymbol{Z}\}.$$

2.2. Weight functions

A function f on G has *weight* k if

$$f(gk(\theta)) = f(g)\,e^{ik\theta}.$$

As $k(\pi) = k(0)$ the weight has to be an even number.

Let W be the differential operator $W = \partial/\partial\theta$. Then f having weight k is equivalent to

$$Wf = ikf.$$

Such functions we call *weight functions*.

A weight function is determined by its values on P. To a function f of weight k we associate functions on $\mathfrak{h}$ in two ways:

$$f^R(z) = f(p(z)),$$

$$f^h(z) = y^{-k/2} f(p(z)).$$

Let $g = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, and $f_1(x) = f(gx)$. Then

$$f_1^h(z) = (cz+d)^{-k} f^h(g \cdot z),$$
$$f_1^R(z) = (cz+d)^{-k} |cz+d|^k f^R(g \cdot z).$$

2.3. Automorphic forms

2.3.1. Definition. A function f: $G \to C$ is called an *automorphic form of weight k and eigenvalue λ* if

(i) $Wf = ikf$,
(ii) $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$,
(iii) $\omega f = \lambda f$,
(iv) $f(p(z)k) \ll y^a$ for $y \to \infty$, uniformly in x and k for some $a \in \boldsymbol{R}$.

The space of automorphic forms of weight k and eigenvalue λ we denote by $\mathscr{F}_k(\lambda)$.

Remark that condition (iii) amounts to an elliptic differential equation for f^R; this implies that all automorphic forms are real analytic functions.

2.3.2. The map $f \mapsto f^R$ gives a bijection between $\mathscr{F}_0(\lambda)$ and the space of real analytic modular forms defined in 1.3.3.

2.3.3. If $f \in M_k$ (see 1.4.1), then

$$F(p(z)k(\theta)) = y^{k/2} f(z) e^{ik\theta}$$

defines $F \in \mathscr{F}_k(\frac{1}{2}k - \frac{1}{4}k^2)$ such that $F^h = f$.

2.3.4. So all modular forms considered in I.1 correspond to automorphic forms on G. For all weights k the map $f \mapsto f^R$ identifies $\mathscr{F}_k(\lambda)$ with the space of automorphic forms considered in [31], [32].

2.4. Examples

2.4.1. By 2.3.2 and 2.3.3 all examples in I.1 give examples of automorphic forms on G.

2.4.2. For k general one has the Eisenstein series of weight k. Put $H_{s,k}(p(z)k(\theta)) = y^{1/2+s} e^{ik\theta}$. Then for $\operatorname{Re} s > \frac{1}{2}$

$$e_{s,k}(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} H_{s,k}(\gamma g)$$

converges absolutely and defines $e_{s,k} \in \mathscr{F}_k(\frac{1}{4} - s^2)$.

2.4.3. The Dedekind eta-function $\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$ is a 24-th root of the form Δ mentioned in 1.4. It satisfies

$$\frac{\eta'(\gamma z)}{\eta(\gamma z)} = (cz+d)^2 \frac{\eta'(z)}{\eta(z)} + \tfrac{1}{2} c(cz+d).$$

Define

$$f(p(z)k(\theta)) = \left(y \frac{\eta'(z)}{\eta(z)} - \tfrac{1}{4} i\right) e^{2i\theta};$$

then some computations show that $f \in \mathscr{F}_2(0)$. So even this nearly automorphic form η'/η is included in definition 2.3.1.

I.3. Differential operators and Lie algebra

Each Lie group has a Lie algebra, so G has one. See for instance Ch. VI § 1 of [18]. In these notes the Lie algebra of G is introduced as a space of differential operators.

3.1. Left-invariant differential operators

Let $L_g f(x) = f(g^{-1}x)$ define the left translation L_g, with $g \in G$, acting on functions on G. A differential operator D is left-invariant if $DL_g = L_g D$ for all $g \in G$. The left-invariant differential operators on G form an algebra $\mathscr{U}$ over C, with composition as the product. One may show that $\mathscr{U}$ is generated by the following three first order differential operators:

$$\begin{aligned} W &= \frac{\partial}{\partial\theta}, \\ E^+ &= e^{2i\theta}\left(2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - i\frac{\partial}{\partial\theta}\right), \\ E^- &= e^{-2i\theta}\left(-2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + i\frac{\partial}{\partial\theta}\right), \end{aligned}$$

subject to the relations

$$\begin{aligned} WE^+ - E^+W &= 2iE^+, \\ WE^- - E^-W &= -2iE^-, \\ E^+E^- - E^-E^+ &= -4iW. \end{aligned}$$

The Casimir operator ω is left- and right-invariant, so $\omega \in \mathscr{U}$. It is given by $\omega = -\frac{1}{4}E^+E^- + \frac{1}{4}W^2 - \frac{i}{2}W$. It generates the center of $\mathscr{U}$.

3.2. The Lie algebra

Let $\mathfrak{g}$ be the linear space in $\mathscr{U}$ spanned by W, E^+ and E^-. One may show that $\mathfrak{g}$ is exactly the space of first order left-invariant differential operators. In 3.1 we have seen that $\mathfrak{g}$ is closed under the bilinear map $(X, Y) \mapsto [X, Y] = XY - YX$. This "product" $[.,.]$ is antisymmetric, non-associative, but satisfies $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. A linear space equipped with this kind of product is called a *Lie algebra*. So $\mathfrak{g}$ is a Lie algebra. It could have been obtained by complexification of the Lie algebra of the Lie group G.

$\mathscr{U}$, as an abstract algebra, may be reconstructed from $\mathfrak{g}$. It is the so-called universal enveloping algebra of $\mathfrak{g}$.

3.3. The action of $\mathfrak{g}$ on automorphic forms

3.3.1. A function $F \in C^\infty(G)$ with weight k satisfies $WF = ikF$. For such F:

$$W(E^\pm F) = E^\pm(WF) + [W, E^\pm]F = i(k\pm 2)F.$$

So $E^\pm$ maps functions with weight k onto functions with weight $k\pm 2$.

It is easily seen that the action of E^+ and E^- preserves conditions (ii) and (iii) in 2.3.1. One may prove that condition (iv) is also respected. So

$$E^\pm\colon \mathscr{F}_k(\lambda) \to \mathscr{F}_{k\pm 2}(\lambda).$$

Up to a constant factor E^+ and E^- are the operators K_k and Λ_k introduced by Maass; see [31].

3.3.2. As an example one may check that $E^\pm e_{s,k} = (1+2s\pm k)\, e_{s,k\pm 2}$, by proving that the same relation holds for $H_{s,k}$, see 2.4.2. So if one knows $e_{s,0}$, then one knows a lot about $e_{s,k}$ for all even k.

3.4. Representations of $\mathfrak{g}$ in spaces of automorphic forms

Let $\mathscr{F}(\lambda) = \bigoplus_{k\,\text{even}} \mathscr{F}_k(\lambda)$. We have seen that $\mathfrak{g}$ acts in $\mathscr{F}(\lambda)$; so $\mathscr{F}(\lambda)$ is a representation space of $\mathfrak{g}$. This means that we have got an additional structure on the space of automorphic forms.

The action of $\mathfrak{g}$ in $\mathscr{F}(\lambda)$ extends to an action of the associative algebra $\mathscr{U}$. In general actions of $\mathscr{U}$ and of $\mathfrak{g}$ amount to the same. As $\mathfrak{g}$ is smaller, we prefer to work with $\mathfrak{g}$. The representation space $\mathscr{F}(\lambda)$ of $\mathfrak{g}$ has two special properties:

(i) The Casimir operator acts as multiplication by λ.

(ii) All elements of the space are finite linear combinations of elements of even weight (i.e. eigenvectors of W with eigenvalues in $2i\mathbf{Z}$).

Condition (i) is true for irreducible $\mathfrak{g}$-spaces, λ depending on the space. (A space V in which $\mathfrak{g}$ acts is irreducible if $\{0\}$ and V are the only $\mathfrak{g}$-invariant subspaces.) If we know which irreducible spaces occur in $\mathscr{F}(\lambda)$ we have got additional insight in this space of automorphic forms.

3.5. List of $\mathfrak{g}$-spaces

Now we enumerate some spaces in which $\mathfrak{g}$ acts, satisfying the conditions (i) and (ii) in 3.4. All possible irreducible spaces occur in this list. These spaces we consider as abstract spaces.

3.5.1. Non-unitary principal series. Let $s \in \mathbf{C}$ and put

$$H(s) = \bigoplus_{k\,\text{even}} \mathbf{C}\cdot\varphi_k \quad\text{with}\quad W\varphi_k = ik\varphi_k,$$

$$E^\pm\varphi_k = (1+2s\pm k)\,\varphi_{k\pm 2}.$$

4.3.4. For the situation of $s = \frac{1}{2}(k-1)$ one has the diagram

$$\begin{tikzcd}
\mathscr{A}(H(\frac{1}{2}(k-1))) \arrow[rr, "T \mapsto T\varphi_k"] \arrow[d, "\text{restriction}"] & & \mathscr{F}_k(\frac{1}{2}k - \frac{1}{4}k^2) \\
\mathscr{A}(D_k^+) \arrow[rr, "\cong", "T \mapsto h \text{ in } 4.3.3"'] & & M_k \arrow[u, hook]
\end{tikzcd}$$

see 2.3.3.

The restriction map is injective for $k \geqslant 4$; for $k = 2$ one may take $T\varphi_0 = 1$, $T\varphi_r = 0$ for $r \neq 0$, to get an element of the kernel. The restriction map is not necessarily surjective.

For $k \geqslant 4$ the map $\mathscr{A}\big(H(\frac{1}{2}(k-1))\big) \to \mathscr{F}_k(\frac{1}{2}k - \frac{1}{4}k^2)$ is surjective. The function f in 2.4.3 is not in the image of $\mathscr{A}\big(H(\frac{1}{2})\big)$.

4.4. Square integrable automorphic models

4.4.1. An automorphic model T of W is called *square integrable* if

$$TW \subset L^2(\Gamma \backslash G).$$

A trivial example is the model T of $H(\frac{1}{2})$ defined by $T\varphi_0 = 1$, $T\varphi_r = 0$ for $r \neq 0$, mentioned in 4.3.4.

4.4.2. On $L^2(\Gamma \backslash G)$ we have the scalar product $\langle f, f_1 \rangle = \int\limits_{\Gamma\backslash G} f(g)\overline{f_1(g)}\, dg$.

For f and f_1 differentiable and $X \in \mathfrak{g}$:

$$\langle X \cdot f, f_1 \rangle + \langle f, X^* \cdot f_1 \rangle = 0.$$

(By $X \mapsto X^*$ is denoted the antilinear map $\mathfrak{g} \to \mathfrak{g}$ for which $W^* = W$, $(E^{\pm})^* = E^{\mp}$.)

If T is a non-zero square integrable automorphic model of W, then we may define a scalar product on W by

$$\langle w, w_1 \rangle = \langle Tw, Tw_1 \rangle.$$

Only a few of the $\mathfrak{g}$-spaces in 3.5 admit a non-trivial scalar product:

4.4.3.	$H(s)$ with $\operatorname{Re} s = 0$	"principal series",
	$H(s)$ with $0 < \lvert s \rvert < \frac{1}{2}$	"complementary series",
	D_k^+ and D_k^- with k even, $k \geqslant 2$	"discrete series",
	D_0	trivial $\mathfrak{g}$-space of dimension one.

(See [18], Ch. VI, § 6.)

4.4.4. The example of 4.4.1 is in fact the trivial model of D_0, composed with $H(\frac{1}{2}) \to H(\frac{1}{2}) \bmod (D_2^+ \oplus D_2^-) \cong D_0$.

In the case of the full modular group there are no square integrable automorphic models of the complementary series.

4.4.5. For the decomposition of $L^2(\Gamma\backslash G)$ it is important to know the square integrable automorphic models, for they give irreducible subspaces of $L^2(\Gamma\backslash G)$.

I.5. Fourier coefficients

5.1. The operators F_n

5.1.1. For each $f \in C^\infty(\Gamma\backslash G)$ and $g \in G$ the function $x \mapsto f(n(x)g)$ has period one and

$$f(n(x)g) = \sum_{n \in \mathbf{Z}} e^{2\pi i n x} \int_0^1 f(n(u)g) e^{-2\pi i n u} du.$$

Each term in this series is an element of $C^\infty(G)$ transforming according to

$$f_n(n(x)g) = e^{2\pi i n x} f_n(g)$$

and $f_n(p(z)k)$ satisfies the same growth conditions for $y \to \infty$ as $f(p(z)k)$. Moreover, the map $f \mapsto f_n$ commutes with the action of $\mathfrak{g}$ by differentiation. So if $T \in \mathscr{A}(W)$ for some $\mathfrak{g}$-space W, then $W \ni w \mapsto (Tw)_n$ is a model of W in some specific space of functions on G.

5.1.2. If h satisfies $h(n(x)g) = e^{iux} h(g)$ then the function $g \mapsto h(a(t)g)$ satisfies the same relation with u replaced by tu. So in 5.1.1 are involved only three different kind of models: u positive, $u = 0$, u negative. It will turn out that $u = 0$, $u = \pm\frac{1}{2}$ is a convenient choice.

5.1.3. We define for $f \in C^\infty(\Gamma\backslash G)$:

$$F_0 f(g) = \int_0^1 f(n(x)g)\,dx,$$

$$F_n f(g) = \int_0^1 f(n(x)a(4\pi|n|)^{-1}g) e^{-2\pi i n x} dx \qquad \text{for} \qquad n \neq 0.$$

The operators F_n are intertwining operators "taking the nth Fourier coefficient".

5.1.4. For each $f \in C^\infty(\Gamma\backslash G)$ we have the Fourier series expansion

$$f(p(z)k) = F_0 f(a(y)k) + \sum_{n \neq 0} e^{2\pi i n x} F_n f(a(4\pi|n|y)k).$$

5.1.5. If T is an automorphic model of W, then $F_n T$ is a model of W in

$$\begin{cases} C^\infty(N\backslash G) & \text{if } n=0; \\ \text{the functions transforming on the left according to the} & \\ \qquad \text{character } n(x)\mapsto e^{i\varepsilon x/2} \text{ of } N & \text{if } \varepsilon n>0,\ \varepsilon=\pm 1. \end{cases}$$

In both cases we have the growth condition

$$|F_n T\varphi(p(z)k)| \ll y^a, \quad y\to\infty \text{ for some } a\in \mathbf{R}.$$

We turn now to the question how such models look like.

5.2. Models in $C^\infty(N\backslash G)$

If T is a model of the g-space W in $C^\infty(N\backslash G)$ and $\varphi_r\in W$ has weight r, then

$$T\varphi_r(p(z)k(\theta)) = f_r(y)e^{ir\theta}$$

for some $f_r\in C^\infty(0,\infty)$. If ω acts as multiplication by λ in W, then $\omega T\varphi_r = \lambda T\varphi_r$ leads to a differential equation for f. The actions of E^+ and E^- give relations between the f_r. Some computations lead to the following results:

5.2.1. The *standard model* $\mathrm{St}(s)\colon H(s)\to C^\infty(N\backslash G)$ is given by

$$\mathrm{St}(s)\varphi_k(p(z)k(\theta)) = y^{1/2+s}e^{ik\theta}.$$

We have in fact already met it in 2.4.2:

$$\mathrm{St}(s)\varphi_k = H_{s,k}.$$

This is the model of the induced representation from NA to G. The definition of $H(s)$ is motivated by this model, hence the name standard model.

5.2.2. For $s\notin\frac{1}{2}+\mathbf{Z}$ also $\mathrm{St}(-s)\iota(s)$ is a model of $H(s)$ in $C^\infty(N\backslash G)$.

5.2.3. For k even, $k\geqslant 2$, the standard model $\mathrm{St}(\frac{1}{2}(k-1))$ by restriction gives models of D_k^+ and D_k^- in $C^\infty(N\backslash G)$; a model of D_{2-k} is obtained by restriction of $\mathrm{St}(-\frac{1}{2}(k-1))$.

5.2.4. The models in 5.2.3 composed with the maps $H(\frac{1}{2}(k-1))\to D_{2-k}$ and $H(-\frac{1}{2}(k-1))\to D_k^+\oplus D_k^-$ give models of $H(\pm\frac{1}{2}(k-1))$ in $C^\infty(N\backslash G)$. The first one may be described as

$$\mathrm{St}_k = \operatorname*{res}_{s\to(k-1)/2}(\mathrm{St}(-s)\iota(s)).$$

5.2.5. Except for the case $s=0$, the models given above span the spaces of models in $C^\infty(N\backslash G)$. They all satisfy the growth condition.

5.3. Whittaker models

5.3.1. For $\varepsilon = 1$ or -1 put

$$\mathscr{W}_\varepsilon = \{f \in C^\infty(G) | \; f(n(x)g) = e^{\frac{1}{2}i\varepsilon x} f(g)\}.$$

This is the space of functions used for the other terms in the Fourier series. $\mathscr{W}_\varepsilon$ is a g-space. The choice of the factor $\frac{1}{2}$ makes that the condition "$\omega - \frac{1}{4} + s^2$ acts as 0" leads to the Whittaker differential equation. Here the growth condition really restricts the possibilities.

5.3.2. For $s \notin \frac{1}{2} + Z$ we find the *Whittaker model*

$$W^\varepsilon(s)\colon \; H(s) \to \mathscr{W}_\varepsilon,$$

$$W^\varepsilon(s)\,\varphi_r\big(p(z)\,k(\theta)\big) = e^{\frac{1}{2}i\varepsilon x}\,\Gamma\big(\tfrac{1}{2} - s - \tfrac{1}{2}\varepsilon r\big)\, W_{\frac{1}{2}\varepsilon r, s}(y)\, e^{ir\theta}.$$

5.3.3. For $k \geqslant 2$, k even:

$$W_k^\varepsilon = \operatorname*{res}_{s \to \frac{1}{2}(k-1)} \; W^\varepsilon(s)\colon \; H\big(\tfrac{1}{2}(k-1)\big) \to \mathscr{W}_\varepsilon$$

is the Whittaker model. It vanishes on D_k^- if $\varepsilon = 1$ and on D_k^+ if $\varepsilon = -1$.

$$W_k^1\,\varphi_k\big(p(z)\,k(\theta)\big) = \frac{1}{(k-1)!}\, y^{\frac{1}{2}k}\, e^{\frac{1}{2}iz}\, e^{ik\theta}.$$

5.3.4. Up to a constant factor these are the only models of $H(s)$, D_k^+ and D_k^- in $\mathscr{W}_\varepsilon$ satisfying the growth condition. The g-space D_{2-k} has no Whittaker models.

5.4. Fourier coefficients

5.4.1. Let $s \notin \frac{1}{2} + Z$, $s \neq 0$. For each $T \in \mathscr{A}(H(s))$ there are complex numbers $a_0(T)$ and $b_n(T)$ such that

$$F_0\, T = a_0(T)\,\mathrm{St}(s) + b_0(T)\,\mathrm{St}(-s)\,\imath(s),$$

$$F_n\, T = b_n(T)\, W^{\operatorname{sign} n}(s), \qquad n \neq 0.$$

5.4.2. Let k even, $k \geqslant 2$. Put for $T \in \mathscr{A}\big(H\big(\tfrac{1}{2}(k-1)\big)\big)$

$$F_0\, T = a_0(T)\,\mathrm{St}\big(\tfrac{1}{2}(k-1)\big) + b_0(T)\,\mathrm{St}_k,$$

$$F_n\, T = b_n(T)\, W_k^{\operatorname{sign} n}, \qquad n \neq 0.$$

5.4.3. For k even, $k \geqslant 2$. For each $T \in \mathscr{A}(D_k^+)$:

$$F_0\, T = a_0(T)\,\mathrm{St}\big(\tfrac{1}{2}(k-1)\big),$$

$$F_n\, T = b_n(T)\, W_k^{\operatorname{sign} n}, \qquad n > 0,$$

$$F_n\, T = 0, \qquad n < 0.$$

5.4.4. The numbers $a_0(T)$ and $b_n(T)$ are the *Fourier coefficients* of T. Remark that they are associated to the automorphic model, not to the individual automorphic forms.

5.5. Fourier coefficients of the Eisenstein model

5.5.1. For $\operatorname{Re} s > \frac{1}{2}$

$$E(s)\varphi_k = e_{s,k}$$

clearly defines a model of $H(s)$, the *Eisenstein model* $E(s) \in \mathscr{A}(H(s))$. We now show how computation of $a_0(E(s))$ and $b_n(E(s))$ leads to the Fourier series expansions of several automorphic forms.

5.5.2. For $f \in C^\infty(N\backslash G)$ define

$$\Theta_0 f(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma g),$$

if this series converges absolutely. This is the case for $f \in \operatorname{St}(s) H(s)$ with $\operatorname{Re} s > \frac{1}{2}$. Clearly $E(s) = \Theta_0 \operatorname{St}(s)$.

5.5.3. If $\Theta_0 f$ is well defined, then the decomposition

$$\Gamma = \Gamma_\infty \cup \bigcup_{\substack{c \geqslant 1 \\ (c,d)=1}} \Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty$$

leads to

$$F_0 \Theta_0 f(g) = f(g) + \sum_{c=1}^{\infty} \sum_{\substack{d \bmod c \\ (c,d)=1}} \int_{-\infty}^{+\infty} f\big(n(a/c)\, a(c^{-2})\, k(\tfrac{1}{2}\pi)\, n(d/c)\, n(x)\, g\big)\, dx.$$

So

$$F_0 \Theta_0 = I + \sum_{c=1}^{\infty} \varphi(c) \cdot d_0(c^{-2})$$

with

$$d_0(t) f(g) = \int_{-\infty}^{+\infty} f\big(a(t)\, k(\tfrac{1}{2}\pi)\, n(x)\, g\big)\, dx$$

and $\varphi(c)$ the number of $d \in [1, c]$ with $(c, d) = 1$.

One may compute

$$d_0(t) \operatorname{St}(s) = 2^{1-2s} t^{1/2+s} \Gamma(2s) \cos \pi s \operatorname{St}(-s)\, \iota(s).$$

So

$$a_0(E(s)) = 1,$$

$$b_0(E(s)) = 2^{1-2s} \Gamma(2s) \cos \pi s \sum_{c=1}^{\infty} \varphi(c)\, c^{-1-2s}.$$

In the modular case there are nicer expressions:

$$b_0(E(s)) = 2^{1-2s}\Gamma(2s)\cos\pi s\,\frac{\zeta(2s)}{\zeta(2s+1)} = \pi^{2s}\zeta(1-2s)/\zeta(1+2s).$$

5.5.4. For $n \neq 0$ similar computations lead to

$$b_n(E(s)) = \pi^{s-\frac{1}{2}}\sigma_{2s}(|n|)\,|n|^{-\frac{1}{2}-s}\cos\pi s/\zeta(1+2s)$$

with

$$\sigma_{2s}(k) = \sum_{d|k} d^{2s}.$$

Remark that $b_n(E(s))$ has zeros at the places where $W^{\operatorname{sign} n}(s)$ has poles.

5.5.5. The Fourier coefficients of $E(s)$ are meromorphic in s. With some care this may be used to get a meromorphic extension of $E(s)$ to the s-plane. It has poles at $s = \frac{1}{2}$ and "around" the line $\operatorname{Re} s = -\frac{1}{4}$.

5.5.6. The Fourier series expansion of $z \mapsto e_{s,0}(p(z))$ is now easily read off from 5.1.4 and 5.5.3,4:

$$e_{s,0}(p(z)) = y^{\frac{1}{2}+s} + \pi^{2s}\frac{\Gamma(\frac{1}{2}-s)\zeta(1-2s)}{\Gamma(\frac{1}{2}+s)\zeta(1+2s)}\,y^{\frac{1}{2}-s} +$$

$$+ \sum_{n\neq 0}\pi^{s-\frac{1}{2}}|n|^{+\frac{1}{2}-s}\sigma_{2s}(|n|)\,\zeta(1+2s)^{-1}\,\Gamma(\tfrac{1}{2}+s)^{-1}\,W_{0,s}(4\pi|n|y)\,e^{2\pi inx}.$$

5.5.7. $$y^{-\frac{1}{2}k}E(\tfrac{1}{2}(k-1))\,\varphi_k(p(z)) = y^{-\frac{1}{2}k}\lim_{s\to\frac{1}{2}(k-1)} E(s)\,\varphi_k(p(z))$$

$$= y^{-\frac{1}{2}k}\Big[1\cdot y^{\frac{1}{2}k} + b_0\big(E(\tfrac{1}{2}(k-1))\big)\cdot 0 +$$

$$+ \sum_{\varepsilon=\pm 1}\sum_{n=1}^{\infty} e^{2\pi i n\varepsilon x}\frac{(-1)^{\frac{1}{2}k}}{\zeta(k)}\,\pi^{\frac{1}{2}k}|n|^{-\frac{1}{2}k}\sigma_{k-1}(n)\,W_k^{\varepsilon}\,\varphi_k\big(a(4\pi ny)\big)\Big]$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^{\frac{1}{2}k}(2\pi)^k\sigma_{k-1}(n)\,\frac{1}{(k-1)!\,\zeta(k)}\,e^{2\pi inz},$$

for k even, $k \geqslant 4$, is the well known Fourier series expansion of $E_k \in M_k$.

5.5.8. The pole of $E(s)$ at $s = \frac{1}{2}$ occurs only in weight zero. So $\lim_{s\to 1/2} E(s)\varphi_2$ is an element of $\mathscr{F}_2(0)$ with Fourier series expansion:

$$\lim_{s\to 1/2} E(s)\,\varphi_2(p(z)) = y + \lim_{s\to 1/2} 2^{1-2s}\Gamma(2s)\cos\pi s\,\frac{\zeta(2s)\,\Gamma(-\frac{1}{2}-s)}{\zeta(2s+1)\,\Gamma(-\frac{1}{2}+s)}\,y^{\frac{1}{2}-s} +$$

$$+\sum_{\varepsilon=\pm 1}\sum_{n=1}^{\infty}\lim_{s\to 1/2}\left(\pi^{s-\frac{1}{2}}\sigma_{2s}(n)\,n^{-\frac{1}{2}-s}\frac{\cos\pi s}{\zeta(1+2s)}\Gamma(\tfrac{1}{2}-s-\varepsilon)\,W_{\varepsilon,s}(4\pi ny)\,e^{2\varepsilon\pi inx}\right)$$

$$=y-\frac{\pi}{3}+\sum_{n=1}^{\infty}\sigma_1(n)\,(-24)\,ye^{2\pi inz}$$

$$=-\frac{\pi}{3}+y-24y\sum_{a=1}^{\infty}\sum_{m=1}^{\infty}ae^{2\pi iamz}$$

$$=-\frac{\pi}{3}+y-24y\sum_{a=1}^{\infty}\frac{-1}{2\pi i}\left(\frac{d}{dz}(1-e^{2\pi iaz})\right)\Big/(1-e^{2\pi iaz})$$

$$=\frac{24}{2\pi i}\left(y\frac{\eta'(z)}{\eta(z)}-\tfrac{1}{4}i\right),$$

compare 2.4.3.

I.6. Cuspidal models

6.1.1. Definition. An automorphic model T of the $\mathfrak{g}$-space W is called a *cuspidal model* if $F_0 T=0$.

The space of cuspidal models of W is denoted by $\mathscr{A}^0(W)$.

6.1.2. One checks easily that under the map $\mathscr{A}(D_k^+)\to M_k$ of the proposition in 4.3.3 the cuspidal models of D_k^+ correspond to the holomorphic cusp forms in S_k.

6.1.3. From the growth of Whittaker functions follows that all functions in the image of a cuspidal model decrease rapidly at the cusp. This implies that each cuspidal model is square integrable. So only the spaces in the list in 4.4.3 may have non-zero cuspidal models.

The $\mathfrak{g}$-space D_0 has no cuspidal models, as it has no Whittaker models.

For the modular group it is known that the complementary series has no cuspidal models; see e.g. [23], Theorem 32, p. 203.

6.2. Dimensions of spaces of automorphic models

6.2.1. The space of models in $C^\infty(N\backslash G)$ is finite-dimensional, so for $F_0 T$ the possibilities are restricted. So $\mathscr{A}^0(W)$ always has finite codimension in $\mathscr{A}(W)$. In fact one may prove that the codimension is at most one (Maass–Selberg relations, see e.g. [32], § 9).

6.2.2. Proposition. *For each W in the list of $\mathfrak{g}$-spaces in* 4.4.3: $\dim\mathscr{A}^0(W)<\infty$.

6.2.3. Proposition. $\{s\in \mathbf{C}\,|\,\mathscr{A}^0(H(s))\neq\{0\}\}$ *is discrete.*

These results are due to Selberg; for a proof see e.g. Ch. XII, § 4 of [18].

6.3. Fourier coefficients of cuspidal models

6.3.1. From the boundedness of the functions in a cuspidal model one derives

$$|b_n(T)| \ll 1 \qquad \text{for} \qquad |n| \to \infty$$

for each cuspidal model T.

For $T \in \mathscr{A}^0(D_k^+)$ this is the estimate $|a_n| \ll n^{\frac{1}{2}k}$ in 1.4.3. Deligne's theorem ([4], Theorem 8.2) amounts to:

6.3.2. THEOREM. *Let $k \geqslant 12$, k even. For each $T \in \mathscr{A}^0(D_k^+)$*

$$|b_n(T)| \ll n^{-\frac{1}{2}+\varepsilon} \qquad \textit{for} \qquad n \to \infty$$

for each $\varepsilon > 0$.

6.3.3. For $T \in \mathscr{A}^0(H(s))$ the same assertion is only a conjecture. It is the Ramanujan–Petersson conjecture for real analytic cusp forms.

I.7. Spectral decomposition

By right translation one gets a unitary representation of G in $L^2(\Gamma \backslash G)$. With help of automorphic models one may describe how this representation is built up from irreducible ones.

7.1. If $T \in \mathscr{A}^0(W)$, $T \neq 0$, then the closure of TW is a G-irreducible subspace of $L^2(\Gamma \backslash G)$.

Let for each W for which $\mathscr{A}^0(W) \neq \{0\}$ a basis $T_{W,1}, \ldots, T_{W,r_W}$ of $\mathscr{A}^0(W)$ be chosen. Then

$$^0L^2(\Gamma \backslash G) = \overline{\bigoplus_W \bigoplus_{j=1}^{r_W} \overline{T_{W,j} W}}$$

describes the decomposition of the *cuspidal part* $^0L^2(\Gamma \backslash G)$ of $L^2(\Gamma \backslash G)$.

7.2. The pole of $E(s)$ at $s = \frac{1}{2}$ has as residue a square integrable model of D_0 in the constant functions. This gives a one-dimensional G-irreducible subspace E of $L^2(\Gamma \backslash G)$.

For general Γ the poles of $E(s)$ in $[0, \frac{1}{2}]$ give square integrable non-cuspidal models, and hence G-irreducible subspaces of $L^2(\Gamma \backslash G)$.

7.3. Finally one is left with the orthogonal complement $^cL^2(\Gamma \backslash G)$ of $^0L^2(\Gamma \backslash G) \oplus E$ in $L^2(\Gamma \backslash G)$. One may describe $^cL^2(\Gamma \backslash G)$ as the direct integral $\int_0^\infty E(it) H(it)\, dt$.

7.4. To prove these results requires hard work. One needs the meromorphic continuation of $E(s)$, which is obtained rather cheaply in the modular case. As possible references we mention [14], [18] Ch. XIII or Ch. XIV, [35].

II. SUM FORMULA OF KUZNETSOV

II.1. Introduction

We use the same notations as in part I. In particular, Γ is the full modular group.

1.1. Spectral decomposition of $L^2(\Gamma\backslash\mathfrak{h})$

1.1.1. The spectral decomposition in I.7 concerns $L^2(\Gamma\backslash G)$. If one considers one particular weight r, one gets from each automorphic model the component of weight r. So for weight zero one gets the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{h})$. As the discrete series does not posses a component in weight zero, it does not occur in this case.

1.1.2. The cuspidal models of the principal series, on the other hand, contribute to the spectral decomposition. Let $T_1, T_2, \ldots$ be an algebraic basis of $\bigoplus_{\substack{\operatorname{Im} s>0 \\ \mathscr{A}^0(H(s))\neq(0)}} \mathscr{A}^0(H(s))$ such that each $T_j\in\mathscr{A}^0(H(s_j))$ with $\operatorname{Im} s_j>0$, $\frac{1}{4}-s_1^2 \leqslant \frac{1}{4}-s_2^2 \leqslant \ldots$ and such that $T_j\varphi_0$ has length 1 in $L^2(\Gamma\backslash G)$ and such that $T_j\varphi_0$ orthogonal to $T_l\varphi_0$ if $j\neq l$. Denote $\psi_j = T_j\varphi_0$.

1.1.3. The spectral decomposition may be written as

$$L^2(\Gamma\backslash\mathfrak{h}) = \overline{\bigoplus_j \mathbf{C}\psi_j} \oplus \int_0^\infty \mathbf{C}e_{it,0}\cdot dt \oplus \mathbf{C}\cdot 1.$$

For sufficiently well behaved f and $g\in L^2(\Gamma\backslash\mathfrak{h})$:

$$\langle f, g\rangle = \sum_{j=1}^\infty \langle f, \psi_j\rangle \langle\psi_j, g\rangle + \frac{1}{2\pi i}\int_0^{i\infty} \eta f(s)\overline{\eta g(\bar{s})}\, ds + \frac{3}{\pi}\langle f, 1\rangle\langle 1, g\rangle$$

with

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)}\,\frac{dx\,dy}{y^2},$$

$$\eta f(s) = \int_{\Gamma\backslash\mathfrak{h}} f(z)\, e_{-s,0}(z)\,\frac{dx\,dy}{y^2}.$$

1.2. Convolution operators

1.2.1. Let $\psi \in C_c^\infty(K\backslash G/K)$. Then an operator T_ψ in $L^2(\Gamma\backslash\mathfrak{h})$ is defined by

$$T_\psi f(g) = f * \psi(g) = \int_G f(gx^{-1})\psi(x)\,dx.$$

(We identify right-K-invariant functions on G with functions on $\mathfrak{h}$.)

1.2.2. T_ψ is described by a kernel:

$$k(z, z') = \sum_{\gamma\in\Gamma} \psi(p(z')^{-1}\gamma p(z)),$$

$$T_\psi f(z) = \int_{\Gamma\backslash\mathfrak{h}} f(z')k(z, z')\frac{dx'\,dy'}{(y')^2}.$$

1.2.3. One may also describe T_ψ as a multiplication operator in the spectral decomposition: From ψ one may compute a holomorphic function $\hat{\psi}$ such that for sufficiently nice $f, g \in L^2(\Gamma\backslash\mathfrak{h})$:

$$\langle T_\psi f, g\rangle = \sum_{j=1}^{\infty} \hat{\psi}(s_j)\langle f, \psi_j\rangle\langle\psi_j, g\rangle + \frac{1}{2\pi i}\int_0^{i\infty} \hat{\psi}(s)\,\eta f(s)\overline{\eta g(s)}\,ds +$$

$$+\frac{3}{\pi}\hat{\psi}(\tfrac{1}{2})\langle f, 1\rangle\langle 1, g\rangle.$$

1.2.4. Selberg's trace formula is obtained by restricting T_ψ to ${}^0L^2(\Gamma\backslash\mathfrak{h}) = \overline{\bigoplus_j C\psi_j}$ and computing its trace with help of 1.2.2 and of 1.2.3. See e.g. [14].

1.3. Sum formula

1.3.1. The kernel k satisfies $k(z+1, z') = k(z, z'+1) = k(z, z')$, so one may take double-sided Fourier coefficients

$$k_{n,m}(y, y') = \int_0^1\int_0^1 e^{2\pi i(mx'-nx)}k(x+iy, x'+iy')\,dx\,dx'.$$

Take $n, m > 0$; we may compute $k_{n,m}$ in two ways, according to 1.2.2 and 1.2.3. So one obtains an equality. Considering the principal term in its asymptotic expansion for $y, y' \to \infty$ one obtains:

1.3.2. Proposition (Sum formula). *Let $\varphi \in C_c^\infty(0, \infty)$ and $n, m \geqslant 1$. Put*

$$\hat{\varphi}(s) = \frac{1}{\sin \pi s}\int_0^\infty (J_{-2s}(y) - J_{2s}(y))\,\varphi(y)\frac{dy}{y},$$

$$\varphi_H(y) = \varphi(y) - \sum_{\substack{k \geq 2 \\ k \text{ even}}} 2(k-1) J_{k-1}(y) \int_0^\infty J_{k-1}(t)\,\varphi(t)\,\frac{dt}{t}.$$

For $c \geq 1$

$$S(n, m; c) = \sum_{x \bmod c}{}' e^{2\pi i(nx + m\bar{x})/c},$$

$x\bar{x} \equiv 1 \pmod{c}$; *this is the well known Kloosterman sum.*

Then

$$\sum_{j=1}^{\infty} \hat{\varphi}(s_j)\, b_n(T_j)\, \overline{b_m(T_j)}\, \pi(\cos \pi s_j)^{-2} +$$

$$+\frac{1}{2\pi i} \int_0^{i\infty} \hat{\varphi}(s)\, n^{-\frac{1}{2}-s} m^{-\frac{1}{2}+s} |\zeta(1+2s)|^{-2} \sigma_{2s}(n)\, \sigma_{-2s}(m)\, ds$$

$$= \frac{1}{2\pi\sqrt{nm}} \sum_{c=1}^{\infty} c^{-1} S(n, m; c)\, \varphi_H(4\pi\sqrt{nm}/c) + \frac{\delta_{n,m}}{4\pi^2 m} \int_0^\infty \varphi(y)\, J_0(y)\, dy.$$

1.3.3. This sum formula has first been published by Kuznetsov [15], [17]. It is difficult to recognize the proof sketched in 1.3.1 in Kuznetsov's papers; he uses the scalar product of Poincaré series, which is in fact the dual formulation of the double-sided Fourier coefficient.

1.3.4. In [1] a similar formula is given, except that the Bessel transformation is not explicit and $\hat{\varphi}$ is the independent test function. The transformation T_ψ is only described as multiplication operator on the spectrum.

1.3.5. As in [15] the Bessel transformation is explicit, its sum formula is better than the one on [1]; so I like to attach Kuznetsov's name to it.

1.3.6. The idea of the proof sketched in 1.3.1 is partly based on an unpublished proof by D. Zagier.

1.3.7. In [15] and [1] the case $nm < 0$ has been considered too; another Bessel transform turns up in this case.

Generalization to Fuchsian groups of the first kind with cusps is given by Proskurin, [26]. For other weights than zero see [29] and [2].

1.3.8. In our sketch of the proof of the sum formula we have cheated a bit. The test function φ is related to ψ by the equation $\hat{\psi}(s) = \hat{\varphi}(s)$. It is not true that compactly supported φ correspond to compactly supported ψ.

Actually, one may prove the sum formula for a wider class of test functions, which is most easily characterized by imposing growth conditions on $\hat{\psi} = \hat{\varphi}$.

1.4. Use of the sum formula, examples:

1.4.1. Information on the growth of the $b_n(T_j)$ for n fixed: Kuznetsov [17], Deshouillers–Iwaniec [5].

1.4.2. Eigenvalues of Hecke operators: [1], [27].

1.4.3. Estimates of Kloosterman sums: Kuznetsov [15], [17], Proskurin [28], [29], Deshouillers–Iwaniec [5].

1.4.4. Arithmetical form of the Selberg trace formula: [16].

1.5. Questions

1.5.1. It would be nice if φ itself, instead of φ_H, would occur in the term with Kloosterman sums. This may be arranged. With some care one may sum the terms with J_{k-1} in the definition of φ_H separately and use the formula for Fourier coefficients of holomorphic Poincaré series in [25]. Then we obtain in the left-hand side of the sum formula an additional term

$$\sum_{k\geqslant 12, k \text{ even}} \hat{\varphi}\left(\tfrac{1}{2}(k-1)\right)\pi^{-1}\left(4\pi\sqrt{mn}\right)^{-k}(k-1)!\sum_{j=1}^{r_k}\overline{a_{k,j}(m)}\,a_{k,j}(n),$$

where the $f_{k,j}(z) = \sum_{n=1}^{\infty} a_{k,j}(n)e^{2\pi i n z}$ form an orthonormal basis of S_k with respect to the Petersson scalar product. The right-hand side of the sum formula becomes

$$\left(2\pi\sqrt{mn}\right)^{-1}\sum_{c=1}^{\infty} c^{-1}S(n, m; c)\,\varphi\left(4\pi\sqrt{mn}/c\right).$$

The δ-term disappears for compactly supported φ.

1.5.2. The appearance of the holomorphic cusp forms in a weight zero situation is a signal that we do not understand what we are doing. Clearly more of the spectrum is involved than the part which occurs in weight zero. We should try to understand the sum formula in $L^2(\Gamma\backslash G)$.

1.5.3. The Bessel functions in the sum formula, and also in Petersson's formula, ask for an explanation.

The Whittaker functions in the Fourier series expansion of automorphic forms arise naturally from the eigenfunction equation $\omega f = (\frac{1}{4}-s^2)f$ for f with prescribed left-N- and right-K-behaviour.

If we could interpret the test function φ as function on the group, the Bessel differential equation might arise in a similar way.

1.5.4. In the next sections we give a proof of the sum formula along representational lines, working in $L^2(\Gamma\backslash G)$. The principal objects in the spectral decomposition are automorphic models. This proof is discussed for n, $m > 0$ for the full modular group. For a more general case see [2].

II.2. Test functions

The basic idea of the proof is to compute the scalar product of two Poincaré series, as is done in Kuznetsov's proof. To build Poincaré series we need a set of auxiliary test functions which will be eliminated later on. The principal test functions correspond to the function φ in 1.3.2.

2.1. Auxiliary test functions

2.1.1. Recall the space $\mathscr{W}_1$ of functions on G defined in I.5.3.1. Let

$$S_0 = \{f \in \mathscr{W}_1 |\ |f| \in C_c(N\backslash G),\ f \text{ is } K\text{-finite on the right}\};$$

the condition of K-finiteness means that f is a sum of weight functions. S_0 is a $\mathfrak{g}$-space by differentiation.

2.1.2. One may define intertwining operators; for $s \notin \frac{1}{2} + Z$:

$$\omega(s)\colon\ S_0 \to H(s),$$

$$\omega(s) f = \sum_{r \text{ even}} \langle f, W^1(-\bar{s})\varphi_r\rangle_{N\backslash G}\,\varphi_r$$

and for $k \geqslant 2$, k even

$$\pi_k\colon\ S_0 \to D_k^+,$$

$$\pi_k f = \sum_{\substack{r \geqslant k \\ r \text{ even}}} \sqrt{k-1}\left(\frac{k+r}{2} - 1\right)!\,\frac{1}{\left(\dfrac{r-k}{2}\right)!}\,\langle f, W_k^1\varphi_r\rangle_{N\backslash G}\,\varphi_r,$$

$\langle .,.\rangle_{N\backslash G}$ means the scalar product obtained by integration over $N\backslash G$ with respect to the quotient measure; similarly for other spaces.

2.1.3. Proposition (spectral decomposition of S_0). *For f, $g \in S_0$:*

$$\langle f, g\rangle_{N\backslash G} = \frac{1}{2\pi i}\int_0^{i\infty} \langle \omega(s) f, \omega(s) g\rangle_s \frac{ds}{|\Gamma(2s)|^2} + \sum_{k \geqslant 2, k \text{ even}} \langle \pi_k f, \pi_k g\rangle_k .$$

$\langle .,.\rangle_s$ denotes the scalar product in $H(s)$, for $\mathrm{Re}\, s = 0$, and $\langle .,.\rangle_k$ one in D_k^+.

This proposition is in fact an inversion theorem for Whittaker transforms.

2.1.4. The space S_0 is too small for our purpose. The easiest way to describe the desired extension, is to work with $\omega(s)$ and π_k. For $f \in S_0$ we may view $\omega(s)f$ as a meromorphic section on each strip $\{s \in \mathbf{C} \mid |\mathrm{Re}\, s| \leqslant \sigma\}$ of the bundle of the $H(s)$. By imposing some growth conditions on sections of this bundle one may describe a larger space of functions S, contained in $\mathscr{W}_1$, and containing S_0.

For details see [2], § 13.5.

2.2. Principal test functions

2.2.1. $Q_0 = \{q \in C^\infty(G) \mid q(n(x)gn(x')) = e^{\frac{1}{2}i(x+x')} q(g),\ |q| \in C_c(N\backslash G/N)\}$ is a space of functions, the elements of which are completely determined by their values on the big cell $NAk(\frac{1}{2}\pi)N$ in the Bruhat decomposition $G = NA \cup NAk(\frac{1}{2}\pi)N$:

$$q(a(y)k(\tfrac{1}{2}\pi)) = y^{1/2} \varphi_q(y^{1/2})$$

for some $\varphi_q \in C_c^\infty(0, \infty)$. This definition of φ_q seems needlessly complicated, but it makes the equation $\omega q = (\frac{1}{4}-s^2)q$ amount to the Bessel differential equation for φ_q.

2.2.2. For $q \in Q_0$ and $f \in S_0$ we put

$$T_q f(g) = q * f(g) = \int_{G\backslash N} q(x) f(x^{-1}g)\, d\dot{x}.$$

This converges and $T_q f \in \mathscr{W}_1$.

2.2.3. PROPOSITION. *For $q \in Q_0$ put*

$$\hat{q}(s) = \frac{-1}{\sin \pi s} \int_0^\infty (J_{2s}(y) - J_{-2s}(y)) \varphi_q(y) \frac{dy}{y};$$

then $\hat{q}$ extends to a holomorphic even function on $\mathbf{C}$. For each f, $g \in S_0$:

$$\langle T_q f, g\rangle = \frac{1}{2\pi i} \int_0^{i\infty} \hat{q}(s) \langle \omega(s) f, \omega(s) g\rangle_s \frac{ds}{|\Gamma(2s)|^2} +$$
$$+ \sum_{\substack{k \geqslant 2,\\ k\,\mathrm{even}}} \hat{q}(\tfrac{1}{2}(k-1)) \langle \pi_k f, \pi_k g\rangle_k.$$

So T_q corresponds to multiplication by $\hat{q}$ in the spectral decomposition of S_0. Estimation of $\hat{q}(s)$ shows that T_q may be extended to S and has its values in S.

2.2.4. By imposing only $|\hat{q}(s)| < (1+|\mathrm{Im}\, s|)^{-a}$ on some strip $|\mathrm{Re}\, s| \leqslant \sigma$, with $\sigma > \frac{1}{2}$ fixed and $a > 2$ fixed and some condition on the $\hat{q}(\frac{1}{2}(k-1))$, we may enlarge Q_0 to a space of functions Q for which everything goes through. The complicated details may be found in [2], § 14.4.

2.3. Bessel inversion

2.3.1. The correspondence $q \mapsto \hat{q}$ may be extended still further. We may replace φ_q by the distribution $\delta(t)$, with $t > 0$, given by

$$\int_0^\infty \delta(t)(y)\,\varphi(y)\,\frac{dy}{y} = \pi t^{1/2}\,\varphi(t^{1/2}).$$

Then T_q corresponds to the operator $d(t)$: $S \to S$

$$d(t)f(g) = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}ix} f\big(k(\tfrac{1}{2}\pi)\,a(t^{-1})\,n(x)\,g\big)\,dx.$$

In the spectral decomposition $d(t)$ corresponds to multiplication by $\delta(t)^\wedge$:

$$\delta(t)^\wedge(s) = \frac{-\pi t^{1/2}}{\sin \pi s}\,\big(J_{2s}(t^{1/2}) - J_{-2s}(t^{1/2})\big).$$

2.3.2. Proposition (Kuznetsov). *For* $q \in Q$:

$$q\big(a(t)\,k(\tfrac{1}{2}\pi)\big) = \frac{1}{2\pi i}\int_0^{i\infty} \hat{q}(s)\,\overline{\delta(t)^\wedge(s)}\,\frac{-2s\sin 2\pi s}{1+\cos 2\pi s}\,ds +$$

$$+\frac{1}{2\pi}\sum_{\substack{k\geq 2,\\ k\,\text{even}}} (k-1)\,\hat{q}\big(\tfrac{1}{2}(k-1)\big)\,\overline{\delta(t)^\wedge\big(\tfrac{1}{2}(k-1)\big)}.$$

In fact this is the inversion theorem for the Bessel transform $\varphi \mapsto \hat{\varphi}$ occurring in the sum formula.

II.3. Poincaré series

3.1. Poincaré operators

3.1.1. Let $n \geqslant 1$. We define a $\mathfrak{g}$-intertwining operator Θ_n: $S \to L^2(\Gamma \backslash G)$ by

$$\Theta_n f(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f\big(a(4\pi n)\,\gamma g\big);$$

we call Θ_n a *Poincaré operator* and $\Theta_n f$ a *Poincaré series*. To get the absolute convergence of the series on compact sets, one needs some estimate of $f\big(p(z)k\big)$ for $y \downarrow 0$. As $g \mapsto f\big(a(4\pi n)g\big)$ is left-Γ_∞-invariant it is clear that $\Theta_n f$ is left-Γ-invariant. The boundedness of $\Theta_n f$ follows from the boundedness of $f\big(p(z)k\big)$ for $y \to \infty$.

3.1.2. Θ_n is adjoint to F_n: For $f \in S$, $g \in C^\infty(\Gamma\backslash G) \cap L^2(\Gamma\backslash G)$:

$$\langle \Theta_n f, g \rangle_{\Gamma\backslash G} = 4\pi n \langle f, F_n g \rangle_{N\backslash G}.$$

3.2. Petersson formula

3.2.1. The decomposition

$$\Gamma = \Gamma_\infty \,\underline{\cup} \bigcup_{\substack{c \geq 1, d \bmod c \\ (c,d)=1}} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty$$

leads to the formula of Petersson for the Fourier coefficients of holomorphic Poincaré series, [25]. The same computations lead to:

3.2.2. PROPOSITION (Petersson formula). *For* $m, n \geq 1$

$$F_n \Theta_m = \delta_{n,m} I + \frac{1}{4\pi n} \sum_{c=1}^{\infty} S(n, m; c)\, d(16\pi^2 nm/c^2).$$

This formula is a straightforward extension of Petersson's result, so we propose the name "Petersson formula".

The $S(n, m; c)$ are the Kloosterman sums. The operator $d(t)$ has been described in 2.3.1.

The convergence of the series is meant in the following way: For each $f \in S$ the series $\sum_{c=1}^{\infty} S(n, m; c)\, d(16\pi^2 nmc^{-2}) f$ converges uniformly on compact sets.

3.3. Scalar product of Poincaré series 1

Take $f, g \in S$ and $m, n \geq 1$. With help of the Petersson formula we get:

$$\langle \Theta_m f, \Theta_n g \rangle = 4\pi n \delta_{m,n} \langle f, g \rangle + \sum_{c=1}^{\infty} S(n, m; c) \langle f, d(16\pi^2 nmc^{-2}) g \rangle.$$

(To interchange the integral and the sum over c one has to use that the Poincaré series are absolutely convergent and that the sum of the absolute values is bounded.)

The expressions in the right-hand side may be expressed in $\omega(s) f$, $\omega(s) g$, $\pi_k f$ and $\pi_k g$ with use of Propositions 2.1.3 and 2.3.1.

3.4. Scalar product of Poincaré series 2

The scalar product $\langle \Theta_m f, \Theta_n g \rangle$ may also be computed with use of the spectral decomposition of $L^2(\Gamma\backslash G)$.

3.4.1. Let for instance $T \in \mathscr{A}^0(H(s))$, $\|T\varphi_0\| = 1$. The projection of $\Theta_n g$

onto $\overline{TH(s)}$ is given by

$$\sum_{r\,\text{even}} \langle \Theta_m f, T\varphi_r\rangle T\varphi_r = 4\pi m \sum_r \langle f, F_m T\varphi_r\rangle T\varphi_r = 4\pi m \sum_r \overline{b_m(T)} \langle f, W^1(s)\varphi_r\rangle T\varphi_r$$
$$= 4\pi m\, \overline{b_m(T)}\, \omega(s) f.$$

As T preserves length the contribution of $\overline{TH(s)}$ to $\langle \Theta_m f, \Theta_n g\rangle$ is $16\pi^2 nmb_n(T)\overline{b_m(T)}\langle \omega(s) f, \omega(s) g\rangle_s$.

3.4.2. In this way one obtains

$$\langle \Theta_m f, \Theta_n g\rangle = 16\pi^2 nm \Bigg[\frac{1}{2\pi i} \int_0^{i\infty} b_n(E(s))\overline{b_m(E(s))}\langle \omega(s) f, \omega(s) g\rangle_s\, ds +$$
$$+ \sum_{j=1}^{\infty} b_n(T_j)\overline{b_m(T_j)}\langle \omega(s_j) f, \omega(s_j) g\rangle_{s_j} +$$
$$+ \sum_{k\,\text{even},\,k\geq 12} \sum_{j=1}^{r_k} b_n(T_{k,j})\overline{b_m(T_{k,j})} \frac{1}{(k-1)(k-1)!} \langle \pi_k f, \pi_k g\rangle_k \Bigg].$$

The T_j have been introduced in 1.1.2. The $T_{k,j}\varphi_k$ correspond to an orthonormal basis of S_k.

II.4. Sum formula

4.1. Two expressions for $\langle \Theta_m T_q f, \Theta_n g\rangle$

In the computations in 3.3 and 3.4 we may replace f by $T_q f$ with $q \in Q$.

Put $\Phi(s) = \hat{q}(s)\langle \omega(s) f, \omega(s) g\rangle_s$, $\Phi_k = \hat{q}\left(\frac{k-1}{2}\right)\langle \pi_k f, \pi_k g\rangle_k$. Then the computations above yield:

$$16\pi^2 nm \Bigg[\frac{1}{2\pi i} \int_0^{i\infty} b_n(E(s))\overline{b_m(E(s))}\,\Phi(s)\, ds + \sum_{j=1}^{\infty} b_n(T_j)\overline{b_m(T_j)}\,\Phi(s_j) +$$
$$+ \sum_{k\geq 12,\,k\,\text{even}} \sum_{j=1}^{r_k} b_n(T_{k,j})\overline{b_m(T_{k,j})} \frac{\Phi_k}{(k-1)(k-1)!} \Bigg]$$
$$= 4\pi n \delta_{n,m} \Bigg[\frac{1}{2\pi i} \int_0^{i\infty} \Phi(s) \frac{ds}{|\Gamma(2s)|^2} + \sum_{k\geq 2,\,k\,\text{even}} \Phi_k \Bigg] +$$

$$+\sum_{c=1}^{\infty} S(n,m;c)\left[\frac{1}{2\pi i}\int_0^{i\infty} \Phi(s)\,\delta\,\overline{\left(\frac{16\pi^2 nm}{c^2}\right)^{\wedge}(s)}\,\frac{ds}{|\Gamma(2s)|^2}+\right.$$

$$\left.+\sum_{k\geq 2,k\,\mathrm{even}} \Phi_k\cdot\delta\,\overline{\left(\frac{16\pi^2 nm}{c^2}\right)^{\wedge}\left(\frac{k-1}{2}\right)}\right].$$

Now take $f,\ g\in S$ in such a way that

$$\langle \omega(s) f,\ \omega(s) g\rangle_s = \tfrac{1}{2}\pi\cdot(\cos\pi s)^{-2} \quad\text{and}\quad \langle \pi_k f,\ \pi_k g\rangle_k = \frac{1}{2\pi}(k-1).$$

Then the equality becomes:

4.2.1. Sum formula. *For each $q\in Q$:*

$$\frac{1}{2\pi i}\int_0^{i\infty} b_n(E(s))\overline{b_m(E(s))}\,\hat q(s)\,\frac{\pi}{(\cos\pi s)^2}\,ds+$$

$$+\sum_{j=1}^{\infty} b_n(T_j)\overline{b_m(T_j)}\,\hat q(s_j)\,\frac{\pi}{(\cos\pi s_j)^2}+$$

$$+\sum_{k\geq 12,k\,\mathrm{even}}\ \sum_{j=1}^{r_k} b_n(T_{k,j})\overline{b_m(T_{j,k})}\,\hat q\left(\frac{k-1}{2}\right)\frac{1}{\pi(k-1)!}$$

$$=\delta_{m,n}\left[-\frac{2}{\pi m}\frac{1}{2\pi i}\int_0^{i\infty}\hat q(s)\,s\tan\pi s\,ds+\sum_{k\geq 2,k\,\mathrm{even}}\frac{k-1}{4\pi^2 m}\,\hat q\left(\frac{k-1}{2}\right)\right]+$$

$$+\frac{1}{8\pi^2 nm}\sum_{c=1}^{\infty} S(n,m;c)\,q\left(a\left(\frac{16\pi^2 nm}{c^2}\right)k(\tfrac{1}{2}\pi)\right).$$

In the last term we have used Proposition 2.3.2. To compare this result with those in 1.3.2 and 1.5.1 use:

The expression for $b_n(E(s))$ in I.5.5.4.

The formula for W_k^1 in I.5.3.3, the Fourier series expansion in I.5.1.4 and the relation in I.2.3.3 for the discrete series term.

The definition of φ_q for the last term.

In general the $\delta_{n,m}$-term does not disappear; but if $q\in Q_0$, then write

$$\frac{1}{2\pi i}\int_0^{i\infty}\hat q(s)\,s\tan\pi s\,ds=\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=0}\frac{-1}{\sin\pi s}\int_0^{\infty} J_{2s}(y)\,\varphi_q(y)\,\frac{dy}{y}\,s\tan\pi s\,ds.$$

$\int_0^{\infty} J_{2s}(y)\,\varphi_q(y)\,\frac{dy}{y}$ satisfies estimates permitting to move off the line of integration to the right, to pick up the k-terms as residues and finally to obtain zero.

4.2.2. If there would be f, $g \in S$ with the properties given above, the proof would be finished. As there are not, there is some non-trivial work left; see [2], §1.4.8 or §16.2, §16.3. The idea is to approximate the desired properties. The problem is to take the limit inside all integrals and infinite sums.

4.3. Comparison

The proof sketched in §1 uses a convolution operator T_ψ in $L^2(\Gamma\backslash\mathfrak{h})$, which is multiplication by $\hat{\psi}$ on the spectrum of $L^2(\Gamma\backslash\mathfrak{h})$.

In the proof in §2.4 we have a convolution operator T_q in S, which is multiplication by $\hat{q}$ on the spectrum of S. By means of the Poincaré operator Θ_m it also gives an operator in $L^2(\Gamma\backslash G)$, which is multiplication by $\hat{q}$ on the spectrum of $L^2(\Gamma\backslash G)$. If we restrict this operator to weight zero we may relate T_ψ and T_q by $\hat{\psi}(s) = \hat{q}(s)$ for $\operatorname{Re} s = 0$. The use of T_q instead of T_ψ makes clear where the Bessel functions come from.

The occurrence of the D_k^+ in the sum formula is a consequence of their occurrence in the spectral decomposition of S.

4.4. Generalizations

4.4.1. If $nm < 0$, one needs $\mathscr{W}_1$ and $\mathscr{W}_{-1}$. In G these spaces are not equivalent. In $G \cup jG$, with $j = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, one has

$$J\colon \mathscr{W}_1 \to \mathscr{W}_{-1}, \qquad Jf(g) = f(jg).$$

Working on $G \cup jG$ one may handle the cases $mn > 0$ and $mn < 0$ more or less simultaneously. For $mn < 0$ the sum formula is

$$\frac{1}{2\pi i}\int_0^{i\infty} \tilde{\varphi}(s)\,|n|^{-1/2-s}|m|^{-1/2+s}|\zeta(1+2s)|^{-2}\cos\pi s\,\sigma_{2s}(|n|)\,\sigma_{-2s}(|m|)\,ds + $$
$$+\sum_{j=1}^{\infty}\tilde{\varphi}(s_j)\,b_n(T_j)\,\overline{b_m(T_j)}\,\frac{\pi}{\cos\pi s_j}$$
$$= \frac{1}{2\pi\sqrt{|mn|}}\sum_{c=1}^{\infty} c^{-1}S(n, m; c)\,\varphi(4\pi\sqrt{|mn|}\,c^{-1})$$

with e.g. $\varphi \in C_c^\infty(0, \infty)$ and

$$\tilde{\varphi}(s) = \frac{4}{\pi}\int_0^\infty \varphi(y)\,K_{2s}(y)\,\frac{dy}{y}, \qquad \varphi(y) = \frac{-4}{2\pi i}\int_0^{i\infty}\tilde{\varphi}(s)\,K_{2s}(y)\,s\sin 2\pi s\,ds$$

(Lebedev transform, see [8], Ch. XII).

4.4.2. Generalization to Fuchsian groups of the first kind, with cusps, meets no essential problem.

4.4.3. To generalize the sum formula to odd weights one has to work on $\mathrm{Sl}_2(\boldsymbol{R})$ or on $\mathrm{Sl}_2(\boldsymbol{R}) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{Sl}_2(\boldsymbol{R})$. For arbitrary real weights, one uses the universal covering group.

III. BILINEAR RELATIONS

In this part we discuss two bilinear relations which are satisfied by the Fourier coefficients of automorphic forms.

III.1. Introduction

In the holomorphic case multiplication gives a bilinear map $M_k \times M_l \to M_{k+l}$. In this way one may, for instance, compute the Fourier coefficients of

$$\Delta = \frac{1}{1728}\left((E_4)^3 - (E_6)^2\right) \in S_{12}.$$

Products of real analytic automorphic forms are not very nice functions. But there are some ways of assigning bilinearly some other objects to pairs of real analytic automorphic forms.

III.2. Periods

2.1. Maass–Selberg relation

2.1.1. Let f, g be function on $\mathfrak{h}$ satisfying $Lf = \lambda f$ and $Lf = \lambda g$. Then

$$\eta = \eta(f, g) = g\,\frac{\partial f}{\partial z}\,dz + f\,\frac{\partial g}{\partial \bar{z}}\,d\bar{z}$$

is a closed differential form on $\mathfrak{h}$, and for $\gamma \in G$:

$$\eta(f, g) \circ \gamma = \eta(f \circ \gamma, g \circ \gamma).$$

If f, $g \in \mathscr{F}_0(\lambda)$ then η is a closed Γ-invariant 1-form on $\mathfrak{h}$; so we may view it as a closed 1-form on $\Gamma \backslash \mathfrak{h}$.

If we integrate η along the contour in the figure, we obtain zero. As the integrals along a and b and along c and d cancel each other, we are left with

$$\int_e \eta = 0.$$

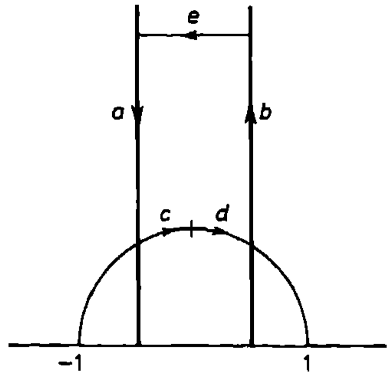

Fig. 4

If one moves e towards the cusp, the integral $\int_e \eta$ tends to a bilinear expression in the Fourier coefficients of order zero of f and g. The resulting equation is the Maass–Selberg relation. (See e.g. [20], p. 331–333 or [32], § 9.)

2.1.2. One may view η as a closed 1-form on $X(\Gamma)$ with a singularity at the cusp. Integrating around the cusp we obtain the residue of η at the cusp. As the sum of all residues has to be zero, we obtain the Maass–Selberg relation.

This procedure also applies to more general Γ and to exponentially growing Γ-invariant eigenfunctions of L.

2.2. Periods

2.2.1. Now let at least one of f and g be a cusp form. The 1-form η on $X(\Gamma)$ still has a singularity at the cusp, but it decreases quickly and the residue vanishes. Moreover, for $z \in \Gamma\backslash\mathfrak{h}$ the integral

$$\int_z^{\text{cusp}} \eta = \lim_{\substack{u \to \text{cusp} \\ u \in \Gamma\backslash\mathfrak{h}}} \int_z^u \eta$$

is well defined. So η determines an element of

$$H_1(X(\Gamma), C)^{\text{dual}} \cong H^1(X(\Gamma), C).$$

2.2.2. For the full modular group Γ it is known that $H^1(X(\Gamma), C) = 0$. As $\{i\infty\} \cup Q$ corresponds to the cusp we get

$$\int_\sigma^{i\infty} \eta(f, g) = 0 \quad \text{for all } \sigma \in Q.$$

2.2.3. Let $\sigma = a/c$, with $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma$. Then $\gamma \cdot \left(-\frac{d}{c} + iy\right) = \frac{a}{c} + \frac{i}{c^2 y}$.

Now

$$\int_{\sigma}^{i\infty} \eta(f, g) = \frac{i}{2} \int_{a/c}^{i\infty} \left(g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}\right) dy.$$

The values of g and $\frac{\partial g}{\partial x}$ on the line $\left\{\frac{a}{c}+iy \mid y>0\right\}$ are related to the values on the line $\left\{-\frac{d}{c}+iy' \mid y'>0\right\}$. Using the Fourier series expansion of f on $\left\{\frac{a}{c}+iy \mid y>0\right\}$ and of g on $\left\{-\frac{d}{c}+iy' \mid y'>0\right\}$ we arrive, for f and g both cusp forms, at:

2.2.4. PROPOSITION. *Let f, $g \in \mathscr{F}_0(\frac{1}{4}-s^2)$, ε, $\theta \in \{1, -1\}$,*

$$f(z) = \sum_{n=1}^{\infty} a_n W_{0,s}(4\pi ny)(e^{2\pi inx} + \varepsilon e^{-2\pi inx}),$$

$$g(z) = \sum_{m=1}^{\infty} b_m W_{0,s}(4\pi my)(e^{2\pi imx} + \theta e^{-2\pi imx}).$$

Then for all a, c, $d \in \mathbf{Z}$, $c \geqslant 1$, $ad \equiv 1 \bmod c$:

$$\sum_{n,m \geqslant 1} (nm)^{1/4} W_{0,2s}(8\pi\sqrt{nm}\, c^{-1}) a_n b_m (e^{2\pi i(na-md)/c} - \varepsilon\theta e^{2\pi i(md-na)/c}) = 0.$$

Remark that the operator T_{-1}, defined by $T_{-1}f(z) = f(-\bar{z})$, maps $\mathscr{F}_0(\frac{1}{4}-s^2)$ into itself; so the restriction to eigenfunctions of T_{-1} in the proposition causes no loss of generality.

In the case that f is a cusp form and $g = e_{s,0}$, one obtains a more complicated expression. This result is similar to Theorem 8 on p. 58 of [15].

2.2.5. Once one has defined η it seems obvious to consider $\int_{a/c}^{i\infty} \eta$ for f and g cusp forms and to arrive at Proposition 2.2.4. I do not know a reference where it has been done.

2.2.6. For general Γ one has $H^1(X(\Gamma), \mathbf{C}) \cong S_2 \oplus \overline{S_2}$. So for each cusp σ, and $\gamma \in \Gamma$

$$\int_{\sigma}^{\gamma\sigma} \eta = \int_{\sigma}^{\gamma\sigma} (h_1\, dz + \overline{h_2}\, d\bar{z})$$

with h_1, $h_2 \in S_2$. The expression in Proposition 2.2.4 is a linear combination of periods of cusp forms of weight 2 and their conjugates in this case, provided $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

If σ and τ correspond to non-equivalent cusps one might want to use the Manin–Drinfeld theorem that $\int_\sigma^\tau$ determines an element of $H_1(X(\Gamma), \mathbf{Q})$, but $\eta - h_1 d_2 - \overline{h_2}\, d\bar{z} = d\varphi$ for some unknown function φ. (See [24], [6] or [19], Ch. IV.)

III.3. Cohomology and automorphic models

In Section III.2 we worked with automorphic forms of weight zero only, contrary to the representational viewpoint of parts I and II. Using automorphic models gives some additional insight.

3.1. Let U be a $\mathfrak{g}$-space; put $U_r = \{u \in U \mid Wu = iru\}$. Define a complex $C^{\cdot}(U)$ by

$$\begin{array}{ccccc} 0 \to C^0(U) & \underset{d^0}{\to} & C^1(U) & \underset{d^1}{\to} C^2(U) \to 0 \\ \| & & \| & \| \\ U_0 & & U_2 \oplus U_{-2} & U_0 \end{array}$$

$$d^0 u = (E^+ u, E^- u), \quad d^1(u_+, u_-) = E^- u_+ - E^+ u_-.$$

Put $H^{\cdot}(U) = h^{\cdot}(C^{\cdot}(U))$, the cohomology of the complex.

3.2. For each $T \in \mathscr{A}(U)$ one may define a map of complexes $T^{\cdot}$: $C^{\cdot}(U) \to \Omega^{\cdot}(\Gamma \backslash \mathfrak{h})$, where $\Omega^{\cdot}(\Gamma \backslash \mathfrak{h})$ is the De Rham complex.

$$T^0 u(z) = -\tfrac{1}{4} i Tu(p(z)),$$

$$T^1(u_+, u_-)(z) = Tu_+(p(z)) y^{-1} dz - Tu_-(p(z)) y^{-1} d\bar{z},$$

$$T^2 u(z) = -\tfrac{1}{4} i\, Tu(p(z)) y^{-2}\, dz\, d\bar{z}.$$

This gives a map $T^{\cdot}$: $H^{\cdot}(U) \to H^{\cdot}(\Gamma \backslash \mathfrak{h}, \mathbf{C})$.

3.3. Let $s \notin \frac{1}{2} + \mathbf{Z}$; take $U = H(s) \otimes_{\mathbf{C}} H(s)$, with $\mathfrak{g}$-action

$$X \cdot \varphi \otimes \psi = (X\varphi) \otimes \psi + \varphi \otimes (X\psi).$$

One may show that $H^1(H(s) \otimes_{\mathbf{C}} H(s))$ has dimension one; a generator is represented by $(\varphi_2 \otimes \varphi_0, \varphi_0 \otimes \varphi_{-2})$. If T_1, $T_2 \in \mathscr{A}(H(s))$ then

$$T_1 \otimes T_2: \ \varphi \otimes \psi \mapsto T_1 \varphi \cdot T_2 \psi$$

defines an automorphic model of $H(s) \otimes_{\mathbf{C}} H(s)$. A computation shows that

$$(T_1 \otimes T_2)^1 (\varphi_2 \otimes \varphi_0, \varphi_0 \otimes \varphi_{-2}) = \frac{4i}{1+2s}\, \eta(T_1 \varphi_0, T_2 \varphi_0).$$

So the cohomology class of $\eta(f, g)$ only depends on the automorphic models associated to f and g.

3.4. For more general Γ one may find $h_1 \in S_2$, $\bar{h}_2 \in \bar{S}_2$ for each pair T_1, $T_2 \in \mathscr{A}(H(s))$ such that

$$\int_{\gamma \cdot i\infty}^{i\infty} \eta(T_1 \varphi_0, T_2 \varphi_0) = \int_{\gamma \cdot i\infty}^{i\infty} (h_1\, dz + \bar{h}_2\, d\bar{z})$$

for all $\gamma \in \Gamma$. Let us identify S_2 and $\overline{S_2}$ with the corresponding subspaces of $L^2(\Gamma \backslash G)$ of weight 2 resp. -2. Then one may show that h_1 resp. $\bar{h}_2$ are, up to a constant factor, equal to the orthogonal projections of $T_1 \varphi_2 \cdot T_2 \varphi_0$ resp. $T_1 \varphi_0 \cdot T_2 \varphi_{-2}$ onto S_2 resp. $\overline{S_2}$.

III.4. Extension of automorphic models

4.1. Introduction

In Section 3.4 we concerned ourselves with the projection of products of two automorphic forms onto subspaces of $L^2(\Gamma \backslash G)$ of discrete series type.

We now ask whether discrete series subspaces of $L^2(\Gamma \backslash G)$ may actually consist of linear combinations of products of automorphic forms. More precisely: May $T_1 \otimes T_2(H(s) \otimes_{\mathbf{C}} H(s_1))$ contain a subspace of type D_k^+, for $T_1 \in \mathscr{A}(H(s))$, $T_2 \in \mathscr{A}(H(s_1))$?

In the L^2-case we see in [30] that the tensor product of e.g. two unitary principal series representations contains discrete series subspaces with multiplicity at most one. As $H(s)$ contains only K-finite elements, $H(s) \otimes_{\mathbf{C}} H(s_1)$ is too small to contain such spaces.

To approach this question it is necessary to embed the $\mathfrak{g}$-space $H(s)$ in larger spaces in which G acts.

4.2. Extension of the standard model

4.2.1. The standard model $\mathrm{St}(s)$ represents $H(s)$ as a space of functions on G satisfying

$$f(p(z)k) = y^{1/2+s} f(k),$$

see I.5.2.1. These functions are determined by their values on K. Let H_0 be the space corresponding to $L^2(K)$. Clearly G acts in H_0 by right translation. This representation is an induced one from P to G. On K it is described by

$$\varrho_s(g) \cdot f(k(\theta)) = j(\theta, g)^{-1/2-s} f\big(k(\Theta(\theta, g))\big),$$

$$j\left(\theta, \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (-a \sin\theta + c\cos\theta)^2 + (-b\sin\theta + d\cos\theta)^2,$$

$$\Theta\left(\theta, \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \arg\{(-b\sin\theta + d\cos\theta) + i(a\sin\theta - c\cos\theta)\}.$$

4.2.2. Let H_∞ be the subspace of H_0 the elements of which are C^∞-functions of θ. It is clearly invariant under $\varrho_s(G)$. The elements of H_∞ are C^∞-vectors for (ϱ_s, H_0), see [18], VI, § 5.

The derived action of $\mathfrak{g}$ in H_∞ we denote also by ϱ_s. The K-finite elements in H_∞ are the polynomials in $e^{2i\theta}$ and $e^{-2i\theta}$. Let $\varphi_r(k(\theta)) = e^{ir\theta}$, then $\varphi_r \in H_\infty$ corresponds to $\mathrm{St}(s)\varphi_r \in \mathrm{St}(s)H(s)$. So the $\mathfrak{g}$-space $H(s)$ may be identified with the space H_K of K-finite elements of (ϱ_s, H_∞).

4.2.3. So we may consider an automorphic model $T \in \mathscr{A}(H(s))$ as a linear map $T\colon H_K \to C^\infty(\Gamma\backslash G)$, intertwining the action of $\varrho_s(\mathfrak{g})$ in H_K and the $\mathfrak{g}$-action in $C^\infty(\Gamma\backslash G)$ by differentiation.

One may ask whether we may extend T to an intertwining operator $H_\infty \to C^\infty(\Gamma\backslash G)$ for the $\mathfrak{g}$-action, or even $H_0 \to C^?(\Gamma\backslash G)$ for $\varrho_s(G)$ in H_0 and right translation in $C^?(\Gamma\backslash G)$.

4.3. Sobolev spaces

4.3.1. For $r \geqslant 0$, r integral, the Sobolev space H_r is the space of those elements in H_0 which posses derivatives up to order r in the L^2-sense; see [18], app. 4. Then $H_\infty = \bigcap_{r\geqslant 0} H_r$ and $\ldots \subset H_r \subset H_{r-1} \subset \ldots \subset H_1 \subset H_0$. From the formulas in 4.2.1 follows that $\varrho_s(G)H_r \subset H_r$.

4.3.2. Proposition. *For each $T \in \mathscr{A}(H(s))$ there is an $r \geqslant 0$ such that the operator $T\colon H_K \to C^\infty(\Gamma\backslash G)$ may be extended to an intertwining operator*

$$T\colon (\varrho_s, H_r) \to (\text{right translation}, C^0(\Gamma\backslash G)).$$

For cuspidal T one may take $r = 3$. For $E(s)$ the number r depends on s.

4.3.3. A proof, using the Fourier series expansion of automorphic models, is given in [3].

4.4. Invariant distributions

4.4.1. The dual space of H_r is denoted H_{-r}; one has $H_1 \subset H_0 \subset H_{-1} \subset H_{-2}\ldots$ The elements of $H_{-\infty} = \bigcup_{r\geqslant 0} H_{-r}$ are called distributions. See [18], app. 4. One may extend ϱ_s to an action of G on $H_{-\infty}$ such that (ϱ_s, H_{-r}) and (ϱ_{-s}, H_r) are contragradient.

4.4.2. Let $T\colon H_r \to C^0(\Gamma\backslash G)$ be an extension of an automorphic model as in Proposition 4.3.2. One may show that $\alpha\colon \varphi \mapsto T\varphi(e)$ is an element of H_{-r}. Moreover, for $\gamma \in \Gamma$:

$$\begin{aligned}(\varphi, \varrho_{-s}(\gamma)\cdot\alpha) &= (\varrho_s(\gamma)^{-1}\varphi, \alpha) = T\varrho_s(\gamma)^{-1}\varphi(e)\\ &= T\varphi(\gamma^{-1}) = T\varphi(e) = (\varphi, \alpha),\end{aligned}$$

so $\alpha \in H_{-r}$ is a $\varrho_{-s}(\Gamma)$-invariant distribution. Conversely, such distributions always give automorphic models by

$$T_\alpha \varphi(g) = (\varrho_s(g) \cdot \varphi, \alpha).$$

So

$$\mathcal{A}(\Gamma, H(s)) \cong (H_{-\infty})^{\varrho_{-s}(\Gamma)} = \{\alpha \in H_{-\infty} | \ \varrho_{-s}(\gamma) \cdot \alpha = \alpha \text{ for all } \gamma \in \Gamma\}.$$

4.4.3. This result is the duality theorem in [9]. Proposition 4.3.2 gives for cusp forms the additional information that r may be taken to be 3.

III.5. Tensor products of automorphic models

Now we return to the question posed in Section 4.1.

5.1. Tensor products

5.1.1. The tensor product of the representations (ϱ_s, H_0) and (ϱ_{s_1}, H_0) of G is realized in $L^2(K \times K)$ by

$$\varrho_s \otimes \varrho_{s_1}(g) \cdot \varphi(k(\theta), k(\eta))$$
$$= j(\theta, g)^{-1/2-s} j(\eta, g)^{-1/2-s_1} \varphi\big(k(\Theta(\theta, g)), k(\Theta(\eta, g))\big).$$

This formula makes sense not only for $H_0^{(2)} = L^2(K \times K)$, but for $H_r^{(2)}$ for all $r \geqslant 0$. Here $H_r^{(2)}$ denotes the two-dimensional Sobolev space; see [18], app. 4.

For $r \leqslant 0$ we define $\varrho_s \otimes \varrho_{s_1}$ to be the contragradient of $\varrho_{-s} \otimes \varrho_{-s_1}$, as we did in the one-dimensional case.

Remark that the action $\varrho_s \otimes \varrho_{s_1}(\mathfrak{g})$ on $H_\infty^{(2)}$ is given by

$$\varrho_s \otimes \varrho_{s_1}(X) \cdot \varphi \otimes \psi = \varrho_s(X) \varphi \otimes \psi + \varphi \otimes \varrho_{s_1}(X) \psi.$$

5.1.2. Take $r \geqslant 0$ and k even, $k > 2r + 2\mathrm{Re}(s+s_1)+1$, $k \geqslant 2$. Then

$$q_k = \sum_{n \in \mathbf{Z}} \cos \pi s_1 \frac{\Gamma(\frac{1}{2}+s_1+n)}{\Gamma(\frac{1}{2}-s+n+\frac{1}{2}k)} \varphi_{2n+k} \otimes \varphi_{-2n}$$

is an element of $H_r^{(2)}$. It has weight k and it generates a $\varrho_s \otimes \varrho_{s_1}(\mathfrak{g})$-subspace of $H_{-\infty}^{(2)}$ isomorphic to D_k^+. Up to a scalar, q_k is the only element of $H_{-\infty}^{(2)}$ with this property.

5.1.3. The $\varrho_s \otimes \varrho_{s_1}(\mathfrak{g})$-space generated by q_k is contained in $H_{r-4}^{(2)}$ if $r \geqslant 4$. For the map F from G into the Banach space $H_{r-4}^{(2)}$ given by $F(g) = \varrho_s \otimes \varrho_{s_1}(g) \cdot q_k$ is differentiable and satisfies $\left(\omega - \frac{i}{2} k + \frac{1}{4} k^2\right) F = 0$. So F is an analytic map, as it satisfies an elliptic differential equation. So q_k is an analytic vector in $(\varrho_s \otimes \varrho_{s_1}, H^{(2)}_{r-4})$, and $\varrho_s \otimes \varrho_{s_1}(\mathfrak{g})$ preserves analyticity.

5.2. The bilinear map

5.2.1. Let $T_1 \in \mathscr{A}(H(s))$ and $T_2 \in \mathscr{A}(H(s_1))$; take $r \geqslant 0$ such that the corresponding distributions α and β are elements of H_{-r}. Define

$$\alpha \otimes \beta = \sum_{m, n \text{ even}} (\alpha, \varphi_{-m})(\beta, \varphi_{-n})\, \varphi_m \otimes \varphi_n,$$

then $\alpha \otimes \beta \in H^{(2)}_{-2r}$. It is invariant under $\varrho_{-s} \otimes \varrho_{-s_1}(\Gamma)$.

5.2.2. In the same way as in Section 4.4 the invariant distribution $\alpha \otimes \beta$ defines an intertwining operator $T_1 \otimes T_2$: $H^{(2)}_{2r} \to C^0(\Gamma \backslash G)$ by

$$(T_1 \otimes T_2 \cdot \chi)(g) = \left(\varrho_s \otimes \varrho_{s_1}(g) \cdot \chi, \alpha \otimes \beta\right).$$

Let k be even, $k > 4r + 2\mathrm{Re}(s + s_1) + 9$. If we restrict $T_1 \otimes T_2$ to the subspace generated by q_k, defined in 5.1.2 and in 5.1.3 shown to be included in $H^{(2)}_{2r}$. we obtain a model of D_k^+ in $C^0(\Gamma \backslash G)$. After checking the growth of the functions involved, we see that it is an automorphic model. So this restriction may be viewed as an element of $\mathscr{A}(D_k^+)$.

5.2.3. In particular

$$h(z) = y^{-\frac{1}{2}k} \cdot (T_1 \otimes T_2 \cdot q_k)(p(z))$$

is an holomorphic form of weight k; it even turns out to be a cusp form.

5.2.4.

$$y^{\frac{1}{2}k} h(z) = \sum_{n \in \mathbf{Z}} \cos \pi s_1 \frac{\Gamma(\frac{1}{2} + s_1 + n)}{\Gamma(\frac{1}{2} - s + n + \frac{1}{2}k)} T_1 \varphi_{2n+k}(p(z))\, T_2 \varphi_{-2n}(p(z)),$$

so indeed h is an infinite sum of products of automorphic forms.

5.2.5. In another way, see [3], § 6, one may show that 5.2.4 defines $h \in M_k$ already under the assumption $k > 4r + 2\mathrm{Re}(s + s_1)$.

5.2.6. So we have defined a bilinear map

$$\mathscr{A}(H(s)) \times \mathscr{A}(H(s_1)) \to M_k.$$

5.3. Fourier coefficients

5.3.1. One might want to describe the bilinear map in 5.2.6 in terms of the Fourier coefficients of the automorphic models involved in it. In [3] this led to some pages of explicit computations.

5.3.2. To state the results put

$$G_k(s, s_1; t, u) = 2\cos \pi s_1 \cdot (k-1)! \left\{ \frac{\Gamma(-2s_1) u^{2s_1} (t+u)^{-\frac{1}{2}k - s_1 + s}}{\Gamma(\frac{1}{2}k - s_1 - s)\Gamma(\frac{1}{2}k - s_1 + s)} \times \right.$$

$$\times {}_2F_1\left[\tfrac{1}{2}k + s_1 - s,\ 1 - \tfrac{1}{2}k + s_1 - s;\ 1 + 2s_1;\ \frac{u}{t+u}\right] +$$

$$+ \frac{\Gamma(2s_1)(t+u)^{-\frac{1}{2}k + s_1 + s}}{\Gamma(\frac{1}{2}k + s_1 - s)\Gamma(\frac{1}{2}k + s_1 + s)} \times$$

$$\left. \times {}_2F_1\left[\tfrac{1}{2}k - s_1 - s,\ 1 - \tfrac{1}{2}k - s_1 - s;\ 1 - 2s_1;\ \frac{u}{t+u}\right]\right\}$$

for $t, u > 0$;

$$G_k(s, s_1; t, u) = 2\cos \pi s_1\, t^{-\frac{1}{2}k + s_1 + s}\, {}_2F_1\left[\tfrac{1}{2}k - s_1 + s,\ \tfrac{1}{2}k - s_1 - s;\ k;\ 1 + \frac{u}{t}\right]$$

for $0 < -u < t$; and

$$G_k(s, s_1; t, u)$$

$$= 2\cos \pi s_1 (-1)^{\frac{1}{2}k} u^{-\frac{1}{2}k + s_1 + s}\, {}_2F_1\left[\tfrac{1}{2}k + s_1 - s,\ \tfrac{1}{2}k - s_1 - s;\ k;\ 1 + \frac{t}{u}\right]$$

for $0 < -t < u$.

For fixed t and u one may extend $G_k(s, s_1; t, u)$ as a holomorphic function of s and s_1.

5.3.3. Proposition. *Let f and g be cusp forms of weight zero;*

$$f(z) = \sum_{n \neq 0} a_n W_{0,s}(4\pi |n| y) e^{2\pi i n x},$$

$$g(z) = \sum_{n \neq 0} b_n W_{0,s_1}(4\pi |n| y) e^{2\pi i n x}.$$

Let $k \geqslant 18$, k even, and put for $m \geqslant 1$

$$c_m = m^{k-1} \sum_{\substack{n \neq 0 \\ n \neq m}} a_n b_m |n|^{1/2 - s} |m-n|^{1/2 - s_1} G_k(s, s_1; 2\pi n, 2\pi(m-n)).$$

Then $\sum_{m=1}^{\infty} c_m e^{2\pi i m z}$ is a holomorphic cusp form of weight k.

5.3.4. Proposition. *Let f be as in 5.3.3. Let $h \in S_{k_1}$, $k_1 \geqslant 12$, k_1 even;*

$$h(z) = \sum_{n=1}^{\infty} d_n e^{2\pi i n z}.$$

Let k be even, $k \geqslant k_1+8$. Put for $m \geqslant 1$:

$$c_m = m^{k-1} \sum_{\substack{n \geqslant 1 \\ n \neq m}} d_n a_{m-n} |m-n|^{1/2+s} n^{1-k_1} G_k\left(\tfrac{1}{2}(k_1-1),\ -s;\ 2\pi n,\ 2\pi(m-n)\right).$$

Then $\sum_{m=1}^{\infty} c_m e^{2\pi i m z}$ is a holomorphic cusp form of weight k.

5.3.5. *Remark.* Take in Proposition 5.3.4:

$$h = \Delta \in S_{12}, \quad k = 26.$$

Then the c_m have to be multiples of the Fourier coefficients p_m of $\Delta \cdot E_{14}$. So we get from 5.3.4 an infinite system of equalities

$$\lambda p_m = m^{25} \sum_{\substack{n \leqslant m-1 \\ n \neq 0}} a_n \cdot \left\{ d_{m-n} |n|^{\frac{1}{2}+s} (m-n)^{-11} \cdot G_{26}\left(11/2,\ -s;\ 2\pi(m-n),\ 2\pi n\right) \right\}$$

in λ and the a_n. All other quantities are known in principle. Moreover, we may assume that f is a normalized eigenfunction of all Hecke operators (not explained in these lectures); then $a_1 = 1$ and $|a_n| \leqslant \sigma_{-1}(n)$. The set of $s \in i\boldsymbol{R}$ for which this infinite system of equalities and inequalities has a solution contains the cuspidal spectrum of $L^2(\Gamma \backslash \mathfrak{h})$.

References

[1] R. W. Bruggeman, *Fourier coefficients of cusp forms*, Invent. Math. 45 (1978), 1–18.

[2] —, *Fourier coefficients of automorphic forms*, Lect. Notes Math. 865, Springer, Berlin–Heidelberg–New York 1981.

[3] —, *Automorphic forms and invariant distributions*, preprint Math. Inst. Utrecht, No. 202, 1981.

[4] P. Deligne, *La conjecture de Weil I*, Publ. Math. Inst. Hautes Et. Scient. 43 (1974), 273–307.

[5] J.-M. Deshouillers and H. Iwaniec, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. 70 (1982), 219–288.

[6] V. G. Drinfeld, *Two theorems on modular curves*, Functional Anal. Appl. 7 (2) (1973), 155–156; transl. from Funktsional Anal. i Prilozhen. 7 (2) (1973), 83–84.

[7] J. Elstrodt, *Die Selbergsche Spurformel für kompakte Riemannsche Flächen*, Jahresber. Deutsch. Math.-Verein. 83 (1981), 45–77.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of integral transforms II*, McGraw-Hill, New York–Toronto–London 1954.

[9] I. Gelfand, M. Graev, I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, Moscow 1966; transl. W. B. Saunders, 1969.

[10] A. Good, *Cusp forms and eigenfunctions of the Laplacian*, Math. Ann. 255 (1981), 523–548.

[11] E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, ibid. 112 (1936), 664–699.

[12] D. A. Hejhal, *Some observations concerning eigenvalues of the Laplacian and Dirichlet*

series; in: *Recent Progress in Analytic Number Theory* (Symp. Durham 1979), Vol. II, Academic Press, London 1981, pp. 95–110.

[13] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL (2); Lect. Notes Math. 114, Springer, Berlin–Heidelberg–New York 1970.

[14] T. Kubota, *Elementary theory of Eisenstein series*, Halsted Press, New York 1973.

[15] N. V. Kuznetsov, *Gipoteza Petersona dlya form vesa nul' i gipoteza Linnika*, Preprint Khab. KNII. DNTs. AN. SSSR. 02–77, 1977.

[16] —, *Arifmeticheskaya forma formuly sleda Sel'berga i raspredelenie norm primitivnykh giperbolicheskikh klassov modulyarnoj gruppy*, Preprint Khab. KNII. DNTs. AN. SSSR, 1978.

[17] —, *Gipoteza Petersona dlya form vesa nul' i gipoteza Linnika. Summy summ Kloostermana*, Mat. Sb. 111 (153) (1980), 334–383.

[18] S. Lang, $\mathrm{Sl}_2(R)$, Addison-Wesley, Reading (Mass.), 1975.

[19] —, *Introduction to modular forms*, Grundl. Math. Wiss. 222, Springer, Berlin–Heidelberg–New York 1976.

[20] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lect. Notes Math. 544; Springer, Berlin–Heidelberg–New York 1976.

[21] J. Lehner, *Discontinuous groups and automorphic functions*, Math. Surveys VIII; AMS, Providence (R.I.), 1964.

[22] H. Maass, *Über eine neue Art von nichtanalytischen automorphe Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichung*, Math. Ann. 121 (1949), 141–183.

[23] —, *Lectures on modular functions of one complex variable*, Tata Inst. of Fund. Research, Bombay 1964.

[24] Ju. I. Manin, *Parabolic points and zeta-functions of modular curves*, Math. USSR Izv. 6 (1) (1972), 19–64; transl. from Izv. Akad. Nauk SSSR, Ser. Mat. 36 (1) 1972.

[25] H. Petersson, *Über die Entwicklungskoeffizienten der automorphen Formen*, Acta Math. 58 (1932), 169–215.

[26] N. V. Proskurin, *Formuly summirovaniya dlya obshchikh summ Kloostermana*, Zap. Nauchn. Seminarov LOMI 82 (1979), 103–135.

[27] —, *Otsenki sobstvennykh chisel operatorov Gekke v prostranstve parabolicheskikh form vesa* 0; ibid. 82 (1979), 136–143.

[28] —, *O gipoteze Yu. V. Linnika*, ibid. 91 (1979), 94–118.

[29] —, *Ob obshchikh summach Kloostermana*, Preprint LOMI R-3-80, Leningrad 1980.

[30] J. Repka, *Tensor products of unitary representations of* $\mathrm{Sl}_2(R)$, Amer. J. Math. 100 (1978), 747–774.

[31] W. Roelcke, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene I*, Math. Ann. 167 (1966), 292–337.

[32] —, — *II*, ibid. 168 (1967), 261–324.

[33] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten and Princeton University Press, 1971.

[34] A. B. Venkov, *Spectral theory of automorphic functions, the Selberg zeta-function, and some problems of analytic number theory and mathematical physics*, Russ. Math. Surv. 34 (3) (1979), 79–153; transl. from Uspekhi Mat. Nauk 34 (3) (1979), 69–135.

[35] —, *Spektral'naya teoriya avtomorfnykh funktsij*, Trudy Mat. Inst. Steklov t. 153, Leningrad 1981.

Presented to the Semester
Elementary and Analytic Theory of Numbers
September 1–November 13, 1982