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THE SHAPE OF A GROUP – CONNECTIONS BETWEEN SHAPE THEORY AND THE HOMOLOGY OF GROUPS*

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§ 1. Introduction

I am going to explain how a number of recent papers can be interpreted as relating ideas in shape theory to various questions in the homological theory of groups. Up to now, the main serious application of shape theory has been its clarifying role in geometric topology. I believe that the ideas discussed here constitute another interesting application, since they provide the correct geometric setting for a range of questions in homological group theory. Of course, to be a successful application, there must be more than just a setting. The results are not all in, but the reader may come to share my view that the results already obtained make further study worthwhile.

I will assume a reasonable acquaintance with shape theory as described in [DS] or [MS]. General references for homological group theory are [B], [SW] and [HS], but I will not assume much knowledge of these.

My thanks to Gary O'Brien and Matt Brin for helping me sort out the ideas in § 6.

§ 2. Geometric aspects of homology of groups

Let G be a group, and let X be a $K(G, 1)$ complex. Assume X has just one vertex. Let $C_*(\tilde{X})$ be the chain complex of cellular chains in the universal cover $\tilde{X}$ [Co]. The covering transformations provide a free left action of G on $C_*(\tilde{X})$ making it a chain complex of free $\mathbf{Z}G$-modules. Since $\tilde{X}$ is contractible, this chain complex is acyclic in positive dimensions. For any left $\mathbf{Z}G$-module M (the "coefficient module"), the cohomology groups $H^*(G; M)$

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can be calculated from the cochain complex $\mathrm{Hom}_{ZG}(C_*(\tilde{X}), M)$. However, this abstraction hides the geometry.

Geometry enters when we make further assumptions on G. We say G is *of type* $\mathscr{F}(n)$ if X can be chosen to have finite n-skeleton, *of type* $\mathscr{F}(\infty)$ if X can be chosen to have finite skeleta, *of type* $\mathscr{F}\mathscr{D}$ if X can be chosen to be finitely dominated, and *of type* $\mathscr{F}$ if X can be chosen to be finite. These are collectively known as *finiteness conditions* on G. $\mathscr{F}(1)$ is "finitely generated", $\mathscr{F}(2)$ is "finitely presented", so the others are higher-dimensional analogues of these two familiar finiteness conditions; analogues of increasing strength. It is a good open question whether $\mathscr{F}\mathscr{D}$ implies $\mathscr{F}$. Finite groups are of type $\mathscr{F}(\infty)$ [B, p. 197], but no group having an element of finite order can be of type $\mathscr{F}\mathscr{D}$ [B, p. 187]. Typical examples of groups of type $\mathscr{F}$ are the fundamental groups of compact aspherical manifolds. For each n, there exist groups of type $\mathscr{F}(n)$ which are not of type $\mathscr{F}(n+1)$ [Bi; p. 37]. I will not discuss the interplay between finiteness conditions on G and decomposability of G as an extension, an amalgamated free product or an HNN extension: it is covered in Chapter 2 of [Bi]. Note that he uses the algebraic condition FP_n rather than my $\mathscr{F}(n)$; for $n \geq 2$ "FP_n+ finitely presented" is equivalent of $\mathscr{F}(n)$, while FP_1 is equivalent to $\mathscr{F}(1)$.

Let G be of type $\mathscr{F}(n)$, $n < \infty$, and let X have finite n-skeleton. Consider the case where the coefficient module is ZG itself. Then $\tilde{X}^n$ is a locally compact complex, and, simply by looking at the definitions, one sees that $H^k(G, ZG) \cong H_c^k(\tilde{X}^n)$ for all $k < n$. (Here, H_c^k denotes cohomology with compact supports.) When G is infinite, $H_c^k(\tilde{X}^n)$ measures k-dimensional cohomology "at the end" of $\tilde{X}^n$. So this must be the geometrical meaning of $H^k(G, ZG)$ – an object of algebraic interest. We now explore this idea.

§ 3. Connectedness at infinity of a locally finite complex

If Y is a countable, locally compact, finite-dimensional, infinite complex (I'm thinking of $Y = \tilde{X}^n$ in § 2), one can pick finite subcomplexes $K_0 \subset K_1 \subset \dots$ whose union is Y. (To avoid technicalities, I'll assume the attaching maps of the cells of Y are PL, and that each $|K_i|$ is a compact subpolyhedron with bicollared frontier: I'm going to omit such details from now on.) By *the end of* Y, I mean the inverse sequence $\varepsilon Y \equiv \{Y \setminus K_i\}$ whose bonds are inclusions. This should not be confused with the *set of ends of* Y which is $\varprojlim_i \{\pi_0(Y \setminus K_i)\}$, abbreviated $\varprojlim_i \pi_0(\varepsilon Y)$.

For our purpose, there is a better way of looking at the set of ends of Y. Two proper maps $f, g\colon X \to Y$ between locally compact complexes are *weakly properly homotopic* if, given a compact subset B of Y, there exist a compact subset A of X and a homotopy $H\colon f \simeq g$ such that $H_t(X \setminus A) \subset Y \setminus B$ for $0 \leq t \leq 1$. If the same H works for all B, then f and g are *properly homotopic*.

It is not hard to see that there is a "canonical" bijection from the set of ends of Y, $\varprojlim_i \pi_0(\varepsilon Y)$, to the set of weak proper homotopy classes of proper maps $[0, \infty) \to Y$. Call a proper map $[0, \infty) \to Y$ a *proper ray*. Then we will also think of the set of ends of Y as being the set of equivalence classes of proper rays under the equivalence relation of "weak proper homotopy".

There are two other more delicate equivalence relations on the proper rays in Y: "proper homology" and (already defined) "proper homotopy". Proper rays f and g are *properly homologous* if, when f and g are oriented towards infinity, there is a proper locally finite singular 2-chain, with $\mathbf{Z}$ coefficients, whose boundary is $f-g+c$, where c is a finite 1-chain. The *set of strong ends* of Y (resp. *the set of strong homological ends* of Y) is the set of proper homotopy classes (resp. the set of proper homology classes) of proper rays in Y.

There are natural surjections

$$\{\text{strong ends of } Y\} \overset{\alpha}{\twoheadrightarrow} \{\text{strong homological ends of } Y\} \overset{\beta}{\twoheadrightarrow} \{\text{ends of } Y\}$$

which in general are not bijections. Examples: embed the solenoid S and the Case–Chamberlin continuum C in ∂B^4. If $Y = B^4 \setminus S$, β is not a bijection. If $Y = B^4 \setminus C$, α is not a bijection.

Thus there are three kinds of connectedness at infinity for Y: (1) one strong end, (2) many strong ends but one strong homological end, (3) many strong homological ends but one end. (Actually, there is a fourth notion: "weak proper homology classes of rays"; but this is easily seen to give back the set of ends.)

§ 4. Connectedness for finitely presented groups

First, let G be infinite and finitely generated (i.e. of type $\mathscr{F}(1)$). Let X be a $K(G, 1)$ complex with finite 1-skeleton. Then $\tilde{X}^1$ is locally compact and is in fact the Cayley graph associated with a finite set of generators of G. The cardinal number of the set of ends of $\tilde{X}^1$ depends only on G, and is 1, 2 or c (see [SW]).

Now, suppose G is infinite and finitely presented ($\mathscr{F}(2)$). The set of strong ends of $\tilde{X}^2$ and the set of strong homological ends of $\tilde{X}^2$ depend (up to canonical bijection) only on G (implicit in [Jo], [M_1]). And the set of ends of $\tilde{X}^2$ "is" the set of ends of $\tilde{X}^1$ as above. So, fixing X, we will regard these sets as invariants of G. We have surjections:

$$\{\text{strong ends of } G\} \overset{\alpha}{\twoheadrightarrow} \{\text{strong homological ends of } G\} \overset{\beta}{\twoheadrightarrow} \{\text{ends of } G\}.$$

THEOREM 4.1 (see [M_1] or [EH]). *The following are equivalent when G is infinite and finitely presented:*

(i) *$\beta \circ \alpha$ is a bijection;*

(ii) $\{\pi_1(\tilde{X}^2 \setminus K_i, r)\}$ *is Mittag–Leffler for any proper base ray* r: $[0, \infty) \to X^2$.

THEOREM 4.2 ([GM]). *The following are equivalent when* G *is infinite and finitely presented*:

(i) β *is a bijection*;

(ii) $\{H_1(\tilde{X}^2 \setminus K_i)\}$ *is Mittag–Leffler*;

(iii) $H^2(G, \mathbf{Z}G)$ *is free abelian.*

Compare these two theorems. Condition (ii) shows that 4.2 is the abelianization of 4.1. Condition (iii) in 4.2 shows that 4.2 pertains to homological group theory. I claim that 4.1 should be thought of as a theorem in "shape theoretical group theory", and that the absence of a Condition 4.1 (iii) analogous to 4.2 (iii) – I know of no suitable analogue – shows the desirability of our approach. If this seems too grandiose a claim, see below: here, I am only looking at connectedness of G, but in later sections I will define and discuss the shape or the n-shape of groups G satisfying appropriate finiteness conditions.

Returning to 4.1 and 4.2, I ask:

QUESTION 4.3. *Do there exist finitely presented infinite groups* G *for which* (a) $\beta\circ\alpha$ *is not a bijection, or* (b) β *is not a bijection*?

As I write, the answers are unknown, but several remarks are needed. First, if G has two ends, then G is a finite extension of $\mathbf{Z}$ (see [SW]) so α and β are both bijections. Secondly, the question of whether $H^2(G, \mathbf{Z}G)$ is always free abelian is quite old. It is stated in [Bi], but I have been told that Hopf conjectured a positive answer (see also [Fa]). By 4.2, this is equivalent (for f.p. infinite G) to Question 4.3 (b). (I would guess that Hopf knew Theorem 4.2.) Thirdly, the case of most interest is where G has one end. Then Question 4.3 becomes:

QUESTION 4.4. *If* G *is a finitely presented one-ended group, is it true that proper rays in* $\tilde{X}^2$ *are* (a) *always properly homotopic*? (b) *always properly homologous*?

Obviously, a positive answer to 4.4 (a) implies a positive answer to 4.4 (b). And here we see another advantage in our point of view. While homological methods (e.g. the Hochschild–Serre spectral sequence when G is a suitable extension of an infinite group by an infinite group) have proved to be of limited use in getting positive answers to 4.4 (b) (see [GM]), a primitive approach to 4.4 (a) has proved quite successful in many cases – see [Ja_1], [Ja_2], [M_1], [M_2], [M_3], [M_4], [M_5]: for many classes of one-ended finitely presented groups G it has been shown that all proper rays in $\tilde{X}^2$ are properly homotopic, hence properly homologous, hence $H^2(G, \mathbf{Z}G)$ is free abelian. By a "primitive approach" I mean: verifying Condition 4.1 (ii) by carefully pushing to infinity loops in the universal cover $\tilde{X}^2$. In the language of shape theory, one is verifying that the inverse sequence $\varepsilon\tilde{X}^2 \equiv \{\tilde{X}^2 \setminus K_i\}$ of CW complexes is "pointed 1-movable".

§ 5. The $(n-1)$-shape of a group of type $\mathscr{F}(n)$

Let G be of type $\mathscr{F}(n)$ where n is a positive integer. Let X be a $K(G, 1)$ complex having one vertex and finite n-skeleton. Then the proper $(n-1)$-type of $\tilde{X}^n$ is an invariant of G: in other words, if X_1 and X_2 are two such, then mutually inverse (cellular) pointed homotopy equivalences $X_1 \underset{g}{\overset{f}{\rightleftarrows}} X_2$ induce proper maps $\tilde{X}_1^n \underset{\tilde{g}}{\overset{\tilde{f}}{\rightleftarrows}} \tilde{X}_2^n$ such that $\tilde{g}\circ\tilde{f}|\tilde{X}_1^{n-1}$ and $\tilde{f}\circ\tilde{g}|\tilde{X}_2^{n-1}$ are properly homotopic to the respective inclusions of $(n-1)$-skeleta into n-skeleta. Because of this, I will call $\varepsilon\tilde{X}^n \equiv \{\tilde{X}^n \setminus K_i\}$ the *pro-$(n-1)$-type* of G.

QUESTION 5.1. *Is it always the case that there is a topological space $E(\tilde{X}^n)$ with which the sequence $\varepsilon\tilde{X}^n$ is associated in the sense of shape theory? And if $E(\tilde{X}^n)$ exists, can it always be chosen to be a compact metric space?*

In all known cases, the answer is yes to both questions, but little is understood. For example, even the answer to the following question is unknown:

QUESTION 5.2. *For $n = 2$, is $\pi_1(\varepsilon\tilde{X}^2) \equiv \{\pi_1(\tilde{X}^2 \setminus K_i)\}$ pro-isomorphic to an inverse sequence of finitely generated groups? Is this true at least when G is of type $\mathscr{F}$?*

A positive answer to Question 5.2 would imply, for example, that the contractible open 3-manifolds described in [Mc] are not universal covers of closed 3-manifolds; in [My] this was shown to be true for some of those open 3-manifolds. In this connection see also [BT].

$E(\tilde{X}^n)$, when it exists, should be called the *$(n-1)$-shape* of G. At any rate, its pro-$(n-1)$-type $\varepsilon\tilde{X}^n$ exists, and its relation to homological group theory is contained in the following generalization of Theorem 4.2 ([GM], [GM_2]):

THEOREM 5.3. *Let G and X be as above.* (i) *for $k \leqslant n$, $H^k(G, ZG)$* mod *torsion is free abelian if and only if $H_{k-1}(\varepsilon\tilde{X}^n)$ is Mittag-Leffler;* (ii) *for $k \leqslant n$, $H^k(G, ZG)$ is torsion free if and only if $H_{k-2}(\varepsilon\tilde{X}^n)$ is pro-torsion free;* (iii) *for G infinite and $k \leqslant n$, $H^k(G, ZG)$ is a torsion group if and only if $\bar{H}_{k-1}(\varepsilon\tilde{X}^n)$ is pro-finite;* (iv) *for G infinite and $k \leqslant n$, $\bar{H}^k(G, ZG)$* mod *torsion is free abelian of finite rank ϱ if and only if $\bar{H}_{k-1}(\varepsilon\tilde{X}^n)$* mod *torsion is stable with free abelian inverse limit of finite rank ϱ.*

(Here $\bar{H}_*$ is reduced homology.)

Thus $H^*(G, ZG)$ is entirely reflected in the pro-homology of $E(\tilde{X}^n)$ or

$\varepsilon\tilde{X}^n$ through dimension n. An example of information about $E(\tilde{X}^n)$ or $\varepsilon\tilde{X}^n$ which can be obtained from the Hochschild–Serre spectral sequence is:

THEOREM 5.4 ([GM]). *Let $H \rightarrowtail G \twoheadrightarrow L$ be a short exact sequence of infinite groups, where L is of type $\mathscr{F}_n$ while H and G are of type $\mathscr{F}_{m+n}$. Let (X_G, X_H, X_L) be $(K(G, 1), K(H, 1), K(L, 1))$ complexes having finite skeleta in dimensions $(m+n, m+n, n)$. Let $\bar{H}_i(\varepsilon\tilde{X}_L^n)$ be pro-trivial for $i \leqslant n-2$ and Mittag–Leffler for $i = n-1$; let $\bar{H}_i(\varepsilon\tilde{X}_H^m)$ be pro-trivial for $i \leqslant m-2$ and Mittag–Leffler for $i = m-1$. Then $\bar{H}_i(\varepsilon\tilde{X}_G^{m+n})$ is pro-trivial for $i \leqslant m+n-2$ and Mittag–Leffler for $i = m+n-1$.*

If H or L, in 5.4, is one-ended, then it follows from [Ja] or [Ho] that $\pi_1(\varepsilon\tilde{X}_G^{m+n})$ is pro-trivial. In that case, the pro-Hurewicz Theorem makes it possible to replace $\bar{H}_i$ by π_i in the conclusion of Theorem 5.4 (assuming $m+n \geqslant 2$). Under much less stringent assumptions it is shown in [M_1–M_5] that $\pi_1(\varepsilon\tilde{X}_G^{m+n})$ is Mittag–Leffler. Then the one-point compactification of $\tilde{X}_G^{m+n}$ is locally $(m+n-1)$-connected [GM].

In the opposite direction (using knowledge of the n-shape of a group to get cohomological information about the group in situations where homological methods fail) there are some results in [GM]. I refer here to higher dimensional versions of what in § 4 was called the "primitive approach".

The whole story is clearer for groups of type $\mathscr{F}$, as I shall now explain.

§ 6. The shape of a group of type $\mathscr{F}$

Let G be of type $\mathscr{F}$ and let X be a finite $K(G, 1)$ complex having one vertex. Then the proper homotopy type of $\tilde{X}$ is an invariant of G: in other words, if X_1 and X_2 are two such, then mutually inverse pointed homotopy equivalences lift to mutually inverse proper homotopy equivalences between $\tilde{X}_1$ and $\tilde{X}_2$. Because of this I will call the contractible infinite locally compact complex $\tilde{X}$ the *proper homotopy type* of G. As before, I pick an exhausting sequence $\{K_i\}$ of nice finite subcomplexes. All the proper homotopy information is contained in the pro-homotopy theory of $\varepsilon\tilde{X} \equiv \{\tilde{X}\setminus K_i\}$ which I call the *pro-homotopy type* of G.

QUESTION 6.1. *Is it always the case that there is a topological space $E(\tilde{X})$ with which the sequence $\{\tilde{X}\setminus K_i\}$ is associated in the sense of shape theory? And if $E(\tilde{X})$ exists, can it always be chosen to be a compact metric space?*

In all known examples, the answer is yes to both questions. $E(\tilde{X})$, when it exists, should be called the *shape* of G.

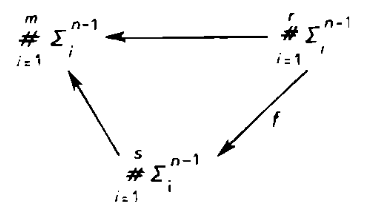
EXAMPLES. Let G be the fundamental group of the closed aspherical n-manifold M. Then $\tilde{M}$ is contractible. In the most familiar cases, $\tilde{M}$ is homeomorphic to $\mathbf{R}^n$, so the shape of G is S^{n-1}. M. Davis [Da] has given examples of groups G and aspherical $K(G, 1)$ closed manifolds M such that the shape of G is not S^{n-1} ($n \geq 4$). In the Davis examples, the shape of G is $E(\tilde{M}) \equiv (\#_{i=1}^{\infty} \Sigma_i^{n-1})^+$ where Σ^{n-1} is an homology $(n-1)$-sphere which bounds a compact contractible n-manifold, $\#_{i=1}^{\infty} \Sigma_i^{n-1}$ is the one-ended[1] countably infinite connected sum of copies of Σ^{n-1}, and $(\ )^+$ denotes one-point-compactification. The inverse system is

$$\varepsilon\tilde{M} \equiv \{\Sigma_1^{n-1} \leftarrow \Sigma_1^{n-1} \# \Sigma_2^{n-1} \leftarrow \Sigma_1^{n-1} \# \Sigma_2^{n-1} \# \Sigma_3^{n-1} \leftarrow \ldots\}$$

where each bond maps the last summand to a ball and is the "identity" on the rest of the space. This compactum is quite interesting. Its pro-homology is that of S^{n-1}. Its pro-π_1 is Mittag–Leffler: in fact each bond is a split epimorphism $\ast_{i=1}^{m+1} \Gamma_i \twoheadrightarrow \ast_{i=1}^{m} \Gamma_i$ where $\ast$ denotes free product and $\Gamma_i = \Gamma = \pi_1(\Sigma)$ for each i (the bond kills the last free factor).

PROPOSITION 6.2. *When Σ^{n-1} is not simply connected, $E(\tilde{M})$ is not movable.*

Proof. Suppose there were a map f making the following diagram commute up to homotopy



where $m < r < s$ and the unmarked arrows are bonds. The bonds have degree 1, so f has degree 1. As is well-known, this implies $f_\#$ is an epimorphism on π_1, contradicting Grushko's Theorem [SW].

Since each bond in pro-$\pi_1(\varepsilon\tilde{M})$ is a split epimorphism, the induced inverse sequence of Whitehead groups is Mittag–Leffler. Hence, by [CS], if Q denotes the Hilbert Cube, $\tilde{M} \times Q$ is homeomorphic to $Q \backslash Z$ where Z is a Z-set copy of $E(\tilde{M})$ in Q. Thus G *acts freely and properly discontinuously, with compact Q-manifold quotient, on the complement of a Z-set in Q.*

If pro-$\pi_1(\varepsilon\tilde{M})$ is stable, one can do better, actually sewing a manifold boundary to $\tilde{M}$. See [S].

The question arises of what variety of examples can occur in the Davis setting. A finitely presented group Γ is the fundamental group of a homology

[1] The end-point compactifications of all countably infinite connected sums of copies of Σ are shape equivalent.

5-sphere which bounds a compact contractible 6-manifold if and only if $H_1(\Gamma, Z) = 0 = H_2(\Gamma, Z)$. A particular case of this is the group of order 120 which is the fundamental group of the Poincaré homology 3-sphere, P. Since $\tilde{P}$ is S^3, P is the 3-skeleton of $K(\pi_1(P), 1)$, so $\pi_1(P)$ has $H_1 = 0 = H_2$, even though P itself does not bound a compact contractible 4-manifold. Thus we get an example of *the shape of a group for which* pro-π_1 *is not pro-torsion-free.* Before [Da], no such example was known.

§ 7. Pseudo-proper homotopy

Let Y be a countable complex each skeleton of which is locally compact. Well-order the vertices of Y. Let K_i be the largest subcomplex of Y whose 0-skeleton consists of the first i vertices in the ordering. I call $\{K_i\}$ a *pseudo-proper filtration* of Y. If $\{K_i'\}$ is a pseudo-proper filtration based on a different ordering of the vertices of Y, it is clear that for each i there exists $j(i)$ such that $K_i' \subset K_{j(i)}$. Thus, for my purposes, $\{K_i\}$ and $\{K_i'\}$ will be equivalent. Note that each K_i has finite type.

Let Y_1 and Y_2 have the properties of Y, above, let $\{K_i\}$ and $\{L_i\}$ be pseudo-proper filtrations of Y_1 and Y_2 respectively. One readily proves:

LEMMA 7.1. *For a cellular map* $f\colon Y_1 \to Y_2$, *the following are equivalent*: (a) $f|Y_1^0\colon Y_1^0 \to Y_2^0$ *is proper*; (b) $f|Y_1^n\colon Y_1^n \to Y_2^n$ *is proper for all* n; (c) *for each* i, *there exists* $j(i)$ *such that* $f^{-1}(L_i) \subset K_{j(i)}$.

A map satisfying (a) or (b) or (c) of Lemma 7.1 is *pseudo-proper*. There is an obvious *pseudo-proper homotopy category* whose objects are complexes having the properties of Y, above.

If $f\colon X_1 \to X_2$ is a cellular map, where X_1 and X_2 are of finite type, and if $\ker(f_\#\colon \pi_1(X_1) \to \pi_1(X_2))$ is finite, then $\tilde{f}\colon \tilde{X}_1 \to \tilde{X}_2$ is pseudo-proper. If f is a homotopy equivalence, $\tilde{f}$ is a pseudo-proper homotopy equivalence.

If $\{K_i\}$ is a pseudo-proper filtration of Y, I define the *end* of Y to be the inverse sequence $\varepsilon Y \equiv \{Y \setminus K_i\}$. Its pro-homotopy type is an invariant of pseudo-proper homotopy.

These notions arise naturally in studying groups of type $\mathscr{F}(\infty)$.

§ 8. The shape of a group of type $\mathscr{F}(\infty)$

Let G be of type $\mathscr{F}(\infty)$ and let X be a $K(G, 1)$ complex of finite type, having one vertex. For example, G could be a finite group, but in that case $\tilde{X}$ would also be of finite type, and the shape of G, in any reasonable definition, would be represented by the empty space.

Of more interest are the groups of type $\mathscr{F}(\infty)$ discussed in [BG_1], [BG_2] and [BG_3]. These are groups F and T which are of type $\mathscr{F}(\infty)$, but not of type $\mathscr{F}$, for which $\pi_i(\varepsilon\tilde{X}^n)$ is pro-trivial whenever $n > i \geq 0$. (The group F is well-known to shape theorists in connection with the problem of splitting homotopy idempotents ("every FANR is a pointed FANR"). See [DS p. 82] and [HH].) It is tempting to consider those groups (and there are lots more!) as having the shape of a point. But § 5 tells us less: it says that *these groups have trivial pro-n-type for each n.*

Define the *pro-homotopy type* of G to be $\varepsilon\tilde{X}$, as defined in § 7. As before, we can ask if there is always a space $E(\tilde{X})$ represented in shape by $\varepsilon\tilde{X}$, and if it can always be chosen to be a compact metric space. Here is another interesting question whose answer is unknown.

QUESTION 8.1. *Does there exist a group of type $\mathcal{F}(\infty)$ having trivial pro-homotopy type (i.e. whose shape is a point)?*

One might guess that the groups F and T have this property, but the method of proof in [BG_2] that the pro-n-type is trivial for all n does not establish that $\{\tilde{X}\setminus K_i\}$ is trivial in pro-homotopy, where $\{K_i\}$ is a pseudo-proper filtration of $\tilde{X}$. The issue is movability; $\{\tilde{X}\setminus K_i\}$ is trivial in pro-homotopy iff it is movable [MS p. 191].

§ 9. The shape theory of groups

The problem is no less than *to describe for each group G of type $\mathcal{F}(\infty)$ its shape*, and to say *which shapes can occur*? These are broad and difficult questions, but their answers should yield important invariants of such groups. Even the simplest kind of connectedness gives the theory of ends of finitely generated groups, whose description by Hopf and Stallings has had profound consequences.

As pointers along the way, the questions posed in this note deserve to be answered: namely Questions 4.3, 4.4, 5.1, 5.2, 6.1 and 8.1. Some of these questions are widely known. My aim in writing this note has been to give them a context.

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