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Domination, independence and irredundance in graphs

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CONTENTS

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1. Introduction

1.1. Purpose and scope. The study of graphs and their various theoretical and real-world applications have led to the study and development of the theory of independence and domination in graphs. In fact, graph theorists have studied independent sets in graphs for a long time, especially in view of their relationships to colorings in graphs. The mathematical study of domination in graphs was begun by König [95], Berge [10, 11, 12] and Ore [111]. Their text-books, the paper by Vizing [156], and the survey papers by Cockayne [34], Cockayne and Hedetniemi [38], Laskar and Walikar [100], and Hedetniemi, Laskar and Pfaff [89] provided the inspiration for many mathematicians working in this field. The concept of irredundance in graphs was first introduced by Cockayne, Hedetniemi and Miller [40] while studying domination in graphs. A firm foundation to the development of irredundance gave Bollobás and Cockayne [20]. During the past 30 years the study of domination has become a significant area of research in graph theory. Currently the domination theory includes a few hundred papers written on domination related problems (for example, the recent domination bibliography compiled by Hedetniemi and Laskar [88] contains 402 citations) and over 70 different types of domination related parameters of graphs have been studied (for example, the paper by Hedetniemi, Hedetniemi and Laskar [87] contains the definitions of 30 domination parameters and some other of them can be found in "Topics on Domination", Discrete Mathematics 86 (1990), edited by S. T. Hedetniemi and R. C. Laskar).

This paper is not a survey paper on domination, independence and irredundance in graphs. Rather, it deals with aspects of the classical cases of domination, independence and irredundance of particular interest to the author. This paper was based on the author's papers [140]–[145] and the papers [117], [126], and [146]–[155] which the author wrote together with E. Prisner of the Hamburg University, P. D. Vestergaard of the Aalborg University, and L. Volkmann of the Technical University of Aachen. The work contains also some new results which have never been published and it includes various references to publications which are beyond the mainstream development. The paper is organized as follows:

Chapter 1 contains some basic graph-theoretic terms used in this paper. Other graph-theoretic terms which are not included in this section will be defined when they are needed (or can be found in [15], [75] or [157]).

In Chapter 2, we introduce the notion of domination, independence and irredundance in graphs. We then give the main properties of independent, dominating and irredundant sets, and general relationships between the independence, domination and irredundance numbers of a graph. The principal results of this chapter are some sufficient conditions for two or more of the domination related parameters to be equal (Sections 2.3 and 2.4).

Chapter 3 deals with graphs in which every maximal independent set of vertices is maximum. Such graphs are called well covered. This chapter offers some general properties of the well covered graphs and characterizations of several subclasses of the well covered graphs.

In Chapter 4, we investigate sequences and sets of integers which are formed for a given graph and a domination related parameter.

1.2. Basic graph-theoretical terms. A *simple graph* G (a *graph* for short) is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a set of two-element subsets of $V(G)$. The set $V(G)$ is the set of *vertices* of G and $E(G)$ is the set of *edges* of G. The cardinality of the vertex set of a graph G is called the *order* of G, while the cardinality of its edge set is the *size* of G. An edge $\{u, v\}$ of G is said to *join* the vertex u to the vertex v and is denoted by uv. We also say that the vertices u and v are *adjacent* and that each of them is *incident* with the edge uv. Two distinct edges are *adjacent* if they are incident with a common vertex; otherwise they are *nonadjacent*. If $uv \in E(G)$, then we say that v is a *neighbour* of u. The set of all neighbours of u is called the *neighbourhood* of u and is denoted by $N_G(u)$. We write $N_G[u]$ instead of $N_G(u) \cup \{u\}$. For a subset X of $V(G)$, we write $N_G(X)$ and $N_G[X]$ instead of $\bigcup_{u\in X} N_G(u)$ and $\bigcup_{u\in X} N_G[u]$, respectively. The *degree* of a vertex u is $|N_G(u)|$ and is denoted by $d_G(u)$. The maximum (resp. minimum) of the degrees of the vertices of G is called the *maximum* (resp. *minimum*) *degree* of G. A vertex of degree zero (one or at least two, resp.) in G is referred to as an *isolated* (*end* or *interior*, resp.) *vertex* of G. An edge uv is an *end edge* of G if u or v is an end vertex of G; otherwise it is an *interior edge* of G. If all the vertices of G have the same degree, say d, then we say that G is *regular of degree d*. A regular graph of degree 3 is called a *cubic graph*. A graph is *complete* if any two of its vertices are adjacent. A complete graph of order n is therefore a regular graph of degree $n-1$ and size $n(n-1)/2$; we denote this graph by K_n. The complete graph having vertex set V is denoted by $K[V]$. The complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ and such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. The complement $\overline{K}_n$ of the complete graph K_n has n vertices and no edges and is referred to as the *totally disconnected graph* of order n.

A graph G_1 is *isomorphic to* a graph G_2 if there exists a bijection $\varphi : V(G_1) \to V(G_2)$, called an *isomorphism*, which preserves adjacency, that is, for all $v, u \in V(G_1)$, $vu \in E(G_1)$ if and only if $\varphi(v)\varphi(u) \in E(G_2)$. It is easy to see that "is isomorphic to" is an equivalence relation on graphs. Therefore, if G_1 is isomorphic to G_2, we may say that G_1 and G_2 are isomorphic. If G_1 and G_2 are isomorphic, we write $G_1 \cong G_2$ or simply $G_1 = G_2$ if there is no danger of confusion. By a *copy* of a graph G we mean a graph isomorphic to G. Two graphs G_1 and G_2 are *disjoint* or *vertex-disjoint* (resp. *edge-disjoint*) if their vertex sets (resp. edge sets) are disjoint.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in such a case, we also say that G is a *supergraph* of H. Any graph isomorphic to a subgraph of G is also referred to as a subgraph of G. A *spanning subgraph* of a graph G is a subgraph containing all the vertices of G. If M is a subset of edges of G, then $G - M$ denotes a spanning subgraph of G with edge set $E(G) - M$. In particular, if $vu \in E(G)$, then $G - \{vu\}$ is called an *edge-deleted subgraph* of G and we write $G - vu$ instead of $G - \{vu\}$. If u and v are nonadjacent vertices of G, then $G + uv$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{uv\}$. For any set X of vertices of G, the *induced subgraph* $G[X]$ of G is the maximal subgraph of G with vertex set X. For a subset X of $V(G)$ and a vertex $v \in V(G)$, we also write $G - X$ and $G - v$ instead of $G[V(G) - X]$ and $G[V(G) - \{v\}]$, respectively. For $v \in V(G)$, $G - v$ is called a *vertex-deleted subgraph* of G. For any set M of edges of G, the *generated subgraph* $G(M)$ of G is the minimal subgraph of G with edge set M, the graph whose vertex set consists of those vertices of G incident with at least one edge of M and whose edge set is M.

A set of pairwise nonadjacent edges of a graph G is called a *matching* in G. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. Clearly, if G has a perfect matching M, then G has an even order and $G(M)$ is a regular spanning subgraph of degree 1 of G. In a graph G, a nonempty subset X of $V(G)$ is said to be *matched* into a subset Y of $V(G) - X$ if there exists a matching M in G such that each edge of M is incident with a vertex of X and a vertex of Y and every vertex of X is incident with an edge of M.

A *path* is a graph P having vertex set $V(P) = \{v_0, v_1, \ldots, v_n\}$ and edge set $E(P) = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$ if $n \geq 1$ or $E(P) = \emptyset$ if $n = 0$. This path P is usually denoted by the sequence $(v_0, v_1, \ldots, v_n)$ of consecutive vertices since the edges present are then evident. The vertices v_0 and v_n are the end vertices of P and n is the *length* of P. We say that P is a $v_0 - v_n$ *path*. Of course, P is also a $v_n - v_0$ path. The symbol P_n denotes an arbitrary path of length n. A vertex u is said to be *joined* to a vertex v in a graph G if there exists a $u - v$ path in G. A graph G is *connected* if any two of its vertices are joined. A graph that is not connected is *disconnected*. A maximal connected subgraph of G is called a *connected component* or simply a *component* of G. A connected regular graph of degree 2 is called a *cycle*. Thus a cycle is a graph C of the form $V(C) = \{v_1, v_2, \ldots, v_n\}$ and $E(C) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. For simplicity this cycle is also denoted by $(v_1, v_2, \ldots, v_n)$, the sequence of consecutive vertices, when it is clear from the context. The number n ($n \geq 3$) is the *length* of C. The symbol C_n denotes an arbitrary cycle of length n. A cycle is *even* if its length is even; otherwise it is *odd*. A cycle of length n is an *n-cycle*; a 3-cycle is also called a *triangle*. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle in G if there is any; otherwise $g(G) = \infty$. A graph G of order at least three is 2-*connected* if and only if any two vertices of G lie on a common cycle. A *unicyclic graph* is a connected graph that contains exactly one cycle. A *tree* is a connected graph with no cycles.

The *distance* $d_G(u, v)$ between two vertices u and v in G is the length of a shortest $u-v$ path. If there is no $u-v$ path, then $d_G(u, v) = \infty$. If X is a nonempty subset of $V(G)$ and $u \in V(G)$, we define $d_G(u, X) = \min_{v \in X} d_G(u, v)$. The *diameter* $d(G)$ of a connected graph G is the maximum distance between two vertices of G, $d(G) = \max_{u,v \in V(G)} d_G(u, v)$.

A graph G is *bipartite* if its vertex set can be partitioned into two sets V_1 and V_2 (called *partite sets*) such that every edge of G joins a vertex of V_1 to a vertex of V_2. A *complete bipartite graph* G is a bipartite graph with partite sets V_1 and V_2 having the added property that if $u \in V_1$ and $v \in V_2$, then $uv \in E(G)$. A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a *star*; its vertex of degree n is called the *center* of $K_{1,n}$.

If G_1 and G_2 are two graphs, then their *union*, denoted by $G_1 \cup G_2$, has $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *disjoint union* of graphs is the union of disjoint copies of the graphs. If a graph G consists of n disjoint copies of a graph H, then we write $G = nH$. The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained from the disjoint union of G_1 and nG_2 (where n is the order of G_1) by joining the ith vertex (of the copy) of G_1 to every vertex in the ith copy of G_2 (see Section 3.2). The *join* $G_1 + G_2$ of graphs G_1 and G_2 is obtained from their disjoint union by joining each vertex (of the copy) of G_1 to each vertex (of the copy) of G_2.

The *line graph* $L(G)$ of a graph G is the graph having vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G are adjacent. The *total graph* $T(G)$ of G is the graph with vertex set $V(G) \cup E(G)$ in which two vertices u and v are adjacent if and only if either u and v are adjacent vertices of G, or u and v are adjacent edges of G, or u is a vertex of G and v is an edge of G incident with u.

A vertex v of a graph G is called a *simplicial vertex* if any two vertices of $N_G(v)$ are adjacent in G. Equivalently, a simplicial vertex is a vertex that appears in exactly one clique of a graph, where a *clique* of a graph G is a maximal complete subgraph of G. A clique of a graph G containing at least one simplicial vertex of G is called a *simplex* of G. Note that if v is a simplicial vertex of G, then $G[N_G[v]]$ is the unique simplex of G containing v. A graph G is said to be *simplicial* if every vertex of G is a simplicial vertex of G or is adjacent to a simplicial vertex of G. Certainly, if G is a simplicial graph and $S_1, \ldots, S_n$ are the simplices of G, then $V(G) = \bigcup_{i=1}^{n} V(S_i)$. A graph G is said to be *chordal* (or *triangulated*) if every cycle of G of length four or more contains a *chord*, i.e., an edge joining two non-consecutive vertices of the cycle. In the literature there are many characterizations of chordal graphs, see Berge [13]–[16], Duchet [51] and

Golumbic [75]. Dirac [47], Lekkerkerker and Boland [101] and Rose [120] have proved that a graph G is chordal if and only if every induced subgraph of G has a simplicial vertex. Certainly, every induced subgraph of a chordal graph is chordal.

A vertex v of a graph G is called a *cut vertex* of G if $G - v$ has more components than G. A connected graph with no cut vertices is called a *block*. A *block of a graph* G is a subgraph of G which is a block itself and which is maximal with respect to that property. A block H of a graph G is called an *end block* of G if H has at most one cut vertex of G. A graph G is called a *block graph* if every block of G is a complete graph. Note that every block graph is a chordal graph.

The words maximal and minimal refer as usual to sets with respect to a prescribed property. Also as usual, the words maximum and minimum refer to the cardinality of a set with a prescribed property.

2. Domination, independence and irredundance in graphs

2.1. Introduction and preliminaries. First we give a few definitions. Let G be a graph and let X be a subset of the vertex set $V(G)$ of G. For every x in X, define

$$I_G(x, X) = N_G[x] - N_G[X - \{x\}],$$

the set of *private neighbours* of the vertex x relative to the set X. If $I_G(x, X) = \emptyset$, then x is said to be *redundant* in X. A set X of vertices containing no redundant vertex is called *irredundant*. It is apparent that irredundance is a hereditary property. The quantities concerning irredundance are the *lower* and *upper irredundance numbers* $\mathrm{ir}(G)$ and $\mathrm{IR}(G)$ of a graph G which are respectively the minimum and maximum cardinalities of maximal irredundant sets of vertices of G.

If X and Y are subsets of $V(G)$, X *dominates* Y if $Y \subseteq N_G[X]$. In particular, if X dominates $V(G)$, then X is called a *dominating set* of G. Equivalently, $X \subseteq V(G)$ is a dominating set of G if any vertex $x \in V(G) - X$ is adjacent to at least one vertex $y \in X$. Certainly, every set containing a dominating set is dominating. The *lower* and *upper domination numbers* $\gamma(G)$ and $\Gamma(G)$ of G are respectively the minimum and maximum cardinalities of minimal dominating sets of G.

A set X of vertices of G is said to be *independent* if no two vertices of X are adjacent in G. Note that every subset of an independent set is independent. The *lower* and *upper independence numbers* $i(G)$ and $\alpha(G)$ of G are respectively the minimum and maximum cardinalities of maximal independent sets of vertices of G.

The parameters $\mathrm{ir}(G)$, $\gamma(G)$, $i(G)$ and $\alpha(G)$ are sometimes referred to as the *irredundance, domination, independent domination* and *independence numbers* of G, respectively.

The concepts of domination and independence in graphs have existed in the literature for a long time. The modern study of domination and independence can be attributed initially to König [95], Berge [10, 11, 12], Ore [111], Liu [104] and Vizing [156]. The independent domination number was introduced by Cockayne and Hedetniemi [37]. The invariants γ and α are well known and they have many applications not only in graph theory, but in game theory, computer science, political science, safeguards analysis, transportation and communication networks, combinatorial optimization and analysis of algorithms as well. The literature includes many papers dealing with the theory of independent sets and the related topics of coding theory (see Ore [111] and Roberts [119]) and graph colorings. The notion of dominance is related to the theory of matchings because any maximal matching in a graph G corresponds to an independent dominating set in the line graph $L(G)$ of G. Applications of kernels (i.e. independent dominating sets) to game theory have been presented in several papers, e.g. see König [95], Neumann and Morgenstern [109], Berge [10, 11, 12, 15], Kummer [96] and Topp [137, 138, 139], to quote a few.

One of the best known problems involving dominating sets is the Five Queens Problem (e.g. see Berge [15] and Ore [111]) in which we are to determine the minimum number of queens to be placed on the 8×8 chessboard so that every square is either occupied by a queen or can be occupied in one move by at least one of the queens. It is easy to see that solutions of this problem are dominating sets in the graph whose vertices are the 64 squares of the chessboard and vertices u and v are adjacent if a queen may move from u to v in one move.

The problem of determining the dominating sets has obvious applications to the location of objects, safeguards or facilities on the vertices of a network, see Roberts [119]. Berge [15] discusses the use of the notion of dominance in devising optimal methods of radar surveillance. In a similar vein, Liu [104] discusses the application of dominance to communication networks. Suppose we have communication links in use between cities, and we want to set up transmitting stations in some of the cities so that every city can receive a message from at least one of the transmitting stations. An acceptable set of locations in which to place transmitting stations corresponds to a dominating set of the network. Irredundant sets in graphs were first defined and studied by Cockayne, Hedetniemi and Miller [40]. The notion of redundancy is also relevant in the context of communication networks, since any redundant vertex in a set can be removed from the set without affecting the totality of vertices that may receive communication from some vertex in the set, see [20] and [89]. The invariants ir and IR seem to have received less attention, although some significant results have been obtained by Allan and Laskar [4], Bollobás and Cockayne [20, 21], Cheston, Hare, Hedetniemi and Laskar [33], Cockayne, Favaron, Payan and Thomason [36], Favaron [60], Golumbic and Laskar [76], Jacobson and Peters [90, 91] and in a few other papers. The bibliography compiled by Hedetniemi and Laskar [88] and survey papers by Cockayne [34], Cockayne and Hedetniemi [38], Hedetniemi, Laskar and

Pfaff [89] and Laskar and Walikar [100] are recommended for further information on this topic.

We shall now briefly mention some results which are concerned with algorithms for computing the lower (upper) irredundance, domination and independence numbers and finding related sets of vertices. The questions how difficult it is to find a minimum (maximum) maximal independent set, a minimum (maximum) maximal irredundant set, a minimum (maximum) minimal dominating set, and the lower (upper) irredundance, domination and independence numbers of a graph have been investigated extensively during the last fifteen years (e.g., see [44], [73], [75] and [93] for extensive references). The problem of finding a minimum cardinality dominating set has been discussed in a large number of papers and it is NP-complete for arbitrary graphs [73]. The problem of determining a minimum dominating set remains NP-complete for comparability graphs, bipartite graphs [46] and split graphs [18, 43]. On the other hand, there are other classes, such as series-parallel graphs [94], k-trees (fixed k) [42], strongly chordal graphs [55] and permutation graphs [57] for which polynomial time algorithms have been designed for solving the minimum cardinality dominating set problem. The minimum cardinality independent dominating set problem is NP-complete for the classes of comparability graphs and bipartite graphs [43], but it can be solved in polynomial time for a number of other classes of graphs, see [54, 55, 57]. The problem of finding a minimum cardinality maximal irredundant set is NP-complete, even for special classes of graphs, such as bipartite graphs [89] and chordal graphs [98], and can be solved in linear time for trees [17] and in polynomial time for weighted interval graphs [19]. It is well known that the problem of determining the upper independence number is NP-complete even for planar graphs with no vertex degree exceeding three [73], but very efficient algorithms for determining the upper independence number have been devised for several classes of perfect graphs [75] and for many other classes of graphs, see [93]. It appears difficult to compute the upper domination and irredundance numbers in general, and we suspect that both the problems are NP-complete. However, for some classes of graphs their determination is reasonable. For example, if G is a circular arc graph, a chordal graph or a bipartite graph, then the upper independence number $\alpha(G)$ can be computed in polynomial time (see [73, 75, 93]) and therefore the upper domination number $\varGamma(G)$ and the upper irredundance number $\mathrm{IR}(G)$ can be determined in polynomial time since $\mathrm{IR}(G) = \varGamma(G) = \alpha(G)$ for such graphs (see [36, 76, 90, 146]).

There are many generalizations of the independence, domination and irredundance numbers of a graph, see survey papers [34, 38, 88, 89, 100] and papers by Acharya [1], Chang and Nemhauser [30, 31], Cockayne, Dawes and Hedetniemi [35], Colbourn, Slater and Stewart [41], Domke, Hedetniemi and Laskar [48], Domke, Hedetniemi, Laskar and Allan [49], Domke, Hedetniemi, Laskar and Fricke [50], Farley and Shacham [58], Fink and Jacobson [69, 70], Golumbic and Laskar [76], Hedetniemi, Hedetniemi and Laskar [87], Meir and Moon [107], Sampathkumar [121, 122, 123], Sampathkumar and Walikar [124], Siemes, Topp and Volkmann [126], Slater [127, 128, 129]. In this paper we consider only some of them. Here is a natural generalization of the concept of domination and independence in graphs (some others will be defined when they are needed).

For a graph G and a positive integer k, a subset $I \subseteq V(G)$ is a *k-packing* of G if $d_G(v, u) > k$ for every pair v and u of distinct vertices from I. The *k-packing number* of G is the number $\alpha_k(G)$ of vertices in any maximum k-packing of G. A subset $C \subseteq V(G)$ is a *k-covering* of G if $d_G(v, C) \leq k$ for every vertex $v \in V(G) - C$. The *k-covering number* of G, denoted as $\gamma_k(G)$, is the number of vertices in any minimum k-covering of G. The k-packing number and the k-covering number were first introduced by Meir and Moon in [107]. In that paper they studied the k-packing and k-covering numbers of trees. Some generalizations of their results and generalizations of the k-packing and k-covering numbers are given in the excellent papers of Chang and Nemhauser [30, 31], Domke, Hedetniemi, Laskar and Allan [49], and in a few other papers. Certainly, the 1-packing number $\alpha_1(G)$ and the 1-covering number $\gamma_1(G)$ are the upper independence number and the lower domination number of a graph G, respectively.

In this section we present various general properties of independent, dominating and irredundant sets, and general relationships between the independence, domination and irredundance numbers of a graph. All these results are very often used in the subsequent sections of this paper. Our first proposition is a generalization of the Berge theorem (see Corollary 2.1.3) and it relates k-packings to k-coverings of a graph. Some other generalizations of the Berge theorem are given by Siemes, Topp and Volkmann [126].

PROPOSITION 2.1.1 [152]. *For a graph G and a subset I of $V(G)$, the following conditions are equivalent*:

(1) *I is a maximal k-packing of G*;
(2) *I is a k-packing and a k-covering of G*;
(3) *I is both a maximal k-packing and a minimal k-covering of G.*

Proof. Let I be a maximal k-packing of G. Clearly, I is a k-covering of G (otherwise there would exist a vertex $v \in V(G) - I$ such that $d_G(v, I) > k$ and $I \cup \{v\}$ would be a k-packing in G).

Let I be a k-packing and a k-covering of G. Then I is a maximal k-packing of G (otherwise I would not be a k-covering). Moreover, for every $u \in I$, the set $I' = I - \{u\}$ cannot be a k-covering of G because $u \notin I'$ and $d_G(u, I') > k$. Thus, I is a minimal k-covering of G.

This suffices to complete the proof of the proposition. ∎

The next three results are immediate consequences of Proposition 2.1.1.

COROLLARY 2.1.1. *For every graph G, $\gamma_k(G) \leq \alpha_k(G)$.* ∎

COROLLARY 2.1.2. *If G is a graph with $\gamma_k(G) = \alpha_k(G)$, then every maximal k-packing I of G is a maximum k-packing and a minimum k-covering.* ∎

Corollary 2.1.3 [12, 15]. *For a graph G and a subset I of $V(G)$, the following conditions are equivalent*:

(1) *I is a maximal independent set of G*;
(2) *I is an independent dominating set of G*;
(3) *I is both a maximal independent and a minimal dominating set of G.* ∎

Ore [111] has proved that a dominating set D in a graph G is minimal if and only if for each vertex $x \in D$ either (i) $N_G(x) \cap D = \emptyset$ or (ii) there exists a vertex $y \in V(G) - D$ such that $N_G(y) \cap D = \{x\}$. This characterization of minimal dominating sets may also be stated in the following form.

Proposition 2.1.2. *Let D be a dominating set in G. Then D is a minimal dominating set in G if and only if $I_G(x, D) \neq \emptyset$ for each $x \in D$.*

Proof. If D is a minimal dominating set in G, then for each $x \in D$, $N_G[x] \cup N_G[D - \{x\}] = N_G[D] = V(G)$, $N_G[D - \{x\}]$ is a proper subset of $V(G)$ and consequently $I_G(x, D) \neq \emptyset$.

Assume D is dominating in G and $I_G(x, D) \neq \emptyset$ for each $x \in D$. Suppose D is not a minimal dominating set. Then for some $x \in D$, $D - \{x\}$ is dominating in G. Therefore $N_G[D - \{x\}] = V(G)$ and, since $N_G[x] \subseteq V(G)$, $I_G(x, D) = \emptyset$, contrary to the hypothesis. ∎

It follows from the definition of an irredundant set and Proposition 2.1.2 that minimal dominating and maximal irredundant sets are related by the following result.

Corollary 2.1.4. *Let X be a dominating set of a graph G. Then X is a minimal dominating set of G if and only if X is a maximal irredundant set of G.* ∎

Since every maximal independent set of a graph is minimal dominating (Corollary 2.1.3) and every minimal dominating set is maximal irredundant (Corollary 2.1.4), it follows immediately from the definition of independence, domination and irredundance numbers that we have the following string of inequalities which was first observed by Cockayne, Hedetniemi and Miller [40].

Proposition 2.1.3. *For any graph G,*

$$\mathrm{ir}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G). \ \blacksquare$$

In general all the six parameters of Proposition 2.1.3 are distinct; Cockayne, Favaron, Payan and Thomason [36] have constructed a graph G with $\mathrm{ir}(G) = 2$, $\gamma(G) = 3$, $i(G) = 4$, $\alpha(G) = 7$, $\Gamma(G) = 9$ and $\mathrm{IR}(G) = 10$. On the other hand, for the corona of graphs G and K_1 all the inequalities of Proposition 2.1.3 turn out to be equalities.

Proposition 2.1.4. *If G is a graph of order n, then*

$$\mathrm{ir}(G \circ K_1) = \gamma(G \circ K_1) = i(G \circ K_1) = \alpha(G \circ K_1) = \Gamma(G \circ K_1) = \mathrm{IR}(G \circ K_1) = n.$$

Proof. Suppose $V(G) = \{v_1, \ldots, v_n\}$ and $V(G \circ K_1) = V(G) \cup \{v_1', \ldots, v_n'\}$, where v_i is the unique neighbour of v_i' in $G \circ K_1$ $(i = 1, \ldots, n)$. Let X be any maximal irredundant set of $G \circ K_1$. By virtue of Proposition 2.1.3, it suffices to show that $|X| = n$. Since X is irredundant, at most one of the vertices v_i and v_i' belongs to X for every $i \in \{1, \ldots, n\}$ (otherwise the set $I_G(v_i', X)$ would be empty and X would not be irredundant). On the other hand, the maximality of X implies that for every $i \in \{1, \ldots, n\}$, v_i or v_i' belongs to X (otherwise $X \cup \{v_i\}$ and $X \cup \{v_i'\}$ would be greater irredundant sets). Consequently, $|X| = n$. ∎

The next result, due to Bollobás and Cockayne [20], will enable us to obtain a few new properties of the irredundant sets and the irredundance numbers of graphs.

THEOREM 2.1.1. *Suppose that X is a maximal irredundant set of a graph G and a vertex u of G is not dominated by X. Then for some $x \in X$,*

(a) *$I_G(x, X) \subseteq N_G(u)$, and*

(b) *for x_1, $x_2 \in I_G(x, X)$ such that $x_1 \neq x_2$, either $x_1x_2 \in E(G)$ or there exist y_1, $y_2 \in X - \{x\}$ such that x_1 is adjacent to each vertex of $I_G(y_1, X)$ and x_2 is adjacent to each vertex of $I_G(y_2, X)$.*

Proof. (a) By maximality of X, $X \cup \{u\}$ is not irredundant in G, so $I_G(x, X \cup \{u\}) = \emptyset$ for some $x \in X \cup \{u\}$. Since u is not dominated by X, $u \in I_G(u, X \cup \{u\})$ and therefore $x \neq u$. Further, since $I_G(x, X \cup \{u\}) = N_G[x] - N_G[X - \{x\}] - N_G[u] = \emptyset$, $I_G(x, X) = N_G[x] - N_G[X - \{x\}] \subseteq N_G[u]$ and therefore $I_G(x, X) \subseteq N_G(u)$ as $u \notin I_G(x, X)$.

(b) Let x_1, x_2 be two nonadjacent vertices of $I_G(x, X)$ and suppose on the contrary that for x_1 or x_2, say for x_1, and for all $y_i \in X - \{x\}$, there exists $z_i \in I_G(y_i, X)$ which is not adjacent to x_1. Then $x_2 \in I_G(x, X \cup \{x_1\})$, $u \in I_G(x_1, X \cup \{x_1\})$, $z_i \in I_G(y_i, X \cup \{x_1\})$ for each $y_i \in X - \{x\}$ and therefore $X \cup \{x_1\}$ is irredundant in G, which contradicts the maximality of X. ∎

By Proposition 2.1.3, $\mathrm{ir}(G) \leq \gamma(G)$ for every graph G. The next theorem, which improves a result of Allan, Laskar and Hedetniemi [5], gives another inequality relating $\gamma(G)$ and $\mathrm{ir}(G)$.

THEOREM 2.1.2. *Let X be a minimum maximal irredundant set in G. If the subgraph $G[X]$ has k isolated vertices and $k < |X|$, then $\gamma(G) \leq 2\,\mathrm{ir}(G) - k - 1$.*

Proof. Let X_0 be the set of isolated vertices of $G[X]$. Since $|X_0| = k < |X|$, $X - X_0 \neq \emptyset$, say $X - X_0 = \{x_1, \ldots, x_n\}$. For each $x_i \in X - X_0$, choose any $x_i' \in I_G(x_i, X)$ and form the set $X' = X \cup \{x_1', \ldots, x_n'\}$. Since $x_i \notin I_G(x_i, X)$, $x_i' \neq x_i$ (for $i = 1, \ldots, n$) and therefore X' is of cardinality $2\,\mathrm{ir}(G) - k$. We show that X' is a dominating set. Suppose that X' is not dominating and let $u \in V(G) - N_G[X']$. Thus, in particular, u is not dominated by X and it follows from Theorem 2.1.1 that $I_G(x, X) \subseteq N_G(u)$ for some $x \in X$. If $x \in X_0$, then $x \in I_G(x, X)$ and u is dominated by x, contrary to our supposition. If $x \in X - X_0$, then $x = x_i$ (for some $i \in \{1, \ldots, n\}$) and u is dominated by x_i', which again contradicts our supposition. Therefore X' is a dominating set. Since X' properly contains a maximal irredundant set X, it follows from Corollary 2.1.4 that X'

is not a minimal dominating set. Therefore, $\gamma(G) < |X'| = 2\,\mathrm{ir}(G) - k$ and $\gamma(G) \le 2\,\mathrm{ir}(G) - k - 1$. ∎

Corollary 2.1.5 [4, 5, 20]. *For any graph G, $\gamma(G) \le 2\,\mathrm{ir}(G) - 1$.*

Proof. Let X be a smallest maximal irredundant set in G. If X is independent, then $\gamma(G) = \mathrm{ir}(G)$ (by Proposition 2.1.3) and therefore $\gamma(G) \le 2\,\mathrm{ir}(G) - 1$. If X is not independent and $G[X]$ has k isolated vertices, then $k < |X|$ and it follows from Theorem 2.1.2 that $\gamma(G) \le 2\,\mathrm{ir}(G) - k - 1 \le 2\,\mathrm{ir}(G) - 1$. ∎

We now give a brief summary of the main results of this chapter.

In §2.2, we study some relationships between the independence, domination and irredundance numbers of a graph and the independence, domination and irredundance numbers of its vertex- and edge-deleted subgraphs. These results are frequently applied in this paper, particularly in the study of feasible sequences of integers in §4.1 and in the study of interpolation properties of the independence, domination and irredundance numbers of a graph.

In §2.3, we analyze some properties of the k-packing and k-covering numbers of a graph. The main result of this section is a characterization of graphs G of order $(k+1)n$ with $\gamma_k(G) = n$. We also characterize bipartite graphs G with $\gamma(G) = \alpha(G)$ and trees T with $\gamma_k(T) = \alpha_k(T)$. We show that $\alpha_k(G) = s_k(G)$ and $\gamma_k(G) = s_{2k}(G)$ for any block graph G, where $s_k(G)$ denotes the smallest integer n for which there exists a partition $V_1, \ldots, V_n$ of the vertex set $V(G)$ in which each set V_i induces a subgraph of diameter at most k.

In §2.4, we briefly mention some sufficient conditions for two or more of the lower and upper independence, domination and irredundance numbers of a graph to be equal. We also give a list of forbidden subgraphs that is sufficient for the equality of $\gamma(G)$ and $i(G)$. Then we show that $\mathrm{ir}(G) = \gamma(G) = i(G)$ for domistable graphs. Finally, we prove that $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$ for all chordal, bipartite and unicyclic graphs.

2.2. Domination parameters of vertex- and edge-deleted subgraphs. In this part of the text we investigate the extent to which the lower and upper irredundance (domination and independence, resp.) number of a graph can vary when an arbitrary vertex or edge of the graph is removed. Such knowledge is not only important in its own right, but also if some results are proven by induction. Consequently, it is desirable to learn as much as possible about such properties. In fact, the main results of this section are required later to prove some of our theorems. The behaviour of some of the independence, domination and irredundance parameters after the removal (or addition) of an edge or a vertex from (to) a graph has already been studied in the existing literature. For example, the graphs G in which $\alpha(G - e) > \alpha(G)$ for any edge e of G have been extensively studied, in particular by Plummer [114], Berge [13, 14, 15], Zykov [162], and others. Harary and Schuster [83] have studied changes of the lower domination number and the lower and upper independence numbers after removal (and addition) of any edge from (to) a graph. Bauer, Harary, Nieminen and Suffel [7], Fink, Jacobson, Kinch and Roberts [72], and Walikar and Acharya [158] have studied the smallest number of edges whose removal renders every minimum dominating set in G a nondominating set in the resulting spanning subgraph. Sumner [133] and Sumner and Blitch [134] have worked on closely related problems and, among other things, they studied graphs G in which $\gamma(G+e) < \gamma(G)$ for any edge e from the complement $\overline{G}$ of G. Brigham, Chinn and Dutton [24] analyze graphs G in which $\gamma(G-v) < \gamma(G)$ for any vertex v of G. In [52], Brigham and Dutton study graphs in which $\gamma(G-e) = \gamma(G)$ for any edge e of G. Recently Haynes, Lawson, Brigham and Dutton [86], among other things, have investigated the changing and unchanging of the upper independence number of a graph G under three different situations: deleting an arbitrary vertex, deleting an arbitrary edge and adding an arbitrary edge from the complement of G. Carrington, Harary and Haynes [29] have investigated similar problems for the lower domination number. Some relationships between the independence, domination and irredundance parameters of a graph and the independence, domination and irredundance parameters of its vertex- and edge-deleted subgraphs were also studied in [62] and [142].

We first focus our attention on vertex-deleted subgraphs of a graph. First of all let us observe that if G is a star of order $n+1$, $G = K_{1,n}$, and if v is the center of G, then $\mathrm{ir}(G) = \gamma(G) = i(G) = 1$ and $\mathrm{ir}(G-v) = \gamma(G-v) = i(G-v) = n$. Consequently, if we delete a vertex v from a graph G, the lower irredundance (domination and independence, resp.) number can increase dramatically and it is impossible to give an upper bound on $\mathrm{ir}(G-v)$ ($\gamma(G-v)$ and $i(G-v)$, resp.) only in terms of $\mathrm{ir}(G)$ ($\gamma(G)$ and $i(G)$, resp.). Our first theorem gives lower bounds on $\gamma(G-v)$ and $i(G-v)$ in terms of $\gamma(G)$ and $i(G)$, respectively, and lower and upper bounds on $\alpha(G-v)$ and $\mathrm{IR}(G-v)$ in terms of $\alpha(G)$ and $\mathrm{IR}(G)$, respectively.

THEOREM 2.2.1. *For any vertex v of a graph G,*

(1) $\gamma(G) - 1 \le \gamma(G-v)$;
(2) $i(G) - 1 \le i(G-v)$;
(3) $\alpha(G) - 1 \le \alpha(G-v) \le \alpha(G)$;
(4) $\mathrm{IR}(G) - 1 \le \mathrm{IR}(G-v) \le \mathrm{IR}(G)$.

Proof. (1) If D is a minimum dominating set of $G-v$, then $D \cup \{v\}$ dominates G and therefore $\gamma(G) \le |D \cup \{v\}| = \gamma(G-v) + 1$.

(2) Let I be a minimum maximal independent set in $G-v$. If $N_G(v) \cap I = \emptyset$, then $I \cup \{v\}$ is a maximal independent set in G and consequently $i(G) \le |I \cup \{v\}| = i(G-v) + 1$. If $N_G(v) \cap I \neq \emptyset$, then I is a maximal independent set in G and again $i(G) \le |I| = i(G-v) < i(G-v) + 1$.

(3) Since every independent set of vertices in $G-v$ is also independent in G, we have $\alpha(G-v) \le \alpha(G)$. In order to prove the inequality $\alpha(G) - 1 \le \alpha(G-v)$, we let I be a maximum independent set of vertices in G. Then $|I| = \alpha(G)$ and in the event $v \notin I$, it is clear that $\alpha(G-v) = \alpha(G)$ and hence $\alpha(G-v) \ge \alpha(G) - 1$.

If $v \in I$, then $I - \{v\}$ is an independent set of vertices in $G - v$ and therefore $\alpha(G - v) \geq |I - \{v\}| = \alpha(G) - 1$.

(4) Any irredundant set of vertices of $G - v$ is also irredundant in G. Hence $\mathrm{IR}(G-v) \leq \mathrm{IR}(G)$. Now suppose that J is a maximum irredundant set of vertices in G. If $v \in J$, then $J - \{v\}$ is irredundant in $G - v$ and $\mathrm{IR}(G - v) \geq |J - \{v\}| = \mathrm{IR}(G) - 1$. Similarly, if $v \notin J$ but J is irredundant in $G - v$, then $\mathrm{IR}(G - v) \geq |J| = \mathrm{IR}(G) \geq \mathrm{IR}(G) - 1$. We therefore examine the situation in which $v \notin J$ and J is not irredundant in $G - v$. In this case the irredundance of J in G implies that there exists exactly one x in $G[J]$ for which $N_G[x] - N_G[J - \{x\}] = \{v\}$. Then $J - \{x\}$ is an irredundant set in $G - v$ and hence $\mathrm{IR}(G - v) \geq |J - \{x\}| = \mathrm{IR}(G) - 1$. This completes the proof. ■

In view of Theorem 2.2.1 it is natural to ask: What relationships, if any, exist between the upper domination number of a graph and the upper domination number of its vertex-deleted subgraph? The following examples show that no particular inequalities hold between these two parameters. For a positive integer n, by A_n we denote the graph which consists of two vertex-disjoint complete graphs with vertices $v_1, v_2, \ldots, v_{n+1}$ and $u_1, u_2, \ldots, u_{n+1}$, respectively, and n additional edges $v_i u_i$ for $i = 1, 2, \ldots, n$. For convenience, we denote $A_n - v_\delta$, where v_δ is a vertex of minimum degree in A_n, by D_n. The graphs A_3 and D_3 are shown in Figure 1. Simple verifications show that graphs A_n and D_n have the following properties.

PROPOSITION 2.2.1. *For every integer $n \geq 2$, $\Gamma(A_n) = 2$ and $\Gamma(D_n) = n$.* ■

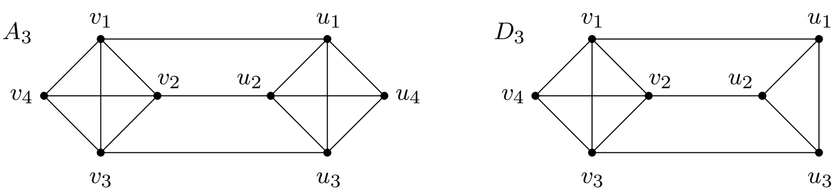

Fig. 1. The graphs A_3 and D_3 of Proposition 2.2.1

Note that for $n \geq 2$, the vertex-deleted subgraph $D_n - v_\Delta$ of D_n is isomorphic to A_{n-1} if v_Δ is any vertex of maximum degree in D_n. From this and from Proposition 2.2.1 it follows that $\Gamma(A_n) = 2$, while $\Gamma(A_n - v_\delta) = \Gamma(D_n) = n$ and, again, $\Gamma(D_n - v_\Delta) = \Gamma(A_{n-1}) = 2$. These examples show that the removal of a vertex need not decrease the upper domination number and may even increase it. Moreover, if v is a vertex of G, then the difference $\Gamma(G) - \Gamma(G - v)$ as well as $\Gamma(G - v) - \Gamma(G)$ can be made arbitrarily large.

In the next theorem, we present the relationship between the lower irredundance number of a graph and the lower irredundance number of its vertex-deleted subgraph. We already know that the deletion of a vertex from a graph can increase the lower irredundance number and that there is no upper bound on $\mathrm{ir}(G - v)$ only in terms of $\mathrm{ir}(G)$. On the other hand, the deletion of a vertex can decrease the lower irredundance number and it follows from Proposition 2.1.3, Theorem 2.2.1(1) and Corollary 2.1.5 that if v is a vertex of a graph G, then $\mathrm{ir}(G) \leq \gamma(G) \leq \gamma(G-v)+1 \leq 2\,\mathrm{ir}(G-v)$. Therefore $\mathrm{ir}(G)/2$ is a lower bound on $\mathrm{ir}(G-v)$. Recently Favaron [62] has proved that if v is a vertex of G such that $\mathrm{ir}(G-v) \geq 2$, then $(\mathrm{ir}(G)+1)/2$ is the best possible lower bound on $\mathrm{ir}(G-v)$. Now it is possible to prove a bit more. The proof of Theorem 2.2.2 given below is a modification of the proof given by Favaron [62].

THEOREM 2.2.2. *If G is a graph of order at least two and v is a vertex of G, then*

$$\mathrm{ir}(G-v) \geq \frac{\mathrm{ir}(G)+\min\{1, |\mathrm{ir}(G)-2|\}}{2}.$$

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a maximal irredundant set of $G-v$, $n = \mathrm{ir}(G-v)$. If $n = 1$, then $1 \leq \mathrm{ir}(G) \leq 2$ and the result is obvious. Thus assume that $n \geq 2$. Certainly, X is an irredundant set in G. If in addition X is a maximal irredundant set of G, then $\mathrm{ir}(G) \leq n \leq 2n-1$ and therefore

$$n \geq \frac{\mathrm{ir}(G)+1}{2} \geq \frac{\mathrm{ir}(G)+\min\{1, |\mathrm{ir}(G)-2|\}}{2}.$$

Similarly, if X is a dominating set of $G-v$, then $\gamma(G-v) = n$ and according to Proposition 2.1.3 and Theorem 2.2.1(1) we have $\mathrm{ir}(G) \leq \gamma(G) \leq \gamma(G-v)+1 = n+1 \leq 2n-1$ which again enforces the result. If the set $X \cup \{v\}$ is irredundant in G, then certainly it is a maximal irredundant set of G and $\mathrm{ir}(G) \leq |X \cup \{v\}| \leq 2n-1$ which implies the result. We have the same result if there exists a vertex $y \in V(G-v) - X$ such that $X \cup \{y\}$ is a maximal irredundant set of G.

We now assume that neither X is a dominating set of $G-v$ nor X or $X \cup \{y\}$ for $y \in V(G) - X$ is a maximal irredundant set of G. Then let Y be a subset of $V(G-v) - X$ of the smallest cardinality such that $|Y| \geq 2$ and $X \cup Y$ is a maximal irredundant set of G, i.e., $I_G(x, X \cup Y) \neq \emptyset$ for each $x \in X \cup Y$.

We assert that $v \in I_G(x_0, X \cup Y)$ for some $x_0 \in X$. First, let us observe that $v \in I_G(x, X \cup Y)$ for some $x \in X \cup Y$; for if $v \notin I_G(x, X \cup Y)$ for each $x \in X \cup Y$, then $X \cup Y$ is irredundant in $G-v$, contrary to the maximality of X in $G-v$. Next, for each $y \in Y$, $v \notin I_G(y, X \cup Y)$; for if there were $y_0 \in Y$ such that $v \in I_G(y_0, X \cup Y)$, then $X \cup (Y - \{y_0\})$ would be irredundant in $G-v$ which again is impossible. Combining the above facts we deduce that $v \in I_G(x_0, X \cup Y)$ for some $x_0 \in X$.

Since X does not dominate all the vertices of $G-v$, the set $U_0 = \{x \in V(G-v) - X : N_{G-v}(x) \cap X = \emptyset\}$ is nonempty, so by Theorem 2.1.1(a) the set $U_1 = \{x \in V(G-v) - X : |N_{G-v}(x) \cap X| = 1\}$ is also nonempty. Denote $U_2 = V(G-v) - X - U_0 - U_1$. By Theorem 2.1.1(a), for each $u \in U_0$, the set $X_u = \{x \in X : I_{G-v}(x, X) \subseteq N_{G-v}(u)\}$ is nonempty. Let M be a subset of X of the smallest cardinality such that $X_u \cap M \neq \emptyset$ for each $u \in U_0$, say $M = \{x_1, x_2, \ldots, x_m\}$. Each vertex x_i of M belongs to X_u for some $u \in U_0$, so $I_{G-v}(x_i, X) \subseteq N_{G-v}(u)$ and therefore $x_i \notin I_{G-v}(x_i, X)$ (as $x_i \notin N_{G-v}(u)$) and x_i

is a nonisolated vertex in the subgraph induced by X in $G-v$. For each $x_i \in M$, we choose any $x_i' \in I_{G-v}(x_i, X)$ and form the set $D = X \cup \{x_1', x_2', \ldots, x_m'\}$ of cardinality $n+m$. Certainly, each vertex of $U_1 \cup U_2$ is adjacent to a vertex of X. Moreover, for each $u \in U_0$, there exists $x_i \in M$ such that $x_i \in X_u$, so u is adjacent to x_i'. We conclude that D is a dominating set of $G-v$ and of G (as v is adjacent to $x_0 \in X \subset D$). Thus, if $m < n$, then the result follows from the inequalities $\mathrm{ir}(G) \le \gamma(G) \le |D| \le 2n-1$. Finally, if $m = n$, then it follows easily from the above and from Theorem 2.1.1(b) that for each $x_i \in X - \{x_0\}$, the set $D - \{x_i\}$ is a dominating set of G and again the result is derived from the inequalities $\mathrm{ir}(G) \le \gamma(G) \le |D - \{x_i\}| = 2n-1$. This completes the proof of the theorem. ■

The next two examples concern the above theorem and they show that this result is the best possible since for every positive integer n there exist a graph G and a vertex v of G such that $\mathrm{ir}(G-v) = n = (\mathrm{ir}(G) + \min\{1, |\mathrm{ir}(G) - 2|\})/2$. For $n = 1$, take $G = K_2$ and any vertex v of G. Then $\mathrm{ir}(G-v) = 1 = (\mathrm{ir}(G) + \min\{1, |\mathrm{ir}(G) - 2|\})/2$. For $n \ge 2$, such a graph G can be constructed as follows (see Figure 2): Take two vertex-disjoint complete graphs K_n and K_n' on vertices $x_1, x_2, \ldots, x_n$ and $x_1', x_2', \ldots, x_n'$, respectively. Now join the vertices x_i and x_i' for $1 \le i \le n$. Add a new vertex v adjacent to x_n and x_n'. Finally, take $n + n(n-1)/2$ additional sets $Y_1, Y_2, \ldots, Y_n, Z_{1,2}, Z_{1,3}, \ldots, Z_{1,n}, Z_{2,3}, \ldots, Z_{n-1,n}$ each with n mutually nonadjacent vertices, join each vertex of Y_i to the vertex x_i' $(1 \le i \le n)$ and each vertex of $Z_{i,j}$ to the vertices x_i and x_j $(1 \le i < j \le n)$. One can verify that $\{x_1, x_2, \ldots, x_{n-1}\} \cup \{x_1', x_2', \ldots, x_n'\}$ and $\{x_1, x_2, \ldots, x_n\}$ are minimum maximal irredundant sets of G and $G-v$, respectively, and therefore $\mathrm{ir}(G-v) = n = (\mathrm{ir}(G) + \min\{1, |\mathrm{ir}(G) - 2|\})/2$.

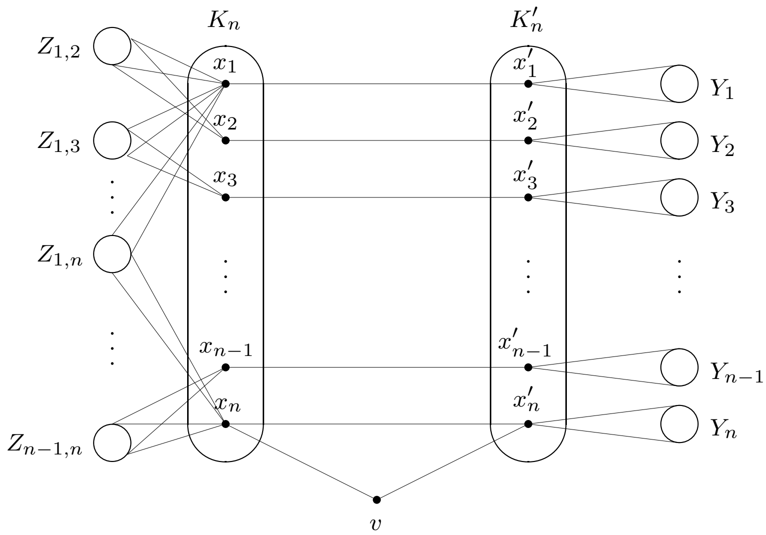

Fig. 2. A graph G in which $\mathrm{ir}(G) = 2n-1 = \mathrm{ir}(G - v - x_n x_n')$ and $\mathrm{ir}(G-v) = n = \mathrm{ir}(G - v x_n)$

The following theorem relates the k-packing and k-covering numbers of a graph and the k-packing and k-covering numbers of its edge-deleted subgraphs.

THEOREM 2.2.3. *For any positive integer k and any edge vu of a graph G,*

(1) $\gamma_k(G) \leq \gamma_k(G - vu) \leq \gamma_k(G) + 1$;
(2) $\alpha_k(G) \leq \alpha_k(G - vu) \leq \alpha_k(G) + 1$.

Proof. (1) If C is a minimum k-covering of $G - vu$, then C is a k-covering of G and therefore $\gamma_k(G) \leq |C| = \gamma_k(G - vu)$. On the other hand, if D is a minimum k-covering of G, then at least one of the sets D, $D \cup \{v\}$, and $D \cup \{u\}$ is a k-covering in $G - vu$ and hence $\gamma_k(G - vu) \leq |D| + 1 = \gamma_k(G) + 1$.

(2) Since every k-packing of G is a k-packing of $G - vu$, so $\alpha_k(G) \leq \alpha_k(G - vu)$.

In order to prove the inequality $\alpha_k(G - vu) \leq \alpha_k(G) + 1$, let I be a maximum k-packing of $G - vu$. If I is also a k-packing in G, then $\alpha_k(G - vu) = |I| \leq \alpha_k(G) \leq \alpha_k(G) + 1$. Thus assume that I is not a k-packing in G. Then there are vertices x, $y \in I$ for which $d_{G-vu}(x, y) > k$, whereas $d_G(x, y) \leq k$. Let I_0 be the set of all such vertices x and y from I, and define $I_v = \{x \in I_0 : d_G(x, v) < d_G(x, u)\}$ and $I_u = \{y \in I_0 : d_G(y, u) < d_G(y, v)\}$. It is easy to observe that the sets I_v and I_u are nonempty and they form a partition of I_0. Note that if x, $y \in I_0$ and $d_G(x, y) \leq k$, then any shortest $x - y$ path passes through the edge vu in G. This implies that $d_G(x, y) > k$ if x and y are different vertices of I_v (I_u, resp.).

We claim that $|I_v| = 1$ or $|I_u| = 1$. Suppose, contrary to our claim, that $|I_v| \geq 2$ and $|I_u| \geq 2$. Let $x_1 \in I_v$ be the vertex nearest v in G. Similarly, let $y_1 \in I_u$ be the vertex nearest u in G. Take any $x_2 \in I_v - \{x_1\}$ and $y_2 \in I_u - \{y_1\}$. It follows from the choice of the vertices x_1 and y_1 that $d_G(x_1, y_1) \leq k$, $d_G(x_2, y_1) \leq k$, $d_G(y_2, x_1) \leq k$, while $d_G(x_1, x_2) > k$ and $d_G(y_1, y_2) > k$. Let P_1 and P_2 be any shortest $x_1 - y_1$ and $x_2 - y_1$ paths in G, respectively. Let x' be the vertex nearest x_1 in P_1 which is also in P_2. Without loss of generality, we assume that the $x' - y_1$ subpaths of P_1 and P_2 are the same. Let P_3 be a shortest $y_2 - x_1$ path in G and let y' be the vertex nearest y_2 in P_3 which is also in P_1 (and P_2). We may assume that the $x' - y'$ subpaths of P_1 and P_3 are the same. Denote $d_G(x', y') = p$, $d_G(x_i, x') = l_i$, and $d_G(y_i, y') = k_i$ for $i = 1, 2$. Since $d_G(x_2, y_1) = l_2 + p + k_1 \leq k < d_G(x_1, x_2) \leq l_1 + l_2$, so $l_1 > k_1 + p$. Therefore $d_G(y_2, x_1) = k_2 + p + l_1 > k_1 + k_2 + 2p \geq d_G(y_1, y_2) + 2p > k + 2p > k$. This contradicts $d_G(y_2, x_1) \leq k$, and our claim follows.

According to the above claim, we may assume that $I_v = \{x_1\}$. Then it is easy to check that $I - \{x_1\}$ is a k-packing in G, so $\alpha_k(G - vu) - 1 = |I - \{x_1\}| \leq \alpha_k(G)$. This completes the proof. ∎

COROLLARY 2.2.1. *Let k be a positive integer. If v and u are two nonadjacent vertices of a graph H, then*

(1) $\gamma_k(H) - 1 \leq \gamma_k(H + vu) \leq \gamma_k(H)$;
(2) $\alpha_k(H) - 1 \leq \alpha_k(H + vu) \leq \alpha_k(H)$.

P r o o f. This follows immediately by applying Theorem 2.2.3 to the edge vu of the graph $G = H + vu$. ■

The next three theorems are counterparts of the last corollary for the lower and upper irredundance, domination, and independence numbers of a graph. The statement (2) of Theorem 2.2.4 and the statements (1) and (2) of Corollary 2.2.2 were proved in [83].

THEOREM 2.2.4. *For every graph G and every edge vu of G,*

(1) $\gamma(G) \leq \gamma(G - vu) \leq \gamma(G) + 1$;
(2) $\alpha(G) \leq \alpha(G - vu) \leq \alpha(G) + 1$;
(3) $2 \leq \Gamma(G - vu) \leq \Gamma(G) + 1$;
(4) $\mathrm{IR}(G) - 1 \leq \mathrm{IR}(G - vu) \leq \mathrm{IR}(G) + 1$.

P r o o f. Since (1) and (2) follow from Theorem 2.2.3, we only prove (3) and (4).

(3) For any edge vu of G, $G - vu$ is not a complete graph and therefore $\Gamma(G - vu) \geq 2$. To prove the inequality $\Gamma(G - vu) \leq \Gamma(G) + 1$, let D be a maximum minimal dominating set of $G - vu$. Certainly, D is a dominating set of G and we consider three cases.

First, if neither v nor u belongs to D, then D is a minimal dominating set of G and therefore $\Gamma(G) \geq |D| = \Gamma(G - vu) \geq \Gamma(G - vu) - 1$.

Assume now that either v or u belongs to D, say $v \in D$ and $u \in V(G) - D$. If D is a minimal dominating set of G, then certainly $\Gamma(G) \geq |D| = \Gamma(G - vu) \geq \Gamma(G - vu) - 1$. Thus assume that D is not a minimal dominating set of G. Then, since D is a minimal dominating set of $G - vu$, there exists a unique vertex $u' \in D - \{v\}$ such that $I_{G-vu}(u', D) = \{u\}$. Now it is easy to observe that $D - \{u'\}$ is a minimal dominating set of G and so $\Gamma(G) \geq |D - \{u'\}| = \Gamma(G - vu) - 1$.

Finally, assume that both v and u belong to D. If $I_{G-vu}(v, D) - \{v\} \neq \emptyset$ and $I_{G-vu}(u, D) - \{u\} \neq \emptyset$, then D is a minimal dominating set of G and $\Gamma(G) \geq |D| \geq \Gamma(G - vu) - 1$. If $I_{G-vu}(v, D) - \{v\} = \emptyset$ or $I_{G-vu}(u, D) - \{u\} = \emptyset$, then $D - \{v\}$ or $D - \{u\}$ is a minimal dominating set of G and $\Gamma(G) \geq \Gamma(G - vu) - 1$.

(4) In order to prove the inequality $\mathrm{IR}(G) - 1 \leq \mathrm{IR}(G - vu)$ (which is obvious if $\mathrm{IR}(G) = 1$), we assume that $\mathrm{IR}(G) \geq 2$ and let X be a maximum irredundant set in G. If the vertices v and u are both either in X or in $V(G) - X$, then we see at once that $I_{G-vu}(x, X) \supseteq I_G(x, X) \neq \emptyset$ for every $x \in X$. Hence X is irredundant in $G - vu$ and therefore $\mathrm{IR}(G - vu) \geq |X| = \mathrm{IR}(G) \geq \mathrm{IR}(G) - 1$. If exactly one of the vertices v and u is in X, say $v \in X$ and $u \notin X$, then it is easy to check that $I_{G-vu}(x, X - \{v\}) = I_G(x, X - \{v\}) \supseteq I_G(x, X) \neq \emptyset$ for every $x \in X - \{v\}$. Thus the set $X - \{v\}$ is irredundant in $G - vu$, so $\mathrm{IR}(G - vu) \geq |X - \{v\}| = \mathrm{IR}(G) - 1$.

We prove the remaining inequality $\mathrm{IR}(G - vu) \leq \mathrm{IR}(G) + 1$ by contradiction. Thus suppose that $\mathrm{IR}(G - vu) > \mathrm{IR}(G) + 1$ and let Y be a maximum irredundant set in $G - vu$. We derive contradictions in three cases.

If none of the vertices v and u belongs to Y, then $I_G(y,Y) = I_{G-vu}(y,Y) \neq \emptyset$ for every $y \in Y$, so Y is irredundant in G and therefore $\mathrm{IR}(G) \geq |Y| = \mathrm{IR}(G-vu)$, contradicting the supposition.

If exactly one of the vertices v and u is in Y, say $v \in Y$ and $u \notin Y$, then $I_G(y, Y-\{v\}) = I_{G-vu}(y, Y-\{v\}) \neq \emptyset$ for every $y \in Y-\{v\}$. We conclude that $Y-\{v\}$ is irredundant in G, from which we see that $\mathrm{IR}(G) \geq |Y-\{v\}| = \mathrm{IR}(G)-1$, a contradiction.

Finally, suppose that the vertices v and u belong to Y. Then $I_G(u, Y-\{v\}) \supseteq I_{G-vu}(u, Y-\{v\}) \supseteq I_{G-vu}(u,Y) \neq \emptyset$ and $I_G(y, Y-\{v\}) = I_{G-vu}(y, Y-\{v\}) - \{v\} \supseteq I_{G-vu}(y,Y)-\{v\} = I_{G-vu}(y,Y) \neq \emptyset$ for every $y \in Y-\{v,u\}$. Consequently, $Y-\{v\}$ is an irredundant set in G, so $\mathrm{IR}(G) \geq |Y-\{v\}| = \mathrm{IR}(G-vu)-1$, our final contradiction. ∎

The following examples show that parts (3) and (4) of Theorem 2.2.4 cannot be improved. For a positive integer n, let H_n denote the graph which consists of two vertex-disjoint complete graphs with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$, respectively, and n additional edges v_iu_i for $i = 1,2,,\ldots,n$. It is no problem to observe that $\Gamma(H_n) = n$ while $\Gamma(H_n - v_iu_i) = 2$ for $i = 1,2,\ldots,n$. For any edge vu of K_n with $n \geq 2$, $\Gamma(K_n - vu) = \Gamma(K_n)+1 = 2$ and $\mathrm{IR}(K_n - vu) = \mathrm{IR}(K_n)+1 = 2$. Finally, the graph G of Figure 3 contains an edge vu such that $\mathrm{IR}(G-vu) = 5$ while $\mathrm{IR}(G) = 6$.

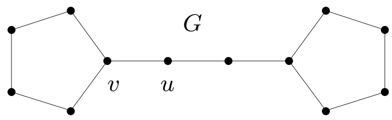

Fig. 3. A graph with $\mathrm{IR}(G-vu) = \mathrm{IR}(G)-1 = 5$

THEOREM 2.2.5. *If vu is an edge of a graph G, then*

$$\min\{2, i(G)\} \leq i(G-vu) \leq i(G)+1.$$

Proof. Note that if $i(G)=1$, then $1 \leq i(G-vu) \leq 2$, so $1 = \min\{2, i(G)\} \leq i(G-vu)$. If $i(G) \geq 2$, then $i(G-vu) \geq 2$ and therefore $2 = \min\{2, i(G)\} \leq i(G-vu)$. On the other hand, if I is a minimum maximal independent set of G, then at least one of the sets I, $I \cup \{v\}$, and $I \cup \{u\}$ is a maximal independent set of $G-vu$, so $i(G-vu) \leq |I|+1 = i(G)+1$. ∎

The restriction imposed by the inequalities of Theorem 2.2.5 cannot be improved in the following sense: For any positive integers n and k with $\min\{2,n\} \leq k \leq n+1$, there exist a graph G and an edge vu in G such that $i(G) = n$ and $i(G-vu) = k$. For $n = k = 1$, the complete graph $G = K_3$ and any edge vu of G have the required properties. For $n \geq 1$ and $k = n+1$, take $G = nK_2$. Then $i(G) = n$ and $i(G-vu) = n+1$ for every edge vu of G. For $n \geq 2$ and $2 \leq k < n+1$, consider the graph G given in Figure 4 and its edge-deleted subgraph $G-vu$. It is easy to check that $i(G) = n$ and $i(G-vu) = k$.

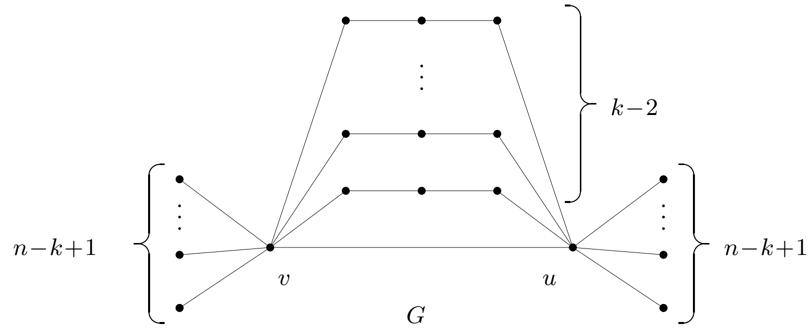

Fig. 4. A graph with $i(G) = n$ and $i(G - vu) = k$ for $2 \le k < n + 1$

We now show that the removal of an edge from a graph increases (decreases, resp.) the lower irredundance number by at most factor 2 (1/2, resp.).

THEOREM 2.2.6. *If vu is an edge of a graph G, then*

$$\frac{\mathrm{ir}(G)+1}{2} \le \mathrm{ir}(G - vu) \le \mathrm{ir}(G) + \max\{1, \mathrm{ir}(G) - 1\}.$$

Proof. According to Proposition 2.1.3, Theorem 2.2.4 and Corollary 2.1.5, $\mathrm{ir}(G) \le \gamma(G) \le \gamma(G - vu) \le 2\,\mathrm{ir}(G - vu) - 1$ and therefore $(\mathrm{ir}(G) + 1)/2 \le \mathrm{ir}(G - vu)$. Similarly, $\mathrm{ir}(G - vu) \le \gamma(G - vu) \le \gamma(G) + 1 \le (2\,\mathrm{ir}(G) - 1) + 1 = 2\,\mathrm{ir}(G)$. Of course, $\mathrm{ir}(G - vu) = 2\,\mathrm{ir}(G)$ if and only if equality holds at each point in the above sequence of inequalities. Furthermore, $2\,\mathrm{ir}(G) = \mathrm{ir}(G) + \max\{1, \mathrm{ir}(G) - 1\}$ if and only if $\mathrm{ir}(G) = 1$. Therefore in order to prove the inequality $\mathrm{ir}(G - vu) \le \mathrm{ir}(G) + \max\{1, \mathrm{ir}(G) - 1\}$ it is enough to assume $\gamma(G - vu) = \gamma(G) + 1$, $\gamma(G) = 2\,\mathrm{ir}(G) - 1$ with $\mathrm{ir}(G) \ge 2$, and then to show that $\mathrm{ir}(G - vu) \le 2\,\mathrm{ir}(G) - 1$.

Let $X = \{x_1, x_2, \ldots, x_n\}$, $n = \mathrm{ir}(G)$, be any minimum maximal irredundant set of G, and let U_0, U_1 and U_2 be subsets of $V(G) - X$, where $U_2 = \{x \in V(G) - X : |N_G(x) \cap X| \ge 2\}$ and $U_i = \{x \in V(G) - X : |N_G(x) \cap X| = i\}$ for $i = 0, 1$. Certainly, the sets X, U_0, U_1, U_2 form a partition of $V(G)$. Since $|X| = n < 2n - 1 = \gamma(G)$, the set U_0 is nonempty and therefore it follows from Theorem 2.1.1 that the set U_1 is nonempty, either. For each $u \in U_0$, define $X_u = \{x \in X : I_G(x, X) \subseteq N_G(u)\}$. Again by Theorem 2.1.1, the set X_u is nonempty for each $u \in U_0$. Let M be a subset of X of the smallest cardinality such that $X_u \cap M \ne \emptyset$ for each $u \in U_0$, say $M = \{x_1, \ldots, x_m\}$. For each $x_i \in M$, $x_i \in X_u$ for some $u \in U_0$, so $I_G(x_i, X) \subseteq U_1 \cap N_G(u)$ and therefore $x_i \notin I_G(x_i, X)$ (as $x_i \notin N_G(u)$) and x_i is not isolated in $G[X]$, the subgraph of G induced by X. For $x_i \in M$, choose any $x_i' \in I_G(x_i, X)$ and define $D = X \cup \{x_1', \ldots, x_m'\}$. It follows from the definition of the sets U_1 and U_2 that every vertex of $U_1 \cup U_2$ is adjacent to a vertex of X. In addition, for each $u \in U_0$, there exists $i \in \{1, \ldots, m\}$ such that $x_i \in X_u$, hence u is adjacent to x_i'. Thus D is a dominating set of G. However, since D contains X, D is not irredundant. Therefore D properly contains a minimal dominating set (by Corollary 2.1.4) and hence $2n - 1 = \gamma(G) < n + m$.

Consequently, $m = n$ and then $M = X$, $G[X]$ is without isolated vertices, and it follows from the above facts and from Theorem 2.1.1(b) that $D - \{x_i\}$ is a minimum dominating set of G for each $x_i \in X$. Furthermore, for any two vertices x_i, $x_j \in X$, $x_i \neq x_j$, there exists a vertex $x \in U_2$ such that $N_G(x) \cap D = \{x_i, x_j\}$, as otherwise the set $D - \{x_i, x_j\}$ would be dominating in G which is impossible.

We now show that one of the vertices v and u belongs to U_0 and the other to U_1 if $\gamma(G - vu) = \gamma(G) + 1$. First it is easy to observe that for every minimum dominating set D' of G we have $|D' \cap \{v, u\}| = 1$. Moreover, if $x \in D' \cap \{v, u\}$ and $y \in \{v, u\} - D'$, then $N_G(y) \cap D' = \{x\}$. Consequently, $\{v, u\}$ cannot be a subset of $U_0 \cup U_2$ as otherwise each of the sets $D - \{x_i\}$, $x_i \in X$, would be a minimum dominating set of $G - vu$ which is impossible. Similarly, neither $|\{v, u\} \cap U_2| = 1$ and $|\{v, u\} \cap (X \cup U_1)| = 1$ nor $|\{v, u\} \cap X| = 1$ and $|\{v, u\} \cap U_1| = 1$ nor $\{v, u\} \subseteq X$ because otherwise at least one of the sets $D - \{x_i\}$, $x_i \in X$, would be a minimum dominating set of $G - vu$ which again is impossible. We now claim that $\{v, u\}$ cannot be a subset of U_1. For if not, then either $\{v, u\} \subseteq I_G(x_k, X)$ for some $x_k \in X$ or $v \in I_G(x_i, X)$ and $u \in I_G(x_j, X)$ for some x_i, $x_j \in X$ with $x_i \neq x_j$. In these cases, if the vertices of $D - X$ are chosen in such a way that $x'_k \in \{v, u\}$, say $x'_k = v$, when $\{v, u\} \subseteq I_G(x_k, X)$ (resp. $x'_i = v$ and $x'_j = u$ if $v \in I_G(x_i, X)$ and $u \in I_G(x_j, X)$), then for the minimum dominating set $D - \{x_l\}$ of G (with $l \neq k$) we have $\{v, u\} \cap (D - \{x_l\}) = \{v\}$ and $\{v, x_k\} \subseteq N_G(u) \cap (D - \{x_l\})$ (resp. $\{v, u\} \subset D - \{x_l\}$) and therefore $\gamma(G - vu) = \gamma(G)$, a contradiction. We therefore have $\{v, u\} \not\subseteq U_1$. Since no vertex of U_0 is adjacent to a vertex of X, it follows from the above and from the assumption $\gamma(G - vu) = \gamma(G) + 1$ that one of the vertices v and u belongs to U_1 and the other to U_0, say $v \in U_1$ and $u \in U_0$.

Let x_i be the unique vertex of $N_G(v) \cap X$. Certainly, $v \in I_G(x_i, X)$. Moreover, v is the unique vertex of $I_G(x_i, X)$, i.e. $v = x'_i$, as otherwise if $x'_i \in I_G(x_i, X) - \{v\}$, then none of the vertices v and u belongs to $D - \{x_l\}$ ($l = 1, \ldots, n$) and therefore $\gamma(G - vu) = \gamma(G)$, a contradiction. Hence we have $I_G(x_i, X) = \{v\}$. We now show that $N_G(u) \cap U_1 = \{v\}$. Suppose on the contrary that $N_G(u) \cap U_1 - \{v\} \neq \emptyset$. Then there exists $x_j \in X - \{x_i\}$ such that $I_G(x_j, X) \subset N_G(u)$. But now for the minimum dominating set $D - \{x_l\}$ of G we have $\{x'_i, x'_j\} \subseteq N_G(u) \cap (D - \{x_l\})$ and consequently $\gamma(G - vu) = \gamma(G)$, a contradiction. It follows that $N_G(u) \cap U_1 = \{v\}$ and in particular $I_G(x_k, X) \cap (N_G[u] - \{v\}) = \emptyset$ for each $x_k \in X$.

In order to complete the proof, we show that $X \cup \{u\}$ is a maximal irredundant set of $G - vu$. Since u is isolated in the subgraph of $G - vu$ induced by $X \cup \{u\}$ and $I_{G-vu}(x_k, X \cup \{u\}) = N_{G-vu}[x_k] - N_{G-vu}[(X - \{x_k\}) \cup \{u\}] = I_G(x_k, X) - (N_G[u] - \{v\}) = I_G(x_k, X) \neq \emptyset$ for each $x_k \in X$, the set $X \cup \{u\}$ is irredundant in $G - vu$. By maximality of X (in G), for every vertex d of $V(G) - X$, there exists some vertex y_d in $X \cup \{d\}$ such that $N_G[y_d] \subseteq N_G[(X \cup \{d\}) - \{y_d\}]$. In particular, for $d \in V(G) - X - \{v, u\}$, there exists $y_d \in X \cup \{d\}$ such that $N_{G-vu}[y_d] = N_G[y_d] \subseteq N_G[(X \cup \{d\}) - \{y_d\}] = N_{G-vu}[(X \cup \{d\}) - \{y_d\}] \subset N_{G-vu}[(X \cup \{d\}) - \{y_d\}] \cup N_{G-vu}[u] = N_{G-vu}[(X \cup \{u, d\}) - \{y_d\}]$. Therefore $X \cup \{u, d\}$ is not irredundant in $G - vu$ for each $d \in V(G) - X - \{v, u\}$. Finally,

$X\cup\{u,v\}$ is not irredundant in $G-vu$ since $I_G(x_i,X)=N_G[x_i]-N_G[X-\{x_i\}]=\{v\}$ and hence $I_{G-vu}(x_i,X\cup\{v,u\})=N_{G-vu}[x_i]-N_{G-vu}[(X-\{x_i\})\cup\{v,u\}]=N_G[x_i]-N_G[X-\{x_i\}]-N_G[\{v,u\}]=\{v\}-N_G[\{v,u\}]=\emptyset$. We conclude that $X\cup\{u\}$ is a maximal irredundant set of $G-vu$. Therefore $\mathrm{ir}(G-vu)\le|X\cup\{u\}|=\mathrm{ir}(G)+1\le 2\,\mathrm{ir}(G)-1$. This completes the proof. ∎

Note that both the lower and upper bounds on the lower irredundance number of an edge-deleted subgraph imposed by the inequalities of Theorem 2.2.6 are attainable in the following sense: For any positive integer n there exist a graph G and an edge vu in G such that $\mathrm{ir}(G-vu)=n=(\mathrm{ir}(G)+1)/2$. Similarly, for a positive integer n there exist a graph F and an edge vu in F such that $\mathrm{ir}(F)=n$ and $\mathrm{ir}(F-vu)=\mathrm{ir}(F)+\max\{1,\mathrm{ir}(F)-1\}$. For $n=1$, let $G=K_1+(K_1\cup K_2)$, and let vu be the unique edge of G such that $d_G(v)=d_G(u)=2$. Then $(\mathrm{ir}(G)+1)/2=\mathrm{ir}(G-vu)=1=\mathrm{ir}(G)+\max\{1,\mathrm{ir}(G)-1\}$. For $n\ge 2$, let G be the graph defined after Theorem 2.2.2, see Figure 2. Then $(\mathrm{ir}(G)+1)/2=n=\mathrm{ir}(G-vx_n)$, as $\{x_1,x_2,\ldots,x_{n-1}\}\cup\{x_1',x_2',\ldots,x_n'\}$ and $\{x_1,x_2,\ldots,x_n\}$ are minimum maximal irredundant sets of G and $G-vx_n$, respectively. Finally, take the subgraph $F=G-v$ of G. Then $\mathrm{ir}(F-x_nx_n')=2n-1=2\,\mathrm{ir}(F)-1=\mathrm{ir}(F)+\max\{1,\mathrm{ir}(F)-1\}$, as $\{x_1,x_2,\ldots,x_n\}$ and $\{x_1,\ldots,x_{n-1}\}\cup\{x_1',\ldots,x_n'\}$ are minimum maximal irredundant sets of F and $F-x_nx_n'$, respectively.

COROLLARY 2.2.2. *If v and u are two nonadjacent vertices of a graph H, then*

(1) $\gamma(H)-1\le\gamma(H+vu)\le\gamma(H)$;
(2) $\alpha(H)-1\le\alpha(H+vu)\le\alpha(H)$;
(3) $\Gamma(H)-1\le\Gamma(H+vu)$;
(4) $\mathrm{IR}(H)-1\le\mathrm{IR}(H+vu)\le\mathrm{IR}(H)+1$;
(5) $i(H)-1\le i(H+vu)$;
(6) $(\mathrm{ir}(H)+\min\{1,|2-\mathrm{ir}(H)|\})/2\le\mathrm{ir}(H+vu)\le 2\,\mathrm{ir}(H)-1$.

Proof. This follows immediately by applying Theorems 2.2.4–2.2.6 to the edge vu of the graph $G=H+vu$. ∎

2.3. Packing and covering numbers. In the rest of this chapter we are mainly interested in classes of graphs for which some of the parameters ir, γ, i, α, Γ, IR, γ_k and α_k are equal. Many results of this type have been given during the last few years and most of them give sufficient conditions, usually in terms of forbidden subgraphs. However, forbidden subgraph characterizations for equality of parameters have been hard to obtain; in fact, it is impossible in general. This is easy to see since the corona of graphs G and K_1 produces the graph $G'=G\circ K_1$ containing G as an induced subgraph and $\mathrm{ir}(G')=\gamma(G')=i(G')=\alpha(G')=\Gamma(G')=\mathrm{IR}(G')$ by Proposition 2.1.4. The same comment applies to the forbidden subgraph characterizations of graphs G for which $\gamma_k(G)=\alpha_k(G)$, see Proposition 2.3.2 in this section.

In 1970, Szamkołowicz [135] (see also [136]) posed the problem of characterizing those graphs for which the domination number is equal to the independence number (see also Problem 1(c) in [100]). Such graphs have been studied in [22, 23], [66] and [146, 148, 149, 150, 152, 153]. In this section, we first give a complete description of connected graphs G of order $(k+1)n$ with $\gamma_k(G) = n$. Then we characterize bipartite graphs G with $\gamma(G) = \alpha(G)$ and trees T with $\gamma_k(T) = \alpha_k(T)$. We go on to show that $\alpha_k(G) = s_k(G)$ for any block graph G, where $s_k(G)$ denotes the smallest integer n for which there exists a partition $V_1, \ldots, V_n$ of the vertex set $V(G)$ in which each set V_i induces a subgraph of diameter at most k. Finally, we prove a theorem from which we can get an effective algorithm for determining the numbers $\alpha_k(G)$, $s_k(G)$, a maximum k-packing, and a decomposition of a block graph G into $s_k(G)$ graphs each of diameter at most k. (Other classes of graphs G for which $\gamma(G) = \alpha(G)$ are given in the next chapter.)

We shall apply the following result due to Meir and Moon [107].

PROPOSITION 2.3.1. *If T is a tree on $p \geq k+1$ vertices, then* $\gamma_k(T) \leq \lfloor p/(k+1) \rfloor$.

Proof. Let $P = (v_0, v_1, \ldots, v_d)$ be any longest path in T. If $d \leq k$, then the vertex v_0 constitutes a k-covering of T and $\gamma_k(T) = 1 \leq \lfloor p/(k+1) \rfloor$. Thus assume $d > k$ and denote

$$D_i = \{v \in V(T) : d_T(v_0, v) = i \,(\mathrm{mod}(k+1))\}$$

for $i = 0, 1, \ldots, k$. We now show that each set D_i is a k-covering of T.

Let z be any vertex of T and suppose that $d_T(v_0, z) = l$. If $l \geq i$, then $i + m(k+1) \leq l < i + (m+1)(k+1)$ for some nonnegative integer m. Let u be the unique vertex of the $v_0 - z$ path such that $d_T(v_0, u) = i + m(k+1)$. Then $u \in D_i$, $d_T(z, u) = d_T(z, v_0) - d_T(u, v_0) = l - i - m(k+1) \leq k$ and therefore $d_T(z, D_i) \leq k$ as required.

If $l < i$, then $d_T(z, v_i) = d_T(z, v_d) - d_T(v_i, v_d) \leq d_T(v_0, v_d) - d_T(v_i, v_d) = d_T(v_0, v_i) = i \leq k$ and again $d_T(z, D_i) \leq d_T(z, v_i) \leq k$ as required.

Since the k-coverings $D_0, D_1, \ldots, D_k$ form a partition of $V(T)$, at least one of them has at most $\lfloor p/(k+1) \rfloor$ vertices. Thus, $\gamma_k(T) \leq \lfloor p/(k+1) \rfloor$. ∎

COROLLARY 2.3.1. *If G is a connected graph on $p \geq k+1$ vertices and T is a spanning tree of G, then $\gamma_k(G) \leq \gamma_k(T) \leq \lfloor p/(k+1) \rfloor$.*

Proof. It follows from Theorem 2.2.3 that $\gamma_k(G) \leq \gamma_k(T)$ for every spanning tree T of G. Consequently, by Proposition 2.3.1, $\gamma_k(G) \leq \gamma_k(T) \leq \lfloor p/(k+1) \rfloor$. ∎

For a graph G and a positive integer k, we denote by $G \circ k$ the graph obtained by taking one copy of G and $|V(G)|$ copies of the path P_{k-1} of length $k-1$, and then joining the ith vertex of G to exactly one end vertex in the ith copy of P_{k-1}. It follows from the definition that $G \circ k$ has exactly $(k+1)|V(G)|$ vertices. If G is without isolated vertices, then $G \circ k$ has exactly $|V(G)|$ end vertices. For a vertex u of G we denote by $\overline{u}$ the only end vertex of $G \circ k$ which is at distance k from u. In addition, for a vertex v of $G \circ k$ we denote by $t(v)$ the unique vertex of G

such that v belongs to the $t(v)-\overline{t}(v)$ path. Note that $G\circ 1$ is the corona $G\circ K_1$ of the graphs G and K_1.

PROPOSITION 2.3.2. *For any graph H of order n, $\gamma_k(H\circ k)=\alpha_k(H\circ k)=n$.*

Proof. Assume that H is a graph on n vertices. Let D and I be a smallest k-covering and a largest k-packing of $H\circ k$, respectively. Let v be a vertex of H. It follows from the minimality of D (the maximality of I, resp.) and the structure of $H\circ k$ that exactly one vertex of the $v-\overline{v}$ path belongs to D (I, resp.). Since the vertices of the $v-\overline{v}$ paths, $v\in V(H)$, form a partition of the vertex set of $H\circ k$, we conclude that $|D|=|I|=n$. This finishes the proof. ∎

According to Corollary 2.3.1, if G is a connected graph of order $(k+1)n$, then $\gamma_k(G)\le n$. The following two theorems characterize connected graphs G of order $(k+1)n$ for which the upper bound is achieved for $\gamma_k(G)$. For $k=1$, these two theorems were first established by Fink, Jacobson, Kinch and Roberts [71]. Theorem 2.3.1 for $k=1$ has also been announced in [100]. The proofs given here are reproduced from the paper by Topp and Volkmann [152].

THEOREM 2.3.1. *Let T be a tree on $(k+1)n$ vertices. Then $\gamma_k(T)=n$ if and only if at least one of the following conditions holds*:

(1) *T is any tree on $k+1$ vertices*;
(2) *$T=R\circ k$ for some tree R on $n\ge 1$ vertices.*

Proof. Let T be a tree on $(k+1)n$ vertices. Since $\gamma_k(T)\ge 1$, it follows from Proposition 2.3.1 that $\gamma_k(T)=1$ if T has $k+1$ vertices. If $T=R\circ k$ for some tree R on n vertices, then $\gamma_k(T)=n$ by Proposition 2.3.2.

Conversely, we shall show that T satisfies the conditions (1) or (2) of the theorem if T is a tree of order $(k+1)n$ with $\gamma_k(T)=n$. We proceed by induction on n. The result is clear for $n=1$. Suppose the result is true for trees of order $(k+1)n$ $(n\ge 1)$ and let T be a tree of order $(k+1)(n+1)$ with $\gamma_k(T)=n+1$. We denote by $d(T)=d$ the diameter of T, and by $P=(v_0,\dots,v_d)$ any longest path in T. Since $\gamma_k(T)=n+1\ge 2$, it follows that $d>2k$; for if $d\le 2k$, then $\{v_l\}$, where $l=\lfloor d/2\rfloor$, would be a smallest k-covering of T and this would contradict the assumption $\gamma_k(T)=n+1\ge 2$. From this we conclude that each component of the graph $T-v_kv_{k+1}$ has at least $k+1$ vertices. Let T_1 (T_2, resp.) be the component of $T-v_kv_{k+1}$ which contains (does not contain, resp.) the vertex v_k. It follows from the choice of P that $d_{T_1}(v,v_k)\le k$ for each $v\in V(T_1)$. Hence $\{v_k\}$ is a k-covering of T_1 and $\gamma_k(T_1)=1$. Now either $T_1=P_k$ or $T_1\ne P_k$; we consider the two cases.

Case 1: $T_1\ne P_k$. In this case, T_2 has less than $(k+1)n$ vertices and $\gamma_k(T_2)<n$ by Proposition 2.3.1. Hence with the vertex v_k, we get $\gamma_k(T)<n+1$, a contradiction. This implies that we have

Case 2: $T_1=P_k$. Then T_2 has $(k+1)n$ vertices and it is easily seen that $\gamma_k(T_2)=n$. Thus, by the induction hypothesis, either T_2 is a tree on $k+1$ vertices if $n=1$ or $T_2=R'\circ k$ for some tree R' on n vertices if $n\ge 2$.

First assume that T_2 has $k+1$ vertices. In this case T has $(k+1)2$ vertices. Since $d = d(T) > 2k$, T is a path on $2k+2$ vertices, $T = P_{2k+1}$. Hence $T = K_2 \circ k$ and T satisfies (2).

Next assume that $T_2 = R' \circ k$, where R' is a tree on $n \geq 2$ vertices. We claim that v_{k+1} is a vertex of the tree R'. Suppose, contrary to our claim, that $v_{k+1} \notin V(R')$. Then v_{k+1} belongs to the $t(v_{k+1}) - \overline{t}(v_{k+1})$ path in T_2 and $v_{k+1} \neq t(v_{k+1})$. It is a simple matter to observe that $(V(R') - \{t(v_{k+1})\}) \cup \{v_k\}$ is a k-covering of T and therefore $\gamma_k(T) \leq |(V(R') - \{t(v_{k+1})\}) \cup \{v_k\}| \leq n$, contradicting our hypothesis. From this we see that $v_{k+1} \in V(R')$. In addition, the subgraph R of T induced by $V(R') \cup \{v_k\}$ is a tree. Because R' is a tree such that $R' \circ k = T_2$, v_k is an end vertex of the path $P_k = T_1$, and $v_k v_{k+1}$ is a unique edge joining a vertex from T_1 to a vertex from T_2, we conclude that $T = R \circ k$. Thus T satisfies the condition (2). The result follows by the principle of induction. ∎

THEOREM 2.3.2. *Let G be a connected graph of order $(k+1)n$. Then $\gamma_k(G) = n$ if and only if at least one of the following conditions holds*:

(1) *G is any connected graph of order $k+1$;*

(2) *$G = C_{2k+2}$;*

(3) *$G = H \circ k$ for some connected graph H of order n.*

Proof. Suppose that G is a connected graph of order $(k+1)n$. It follows easily from Corollary 2.3.1, simple observation, and Proposition 2.3.2 that $\gamma_k(G) = 1$ if G has $k+1$ vertices, $\gamma_k(G) = 2$ if $G = C_{2k+2}$, and $\gamma_k(G) = n$ if $G = H \circ k$ and H has n vertices, respectively.

It clearly suffices to prove the converse for $n \geq 2$. Assume that G is a connected graph of order $(k+1)n$ such that $\gamma_k(G) = n$. We first prove that $G = C_{2k+2}$ or $G = P_{2k+1} = K_2 \circ k$ if $n = 2$. Suppose on the contrary that G is different from C_{2k+2} and P_{2k+1}. Then G has a spanning tree, say T, which is not a path. Since T is not a path and has $2k+2$ vertices, its diameter $d(T) = d$ is not greater than $2k$. Let $P = (v_0, \ldots, v_d)$ be any longest path in T and $l = \lfloor d/2 \rfloor$. Then $d_T(v, v_l) \leq k$ for each vertex v of T and therefore $\{v_l\}$ is a k-covering of T. This implies that $\{v_l\}$ is a k-covering of G and $\gamma_k(G) = 1$, which is impossible. Thus, $G = C_{2k+2}$ or $G = K_2 \circ k$, and G has the desired properties.

The proof will be completed by showing that $G = H \circ k$ for some connected graph H if $n \geq 3$. In order to get this, let T be a spanning tree of G. It follows from Corollary 2.3.1 that $\gamma_k(T) = n$. Then, by Theorem 2.3.1, $T = R \circ k$ for some tree R of order n. Moreover, the set $V(R)$ containing n vertices is a smallest k-covering of G. Let H be the subgraph of G induced by $V(R)$. We claim that $G = H \circ k$. Suppose on the contrary that $G \neq H \circ k$. Then G contains two vertices $v \in V(G) - V(H)$ and $u \in V(G)$ such that $vu \in E(G) - E(H \circ k)$. There are two cases to consider.

Case 1: $t(v) = t(u)$. Then $k \geq 2$ and vu is a chord of the $t(v) - \overline{t}(v)$ path. Choose any neighbour z of $t(v)$ in R. Certainly, each vertex of the $t(v) - \overline{t}(v)$ path is at distance at most k from z. This makes it obvious that the set $V(R)-\{t(v)\}$ of order $n-1$ is a k-covering of G, a contradiction.

Case 2: $t(v)\neq t(u)$. First suppose that $d_T(v,t(v))=d_T(u,t(u))$. Since $n\geq 3$ and R is connected, there is a vertex $z\in V(R)-\{t(v),t(u)\}$ which is adjacent to $t(v)$ or $t(u)$, say z is adjacent to $t(v)$ in R. It is easy to verify that each vertex of the $t(v)-\overline{t}(v)$ path is at distance at most k from z or u. Then it is easily seen that the set $(V(R)-\{t(v),t(u)\})\cup\{u\}$ containing $n-1$ vertices is a k-covering of G, a contradiction. Therefore $d_T(v,t(v))\neq d_T(u,t(u))$ and if without loss of generality $d_T(v,t(v))>d_T(u,t(u))$, then we choose any neighbour z of $t(v)$ in R. It is again easy to observe that each vertex of the $t(v)-\overline{t}(v)$ path is at distance at most k from z or $t(u)$ and then one can check that the set $V(R)-\{t(v)\}$ of order $n-1$ is a k-covering of G, a contradiction.

Since both Case 1 and Case 2 lead to contradictions, it follows that $G=H\circ k$, which completes the proof. ∎

The equivalence of the statements (1) and (3) of the next theorem is the content of a theorem established by Fink, Jacobson, Kinch and Roberts [71] and it follows from Theorem 2.3.2. In [146], Topp and Vestergaard have given an independent and considerably shorter proof of this equivalence. The technique of this proof can be used to obtain slightly more general results.

Theorem 2.3.3. *Let G be a connected graph of order $2n$. Then the following statements are equivalent*:

(1) $G=C_4$ *or* $G=H\circ K_1$ *for some connected graph* H;
(2) $\mathrm{ir}(G)=n$;
(3) $\gamma(G)=n$.

Proof. The implication (1)⇒(2) is obvious if $G=C_4$ and follows from Proposition 2.1.4 if $G=H\circ K_1$. The implication (2)⇒(3) follows from Proposition 2.1.3 and the observation that $\gamma(G)\leq |V(G)|/2=n$ for a graph G without isolated vertices.

To prove the implication (3)⇒(1), assume G is a connected graph of order $2n$ with $\gamma(G)=n$. Let D be a minimum dominating set of G. Then $|D|=n$ and $\overline{D}=V(G)-D$ is another minimum dominating set of G. It follows from the König–Hall theorem (see [15, p. 132]) that G has a perfect matching M between D and $\overline{D}$; otherwise there exists a subset S of D such that $|N_G(S)\cap\overline{D}|<|S|$ and then $D'=(D-S)\cup(N_G(S)\cap\overline{D})$ is dominating in G with $|D'|<n$. Let $M=\{v_1u_1,\ldots,v_nu_n\}$ be a perfect matching between $D=\{v_1,\ldots,v_n\}$ and $\overline{D}=\{u_1,\ldots,u_n\}$. If every edge of M is an end edge of G, then certainly $G=H\circ K_1$ with $H=K_1$ if $G=K_2$ or $H=G-\Omega$ otherwise, where Ω is the set of end vertices of G. Clearly H is connected since G is connected. Thus assume that M contains a non-end edge of G. Let v_iu_i be any such edge. Then the sets $A=N_G(v_i)-\{u_i\}$ and $B=N_G(u_i)-\{v_i\}$ are nonempty, say $x\in A$ and $y\in B$. Moreover, $A\cap B=\emptyset$; for if there were $t\in A\cap B$, then v_i and u_i would be dominated by t, and $D'=D-\{v_i\}$

or $D' = D - \{u_i\}$ would be dominating in G with $|D'| < n$. Observe next that $A = \{x\}$, $B = \{y\}$, x and y are adjacent and $xy \in M$; otherwise there are $x' \in A$, $y' \in B$, distinct edges $v_k u_k, v_l u_l \in M - \{v_i u_i\}$ such that $x' \in \{v_k, u_k\}$ and $y' \in \{v_l, u_l\}$, and then $D' = (D - \{v_i, v_k, v_l\}) \cup \{x', y'\}$ is dominating in G with $|D'| < n$. Consequently, xy is another non-end edge from M and it has the same properties as $v_i u_i$. Thus, since G is connected and $N_G(x) - \{y\} = \{v_i\}$ and $N_G(y) - \{x\} = \{u_i\}$, G is a 4-cycle with $V(G) = \{v_i, u_i, x, y\}$. This completes the proof of the theorem. ∎

Let G be a nontrivial connected graph of order p and let $\varepsilon(G)$ denote the maximum number of end edges in a spanning forest of G. In [110], Nieminen proved that $\gamma(G) + \varepsilon(G) = p$. Consequently, $\varepsilon(G) \geq p/2$ (since $\gamma(G) \leq p/2$) and it follows from Theorem 2.3.3 that this lower bound for $\varepsilon(G)$ is attained if and only if $G = C_4$ or $G = H \circ K_1$ for some connected graph H. Similar remarks may be given for other Gallai-type results (see [39]) which involve the domination number.

As we have already mentioned, the structure of graphs with equal 1-packing and 1-covering numbers has been studied in [22, 23], [66] and [146, 148, 149, 150, 152, 153] (see also the next chapter in this paper and Theorem 3.1.10 in [100]). It follows from Proposition 2.1.4 that if $G = H \circ K_1$, then $\gamma(G) = \alpha(G)$. The next result shows that in the class of connected bipartite graphs, except for K_1 and C_4, the converse implication is true.

COROLLARY 2.3.2 [149]. *If G is a connected bipartite graph, then $\gamma(G) = \alpha(G)$ if and only if $G = K_1$, $G = C_4$, or $G = H \circ K_1$ for some connected bipartite graph H.*

Proof. The sufficiency is obvious if $G \in \{K_1, C_4\}$ and follows from Proposition 2.1.4 if $G = H \circ K_1$ for some graph H. Conversely, assume that G is a connected bipartite graph with $\gamma(G) = \alpha(G)$ and $G \neq K_1$. Let V_1 and V_2 be partite sets of G. Clearly, each of the sets V_1 and V_2 is both independent and dominating in G and so

$$\alpha(G) \geq \max\{|V_1|, |V_2|\} \geq \min\{|V_1|, |V_2|\} \geq \gamma(G).$$

Hence $\alpha(G) = |V_1| = |V_2| = \gamma(G)$ and it follows from Theorem 2.3.3 that either $G = C_4$ or $G = H \circ K_1$ for some connected graph H. In the latter case H is bipartite since G is bipartite. ∎

Corollary 2.3.2 gives a solution to the Szamkołowicz problem (and to Problem 1(c) of Laskar and Walikar [100]) for bipartite graphs. It follows from Corollary 2.3.2 and from a result of Payan and Xuong [112] that the graphs characterized in Corollary 2.3.2 are exactly those connected bipartite graphs G for which $\gamma(G)\gamma(\overline{G}) = |V(G)|$, so they also form a solution to Problem 1(e) of Laskar and Walikar [100] for connected bipartite graphs. The next result due to Borowiecki [22, 23] easily follows from Corollary 2.3.2.

Corollary 2.3.3. *If T is a tree, then $\gamma(T) = \alpha(T)$ if and only if $T = K_1$ or $T = R \circ K_1$ for some tree R.* ■

The result of Corollary 2.3.3 has also been obtained by Walikar, Acharya and Sampathkumar, see Theorem 3.1.10 in [100]. We now give a generalization of the last result.

Theorem 2.3.4 [152]. *If T is a tree, then $\gamma_k(T) = \alpha_k(T) = n$ if and only if one of the following statements holds*:

(1) *T is a tree of diameter at most k and $n = 1$;*

(2) *There exists a decomposition of T into n subgraphs $T_1, \ldots, T_n$ in such a way that*

(a) *T_i is a tree of diameter k $(i = 1, \ldots, n)$, and*

(b) *for each $i \in \{1, \ldots, n\}$, there exists a vertex $u_i \in V(T_i) - V(T_0)$ such that $d_T(u_i, V(T_0)) = k$, where T_0 is the subgraph of T generated by the edges which do not belong to any of the trees $T_1, \ldots, T_n$.*

Proof. Let T be a tree such that $\gamma_k(T) = \alpha_k(T) = n$. It is obvious that the diameter $d(T) = d$ of T is not greater than k if $n = 1$. Thus assume $n \geq 2$ and let $P = (v_0, \ldots, v_d)$ be any longest path in T. An analysis similar to that in the proof of Theorem 2.3.1 shows that $d > 2k$. Let T' (T'', resp.) be the component of $T - v_k v_{k+1}$ which contains (does not contain, resp.) the vertex v_k.

First we claim that $\gamma_k(T') = \alpha_k(T') = 1$. It follows from the choice of P that $\{v_k\}$ is a k-covering of T' and therefore $\gamma_k(T') = 1 \leq \alpha_k(T')$. There remains to prove that $\alpha_k(T') = 1$. By contradiction, suppose that $\alpha_k(T') = m > 1$. Let I' be a k-packing of T' such that $|I'| = m$ and let I be a maximal k-packing of T such that $I' \subseteq I$. By Corollary 2.1.2, $|I| = n$ and I is a minimum k-covering of T. On the other hand, it is seen at once that the set $(I - I') \cup \{v_k\}$ is a k-covering of T and $|(I - I') \cup \{v_k\}| \leq n - m + 1 < \gamma_k(T)$, a contradiction. This implies our claim. By the way, since $\alpha_k(T') = 1$ and T' contains the $v_0 - v_k$ path of length k, the diameter $d(T') = k$ and $d_T(v_0, v_k) = k$.

Now we claim that $\gamma_k(T'') \geq n - 1$; for if $\gamma_k(T'') < n - 1$, then for any minimum k-covering I'' of T'' the set $I'' \cup \{v_k\}$ would be a k-covering of T and $\gamma_k(T) \leq |I'' \cup \{v_k\}| < n$ which is impossible. Furthermore, $\alpha_k(T'') \leq n - 1$; for if $\alpha_k(T'') > n - 1$, then for any maximum k-packing J'' of T'' the set $J'' \cup \{v_0\}$ would be a k-packing of T and $\alpha_k(T) \geq |J'' \cup \{v_0\}| > n$ which is also impossible. Hence, by Corollary 2.1.1, we get $\gamma_k(T'') = \alpha_k(T'') = n - 1$.

After the above observation, by induction on n, we prove that T has property (2). First, if $n = 2$, then $\gamma_k(T'') = \alpha_k(T'') = 1$ and, since T'' contains the $v_{k+1} - v_d$ path of length at least k, we exactly have $d(T'') = k$ and $d_T(v_d, v_{k+1}) = k$. One sees immediately that the decomposition $T_1 = T'$, $T_2 = T''$ of T with $u_1 = v_0$ and $u_2 = v_d$ satisfies (2). Second, if $n \geq 3$, then the induction hypothesis implies that there exists a decomposition $T_1, \ldots, T_{n-1}$ of T'' into $n - 1$ trees with property (2). For convenience, let T_0'' (resp., T_0) denote the subgraph of T'' (resp., T)

generated by the edges which do not belong to any of the trees $T_1, \dots, T_{n-1}$ (resp., $T_1, \dots, T_{n-1}, T_n = T'$). We shall prove that the trees $T_1, \dots, T_n$ have property (2) in T. Certainly, $T_1, \dots, T_n$ form a decomposition of T into n trees of diameter k. In order to prove that this decomposition satisfies the condition (b) of (2), without loss of generality we can assume that the vertex v_{k+1} belongs to the tree T_{n-1}. Then, since $d_T(v_0, V(T_0)) = d_T(v_0, v_k) = k$ and there exists $u_i \in V(T_i) - V(T_0'')$ such that $d_T(u_i, V(T_0'')) = k$ $(i = 1, \dots, n-1)$, it suffices to show that $d_T(\overline{u}_{n-1}, V(T_0)) = k$ for some vertex $\overline{u}_{n-1} \in V(T_{n-1}) - V(T_0) = V(T_{n-1}) - (V(T_0'') \cup \{v_{k+1}\})$. Suppose on the contrary that $d_T(v, V(T_0)) < k$ for each $v \in V(T_{n-1})$. Then $d_T(v, N_T(V(T_{n-1})) - V(T_{n-1})) \le k$ for each $v \in V(T_{n-1})$. Since T is a tree, no two vertices of the set $N_T(V(T_{n-1})) - V(T_{n-1})$ $(\subset V(T_0) - V(T_{n-1}))$ belong to the same tree T_i $(i \in \{1, \dots, n\} - \{n-1\})$. Hence there exists a superset of $N_T(V(T_{n-1})) - V(T_{n-1})$, say I, such that $|I \cap V(T_i)| = 1$ for $i = 1, \dots, n$. Let z_i denote a unique vertex of I which belongs to the tree T_i $(i = 1, \dots, n)$. We shall prove that $I - \{z_{n-1}\}$ is a k-covering of T. Let v be any vertex of T. If $v \in V(T_{n-1})$, then $d_T(v, I - \{z_{n-1}\}) = d_T(v, N_T(V(T_{n-1})) - V(T_{n-1})) \le k$. If $v \in V(T_i)$ for some $i \in \{1, \dots, n\} - \{n-1\}$, then $d_T(v, I - \{z_{n-1}\}) \le d_T(v, z_i) \le k$ since v, $z_i \in V(T_i)$ and T_i is a tree of diameter k. This implies that the set $I - \{z_{n-1}\}$ containing $n-1$ vertices is a k-covering of T. This contradicts $\gamma_k(T) = n$ and therefore our assertion follows. This proves the necessity of the conditions.

The sufficiency is obvious if the diameter of T is not greater than k. If the diameter of T is greater than k, then assume that we have a decomposition of T into trees $T_1, \dots, T_n$ satisfying (2). We shall prove that $\gamma_k(T) = \alpha_k(T) = n$. Let I be any maximum k-packing of T. Since the distance between any two vertices of T_i is not greater than k (by (a)), at most one vertex of T_i belongs to I $(i = 1, \dots, n)$. Therefore $n \ge |I| = \alpha_k(T)$. On the other hand, let J be any minimum k-covering of T. It follows from the property (b) of the decomposition $T_1, \dots, T_n$ of T that there is a vertex u_i in T_i such that $d_T(u_i, V(T) - V(T_i)) > k$ $(i = 1, \dots, n)$. Consequently, since $d_T(u_i, J) \le k$, at least one vertex of T_i belongs to J and therefore $\gamma_k(T) = |J| \ge n$. Hence, by Corollary 2.1.1, $\gamma_k(T) = \alpha_k(T) = n$ and this completes the proof. ∎

The following theorem extends the last theorem to block graphs and it has recently been proved by Hatting and Henning [85].

THEOREM 2.3.4′. *If G is a block graph, then $\gamma_k(G) = \alpha_k(G) = n$ if and only if one of the following statements holds*:

(1) *G has diameter at most k and $n = 1$;*

(2) *There exists a decomposition of G into n subgraphs $G_1, \dots, G_n$ in such a way that*

(a) *G_i is a block graph of diameter k $(i = 1, \dots, n)$,*

(b) *for each $i \in \{1, \dots, n\}$, there exists a vertex $u_i \in V(G_i) - V(G_0)$ such that $d_G(u_i, V(G_0)) = k$, where G_0 is the subgraph of G generated by the edges which do not belong to any of the subgraphs $G_1, \dots, G_n$, and*

(c) *there is at most one edge with one end in* $V(G_i)$ *and the other end in* $V(G_j)$ *for* $1 \le i < j \le n$. ∎

The *clique covering number* $\theta(G)$ of a graph G is the smallest integer n for which there exists a partition $V_1, \dots, V_n$ of the vertex set $V(G)$ such that each V_i induces a complete subgraph of G. It is easy to observe that $\alpha_1(G) \le \theta(G)$ for every graph G. In [77], Hajnál and Suranýi proved the following result.

Proposition 2.3.3. *For any chordal graph* G, $\alpha_1(G) = \theta(G)$.

Proof. Let G be a chordal graph and suppose that $\alpha_1(H) = \theta(H)$ for all smaller chordal graphs H. Let x be a simplicial vertex of G. Then $G - N_G[x]$ is a smaller chordal graph and therefore $\alpha_1(G - N_G[x]) = \theta(G - N_G[x])$. On the other hand, $\alpha_1(G - N_G[x]) = \alpha_1(G) - 1$ since every maximal independent set of G has exactly one vertex in $N_G[x]$. Similarly, $\theta(G - N_G[x]) = \theta(G) - 1$ because every minimal covering of G by cliques must necessarily contain the clique $G[N_G[x]]$ to cover the vertex x. Thus, $\alpha_1(G) = \theta(G)$. ∎

For a graph G and a positive integer k, we denote by G^k the kth *power* of G, the graph with the same vertices as G, two vertices being adjacent in G^k when their distance in G is at most k. In [51], Duchet has proved that if G^k is a chordal graph, then G^{k+2} is chordal. This result applied to block graphs implies the next result due to Jamison (see Corollary 6.9 in [51]); the same result may also be obtained from [6, Th. 1] and [30, Th. 2.2].

Proposition 2.3.4. *If* G *is a block graph, then* G^k *is chordal for each integer* $k \ge 1$. ∎

Proposition 2.3.5. *For any graph* G, $\alpha_k(G) \le s_k(G)$.

Proof. Assume that I is a maximum k-packing of G. Let $G_1, \dots, G_s$ be a decomposition of G into $s = s_k(G)$ graphs each of diameter at most k. Since $d(G_i) \le k$, $|I \cap V(G_i)| \le 1$ for $i = 1, \dots, s$. Therefore $\alpha_k(G) = |I| = |I \cap V(G)| = |I \cap \bigcup_{i=1}^{s} V(G_i)| = \sum_{i=1}^{s} |I \cap V(G_i)| \le s = s_k(G)$. ∎

Theorem 2.3.5 [152]. *For any block graph* G, $s_k(G) = \alpha_k(G)$.

Proof. It follows from the definition of G^k that two vertices in G^k are not adjacent if and only if their distance in G is greater than k. This implies that a subset I of $V(G) = V(G^k)$ is a maximum k-packing in G if and only if it is a maximum 1-packing in G^k. Hence $\alpha_k(G) = \alpha_1(G^k)$. Moreover, since a subset X of $V(G)$ induces in G a subgraph of diameter at most k if and only if it induces a complete subgraph in G^k, we have $s_k(G) = \theta(G^k)$. The rest follows from Propositions 2.3.3 and 2.3.4. ∎

The next result for trees when $k = 1$ has been mentioned in [2].

Corollary 2.3.4 [152]. *For any block graph* G, $s_{2k}(G) = \gamma_k(G)$.

Proof. Since $\alpha_{2k}(G) = \gamma_k(G)$ for any block graph G (see [30, Th. 4.1], [49, Th. 4], and [107, Th. 9] (for trees)), the result follows from Theorem 2.3.5. ∎

THEOREM 2.3.6 [152]. *Let T be a block graph with the diameter $d(T) = d \geq k+1$ and $s_k(T) = \alpha_k(T) = n$. Assume that $P = (v_0, v_1, \ldots, v_d)$ is any longest path without chords in T, let T_i be that connected component of $T - (\{v_{i-1}\} \cup (N_T(v_{i+1}) - \{v_i\}))$ which contains the vertex v_i of P; in addition, let T_0 be the subgraph induced by the vertex v_0, and $T_d = T - \bigcup_{i=0}^{d-1} V(T_i)$. Assume that i_0 is the greatest integer i such that $d_T(v_0, v) \leq k$ for each vertex $v \in \bigcup_{l=0}^{i} V(T_l)$, and denote by T' and T'' the subgraph of T induced by $\bigcup_{i=0}^{i_0} V(T_i)$ and $\bigcup_{i=i_0+1}^{d} V(T_i)$, respectively. Then $d(T') \leq k$, $s_k(T') = \alpha_k(T') = 1$, and $s_k(T'') = \alpha_k(T'') = n-1$.*

Proof. For convenience, let $V_0(T_i)$ be the set of vertices $v \in V(T_i) - \{v_i\}$ such that the shortest $v - v_0$ path joining v with v_0 does not contain the vertex v_i ($i \in \{1, \ldots, d\}$). By $V_1(T_i)$ we denote the set $V(T_i) - V_0(T_i)$.

First we prove that $d(T') \leq k$. To prove this, it would suffice to show that for any pair $a, b \in V(T') - \{v_0\}$ we have $d_T(a, b) \leq k$. Without loss of generality we can assume that $a \in V(T_s)$, $b \in V(T_t)$ and $s \leq t \leq i_0$. It follows from the choice of P and i_0 that $d_T(v_s, a) \leq d_T(v_s, v_0)$ and $d_T(b, v_0) \leq k$. Therefore we have $d_T(b,a) = d_T(b, v_{t-1}) + d_T(v_{t-1}, v_s) + d_T(v_s, a) \leq d_T(b, v_{t-1}) + d_T(v_{t-1}, v_s) + d_T(v_s, v_0) = d_T(b, v_0) \leq k$ if $s < t$. If $s = t$, then we distinguish two cases.

Case 1: *Either $a \in V_0(T_s)$ and $b \in V_1(T_s)$ or a, $b \in V_1(T_s)$*. Then $d_T(b,a) \leq d_T(b, v_s) + d_T(v_s, a) \leq d_T(b, v_s) + d_T(v_s, v_0) = d_T(b, v_0) \leq k$.

Case 2: $a, b \in V_0(T_s)$. Let a' (b', resp.) be the neighbour of v_s which belongs to the shortest $v_s - a$ ($v_s - b$, resp.) path. Certainly, $d_T(b', a') \leq 1 = d_T(b', v_{s-1})$ and $d_T(a', a) \leq d_T(v_{s-1}, v_0)$. Therefore $d_T(b,a) \leq d_T(b,b') + d_T(b',a') + d_T(a',a) \leq d_T(b,b') + d_T(b', v_{s-1}) + d_T(v_{s-1}, v_0) = d_T(b, v_0) \leq k$. This implies that $d(T') \leq k$. Hence $s_k(T') = 1 = \alpha_k(T')$ and, in addition, $\alpha_k(T'') \geq n-1$.

Next we prove that $\alpha_k(T'') = n-1$. In order to prove this, for a maximum k-packing J of T'', we denote by $J(v_0)$ the subset of J, where $J(v_0) = \{v \in J : d_T(v_0, v) \leq k\}$. First, let us observe that if there were different vertices a and b in $J(v_0)$, then (in a similar manner as we have proved that $d(T') \leq k$) we would get $d_T(a,b) \leq k$, which is impossible since $J(v_0)$ is a subset of a k-packing of T''. This implies that $|J(v_0)| \leq 1$ for any maximum k-packing J of T''. We claim that $J(v_0) = \emptyset$ for some maximum k-packing J of T''. Let J be a maximum k-packing of T''. If $J(v_0) = \emptyset$, then we are done. On the other hand, if $J(v_0) \neq \emptyset$, let a be the unique element of $J(v_0)$, and assume that $a \in V(T_s)$ for some $s > i_0$. Since $d_T(a, v_0) \leq k$, the choice of P implies that $d_T(a, v) \leq k$ for each vertex $v \in \bigcup_{i=0}^{s} V(T_i)$ ($v \in \bigcup_{i=0}^{s-1} V(T_i) \cup V_0(T_s)$, resp.) if $a \in V_1(T_s)$ ($a \in V_0(T_s)$, resp.). Hence $J - \{a\} \subset \bigcup_{i=s+1}^{d} V(T_i)$ ($J - \{a\} \subset V_1(T_s) \cup \bigcup_{i=s+1}^{d} V(T_i)$, resp.) if $a \in V_1(T_s)$ ($a \in V_0(T_s)$, resp.). It follows from the definition of i_0 that there exists a vertex $u_0 \in V(T_{i_0+1})$ such that $d_T(v_0, u_0) = k+1$. We shall prove that $J_0 = (J - \{a\}) \cup \{u_0\}$ is a maximum k-packing of T''. Since $|J_0| = |J|$ and $J - \{a\}$ is a subset of a k-packing, it remains to show that $d_T(u_0, x) > k$ for each $x \in J - \{a\}$. For convenience, we let $d_T(v_{i_0+1}, u_0) = l_1$, $d_T(v_{i_0+1}, v_s) = l_2$, and $d_T(v_s, a) = l_3$. Then $d_T(v_{i_0}, u_0) \leq l_1 + 1$, $l_3 \leq d_T(v_{s-1}, a)$, and $d_T(v_{i_0}, v_{s-1}) = l_2$.

Since $d_T(v_0, u_0) = d_T(v_0, v_{i_0}) + d_T(v_{i_0}, u_0) = k+1 > k \geq d_T(v_0, a) = d_T(v_0, v_{i_0}) + d_T(v_{i_0}, v_{s-1}) + d_T(v_{s-1}, a)$, it follows that $l_1 + 1 \geq d_T(v_{i_0}, u_0) > d_T(v_{i_0}, v_{s-1}) + d_T(v_{s-1}, a) \geq l_2 + l_3$ and this implies that $l_1 + l_2 \geq l_3$. Let x be any vertex from $J - \{a\}$. Then $d_T(x, a) = d_T(x, v_s) + l_3 > k$ and therefore $d_T(x, u_0) = d_T(x, v_s) + l_1 + l_2 \geq d_T(x, v_s) + l_3 > k$. This implies that the set J_0 is a maximum k-packing of T'' and certainly $J_0(v_0) = \emptyset$. Furthermore, we have $\alpha_k(T'') = |J_0| = n - 1$; for if it were $|J_0| \geq n$, then the set $J_0 \cup \{v_0\}$ would be a k-packing in T with $|J_0 \cup \{v_0\}| > n = \alpha_k(T)$ which is impossible. Then, by Theorem 2.3.5, we have $s_k(T'') = n - 1$. This completes the proof. ∎

2.4. Conditions for equalities of domination parameters. As pointed out before, various authors have found sufficient conditions for two or more of the lower and upper independence, domination and irredundance numbers of a graph to be equal. Of specific importance to the present section are conditions under which equality of the lower domination and independence numbers occurs. In this respect Allan and Laskar have proved in [3] that if G is a $K_{1,3}$-free graph (i.e. G has no induced subgraph isomorphic to $K_{1,3}$), then $\gamma(G) = i(G)$. This extends an earlier result by Mitchell and Hedetniemi [108] that if G is the line graph of a tree, then $\gamma(G) = i(G)$, and a result by Cockayne, Hedetniemi and Miller [40] that if G is the middle graph of a graph (that is, $G = L(H \circ K_1)$ for some graph H), then $\mathrm{ir}(G) = \gamma(G) = i(G)$. A simple and short proof of the Allan–Laskar theorem is due to Sumner [133]. The Allan–Laskar theorem has been further generalized by Bollobás and Cockayne [20] and Zverovich and Zverovich [161]. They have proved that if G has no induced subgraph isomorphic to $K_{1,k}$ with $k \geq 3$, then $i(G) \leq (k-2)\gamma(G) - (k-3)$. Bollobás and Cockayne [20] have also proved that if G does not have two induced subgraphs isomorphic to P_4 with vertex sequences (a_i, b_i, c_i, d_i), $i = 1, 2$, where $b_1, b_2, c_1, c_2, d_1, d_2$ are distinct and $a_i \notin \{c_1, c_2, d_1, d_2\}$ for $i = 1, 2$, then $\mathrm{ir}(G) = \gamma(G)$. Favaron [60] has improved this result showing that if G has no induced subgraph isomorphic to one of the six graphs G_i of Figure 5, then $\mathrm{ir}(G) = \gamma(G)$. Favaron [60] has also shown that for any graph G that does not contain either $K_{1,3}$ or the graph G_1 of Figure 6 as an induced subgraph, $\mathrm{ir}(G) = \gamma(G) = i(G)$.

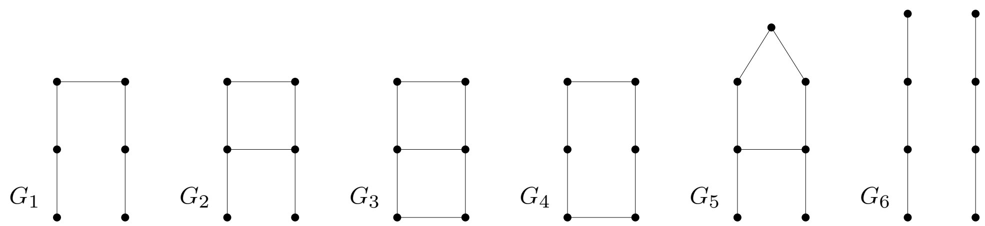

Fig. 5. Forbidden subgraphs for $\mathrm{ir}(G) = \gamma(G) = i(G)$

Laskar and Pfaff [98, 99] have also given three results which guarantee the equality $\mathrm{ir}(G) = \gamma(G)$ or $\mathrm{ir}(G) = \gamma(G) = i(G)$ for chordal and split graphs; a graph G is a *split graph* if both G and its complement $\overline{G}$ are chordal. Laskar and Pfaff have shown that if G is connected and it is a split graph or the complement of a bipartite graph or G is chordal and it contains neither G_1 nor G_2 of Figure 6 as an induced subgraph, then $\mathrm{ir}(G) = \gamma(G)$. Also, they show for any graph G that does not contain either $K_{1,3}$ or G_3 of Figure 6, where the dotted edges of G_3 are the only extra edges allowed, $\mathrm{ir}(G) = \gamma(G) = i(G)$. A recent paper by Jacobson, Peters and Rall [92] gives several sufficient conditions on G such that $\mathrm{ir}(G) = \gamma(G)$ (and some sufficient conditions for equality of the lower n-dependence and lower n-irredundance numbers). Finally, recent papers by Harary and Livingston [79, 80] provide forbidden subtree characterizations of the trees and caterpillars T for which $\gamma(T) = i(T)$.

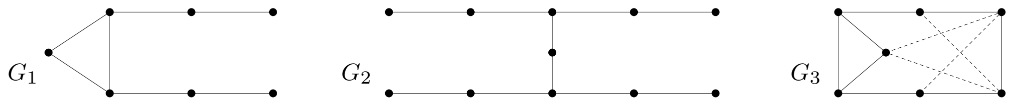

Fig. 6. Forbidden subgraphs for $\mathrm{ir}(G) = \gamma(G)$ and $\mathrm{ir}(G) = \gamma(G) = i(G)$

We begin with inequalities which relate the domination number, the independent domination number and the independence number to one another in graphs that do not contain some forbidden graphs. For integers $n \ge 2$ and $m \ge 2$, the *double star* $S_{n,m}$ is the graph formed from the graph K_2 by attaching $n-1$ pendant edges at one end vertex of K_2 and $m-1$ pendant edges at the other. The following theorem was previously proved by Zverovich and Zverovich [161]; we give a somewhat different proof of this theorem.

THEOREM 2.4.1. *If G has no induced subgraph isomorphic to $S_{k,k}$ $(k \ge 3)$, then*

$$i(G) \le (k-2)\gamma(G) - (k-3).$$

P r o o f. For a subset X of vertices of G, let $n(X)$ denote the number of nonisolated vertices of $G[X]$. Let D_1 be a minimum dominating set of G. We consider two cases.

C a s e 1: $n(D_1) = 0$. Then D_1 is an independent dominating set of G and so $i(G) \le |D_1| = \gamma(G) \le (k-2)\gamma(G) - (k-3)$.

C a s e 2: $n(D_1) = l > 0$. Then $2 \le l \le \gamma(G)$. Let $D_1, D_2, \ldots$ be a sequence of minimal dominating sets of G defined as follows: For $i \ge 1$, if $n(D_i) > 0$, then let D_i' be the set of nonisolated vertices of $G[D_i]$. Since $S_{k,k}$ is not an induced subgraph of G, D_i' contains a vertex v_i such that $\alpha(G[I_G(v_i, D_i)]) \le k-2$. Let I_i be a maximal independent set of $G[I_G(v_i, D_i)]$. Then $\overline{D}_{i+1} = (D_i - \{v_i\}) \cup I_i$ is a dominating set of G. Now D_{i+1} is defined to be a minimal subset of $\overline{D}_{i+1}$ dominating G. Certainly, $|D_{i+1}| \le |D_i| + (k-3)$ and $n(D_{i+1}) < n(D_i)$. In addition, since $n(D_1) > n(D_2) > \ldots$, there exists an integer m, $2 \le m \le l-1$, such that $n(D_m) = 0$ while $n(D_{m-1}) > 0$. Thus D_m is an independent dominating set of G and therefore

$$
\begin{aligned}
i(G) \le |D_m| &\le |D_{m-1}| + (k-3) \\
&\le |D_{m-2}| + 2(k-3) \\
&\vdots \\
&\le |D_1| + (m-1)(k-3) \\
&= \gamma(G) + (m-1)(k-3) && (|D_1| = \gamma(G)) \\
&\le \gamma(G) + (l-2)(k-3) && (m \le l-1) \\
&\le \gamma(G) + (\gamma(G)-2)(k-3) && (l \le \gamma(G)) \\
&= (k-2)\gamma(G) - 2(k-3) \\
&\le (k-2)\gamma(G) - (k-3). \ \blacksquare
\end{aligned}
$$

Since every $K_{1,k}$-free graph is $S_{k,k}$-free, we have the following result due to Bollobás and Cockayne [20].

Corollary 2.4.1. *If G has no induced subgraph isomorphic to $K_{1,k}$ $(k \ge 3)$, then*

$$i(G) \le (k-2)\gamma(G) - (k-3). \ \blacksquare$$

From Proposition 2.1.3 and Corollary 2.4.1 (for $k = 3$), we immediately have the following corollary proved in [3, 20, 133, 151, 161].

Corollary 2.4.2. *If G has no induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G) = i(G)$.* $\blacksquare$

For any graph G, we have $\gamma(G) \le i(G) \le \alpha(G)$ and it is easy to observe that in general the gap between any two elements of this inequality may be arbitrary large. However, the next theorem shows that for $K_{1,k}$-free graphs, the independence number and the independent domination number can be bounded in terms of the domination number.

Theorem 2.4.2. *If G has no induced subgraph isomorphic to $K_{1,k}$ $(k \ge 3)$, then*

$$\gamma(G) \le \alpha(G) \le (k-1)\gamma(G).$$

Proof. Since the inequality $\gamma(G) \le \alpha(G)$ is obvious, we prove that $\alpha(G) \le (k-1)\gamma(G)$. Let D and I be respectively a minimum dominating set and a maximum independent set of G. Then $|D| = \gamma(G)$ and $|I| = \alpha(G)$. Since D is dominating, $V(G) = \bigcup_{v\in D} N_G[v]$. On the other hand, since G is $K_{1,k}$-free, for every $v \in D$, the set $N_G[v]$ contains at most $k-1$ independent vertices and therefore $|I \cap N_G[v]| \le k-1$. Thus $\alpha(G) = |I| = |I \cap V(G)| = |I \cap \bigcup_{v\in D} N_G[v]| \le \sum_{v\in D} |I \cap N_G[v]| \le (k-1)|D| = (k-1)\gamma(G)$. $\blacksquare$

The next two corollaries have been announced by Sumner [133].

Corollary 2.4.3. *If G has no induced subgraph isomorphic to $K_{1,k}$ $(k \ge 3)$, then $i(G) \ge \alpha(G)/(k-1)$.*

Proof. Since $\alpha(G) \le (k-1)\gamma(G)$ (by Theorem 2.4.2) and $\gamma(G) \le i(G)$, the inequality $i(G) \ge \alpha(G)/(k-1)$ is obvious. $\blacksquare$

COROLLARY 2.4.4. *If G has no induced subgraph isomorphic to $K_{1,k}$ $(k \geq 3)$, then*

$$\gamma(G) \geq \frac{\alpha(G) + (k-1)(k-3)}{(k-1)(k-2)}.$$

Proof. By Theorem 2.4.1 and Corollary 2.4.3, $(k-2)\gamma(G) - (k-3) \geq i(G) \geq \alpha(G)/(k-1)$ and this implies the result. ∎

Motivated by the Allan–Laskar theorem, we now give a list of forbidden subgraphs (from a paper by Topp and Volkmann [151]) that is sufficient for $\gamma(G) = i(G)$. We also show that $\mathrm{ir}(G) = i(G)$ if every minimal dominating set of G is independent. In the second part of the section we show that $\alpha(G) = \mathrm{IR}(G)$ for all chordal and unicyclic graphs.

THEOREM 2.4.3. *If a graph G contains no induced subgraph isomorphic to one of the graphs $H_1, \ldots, H_{14}$ of Figure 7, then $\gamma(G) = i(G)$.*

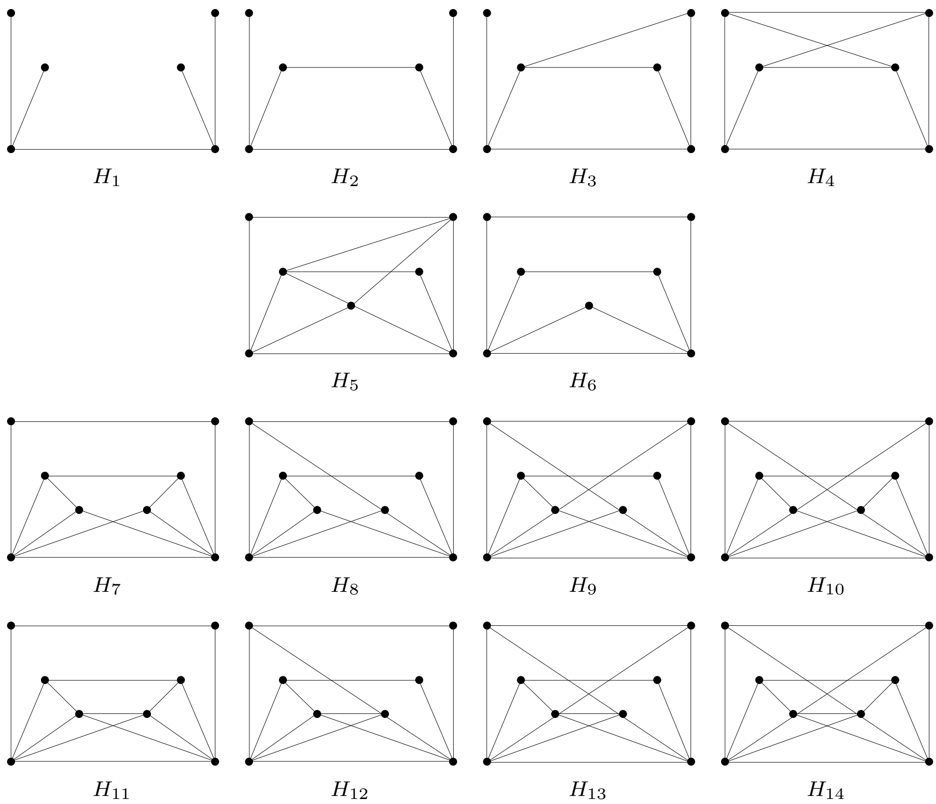

Fig. 7. The forbidden subgraphs for Theorem 2.4.3

Proof. Assume that none of the graphs $H_1, \ldots, H_{14}$ is an induced subgraph of G. We will show that $\gamma(G) = i(G)$. Since $\gamma(G) \leq i(G)$ (Proposition 2.1.3), it is sufficient to prove that in G there is a minimum dominating set which is independent, that is, there exists an independent dominating set of the cardinality $\gamma(G)$. Suppose on the contrary that each minimum dominating set of G is not independent. Let D_0 be a minimum dominating set of G such that $e(G[D_0])$ is the minimum number taken over all minimum dominating sets of G, where $e(G[X])$ denotes the number of edges in the subgraph induced by $X \subseteq V(G)$. Take two adjacent vertices x_1, x_2 from D_0 and the sets

$$I_i = \{v \in V(G) - D_0 \colon N_G(v) \cap D_0 = \{x_i\}\} \quad (i = 1, 2),$$

and

$$I_{1,2} = \{v \in V(G) - D_0 \colon N_G(v) \cap D_0 = \{x_1, x_2\}\}.$$

Since every minimum dominating set is minimal, it follows from Proposition 2.1.2 that the sets I_1, I_2 are nonempty and disjoint. We derive contradictions in two cases.

Case 1: *For $i = 1$ or 2, there exists a vertex $v_i \in I_i$ such that $I_i \subset N_G[v_i]$.* Then, it is easy to see that the set $D_1 = (D_0 - \{x_i\}) \cup \{v_i\}$ (for $i = 1$, 2) is a minimum dominating set of G and $e(G[D_1]) < e(G[D_0])$, contradicting the choice of D_0.

Case 2: *For $i = 1, 2$ and every $y \in I_i$, $I_i \not\subset N_G[y]$.* Then in I_i $(i = 1, 2)$ there are nonadjacent vertices. Let v_1, v_2 and u_1, u_2 be nonadjacent vertices from I_1 and I_2, respectively. From the fact that the subgraph $G[\{x_1, x_2, v_1, v_2, u_1, u_2\}]$ is not isomorphic to H_4 it follows that there exist $v \in \{v_1, v_2\} \subseteq I_1$ and $u \in \{u_1, u_2\} \subseteq I_2$ such that $vu \notin E(G)$.

We now claim that $I_1 \cup I_2 \subset N_G[\{v, u\}]$ if $v \in I_1$, $u \in I_2$ and $vu \notin E(G)$. For if not, then there exist vertices $v_0 \in I_1$ and $u_0 \in I_2$ such that $v_0u_0 \notin E(G)$ and the set $(I_1 \cup I_2) - N_G[\{v_0, u_0\}]$ is not empty. Without loss of generality we may assume that $I_1 - N_G[\{v_0, u_0\}] \neq \emptyset$. Take any vertex $\overline{v}$ from $I_1 - N_G[\{v_0, u_0\}]$ and any vertex $\overline{u}$ from $I_2 - N_G[u_0]$. Then, since x_1x_2, x_1v_0, $x_1\overline{v}$, x_2u_0, $x_2\overline{u} \in E(G)$ and v_0u_0, $v_0\overline{v}$, $u_0\overline{u}$, $u_0\overline{v} \notin E(G)$, the induced subgraph $G[\{x_1, x_2, v_0, \overline{v}, u_0, \overline{u}\}]$ of G is isomorphic to one of the graphs H_1, H_2, H_3, a contradiction. This contradiction shows that $I_1 \cup I_2 \subset N_G[\{v, u\}]$ whenever $v \in I_1$, $u \in I_2$ and $vu \notin E(G)$.

Next we show that there exist vertices $v_0 \in I_1$, $u_0 \in I_2$ such that $v_0u_0 \notin E(G)$ and $I_{1,2} \subset N_G(\{v_0, u_0\})$. Suppose to the contrary that the set $I_{1,2} - N_G(\{v, u\})$ is not empty for every $v \in I_1$, $u \in I_2$ if $vu \notin E(G)$. It is easy to see that for nonadjacent vertices $v \in I_1$, $u \in I_2$ and for any vertices $\overline{v} \in I_1 - N_G[v]$ and

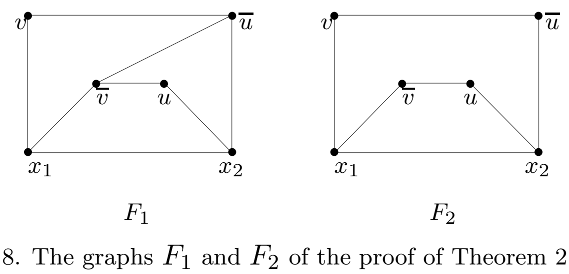

Fig. 8. The graphs F_1 and F_2 of the proof of Theorem 2.4.3

$\overline{u} \in I_2 - N_G[u]$, the subgraph $A = G[\{x_1, x_2, v, \overline{v}, u, \overline{u}\}]$ is isomorphic to one of the graphs F_1, F_2 in Figure 8, as otherwise A would be isomorphic to one of the forbidden graphs H_1, H_2, H_3. We distinguish two subcases.

Subcase 2.1: *A is isomorphic to F_1.* Then for any $x \in I_{1,2} - N_G(\{v,u\})$, the subgraph $G[V(A) \cup \{x\}]$ is isomorphic to H_5 if $|\{\overline{v}, \overline{u}\} \cap N_G(x)| = 2$ or $G[V(A) \cup \{x\}]$ contains H_2 or H_3 as an induced subgraph if $|\{\overline{v}, \overline{u}\} \cap N_G(x)| \le 1$, a contradiction.

Subcase 2.2: *A is isomorphic to F_2.* First let us observe that if there exists a vertex $x \in I_{1,2} - (N_G(\{v,u\}) \cup N_G(\{\overline{v}, \overline{u}\}))$, then the subgraph $G[V(A) \cup \{x\}]$ is isomorphic to H_6, contradicting the hypothesis of the theorem. Thus assume that the set $I_{1,2} - (N_G(\{v,u\}) \cup N_G(\{\overline{v}, \overline{u}\}))$ is empty. Since the sets $I_{1,2} - N_G(\{v,u\})$ and $I_{1,2} - N_G(\{\overline{v}, \overline{u}\})$ are not empty and $I_{1,2} \subset N_G(\{v,u\}) \cup N_G(\{\overline{v}, \overline{u}\})$, the sets $(I_{1,2} - N_G(\{v,u\})) \cap N_G(\{\overline{v}, \overline{u}\})$ and $(I_{1,2} - N_G(\{\overline{v}, \overline{u}\})) \cap N_G(\{v,u\})$ are nonempty and disjoint. For $y \in (I_{1,2} - N_G(\{v,u\})) \cap N_G(\{\overline{v}, \overline{u}\})$ and $z \in (I_{1,2} - N_G(\{\overline{v}, \overline{u}\})) \cap N_G(\{v,u\})$ we consider the subgraph $G[V(A) \cup \{y,z\}]$. It is evident that $G[V(A) \cup \{y,z\}]$ is isomorphic to one of the graphs $H_7, \ldots, H_{10}$ ($H_{11}, \ldots, H_{14}$, resp.) if $yz \notin E(G)$ ($yz \in E(G)$, resp.). Again, we have obtained a contradiction to the hypothesis of the theorem and therefore we shall suppose that there exist vertices $v \in I_1$, $u \in I_2$ such that $vu \notin E(G)$ and $I_{1,2} \subset N_G(\{v,u\})$.

The proof may now be completed. It follows from the above established observations that there exist vertices $v \in I_1$, $u \in I_2$ such that $vu \notin E(G)$ and $I_1 \cup I_2 \cup I_{1,2} \subset N_G[\{v,u\}]$. Then consider the set $D_1 = (D_0 - \{x_1, x_2\}) \cup \{v,u\}$. Let $x \in V(G) - D_1 = P \cup R$, where $P = V(G) - (D_0 \cup I_1 \cup I_2 \cup I_{1,2})$ and $R = (I_1 \cup I_2 \cup I_{1,2} \cup \{x_1, x_2\}) - \{v,u\}$. The fact that D_0 is a dominating set of G and the definitions of the sets I_1, I_2, and $I_{1,2}$ imply that $N_G(x) \cap (D_0 - \{x_1, x_2\}) \neq \emptyset$ and therefore $N_G(x) \cap D_1 \neq \emptyset$ for each $x \in P$. From the choice of the vertices v and u we have $N_G(x) \cap \{v,u\} \neq \emptyset$ for each $x \in R$. Hence D_1 is a dominating set of G. Since $|D_1| = |D_0|$ and $N_G(\{v,u\}) \cap D_1 = \emptyset$, D_1 is a minimum dominating set of G with $e(G[D_1]) < e(G[D_0])$. Again, we have obtained a contradiction to the choice of D_0. This contradiction completes the proof. ∎

Sumner [133] defines a graph G to be *domination perfect* if for each induced subgraph H of G, $\gamma(H) = i(H)$. It follows from Theorem 2.4.3 that if G has no induced subgraph isomorphic to any of the graphs $H_1, \ldots, H_{14}$ of Figure 7, then G is domination perfect. The converse implication is not true as H_{10} is domination perfect itself. The same theorem implies that the characterization of domination perfect graphs offered in [161] is not correct. (According to Theorem 3 of [161], a graph G is domination perfect if and only if G does not contain as an induced subgraph any of $H_1, \ldots, H_4$ in Figure 7. However, the graph H_6 of Figure 7 is the smallest counterexample to this characterization, since H_6 is not domination perfect and it does not contain as an induced subgraph any of $H_1, \ldots, H_4$.) Theorem 2.4.3 immediately implies some next results about domination perfect graphs.

Corollary 2.4.5. *Let G be a graph of girth at least four. Then G is domination perfect if and only if G contains no induced subgraph isomorphic to one of the four graphs H_1, H_2, H_3, H_4 of Figure* 7.

Proof. The necessity follows from the observation that $H_1, \ldots, H_4$ are not domination perfect. The sufficiency follows from Theorem 2.4.3 and the observation that of the graphs of Figure 7, only $H_1, \ldots, H_4$ are of girth at least four. ∎

The proofs of the next two corollaries are similar to that of Corollary 2.4.5.

Corollary 2.4.6. *A graph of girth at least five is domination perfect if and only if it does not contain H_1 as an induced subgraph.* ∎

Corollary 2.4.7 [133]. *A chordal graph is domination perfect if and only if it does not contain H_1 as an induced subgraph.* ∎

Corollary 2.4.8 [133]. *A graph G is domination perfect if and only if $\gamma(H) = i(H)$ for every induced subgraph H of G with $\gamma(H) = 2$.*

Proof. The necessity is obvious. To prove the sufficiency, assume that a graph G is not domination perfect. We may assume that $\gamma(G) < i(G)$ while $\gamma(F) = i(F)$ for every proper induced subgraph F of G. Take a minimum dominating set D_0 of G, adjacent vertices $x_1, x_2 \in D_0$, sets I_1, I_2, $I_{1,2}$ as in the proof of Theorem 2.4.3, and consider the subgraph H induced by $I_1 \cup I_2 \cup I_{1,2} \cup \{x_1, x_2\}$. Observe that $\gamma(H) = 2 < i(H)$, for otherwise we could find a minimum dominating set D_1 of G such that $G[D_1]$ has fewer edges than $G[D_0]$. ∎

Corollary 2.4.9. *If a graph G has no induced subgraph isomorphic to one of the six graphs H_1, H_2, H_3, H_4, F_1, F_2 of Figures* 7 *and* 8, *then G is domination perfect.*

Proof. The result is immediate from Theorem 2.4.3, since G (and every induced subgraph of G) does not have an induced subgraph isomorphic to one of the graphs $H_5, \ldots, H_{14}$ if it does not have an induced subgraph isomorphic to F_1 or F_2. ∎

Corollary 2.4.10. *If G is a graph in which no two induced subgraphs isomorphic to $K_{1,3}$ have a common edge and different centers*, *then G is domination perfect.*

Proof. Under these conditions on G, no induced subgraph of G contains any of the graphs $H_1, \ldots, H_{14}$ as an induced subgraph and the result follows from Theorem 2.4.3. ∎

The *subdivision graph* $S(G)$ of a graph G is a graph with the property that there exists a one-to-one correspondence between its vertices and the elements of G such that two vertices of $S(G)$ are adjacent if and only if the corresponding elements of G are an edge and an incident vertex. In other words, $S(G)$ is a graph obtained from G by inserting a new vertex on each edge of G. The next result is immediate from Corollary 2.4.10.

Corollary 2.4.11. *For any graph G, the subdivision graph $S(G)$ is domination perfect.* ■

For a graph G, let $C_3(G)$ denote the set $\{v \in V(G) : \alpha(G[N_G(v)]) \geq 3\}$. We say that a graph G is *almost $K_{1,3}$-free* if the set $C_3(G)$ is independent and $\gamma(G[N_G(v)]) \leq 2$ for every $v \in C_3(G)$. Certainly, every $K_{1,3}$-free graph is almost $K_{1,3}$-free. For any graph H_i of Figure 7, the set $C_3(H_i)$ is not independent and therefore an almost $K_{1,3}$-free graph contains no induced subgraph isomorphic to any of the graphs $H_1, \ldots, H_{14}$ of Figure 7. Thus, from Theorem 2.4.3, we immediately get the following generalization of Corollary 2.4.2.

Corollary 2.4.12. *Every almost $K_{1,3}$-free graph is domination perfect.* ■

The next two theorems give other instances in which the lower irredundance, domination and independence numbers are equal.

Theorem 2.4.4. *If X is a smallest maximal irredundant set in G and X is independent, then* $\mathrm{ir}(G) = \gamma(G) = i(G)$

Proof. Because of Proposition 2.1.3, it suffices to show that $\mathrm{ir}(G) = i(G)$. Suppose on the contrary that $\mathrm{ir}(G) \neq i(G)$. Then $|X| = \mathrm{ir}(G) < i(G)$ and therefore X is not a maximal independent set in G. But then $V(G) - N_G[X] \neq \emptyset$ and for any $x \in V(G) - N_G[X]$, the set $X \cup \{x\}$ is independent and therefore irredundant in G, contrary to the maximality of X. ■

As pointed out before, every maximal independent set of a graph G is a minimal dominating set of G (see Corollary 2.1.3). The converse is generally not true. Benzaken and Hammer [9] define a graph G to be *domistable* if every minimal dominating set of G is independent. It would be a challenging and worth investigating problem to characterize domistable graphs. Benedetti and Mason [8] give some examples of domistable graphs, and some conditions for domistability. It follows from Proposition 2.1.3 and the definition of a domistable graph that every domistable graph G, in particular, satisfies the equalities

$$\gamma(G) = i(G) \quad \text{and} \quad \alpha(G) = \Gamma(G).$$

We now show that for domistable graphs, $\mathrm{ir}(G) = \gamma(G)$. (We do not know if a similar result is true for the upper domination number $\Gamma(G)$ and the upper irredundance number $\mathrm{IR}(G)$ of a domistable graph G.)

Theorem 2.4.5. *If G is a domistable graph, then* $\mathrm{ir}(G) = \gamma(G) = i(G)$.

Proof. Assume that G is domistable and suppose that $\mathrm{ir}(G) \neq \gamma(G) = i(G)$, so $\mathrm{ir}(G) < \gamma(G) = i(G)$. Let $X = \{x_1, \ldots, x_n\}$ be a smallest maximal irredundant set in G. Since $|X| = \mathrm{ir}(G) < \gamma(G)$, X does not dominate all the vertices of G and therefore the set $U_0 = \{x \in V(G) - X : N_G(x) \cap X = \emptyset\}$ is nonempty. Then, by Theorem 2.1.1(a), the set $U_1 = \{x \in V(G) - X : |N_G(x) \cap X| = 1\}$ is nonempty, either. Denote $U_2 = V(G) - X - U_0 - U_1$. Certainly, each vertex of $U_1 \cup U_2$ is adjacent to a vertex of X. By Theorem 2.1.1(a), for each $u \in U_0$, the set $X_u = \{x \in X : I_G(x, X) \subseteq N_G(u)\}$ is nonempty. Let M be a subset of X of the smallest cardinality such that $X_u \cap M \neq \emptyset$ for each $u \in U_0$, say $M = \{x_1, \ldots, x_m\}$, $m \leq n$. Each vertex x_i of M belongs to X_u for some $u \in U_0$, so $I_G(x_i, X) \subseteq N_G(u)$ and therefore $x_i \notin I_G(x_i, X)$. For each $x_i \in M$, we choose any $x_i' \in I_G(x_i, X)$ and form the set $M' = \{x_1', \ldots, x_m'\}$. Note that for each $u \in U_0$, there exists $x_i \in M$ such that $x_i \in X_u$, so u is adjacent to x_i'. We conclude that the set M' dominates U_0. Let M'' be a minimal subset of M' which dominates U_0, say $M'' = \{x_1, \ldots, x_p\}$, $p \leq m$. Then $D = X \cup M''$ is a dominating set of G. However, since D contains X, it follows from Corollary 2.1.4 that D properly contains a minimal dominating set D' of G. It follows from the choice of M'' that for each $x_i' \in M''$, the set $M'' - \{x_i'\}$ does not dominate U_0 and therefore $D - \{x_i'\}$ is not dominating in G. This enables M'' to be a subset of D'. Further, since G is domistable and D' is a minimal dominating set of G, D' is independent. Consequently, $D' \subseteq (X - \{x_1, \ldots, x_p\}) \cup \{x_1' \ldots, x_p'\}$ because $\{x_1', \ldots, x_p'\} \subseteq D'$ and each x_i is adjacent to x_i', $i = 1, \ldots, p$. Thus, $\gamma(G) \leq |D'| \leq |(X - \{x_1, \ldots, x_p\}) \cup \{x_1' \ldots, x_p'\}| = \mathrm{ir}(G)$, contrary to our supposition. This completes the proof of the theorem. ∎

Sufficient conditions for equality of some of the upper independence, domination and irredundance numbers of a graph have been presented in [33, 36, 40, 60, 76, 90, 91, 92, 146]. In [40], Cockayne, Hedetniemi and Miller have observed that if G is the middle graph of a graph (that is, $G = L(H \circ K_1)$ for some graph H), then $\alpha(G) = \mathit{\Gamma}(G) = \mathrm{IR}(G)$. Favaron [60] shows that if G is $K_{1,3}$-free and it contains neither G_1 of Figure 6 nor A_3 of Figure 1 as an induced subgraph, then $\mathit{\Gamma}(G) = \mathrm{IR}(G)$. Cockayne, Favaron, Payan and Thomason [36] present several sufficient conditions for equality of some of the upper parameters. In particular, they prove the following theorem.

THEOREM 2.4.6. *If G is a bipartite graph, then* $\alpha(G) = \mathit{\Gamma}(G) = \mathrm{IR}(G)$.

Proof. Let R and S be the defining sets of the bipartite graph G. Suppose X is a maximum irredundant set of G and let U be the set of isolated vertices of $G[X]$. If $X = U$, then X is independent, $\alpha(G) \geq |X| = \mathrm{IR}(G)$ and the result follows from Proposition 2.1.3. If $X \neq U$, then we define

$$A = U \cap R, \quad B = (X \cap R) - U, \quad C = U \cap S \quad \text{and} \quad D = (X \cap S) - U.$$

In this case the sets B and D are nonempty and each vertex of D is adjacent to a vertex of B (and vice versa). Irredundance implies that $d \notin I_G(d, X)$ for each $d \in D$. Moreover, the sets $I_G(d, X)$, $d \in D$, are nonempty and disjoint subsets of $(V(G) - X) \cap R$ and no vertex of $\bigcup_{d \in D} I_G(d, X)$ is adjacent to a vertex of C. Consequently, $A \cup B \cup C \cup \bigcup_{d \in D} I_G(d, X)$ is an independent set of G. Therefore $\alpha(G) \geq |A| + |B| + |C| + \sum_{d \in D} |I_G(d, X)| \geq |A| + |B| + |C| + |D| = \mathrm{IR}(G)$ and again the result follows from Proposition 2.1.3. ∎

Cheston, Hare, Hedetniemi and Laskar [33] show that if G is a simplicial graph, then $\alpha(G) = \mathit{\Gamma}(G)$. Also, they show that $\alpha(G) = \mathit{\Gamma}(G) = \mathrm{IR}(G)$ for any edge simplicial graph G; an edge simplicial graph is a graph in which every edge belongs to a simplex. A recent result by Golumbic and Laskar [76] shows that the same holds for circular arc graphs. (A graph is a circular arc graph if it can be represented as the intersection graph of arcs on a circle.) Jacobson and Peters [90, 91] and Jacobson, Peters and Rall [92] present several conditions which involve or do not involve forbidden subgraph characterizations of graphs G for which $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$. In [91], Jacobson and Peters survey a wide variety of families of graphs G for which $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$. The following two theorems due to Jacobson and Peters [90] demonstrate the equality of $\alpha(G)$ and $\mathrm{IR}(G)$ for chordal graphs and for graphs which do not contain either $K_{1,3}$, C_4 or $K_3 \circ 2$ as an induced subgraph (the definition of $G \circ k$ has been given before Proposition 2.3.2); we present new proofs of these theorems.

THEOREM 2.4.7. *If G is a chordal graph, then $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$.*

Proof. Assume that the result is not true for some chordal graph. Let G be a smallest chordal graph with $\alpha(G) < \mathrm{IR}(G)$. The choice of G implies that G is connected and noncomplete. Further, for each $v \in V(G)$, since $G - v$ is chordal, we have $\alpha(G-v) = \mathrm{IR}(G-v)$. From this and from Theorem 2.2.1 it follows that $\mathrm{IR}(G - v) = \mathrm{IR}(G) - 1$ (and $\alpha(G - v) = \alpha(G)$) for each $v \in V(G)$. Let X be any largest irredundant set in G, X' be the set of isolated vertices in $G[X]$ and $X'' = X - X'$. Further, define $U_i = \{x \in V(G) - X : |N_G(x) \cap X| = i\}$ for $i = 0$, 1, and $U_2 = \{x \in V(G) - X : |N_G(x) \cap X| \geq 2\}$. We note that $U_0 = U_2 = \emptyset$, for otherwise X would be irredundant in $G - v$ for each $v \in U_0 \cup U_2$. Moreover, we have $X' = \emptyset$; otherwise $N_G(X')$ would be a nonempty subset of U_1 and X would be irredundant in $G - v$ for $v \in N_G(X')$. Thus, $V(G) = X \cup U_1$ and $X = X''$. Since no vertex of X is isolated in $G[X]$, $I_G(x, X)$ is a nonempty subset of U_1 for each $x \in X$. In addition, for each $x \in X$, $I_G(x, X)$ has exactly one vertex; for if there were $x \in X$ with $|I_G(x, X)| \geq 2$, then X would be irredundant in $G - v$ for each $v \in I_G(x, X)$. Thus, each vertex of X is adjacent to exactly one vertex of U_1 and vice versa. We note that no vertex of $G[U_1]$ is isolated; for if v were isolated in $G[U_1]$ and x were the unique neighbour of v in X, then $(X - \{x\}) \cup \{v\}$ would be an irredundant set of cardinality $\mathrm{IR}(G)$ in $G - x$. Consequently, each vertex of X (resp. U_1) is adjacent to at least two nonadjacent vertices—one in U_1 (resp. X) and the other in X (resp. U_1). Thus, no vertex of G is simplicial and therefore G is not a chordal graph, contrary to our assumption, and the result follows. ■

Since a block graph is a chordal graph, we have the following corollary for a block graph.

COROLLARY 2.4.13. *If G is a block graph, then $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$.* ■

THEOREM 2.4.8. *If a graph G does not contain either $K_{1,3}$, C_4 or $K_3 \circ 2$ as an induced subgraph, then $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$.*

Proof. Suppose it is not true and let G be a smallest graph that does not contain either $K_{1,3}$, C_4 or $K_3 \circ 2$ as an induced subgraph and for which $\alpha(G) < \mathrm{IR}(G)$. By the choice of G, G is connected, noncomplete and $\alpha(G-v) = \mathrm{IR}(G-v)$

for any vertex v of G. Consequently, by Theorem 2.2.1, we have $\mathrm{IR}(G-v) = \mathrm{IR}(G)-1 = \alpha(G) = \alpha(G-v)$ for each $v \in V(G)$. Let X be any largest irredundant set in G. For any $v \in V(G) - X$, $|X| = \mathrm{IR}(G) = \mathrm{IR}(G-v) - 1$ and therefore X is not irredundant in $G - v$. Let X' be the set of isolated vertices in $G[X]$ and define $X'' = X - X'$, $U_i = \{x \in V(G) - X : |N_G(x) \cap X| = i\}$ for $i = 0, 1$, and $U_2 = \{x \in V(G) - X : |N_G(x) \cap X| \geq 2\}$. Note that $U_0 = U_2 = \emptyset$; otherwise, for every $v \in U_0 \cup U_2$, X is irredundant in $G - v$ and consequently $|X| \leq \mathrm{IR}(G-v) = \mathrm{IR}(G) - 1 = |X| - 1$ which is impossible. Moreover, $X' = \emptyset$; otherwise $N_G(X')$ would be a nonempty subset of U_1 and X would be irredundant in $G - v$ for each $v \in N_G(X')$. Thus, $V(G) = X \cup U_1$ and $X = X''$. Since no vertex of X is isolated in $G[X]$, $I_G(x, X)$ is a nonempty subset of U_1 for each $x \in X$. In addition, for each $x \in X$, $I_G(x, X)$ has exactly one vertex; for if there were $x \in X$ with $|I_G(x, X)| \geq 2$, then X would be irredundant in $G - v$ for each $v \in I_G(x, X)$. Thus, each vertex of X is adjacent to exactly one vertex of U_1 and vice versa. This implies that U_1 is another largest irredundant set in G. Note that no vertex of $G[U_1]$ is isolated; for if v were isolated in $G[U_1]$ and x were the unique neighbour of v in X, then $(X - \{x\}) \cup \{v\}$ would be an irredundant set of cardinality $\mathrm{IR}(G)$ in $G - x$. Consequently, each vertex of X (resp. U_1) is adjacent to at least two nonadjacent vertices—one in U_1 (resp. X) and the other in X (resp. U_1). Certainly, G is not a cycle (otherwise $\alpha(G) = |V(G)|/2 = \mathrm{IR}(G)$) and therefore G has a vertex x_0 of degree at least three. We may assume that x_0 belongs to X. Let x_1 and x_2 be distinct neighbours of x_0 in X. For $i = 0, 1, 2$, let y_i be the unique element of $I_G(x_i, X)$. Since $K_{1,3}$ is not an induced subgraph of G, the vertices x_1 and x_2 are adjacent. Similarly, since C_4 is not an induced subgraph of G, the vertices y_0, y_1 and y_2 are mutually nonadjacent. For $i = 0, 1, 2$, let y_i' be a neighbour of y_i in U_1. Again, since $K_{1,3}$ is not an induced subgraph of G, the vertices y_0', y_1' and y_2' are distinct and mutually nonadjacent. But now the vertices x_0, x_1, x_2, y_0, y_1, y_2, y_0', y_1', y_2' induce a graph isomorphic to $K_3 \circ 2$ in G, and we have a final contradiction. ∎

The next theorem shows that the conclusion of the last theorem is also true for unicyclic graphs.

THEOREM 2.4.9. *If G is a unicyclic graph, then* $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$.

Proof. It is not difficult to verify the result for cycles. Thus let G be a unicyclic graph of order n ($n \geq 4$), $G \neq C_n$, and suppose that the result is true for trees (see Theorems 2.4.6 and 2.4.7 or Corollary 2.4.13) and for unicyclic graphs of order less than n. Since $\alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G)$, it suffices to show that $\alpha(G) \geq \mathrm{IR}(G)$. Let $\Omega(G)$ be the set of end vertices of G and let x be any farthest vertex from the unique cycle of G. Certainly, $x \in \Omega(G)$. Let x' be the unique neighbour of x and denote $A = N_G(x') \cap \Omega(G)$. Suppose X is a maximum irredundant set of G. The maximality of X implies that $A \subset X$ if $A \cap X \neq \emptyset$. Similarly, if $A \cap X = \emptyset$ but $x' \in X$, then $(X - \{x'\}) \cup A$ is a maximum irredundant set and it contains the vertices of A. Finally, suppose that $A \cap X = \emptyset$

and $x' \notin X$. Then there is exactly one vertex y in X such that $I_G(y, X) = \{x'\}$ and consequently $(X - \{y\}) \cup A$ is a maximum irredundant set containing all the vertices of A. Therefore we henceforth suppose that X contains the vertices of A.

Consider the graph $H = G - (A \cup \{x'\})$. It is no problem to observe that $X - A$ is a maximum irredundant set of H, so $\mathrm{IR}(H) = \mathrm{IR}(G) - |A|$. Further, since H is a tree or a unicyclic graph of order less than n, the induction hypothesis implies that $\alpha(H) = \Gamma(H) = \mathrm{IR}(H)$. In addition, if I is a maximum independent set in H, then $I \cup A$ is independent in G and therefore $\alpha(G) \geq |I \cup A| = \alpha(H) + |A| = \mathrm{IR}(H) + |A| = \mathrm{IR}(G)$. This completes the proof. ∎

The conclusion of Theorem 2.4.9 is not true for connected graphs with two or more cycles. In fact, the graph G of Figure 3 has two cycles and $\alpha(G) = 5$ while $\Gamma(G) = \mathrm{IR}(G) = 6$.

3. Well covered graphs

3.1. Introduction and preliminary results. A graph G is called *well covered* if every maximal independent set of vertices in G is a maximum independent set. A graph G is said to be *well dominated* if every minimal dominating set in G is a minimum dominating set. By analogy to these concepts, a graph G is *well irredundant* if every maximal irredundant set in G is a maximum irredundant set. Equivalently, a graph G is well covered (dominated, irredundant, resp.) if $i(G) = \alpha(G)$ ($\gamma(G) = \Gamma(G)$, $\mathrm{ir}(G) = \mathrm{IR}(G)$, resp.). A graph G is *very well covered* if it is a well covered graph without isolated vertices and $\alpha(G) = |V(G)|/2$. It follows from the Proposition 2.1.3 that every well irredundant graph is well dominated, and every well dominated graph is well covered. The converse is not necessarily true. For example, the graph G_1 in Figure 9 is well dominated but not well irredundant as $I_1 = \{v_1, v_4, v_7\}$ and $I_2 = \{v_3, v_5\}$ are both maximal irredundant sets in G_1. On the other hand, the graph G_2 in Figure 9 is well covered but not well dominated since $D_1 = \{u_1, u_3, u_6\}$ and $D_2 = \{u_2, u_5\}$ are minimal dominating sets of different cardinalities in G_2.

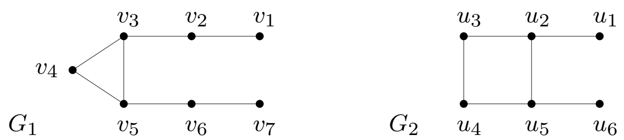

Fig. 9. The graph G_1 (G_2, resp.) is well dominated (covered, resp.) but not well irredundant (dominated, resp.)

The concept of well covered graphs was introduced by Plummer [115] and generalized by Favaron and Hartnell [63] and Currie and Nowakowski [45]. Some interest in these graphs is motivated by the fact that a maximum independent set can always be found efficiently in a well covered graph, whereas the independence set problem is NP-complete for general graphs, as we have mentioned in the second chapter. The well covered and well dominated graphs have been studied in a few papers. For example, Staples [130, 131] studied the properties of the W_n classes of graphs, where a graph G belongs to class W_n if $|V(G)| \geq n$ and every n disjoint independent sets in G are contained in n disjoint maximum independent sets. The W_n classes form a descending chain $W_1 \supseteq W_2 \supseteq \ldots$ and W_1 is the class of well covered graphs. Staples [130] and later Favaron [59] gave a characterization of very well covered graphs. These graphs include bipartite well covered graphs which were also characterized by Ravindra [118]. The cubic, planar, 3-connected graphs which are well covered have been characterized in [28] by Campbell and Plummer. Finbow and Hartnell [64] characterized well covered graphs of girth at least 8. Recently Finbow, Hartnell, and Nowakowski in [67] and [68] have extensively described the well covered graphs of girth at least 5 and the well covered graphs containing neither a cycle C_4 nor a cycle C_5 as a subgraph. The well dominated graphs of girth at least five and the well dominated bipartite graphs are characterized in [66] again by Finbow, Hartnell and Nowakowski. Topp and Volkman [154] studied the well coveredness of products of graphs. In [150], they have also given structural characterizations of the well covered and well dominated block graphs and unicyclic graphs. The well irredundant graphs were defined and studied in [146]. Berge [14], among other things, presents some relationships between the class of well covered graphs and some other classes of graphs. Other subclasses of the well covered graphs were studied in [45, 63, 141, 148]. Various approaches to the problem of characterizing families of well covered graphs have been tried and the reader is referred to an article by Plummer [116] for an excellent survey of progress.

The main objectives of this chapter are to study various general properties and various subclasses of well covered graphs. The following is a summary of the results presented in this chapter.

In §3.1 (an introductory section), we first give several general properties of maximal independent sets and then we prove theorems due to Staples, Ravindra and Favaron which characterize the very well covered graphs. Three theorems which concern well covered (and well dominated) graphs of girth at least five and well covered cubic, 3-connected, planar graphs (due to Finbow, Hartnell and Nowakowski, and Campbell and Plummer, resp.) are given without proofs due to lack of space. All these results are used in the subsequent sections of this chapter. We also present a subclass of the well covered graphs introduced by Finbow and Hartnell.

In §3.2, we study the well coveredness of graphs formed from other graphs by various operations.

In §3.3, we characterize well covered and well dominated graphs within the following families: simplicial graphs, chordal graphs, circular arc graphs, k-trees, and $C_{(n)}$-trees.

In §3.4, we investigate edge and total versions of the well coveredness.

In §3.5, we show that there are exactly five well covered generalized Petersen graphs.

In §3.6, we investigate the well irredundance of bipartite graphs, chordal graphs, and graphs of girth at least five.

We begin with a useful characterization of maximum independent sets of vertices in a graph. This characterization is due to Berge [15].

PROPOSITION 3.1.1. *Let I be an independent set in a graph G. Then I is a maximum independent set of G if and only if every independent subset S of $V(G) - I$ can be matched into I.*

Proof. Assume that I is a maximum independent set of G and S is an independent subset of $V(G)-I$. Note that $|N_G(A) \cap I| \geq |A|$ for every set $A \subseteq S$; for if there were $A \subseteq S$ with $|N_G(A) \cap I| < |A|$, then $(I - N_G(A)) \cup A$ would be a larger independent set in G which is impossible. From this and from König–Hall's Theorem (see [15, p. 132]) it follows that S can be matched into I.

Conversely, assume that every independent subset $S \subseteq V(G) - I$ can be matched into I and suppose indirectly that I is not a maximum independent set. Then there is an independent set J in G with $|J-I| > |I-J|$ and, certainly, the set $J - I$ cannot be matched into I, a contradiction. ∎

As in [15], a vertex x of a graph G is called a *critical vertex* of G if $\alpha(G-x) \neq \alpha(G)$, or equivalently, if every maximum independent set of G contains x. Note that every isolated vertex is a critical vertex.

PROPOSITION 3.1.2 [15]. *If a graph G has no critical vertex, then every independent set J of G can be matched into $V(G) - J$.*

Proof. We proceed by induction on $|J|$. Since G is without critical vertices, the result is trivial if $|J| = 1$. Suppose the result is true for sets of cardinality at most $p-1$ and let J be an independent set with $|J| = p > 1$. Take any vertex $v \in J$. Since v is not a critical vertex, there exists a maximum independent set I which does not contain v. By Proposition 3.1.1, $J - I$ can be matched into I and so into $I - J$. By the induction hypothesis, $J \cap I$ can be matched into $V(G) - (J \cap I)$ and so into $V(G) - (J \cup I)$. These two matchings give a matching of J into $V(G) - J$. ∎

COROLLARY 3.1.1 [15]. *If a graph G has no critical vertex, then $\alpha(G) \leq |V(G)|/2$. Moreover, if $\alpha(G) = |V(G)|/2$, then G has a perfect matching.*

Proof. Let I be a maximum independent set in G. By Proposition 3.1.2, I can be matched into $V(G)-I$ and therefore $\alpha(G) = |I| \leq |V(G)-I| = |V(G)|-\alpha(G)$. Hence, $\alpha(G) \leq |V(G)|/2$. Certainly, if $\alpha(G) = |V(G)|/2$, then any matching between I and $V(G) - I$ is a perfect matching in G. ∎

PROPOSITION 3.1.3. *If G is a well covered graph without isolated vertices, then G has no critical vertices.*

P r o o f. Let x be any vertex of G. It is enough to show that there exists a maximum independent set in G that does not contain x. Let y be any neighbour of x and let I be any maximal independent set that contains y. Certainly, $x \notin I$. In addition, since G is well covered and I is a maximal independent set in G, I is a maximum independent set in G. This implies the result. ∎

COROLLARY 3.1.2. *If G is a well covered graph without isolated vertices, then* $\alpha(G) \leq |V(G)|/2$.

P r o o f. The result is immediate from Corollary 3.1.1 and Propositions 3.1.3. ∎

We now prove the first characterization of the very well covered graphs. The following theorem is a slight modification of the result due to Staples [130] and later to Favaron [59].

THEOREM 3.1.1. *Let G be a connected graph of order $n \geq 2$. Then G is very well covered if and only if G has a perfect matching M and for every edge $vu \in M$,*

(1) *vu does not belong to a triangle and*
(2) *every vertex of $N_G(v)$ is adjacent to every vertex of $N_G(u)$.*

P r o o f. Assume G is a very well covered graph. Then $\alpha(G) = n/2$ and G has a perfect matching (see Corollary 3.1.1). Let I be any maximal (and hence maximum) independent set in G and let M be any perfect matching of G. Certainly, every edge from M has exactly one of its vertices in I and therefore in every maximal independent set in G. Take any edge vu from M. Observe that vu does not belong to a triangle in G; for if there were a vertex y adjacent to both v and u, then every maximal independent set containing y would contain none of the vertices v and u of the edge vu from M which is impossible. Similarly, every vertex of $N_G(v)$ is adjacent to every vertex of $N_G(u)$; for if there were nonadjacent vertices $y \in N_G(v)$ and $z \in N_G(u)$, then every maximal independent set containing y and z would contain none of the vertices v and u which again is impossible.

Assume now that G has a perfect matching M such that the conditions (1) and (2) are satisfied for every edge of M. Let I be any maximal independent set in G. Then $|I| \leq |M| = n/2$ and it is enough to show that $|I| = |M|$. Suppose also that $|I| < |M|$. Then there is an edge vu in M with $v, u \in V(G) - I$. Since I is a maximal independent set in G, the vertices v and u are adjacent to some vertices of I, say $v' \in N_G(v) \cap I$ and $u' \in N_G(u) \cap I$. It follows from (1) and (2) that the vertices v' and u' are different and adjacent, contrary to the independence of I. Consequently, $|I| = |M| = n/2$ which completes the proof. ∎

It is obvious from the definition of a very well covered graph that every very well covered graph is well covered. The converse implication is true for bipartite graphs.

PROPOSITION 3.1.4. *If G is a connected bipartite graph of order $n \geq 2$, then G is well covered if and only if G is very well covered.*

P r o o f. Assume that G is a well covered connected bipartite graph of order $n \geq 2$. Let V_1 and V_2 be partite sets of vertices of G. Since both V_1 and V_2 are maximal independent sets in G, we have

$$\alpha(G) \geq \max\{|V_1|, |V_2|\} \geq \min\{|V_1|, |V_2|\} \geq i(G)$$

and therefore $\alpha(G) = |V_1| = |V_2| = i(G) = n/2$, as $\alpha(G) = iG)$ and $|V_1|+|V_2| = n$. Hence G is very well covered. The converse implication is obvious. ■

For bipartite graphs we have the following two immediate consequences of Theorem 3.1.1 and Proposition 3.1.4. These two results were originally due to Ravindra [118]; the second one describes the structure of well covered trees.

COROLLARY 3.1.3. *A connected bipartite graph G of order $n \geq 2$ is well covered if and only if G has a perfect matching M and for every $vu \in M$, the induced subgraph $G[N_G(\{v, u\})]$ is a complete bipartite graph.*

P r o o f. The result follows from Theorem 3.1.1, Proposition 3.1.4 and the fact that a bipartite graph does not have any triangle. ■

COROLLARY 3.1.4. *A tree T is well covered if and only if every interior vertex of T is adjacent to exactly one end vertex of T.*

P r o o f. Assume that T is a well covered tree, $T \neq K_1$. Then, by Corollary 3.1.3, T has a perfect matching M and so every interior vertex of T is adjacent to at most one end vertex of T. On the other hand, if v is an interior vertex of T, then there is a vertex u in T such that $vu \in M$. Since $G[N_G(\{v, u\})]$ is a complete bipartite graph and T has no cycles, $G[N_G(\{v, u\})]$ is a star and u is an end vertex in T. Thus, every interior vertex of T is adjacent to exactly one end vertex of T.

If $T \neq K_1$ and every interior vertex of T is adjacent to exactly one end vertex of T, then the end edges of T form a perfect matching of T and for every end edge vu of T, the subgraph $G[N_G(\{v, u\})]$ is complete bipartite. Thus, T is well covered by Corollary 3.1.3. ■

We remark that Corollary 3.1.4 may also be stated in the form "A tree T is well covered if and only if $T = K_1$ or $T = R \circ K_1$ for some tree R."

The following simple property of the well covered graphs was first observed by Campbell and Plummer [28]; we present a somewhat different proof here.

PROPOSITION 3.1.5. *If G is a well covered graph, then for each independent set I in G, $G - N_G[I]$ is a well covered graph.*

P r o o f. Suppose on the contrary that $G - N_G[I]$ is not well covered for some independent set I of G. Then there are maximal independent sets I_1 and I_2 in $G - N_G[I]$ with $|I_1| \neq |I_2|$. But then, since no vertex of I is adjacent to a vertex of $V(G) - N_G[I]$ in G, it is easy to observe that $I_1 \cup I$ and $I_2 \cup I$ are maximal independent sets of different cardinalities in G, contradicting the well coveredness of G. ■

It follows from Proposition 3.1.5 that if G is a well covered graph, then $G - N_G[v]$ is a well covered graph for every vertex v of G. This condition often provides a quick means for showing that a given graph fails to be well covered. On the other hand, that this condition is not sufficient for any graph to be well covered, may be seen by considering a star $K_{1,n}$ with $n \geq 2$. In the next theorem we will show that for a $K_{2,3}$-free graph G with $i(G) > 1$, this condition is sufficient for the well coveredness of G.

THEOREM 3.1.2. *Let G be a graph with $i(G) > 1$ and assume that no induced subgraph of G is isomorphic to $K_{2,3}$. Then G is well covered if and only if for every vertex v of G, $G - N_G[v]$ is a well covered graph.*

Proof. The "only if" part of the theorem follows from Proposition 3.1.5. To prove the "if" part, assume that G is a $K_{2,3}$-free graph with $i(G) \geq 2$, $G - N_G[v]$ is a well covered graph for every vertex v of G, and suppose to the contrary that G is not well covered. Then G possesses maximal independent sets of different cardinality. First we claim that any two maximal independent sets (of G) of different cardinality are disjoint. Suppose, to the contrary, that there are maximal independent sets I_1 and I_2 in G such that $|I_1| \neq |I_2|$ and $I_1 \cap I_2 \neq \emptyset$. Then for every $v_0 \in I_1 \cap I_2$, the sets $I_1 - \{v_0\}$ and $I_2 - \{v_0\}$ are maximal independent sets in $G - N_G[v_0]$ and $|I_1 - \{v_0\}| \neq |I_2 - \{v_0\}|$. This contradicts the well coveredness of $G - N_G[v_0]$ and proves our claim.

Let J_1 and J_2 be two maximal independent sets of G with $|J_1| \neq |J_2|$, say $|J_1| < |J_2|$. We now claim that $G[J_1 \cup J_2]$ is a complete bipartite graph. Since the sets J_1 and J_2 are independent and disjoint, it suffices to show that every vertex of J_1 is adjacent to every vertex of J_2. Suppose to the contrary that there are nonadjacent vertices v and u in $G[J_1 \cup J_2]$ such that $v \in J_1$ and $u \in J_2$. Let I be any maximal independent set of G that contains v and u. Then $I \cap J_1 \neq \emptyset$, $I \cap J_2 \neq \emptyset$, and $|I| \neq |J_1|$ or $|I| \neq |J_2|$, a contradiction to the first claim. Thus, $G[J_1 \cup J_2]$ is a complete bipartite graph. From this and from the inequalities $2 \leq i(G) \leq |J_1| < |J_2|$ it follows that the complete bipartite graph $K_{2,3}$ is an induced subgraph of G. This contradicts our assumption that G is a $K_{2,3}$-free graph. With this contradiction the theorem is established. ■

We shall now briefly mention three interesting and important theorems concerning some subclasses of the well covered graphs. These three theorems are deep and their proofs are difficult and long and therefore we refer the readers who are interested in this topic to the original papers. In [28], Campbell and Plummer gave the following characterization of cubic, 3-connected, planar, well covered graphs. (Campbell, Ellingham and Royle [27] have recently characterized all well covered cubic graphs.)

THEOREM 3.1.3. *There are exactly four cubic, planar, 3-connected, well covered graphs and they are shown in Figure 10.* ■

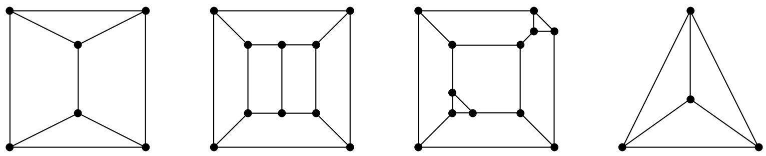

Fig. 10. The four graphs of Theorem 3.1.3

A series of papers by Finbow and Hartnell [64, 65] and Finbow, Hartnell and Nowakowski [66, 67, 68] has been devoted to characterizations of well covered and well dominated graphs of girth at least five, and well covered graphs containing neither a cycle C_4 nor a cycle C_5 as a subgraph. We need the following definitions. A cycle C of a graph G is said to be basic if C is of length 5 and does not contain two adjacent vertices of degree three or more. Let $\mathcal{PC}$ be the family of graphs defined as follows: A graph G belongs to the family $\mathcal{PC}$ if its vertex set can be partitioned into two subsets, say $V_{\mathcal{P}}$ and $V_{\mathcal{C}}$, where $V_{\mathcal{P}}$ consists of the vertices incident with the end edges of G and, in addition, the end edges form a perfect matching of the subgraph $G[V_{\mathcal{P}}]$ induced by $V_{\mathcal{P}}$, while $V_{\mathcal{C}}$ consists of the vertices of the basic 5-cycles and the vertex sets of the basic 5-cycles form a partition of $V_{\mathcal{C}}$. It is possible that one of the sets $V_{\mathcal{P}}$ and $V_{\mathcal{C}}$ is empty. Note that the subgraph $G[V_{\mathcal{P}}]$ is the corona of some graph H and K_1. If $G \in \mathcal{PC}$ and the set $V_{\mathcal{P}}$ ($V_{\mathcal{C}}$, resp.) is empty, then we say that G belongs to the family $\mathcal{C}$ ($\mathcal{P}$, resp.). Notice that if a graph G of order p belongs to the family $\mathcal{C}$, then G contains at least $3p/5$ vertices of degree two. Figure 11 contains a graph G which belongs to $\mathcal{PC}$.

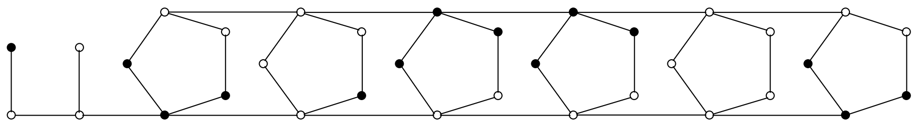

Fig. 11. A graph of the family $\mathcal{PC}$

The following two structural characterizations of well covered and well dominated graphs of girth at least five are the keys to some of our theorems.

THEOREM 3.1.4 [67]. *Let G be a connected graph of girth at least five. Then G is well covered if and only if either G belongs to the family $\mathcal{PC}$, or $G = K_1$, or G is isomorphic to one of the five graphs Q_{13}, P_{13}, C_7, P_{10}, or P_{14} in Figures* 12 *and* 13. ∎

THEOREM 3.1.5 [66]. *If a graph G belongs to the family $\mathcal{PC}$, then G is well dominated if and only if for every pair of basic* 5*-cycles there is either no edge joining them, exactly two edges and they are vertex disjoint, or four edges.* ∎

COROLLARY 3.1.5. *Let G be a connected graph of girth at least five. Then G is well dominated if and only if either $G = K_1$, or G is isomorphic to one of the three graphs C_7, P_{10}, and P_{14} in Figure* 13, *or G belongs to the family $\mathcal{PC}$ and* *for every pair of basic* 5*-cycles there is either no edge joining them or exactly two edges and they are vertex disjoint.*

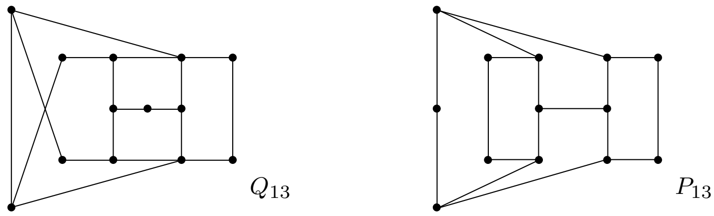

Fig. 12. The graphs Q_{13} and P_{13} of Theorem 3.1.4

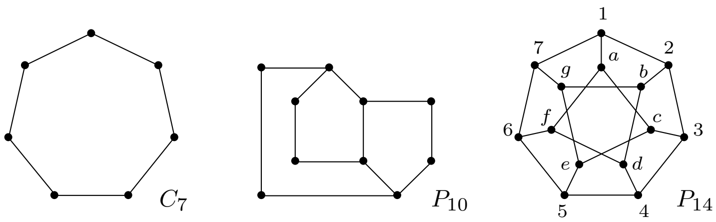

Fig. 13. The graphs C_7, P_{10} and P_{14} of Theorem 3.1.4

Proof. Since every well dominated graph is well covered and the graphs Q_{13} and P_{13} of Figure 12 are not well dominated, the result follows from Theorems 3.1.4 and 3.1.5 and from the fact that no two basic 5-cycles are joined by four edges in a graph of girth at least five. ∎

COROLLARY 3.1.6 [66]. *A connected graph of girth at least five is well covered if and only if it is well dominated.* ∎

Two new subclasses of the well covered graphs were introduced and studied in [65] by Finbow and Hartnell. First we recall some terminology from [65]. A dominating set D of a graph G is defined to be *locating* ([41, 127, 128, 129]) if $N_G(v) \cap D \neq N_G(u) \cap D$ for every pair of vertices $v, u \in V(G) - D$. In [65], Finbow and Hartnell call a graph G to be an EDL graph if every dominating set of G is locating. They also refer to a graph G as an EIDL graph if every independent dominating set of G is locating. Certainly, every EDL graph is an EIDL graph. As the graph G_2 in Figure 9 shows, the converse is, in general, not true. In what follows, it is helpful to note that if G is an EIDL graph and I is an independent set of vertices in G, then $G - N_G[I]$ is also an EIDL graph. (The formal proof of this fact is similar to the proof of Proposition 3.1.5.) The following relationship between the EIDL and well covered graphs was observed by Finbow and Hartnell [65].

THEOREM 3.1.6. *Every EIDL graph is well covered.*

Proof. Suppose that G is an EIDL graph which is not well covered. Among the pairs (S, T) of the maximal independent subsets of $V(G)$ with $|S| > |T|$, choose one, say (H,K), such that $H-K$ has the smallest cardinality. We complete the proof by showing that there exists a maximal independent set K' in G with $|K'| \le |K|$ and $|H-K'| < |H-K|$.

Take any $x \in H-K$. Observe that $N_G(x)\cap K \neq \emptyset$. Moreover, if $|N_G(x)\cap K| = 1$, say $N_G(x)\cap K = \{y\}$, then $K' = (K-\{y\})\cup\{x\}$ is the independent set required to complete the proof: indeed K' is an independent set in G and, in addition, it is a maximal independent set, for if there were $v \in V(G)-K'$ such that $N_G(v)\cap K' = \emptyset$, then it would be $N_G(v)\cap K = \{y\} = N_G(x)\cap K$ which is impossible in an EIDL graph.

The proof can thus be completed by showing that for each $x \in H-K$ with $|N_G(x)\cap K| \ge 2$ there is a maximal independent set K'' such that $|K''| \le |K|$, $|H-K''| \le |H-K|$, and $|N_G(x)\cap K''| < |N_G(x)\cap K|$. Consider the subgraph $G'' = G-N_G[K-N_G(x)]$. Certainly, G'' is an EIDL graph and the set $A=\{x\}\cup(N_G(x)\cap K)$ is a subset of $V(G'')$. Moreover, $V(G'')-A \neq \emptyset$, for otherwise $\{x\}$ would be a maximal independent set in G'' such that $N_{G''}(v)\cap\{x\} = N_{G''}(u)\cap\{x\} = \{x\}$ for any two vertices $v,u \in N_G(x)\cap K$. Observe that for every $y \in V(G'')-A$, the set $N_G(y)\cap K$ is nonempty and it is a proper subset of $N_G(x)\cap K$, for otherwise $N_G(y)\cap K = N_G(x)\cap K$. Among the vertices of $V(G'')-A$, choose one, say y_0, such that $N_G(y_0)\cap K$ has the smallest cardinality. Then, as it is easy to check, $K'' = (K-N_G(y_0))\cup\{y_0\}$ is the required maximal independent set in G. ∎

Complete graphs of order at least three show that not every well covered graph is an EIDL graph. However, this is not the case if we restrict our attention to well covered graphs of girth at least five.

THEOREM 3.1.7 [65]. *Let G be a graph of girth at least five. Then G is an EIDL graph if and only if G is well covered.*

Proof. Assume that G is a well covered graph of girth at least five and suppose on the contrary that G is not an EIDL graph. Then there exist a maximal independent set I in G and different vertices $v,u \in V(G)-I$ such that $N_G(v)\cap I = N_G(u)\cap I$. It follows from the girth restriction that $N_G(v)\cap I = \{x\} = N_G(u)\cap I$ for some $x \in I$. But then $(I-\{x\})\cup\{v,u\}$ is a greater independent set in G which is impossible in a well covered graph. The second part of the result follows from Theorem 3.1.5. ∎

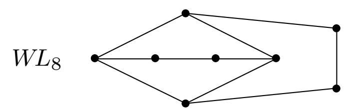

Fig. 14. The graph WL_8 is an EDL graph

A structural characterization of the well covered graphs of girth at least five and therefore a structural characterization of the EIDL graphs of girth at least five is given in Theorem 3.1.4. Finally, Finbow and Hartnell [65] gave a representation theorem of the EDL graphs: a connected graph G is an EDL graph if and only if

$G \in \{K_1, C_7, WL_8\} \cup \{H \circ K_1 : H$ is a connected graph$\}$, where the graph WL_8 is given in Figure 14.

3.2. The well coveredness of products of graphs. Many techniques for building various families of well covered graphs have been provided by Staples [130, 131], Campbell [34], and recently by Gasquoine, Hartnell, Nowakowski, and Whitehead [74], Pinter [113], and Whitehead [159]. In this section, we study the following types of graph products with respect to the well covered and the very well covered properties: the corona, the join, the disjunction, the conjunction, the lexicographic product, and the cartesian product of graphs. Conditions for these products of graphs to be (very) well covered are established based upon the factors. The products of graphs used here can be found in the literature under various aliases. To avoid confusion, we state the definitions explicitly. All the results of this section are taken from the paper by Topp and Volkmann [154].

The corona of graphs. For a graph G and a family $\mathcal{H} = \{H_v : v \in V(G)\}$ of graphs indexed by the vertices of G, the *corona* $G \circ \mathcal{H}$ of G and $\mathcal{H}$ is the disjoint union of G and H_v, $v \in V(G)$, with additional edges joining each vertex v of G to all vertices of H_v. If all the graphs of the family $\mathcal{H}$ are isomorphic to one and the same graph H then we shall write $G \circ H$ instead of $G \circ \mathcal{H}$.

The following results specify when the corona $G \circ \mathcal{H}$ is a (very) well covered graph.

THEOREM 3.2.1. *Let G be a graph, and let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of nonempty graphs indexed by the vertices of G. Then the corona $G \circ \mathcal{H}$ is a well covered graph if and only if $\mathcal{H}$ consists of complete graphs.*

Proof. Assume that $G \circ \mathcal{H}$ is a well covered graph. For every vertex $v \in V(G)$, let I_v be any maximum independent set in H_v. It is easy to see that $I = \bigcup_{v \in V(G)} I_v$ is a maximal (and thus, maximum) independent set in $G \circ \mathcal{H}$. We claim that H_v is a complete graph for every $v \in V(G)$. Suppose that H_{v_0} is not a complete graph for some $v_0 \in V(G)$. Then $|I_{v_0}| > 1$ and by removing I_{v_0} from I and replacing it by $\{v_0\}$, we form the set I' which is also a maximal independent set in $G \circ \mathcal{H}$ but which is smaller than I, a contradiction. This implies that the above condition is necessary for the corona $G \circ \mathcal{H}$ to be well covered.

We now assume that each graph of the family $\mathcal{H}$ is complete. Let I be a maximal independent set in $G \circ \mathcal{H}$. It follows from the definition of $G \circ \mathcal{H}$ and the choice of I that either $v \in I$ or $|I \cap V(H_v)| = 1$ for every $v \in V(G)$; for if there were a vertex v_0 in G such that $v_0 \notin I$ and $I \cap V(H_{v_0}) = \emptyset$, then, for any $x \in V(H_{v_0})$, the set $I \cup \{x\}$ would be a larger independent set in $G \circ \mathcal{H}$ which is impossible. This implies that each maximal independent set in $G \circ \mathcal{H}$ has exactly $|V(G)|$ elements. Hence $G \circ \mathcal{H}$ is well covered. ∎

COROLLARY 3.2.1. *For any graph G and a positive integer n, the corona $G \circ K_n$ is a well covered graph.* ∎

The above theorem and its proof immediately yield the next corollary.

COROLLARY 3.2.2. *For a graph G and a family $\mathcal{H}$ of nonempty graphs indexed by the vertices of G, the corona $G \circ \mathcal{H}$ is very well covered if and only if $G \circ \mathcal{H} = G \circ K_1$.* ∎

The lexicographic product. For a graph G and a family $\mathcal{H} = \{H_v : v \in V(G)\}$ of nonempty graphs indexed by the vertices of G, the *lexicographic product* $G[\mathcal{H}]$ of G and $\mathcal{H}$ is the graph having vertex set $V(G[\mathcal{H}]) = \bigcup_{v \in V(G)}\{(v,u) : u \in V(H_v)\} = \bigcup_{v \in V(G)}\{v\} \times V(H_v)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G[\mathcal{H}]$ are adjacent whenever either $[v_1u_1 \in E(G)]$ or $[v_1 = u_1$ and $v_2u_2 \in E(H_{v_1})]$. If all the graphs of the family $\mathcal{H}$ are isomorphic to one and the same graph H then we shall write $G[H]$ instead of $G[\mathcal{H}]$. For a subset S of $V(G[\mathcal{H}])$, we denote $\pi_G(S) = \{x \in V(G) : \exists_{y \in V(H_x)}(x,y) \in S\}$ and $\pi_{H_x}(S) = \{y \in V(H_x) : (x,y) \in S\}$ for every $x \in \pi_G(S)$.

The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is defined as the disjoint union of G_1 and G_2 with additional edges linking each vertex of G_1 with each vertex of G_2. It is obvious that the join $G_1 + G_2$ is isomorphic to the lexicographic product $K_2[\{G_1, G_2\}]$.

In this subsection we establish some necessary and sufficient conditions for the (very) well coveredness of the lexicographic product of graphs. Our first proposition describes the maximal independent sets in the lexicographic product of graphs.

PROPOSITION 3.2.1. *Let G be a graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ a family of nonempty graphs indexed by the vertices of G. A subset S of $V(G[\mathcal{H}])$ is a maximal independent set in $G[\mathcal{H}]$ if and only if $\pi_G(S)$ is a maximal independent set in G, and for every $v \in \pi_G(S)$, the set $\pi_{H_v}(S)$ is a maximal independent set in the graph H_v.*

Proof. Assume that the set $S \subseteq V(G[\mathcal{H}])$ is a maximal independent set in $G[\mathcal{H}]$. It is obvious from the definition of the lexicographic product that the set $\pi_G(S)$ is independent in G, and for every $v \in \pi_G(S)$, the set $\pi_{H_v}(S)$ is independent in H_v. We claim that $\pi_G(S)$ is a maximal independent set in G and $\pi_{H_v}(S)$ is a maximal independent set in H_v for $v \in \pi_G(S)$. First suppose that $\pi_G(S)$ is not a maximal independent set in G. Then there is $v_0 \in V(G) - \pi_G(S)$ such that the set $\pi_G(S) \cup \{v_0\}$ is independent in G. Hence, for every $x \in V(H_{v_0})$, the set $S \cup \{(v_0, x)\}$ would be a greater independent set in $G[\mathcal{H}]$, a contradiction. Similarly, the set $\pi_{H_v}(S)$ (for $v \in \pi_G(S)$) is a maximal independent set in H_v, as otherwise there is $x \in V(H_v) - \pi_{H_v}(S)$ such that $\pi_{H_v}(S) \cup \{x\}$ is independent in H_v and then $S \cup \{(v,x)\}$ would be a greater independent set in $G[\mathcal{H}]$, which is impossible.

On the other hand, if $\pi_G(S)$ is a maximal independent set in G and $\pi_{H_v}(S)$ is a maximal independent set in H_v, $v \in \pi_G(S)$, then S is a maximal independent set in $G[\mathcal{H}]$; for if not, then there is a vertex $(v_0, x_0) \in V(G[\mathcal{H}]) - S$ such that

$S \cup \{(v_0, x_0)\}$ is independent in $G[\mathcal{H}]$ and then $\pi_G(S \cup \{(v_0, x_0)\}) = \pi_G(S) \cup \{v_0\}$ or $\pi_{H_{v_0}}(S \cup \{(v_0, x_0)\}) = \pi_{H_{v_0}}(S) \cup \{x_0\}$ is a greater independent set in G or in H_{v_0}, respectively, which is impossible. This completes the proof. ∎

We are now ready to show conditions for the lexicographic product of graphs to be well covered. (Pinter, in his Ph.D. thesis [113], was able to employ the following theorem to obtain infinite families of W_2 graphs and well covered graphs G for which $G - e$ is also well covered for each edge $e \in E(G)$.)

THEOREM 3.2.2. *Let G be a graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ a family of nonempty graphs indexed by the vertices of G. Then the lexicographic product $G[\mathcal{H}]$ is a well covered graph if and only if G and $\mathcal{H}$ satisfy the following two conditions*:

(1) *each graph H_v of the family $\mathcal{H}$ is well covered*,

(2) $\sum_{v \in S_G} \alpha(H_v) = \sum_{u \in S'_G} \alpha(H_u)$ *for any two maximal independent sets S_G and S'_G of G.*

Proof. We begin by assuming that $G[\mathcal{H}]$ is a well covered graph. First we claim that every graph H_v from $\mathcal{H}$ is well covered. For if not, let H_{v_0} be a counterexample. Then H_{v_0} has two maximal independent sets of different cardinality, say I_{v_0} and I'_{v_0}. Let $S_G \subseteq V(G) - \{v_0\}$ be such that $S_G \cup \{v_0\}$ is a maximal independent set in G. For every $v \in S_G$, let I_v be any maximal independent set in H_v. Since $|I_{v_0}| \neq |I'_{v_0}|$, Proposition 3.2.1 implies that $\bigcup_{v \in S_G} \{(v, x) : x \in I_v\} \cup \{(v_0, y) : y \in I_{v_0}\}$ and $\bigcup_{v \in S_G} \{(v, x) : x \in I_v\} \cup \{(v_0, t) : t \in I'_{v_0}\}$ are maximal independent sets of different cardinality in $G[\mathcal{H}]$, which contradicts our assumption. Hence, each graph of the family $\mathcal{H}$ is well covered if the graph $G[\mathcal{H}]$ is well covered.

Let S_G and S'_G be two maximal independent sets in G. We now claim that $\sum_{v \in S_G} \alpha(H_v) = \sum_{v \in S'_G} \alpha(H_v)$. To prove this, let J_v be any maximum independent set in H_v for $v \in S_G \cup S'_G$. Proposition 3.2.1 and the assumption on $G[\mathcal{H}]$ imply that $S = \bigcup_{v \in S_G} \{(v, x) : x \in J_v\}$ and $S' = \bigcup_{v \in S'_G} \{(v, x) : x \in J_v\}$ are maximum independent sets in $G[\mathcal{H}]$. Hence $|S| = |S'|$ and then from the observation $|\{(v, x) : x \in J_v\}| = |J_v| = \alpha(H_v)$ (for $v \in S_G \cup S'_G$) we have $\sum_{v \in S_G} \alpha(H_v) = |S| = |S'| = \sum_{v \in S'_G} \alpha(H_v)$, and our assertion follows.

For the converse, assume G and $\mathcal{H}$ satisfy the conditions (1) and (2). We shall prove that $G[\mathcal{H}]$ is a well covered graph. For this purpose, assume that S is a maximal independent set in $G[\mathcal{H}]$. Then, by Proposition 3.2.1, $\pi_G(S)$ is a maximal independent set in G and $\pi_{H_v}(S)$ is a maximal independent set in H_v for every $v \in \pi_G(S)$. Since $S = \bigcup_{v \in \pi_G(S)} \{(v, x) : x \in \pi_{H_v}(S)\}$ and $|\pi_{H_v}(S)| = \alpha(H_v)$ (by (1)), $|S| = \sum_{v \in \pi_G(S)} |\{(v, x) : x \in \pi_{H_v}(S)\}| = \sum_{v \in \pi_G(S)} |\pi_{H_v}(S)| = \sum_{v \in \pi_G(S)} \alpha(H_v)$. Consequently, by (2), any two maximal independent sets in $G[\mathcal{H}]$ have the same cardinality and therefore $G[\mathcal{H}]$ is a well covered graph. ∎

COROLLARY 3.2.3. *The lexicographic product $G[H]$ of two nonempty graphs G and H is a well covered graph if and only if G and H are well covered graphs*; *if graphs G and H are nonempty and one of them is without isolated vertices*, *then* *the lexicographic product $G[H]$ is very well covered if and only if exactly one of G and H is very well covered and the second is totally disconnected*, *i.e.*, *without edges.*

Proof. The first part of the assertion easily follows from Theorem 3.2.2. Thus we shall only prove the second part. Let $a = |V(G)|$ and $b = |V(H)|$.

We first assume that $G[H]$ is very well covered. Then G and H are well covered (by the first part of the corollary), and $\alpha(G[H]) = |V(G[H])|/2 = ab/2$. Moreover, it follows from Proposition 3.2.1 that $\alpha(G[H]) = \alpha(G)\alpha(H)$. Since G or H is without isolated vertices, Corollary 3.1.1 implies that $\alpha(G) \leq a/2$ or $\alpha(H) \leq b/2$. Therefore $ab/2 = \alpha(G)\alpha(H) \leq (a/2)\alpha(H)$ or $ab/2 = \alpha(G)\alpha(H) \leq \alpha(G)b/2$. This makes it obvious that $\alpha(H) = b$ and $\alpha(G) = a/2$ or $\alpha(G) = a$ and $\alpha(H) = b/2$. From this it may be concluded that H is totally disconnected and G is very well covered or vice versa, as claimed.

Finally, if G is very well covered and H is totally disconnected (or G is totally disconnected and H is very well covered), then $\alpha(G) = a/2$, $\alpha(H) = b$ (or $\alpha(G) = a$, $\alpha(H) = b/2$) and $G[H]$ is well covered. Moreover, since $\alpha(G[H]) = \alpha(G)\alpha(H) = ab/2 = |V(G[H])|/2$, $G[H]$ is very well covered. ∎

COROLLARY 3.2.4. *The join $G+H$ of two nonempty graphs G and H is a well covered graph if and only if G and H are well covered graphs and $\alpha(G) = \alpha(H)$; $G + H$ is very well covered if and only if both G and H are totally disconnected and have the same number of vertices.*

Proof. The first part of the assertion immediately follows from Theorem 3.2.2, since $G + H$ is isomorphic to $K_2[\{G, H\}]$.

In order to prove the second part, assume first that G and H are totally disconnected and each of them has n vertices. Then $G + H$ is isomorphic to the bipartite complete graph $K_{n,n}$. Since $K_{n,n}$ is very well covered, $G + H$ is very well covered.

Now assume that $G + H$ is very well covered. Then at once $\alpha(G + H) = |V(G + H)|/2 = |V(G)|/2 + |V(H)|/2$ and $\alpha(G + H) = \alpha(G) = \alpha(H)$. Since $\alpha(G) \leq |V(G)|$ and $\alpha(H) \leq |V(H)|$, so we have $\alpha(G) = |V(G)| = |V(H)| = \alpha(H)$, and thus G and H are totally disconnected graphs of the same order. ∎

The disjunction of graphs. In this subsection the (very) well coveredness of a disjunction graph is established based upon the (very) well coveredness of the factors. The *disjunction* $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph having vertex set $V(G_1 \vee G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G_1 \vee G_2$ are adjacent whenever $v_1u_1 \in E(G_1)$ or $v_2u_2 \in E(G_2)$. For a subset S of $V(G_1 \vee G_2)$, we denote by $\pi_{G_1}(S)$ and $\pi_{G_2}(S)$ the projections of S onto $V(G_1)$ and $V(G_2)$ respectively, so $\pi_{G_1}(S) = \{x \in V(G_1) : \exists_{y \in V(G_2)}(x, y) \in S\}$ and $\pi_{G_2}(S) = \{y \in V(G_2) : \exists_{x \in V(G_1)}(x, y) \in S\}$.

The next four properties of independent sets in a disjunction graph will help provide a well coveredness criterion for the disjunction of two graphs.

PROPOSITION 3.2.2. *If $I_i \subseteq V(G_i)$ is an independent set in a graph G_i ($i = 1, 2$), then $I_1 \times I_2$ is an independent set in $G_1 \vee G_2$.*

Proof. Since the set I_i is independent in G_i, $N_{G_i}(v_i) \subseteq V(G_i) - I_i$ for each vertex $v_i \in I_i$ ($i = 1, 2$). Hence $N_{G_1 \vee G_2}((v_1, v_2)) = (N_{G_1}(v_1) \times V(G_2)) \cup (V(G_1) \times N_{G_2}(v_2)) \subseteq ((V(G_1) - I_1) \times V(G_2)) \cup (V(G_1) \times (V(G_2) - I_2)) = V(G_1 \vee G_2) - (I_1 \times I_2)$ for each $(v_1, v_2) \in I_1 \times I_2$, and therefore the set $I_1 \times I_2$ is independent in $G_1 \vee G_2$. ∎

PROPOSITION 3.2.3. *If a set $I \subseteq V(G_1 \vee G_2)$ is independent in $G_1 \vee G_2$, then the set $\pi_{G_i}(I)$ is independent in G_i ($i = 1, 2$).*

Proof. Let v_1, v_2 be any two vertices from $\pi_{G_1}(I)$. We claim that they are nonadjacent; for if not, then vertices $(v_1, v_1'), (v_2, v_2') \in I$ (for some $v_1', v_2' \in V(G_2)$) would be adjacent in $G_1 \vee G_2$, which is impossible. This implies that the set $\pi_{G_1}(I)$ is independent in G_1. We conclude similarly that $\pi_{G_2}(I)$ is an independent set in G_2. ∎

PROPOSITION 3.2.4. *If $I_i \subseteq V(G_i)$ is a maximal independent set in G_i ($i = 1, 2$), then $I_1 \times I_2$ is a maximal independent set in $G_1 \vee G_2$.*

Proof. By Proposition 3.2.2, the set $I_1 \times I_2$ is independent in $G_1 \vee G_2$. We claim that $I_1 \times I_2$ is a maximal independent set in $G_1 \vee G_2$. Suppose to the contrary that $I_1 \times I_2$ is a proper subset of some independent set I in $G_1 \vee G_2$. Then the set $\pi_{G_i}(I)$ is independent in G_i (by Proposition 3.2.3) and $I_i \subseteq \pi_{G_i}(I)$ for $i = 1, 2$. Since $|I_i| \le |\pi_{G_i}(I)|$ ($i = 1, 2$) and $|I_1 \times I_2| < |I| \le |\pi_{G_1}(I) \times \pi_{G_2}(I)|$, $|I_1| < |\pi_{G_1}(I)|$ or $|I_2| < |\pi_{G_2}(I)|$ and therefore at least one of the sets I_1 and I_2 is not a maximal independent set in G_1 and G_2, respectively, a contradiction. ∎

PROPOSITION 3.2.5. *If $I \subseteq V(G_1 \vee G_2)$ is a maximal independent set in $G_1 \vee G_2$, then $I = \pi_{G_1}(I) \times \pi_{G_2}(I)$ and $\pi_{G_i}(I)$ is a maximal independent set in G_i ($i = 1, 2$).*

Proof. Assume that I is a maximal independent set in $G_1 \vee G_2$. By Proposition 3.2.3, $\pi_{G_i}(I)$ is an independent set in G_i ($i = 1, 2$). Let I_i be an independent set in G_i such that $\pi_{G_i}(I) \subseteq I_i$ ($i = 1, 2$). Then $\pi_{G_1}(I) \times \pi_{G_2}(I)$ and $I_1 \times I_2$ are independent sets in $G_1 \vee G_2$ by Proposition 3.2.2. Since $I \subseteq \pi_{G_1}(I) \times \pi_{G_2}(I) \subseteq I_1 \times I_2$, from the maximality of I we have $I = \pi_{G_1}(I) \times \pi_{G_2}(I) = I_1 \times I_2$. In addition, $\pi_{G_1}(I) = I_1$ and $\pi_{G_2}(I) = I_2$. Consequently, $\pi_{G_1}(I)$ and $\pi_{G_2}(I)$ are maximal independent sets in G_1 and G_2, respectively. ∎

With the above, the main result of this subsection falls out quite quickly.

THEOREM 3.2.3. *The disjunction $G_1 \vee G_2$ of graphs G_1 and G_2 is a well covered graph if and only if the graphs G_1 and G_2 are well covered.*

Proof. Assume G_1 and G_2 are well covered graphs. In order to prove the sufficiency, it is enough to show that every maximal independent set in $G_1 \vee G_2$ has $\alpha(G_1)\alpha(G_2)$ elements. Let $I \subseteq V(G_1 \vee G_2)$ be any maximal independent set in $G_1 \vee G_2$. Then by Proposition 3.2.5, $I = \pi_{G_1}(I) \times \pi_{G_2}(I)$, and $\pi_{G_1}(I)$ and

$\pi_{G_2}(I)$ are maximal independent sets in G_1 and G_2, respectively. Consequently, by hypothesis, $|\pi_{G_1}(I)| = \alpha(G_1)$, $|\pi_{G_2}(I)| = \alpha(G_2)$ and therefore $|I| = \alpha(G_1)\alpha(G_2)$.

On the other hand assume that $G_1 \vee G_2$ is well covered and suppose on the contrary that G_1 or G_2 is not well covered. Without loss of generality, we may assume that G_1 is not well covered. Then G_1 has two maximal independent sets of different cardinality, say I_1 and I_1'. Let I_2 be a maximal independent set in G_2. Then by Proposition 3.2.4, $I_1 \times I_2$ and $I_1' \times I_2$ are maximal independent sets of different cardinality in $G_1 \vee G_2$, a contradiction. This proves the necessity and completes the proof of the theorem. ∎

Corollary 3.2.5. *If graphs G_1 and G_2 are nonempty and one of them is without isolated vertices, then the disjunction $G_1 \vee G_2$ is very well covered if and only if exactly one of G_1 and G_2 is very well covered and the second is totally disconnected.*

The proof of Corollary 3.2.5 is similar to the proof of the second part of Corollary 3.2.3, so it will be omitted.

The conjunction of graphs. The *conjunction* $G_1 \wedge G_2$ of graphs G_1 and G_2 is the graph having vertex set $V(G_1 \wedge G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G_1 \wedge G_2$ are adjacent if $v_1u_1 \in E(G_1)$ and $v_2u_2 \in E(G_2)$.

In this subsection we study conditions for the well coveredness of conjunction graphs. We begin with a simple observation.

Proposition 3.2.6. *Let G_1 and G_2 be graphs without isolated vertices. If I_1 and I_2 are maximal independent sets in G_1 and G_2 respectively, then $I_1 \times V(G_2)$ and $V(G_1) \times I_2$ are maximal independent sets in $G_1 \wedge G_2$.*

Proof. Assume that I_1 is a maximal independent set in G_1, and G_2 has no isolated vertex. Then $N_{G_1}(v) \cap I_1 = \emptyset$ ($\neq \emptyset$, resp.) if $v \in I_1$ ($v \in V(G_1) - I_1$, resp.), and $N_{G_2}(u) \neq \emptyset$ for $u \in V(G_2)$. Thus $N_{G_1 \wedge G_2}((v,u)) \cap (I_1 \times V(G_2)) = (N_{G_1}(v) \cap I_1) \times N_{G_2}(u) = \emptyset$ ($\neq \emptyset$, resp.) if $(v,u) \in I_1 \times V(G_2)$ ($(v,u) \notin I_1 \times V(G_2)$, resp.). Hence $I_1 \times V(G_2)$ is a maximal independent set in $G_1 \wedge G_2$. Likewise, $V(G_2) \times I_2$ is a maximal independent set in $G_1 \wedge G_2$. ∎

The next theorem gives some necessary conditions for the conjunction of two graphs to be well covered.

Theorem 3.2.4. *If G_1 and G_2 are graphs without isolated vertices and $G_1 \wedge G_2$ is a well covered graph, then*

(1) *G_1 and G_2 are well covered and*
(2) $\alpha(G_1)|V(G_2)| = \alpha(G_2)|V(G_1)|$.

Proof. Let I_i be any maximal independent set in G_i ($i = 1, 2$). By Proposition 3.2.6, $I_1 \times V(G_2)$ and $V(G_1) \times I_2$ are maximal independent sets in $G_1 \wedge G_2$. Since $G_1 \wedge G_2$ is well covered, the sets $I_1 \times V(G_2)$ and $V(G_1) \times I_2$ have the same cardinality and therefore $|I_1||V(G_2)| = |I_2||V(G_1)|$. This implies that $|I_i| = \alpha(G_i)$ ($i = 1, 2$) and then the result follows. ∎

The implication in Theorem 3.2.4 cannot be reversed. This can be seen with the aid of the cycle C_5 of length 5. The graphs $G_1 = G_2 = C_5$ have the properties (1) and (2) of Theorem 3.2.4, and it is easy to check that $C_5 \wedge C_5$ is not a well covered graph. However, for very well covered graphs the converse of Theorem 3.2.4 is true. The following proposition is useful to prove that fact.

PROPOSITION 3.2.7. *Let $v_1, \ldots, v_{2n}$ and $u_1, \ldots, u_{2m}$ be the vertices of graphs G_1 and G_2, respectively. If the edges $v_{2i-1}v_{2i}$ $(i = 1, \ldots, n)$ and $u_{2j-1}u_{2j}$ $(j = 1, \ldots, m)$ form a perfect matching in G_1 and G_2 respectively, then the edges $(v_{2i-1}, u_{2j-1})(v_{2i}, u_{2j})$ and $(v_{2i}, u_{2j-1})(v_{2i-1}, u_{2j})$ $(i = 1, \ldots, n; j = 1, \ldots, m)$ form a perfect matching of the graph $G_1 \wedge G_2$.*

Proof. The proof is immediate. ■

The following theorem and its corollaries will establish where the class of very well covered conjunction graphs belongs to the world of the well covered graphs.

THEOREM 3.2.5. *Let G_1 and G_2 be graphs without isolated vertices. Then the graph $G_1 \wedge G_2$ is very well covered if and only if G_1 and G_2 are very well covered.*

Proof. Let $G_1 \wedge G_2$ be a very well covered graph. By Theorem 3.2.4, G_1 and G_2 are well covered. Clearly, G_1 and G_2 are very well covered; for if not, there exists a maximal independent set I_1 in G_1 (or I_2 in G_2) such that $|I_1| \neq |V(G_1)|/2$ (or $|I_2| \neq |V(G_2)|/2$) and then $|I_1 \times V(G_2)| = |I_1||V(G_2)| \neq |V(G_1)||V(G_2)|/2 = |V(G_1 \wedge G_2)|/2$ (or $|V(G_1) \times I_2| \neq |V(G_1 \wedge G_2)|/2$), which is impossible since $I_1 \times V(G_2)$ (or $V(G_1) \times I_2$) is a maximal independent set in $G_1 \wedge G_2$. Hence, G_1 and G_2 are very well covered if $G_1 \wedge G_2$ is very well covered.

Conversely, assume that the graphs G_1 and G_2 are very well covered. For $i = 1, 2$, let M_i be a perfect matching of G_i that has the properties (1) and (2) of Theorem 3.1.1 in G_i. Assume that $M_1 = \{v_{2i-1}v_{2i} : i = 1, \ldots, n\}$ and $M_2 = \{u_{2j-1}u_{2j} : j = 1, \ldots, m\}$. By Proposition 3.2.7,

$$M = \{(v_{2i-1}, u_{2j-1})(v_{2i}, u_{2j}),\ (v_{2i}, u_{2j-1})(v_{2i-1}, u_{2j}) : \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, m\}$$

is a perfect matching of $G_1 \wedge G_2$ and in order to prove that $G_1 \wedge G_2$ is very well covered it is enough to show that M satisfies the conditions of Theorem 3.1.1 in $G_1 \wedge G_2$.

First we claim that no edge of M belongs to a triangle in $G_1 \wedge G_2$. Let (v, u) be any vertex of $G_1 \wedge G_2$. It follows from the property (1) of M_1 and M_2 that $\{v_{2i-1}, v_{2i}\} \not\subseteq N_{G_1}(v)$ $(i = 1, \ldots, n)$ and $\{u_{2j-1}, u_{2j}\} \not\subseteq N_{G_2}(u)$ $(j = 1, \ldots, m)$. Hence, neither $\{(v_{2i-1}, u_{2j-1}),\ (v_{2i}, u_{2j})\}$ nor $\{(v_{2i-1}, u_{2j}), (v_{2i}, u_{2j-1})\}$ is a subset of $N_{G_1 \wedge G_2}((v, u))$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ and therefore no edge of M belongs to a triangle in $G_1 \wedge G_2$.

Finally, we claim that the matching M has the property (2) (of Theorem 3.1.1) in $G_1 \wedge G_2$. Since M_1 and M_2 have the property (2) in G_1 and G_2 respectively, every vertex $v \in N_{G_1}(v_{2i-1})$ is adjacent to every vertex $v' \in N_{G_1}(v_{2i})$ $(i = 1, \ldots, n)$ in

G_1, and every vertex $u \in N_{G_2}(u_{2j-1})$ is adjacent to every vertex $u' \in N_{G_2}(u_{2j})$ $(j = 1, \ldots, m)$ in G_2. This combined with the definition of the conjunction of graphs implies that every vertex $(v, u) \in N_{G_1 \wedge G_2}((v_{2i-1}, u_{2j-1}))$ is adjacent to every vertex $(v', u') \in N_{G_1 \wedge G_2}((v_{2i}, u_{2j}))$, and every $(v, u') \in N_{G_1 \wedge G_2}((v_{2i-1}, u_{2j}))$ is adjacent to every $(v', u) \in N_{G_1 \wedge G_2}((v_{2i}, u_{2j-1}))$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$. This implies the desired claim and finishes the proof. ∎

Corollary 3.2.6. *Let G_1 and G_2 be graphs without isolated vertices. If at least one of G_1 and G_2 is very well covered, then the following statements are equivalent*:

(1) $G_1 \wedge G_2$ *is well covered*,
(2) $G_1 \wedge G_2$ *is very well covered*,
(3) *both* G_1 *and* G_2 *are very well covered.*

Proof. We have already proved that (2) and (3) are equivalent, and since (2) trivially implies (1), it suffices to prove that (1) implies (3). Let us assume that $G_1 \wedge G_2$ is well covered and G_2 is very well covered. By Theorem 3.2.4, G_1 is well covered and $\alpha(G_1)|V(G_2)| = \alpha(G_2)|V(G_1)| = |V(G_1)||V(G_2)|/2$. Thus $\alpha(G_1) = |V(G_1)|/2$ and hence G_1 is very well covered. ∎

There is an analogous result for bipartite graphs.

Corollary 3.2.7. *Let G_1 and G_2 be graphs without isolated vertices. If at least one of G_1 and G_2 is bipartite, then the following statements are equivalent*:

(1) $G_1 \wedge G_2$ *is well covered*,
(2) $G_1 \wedge G_2$ *is very well covered*,
(3) G_1 *and* G_2 *are very well covered.*

Proof. By Theorem 3.2.5, (2) and (3) are equivalent. Our assumption on G_1 and G_2 imply that $G_1 \wedge G_2$ is a bipartite graph without isolated vertices, so (1) and (2) are equivalent by Proposition 3.1.4. ∎

The above results give rise to some interesting observations. For example, if both G_1 and G_2 are graphs without isolated vertices, then: (a) $G_1 \wedge G_2$ is very well covered if and only if both G_1 and G_2 are very well covered; (b) $G_1 \wedge G_2$ is not well covered if exactly one of G_1 and G_2 is very well covered; (c) G_1 and G_2 are well covered but not very well covered if $G_1 \wedge G_2$ is well covered but not very well covered.

As we have already admitted, it is possible that $G_1 \wedge G_2$ is not well covered whereas G_1 and G_2 are well covered. It appears difficult to find general theorems for the cases where each of the graphs G_1, G_2 and $G_1 \wedge G_2$ is well but not very well covered.

We conclude this section with well covered conjunctions of complete graphs and cycles.

PROPOSITION 3.2.8. *The conjunction* $K_n \wedge K_m$ *of complete graphs* K_n *and* K_m $(n, m \geq 2)$ *is a well covered graph if and only if* $n = m$; $K_n \wedge K_m$ *is a very well covered graph if and only if* $n = m = 2$.

P r o o f. The necessity of the first part follows immediately by applying Theorem 3.2.4 to $K_n \wedge K_m$. On the other hand, assume that I is a maximal independent set in $K_n \wedge K_n$ and $(v, u) \in I$. Since $N_{K_n \wedge K_m}((v, u)) \cap I = \emptyset$ and $N_{K_n \wedge K_m}((v, u)) = (V(K_n) - \{v\}) \times (V(K_n) - \{u\})$, the maximality of I implies that either $I = \{v\} \times V(K_n)$ or $I = V(K_n) \times \{u\}$. Therefore every maximal independent set in $K_n \wedge K_n$ has exactly n elements, so $K_n \wedge K_n$ is well covered. Since K_2 is the only complete very well covered graph, Theorem 3.2.5 implies that $K_n \wedge K_m$ is very well covered if and only if $n = m = 2$. ∎

PROPOSITION 3.2.9. *The conjunction* $C_n \wedge C_m$ *of cycles* C_n *and* C_m *is a well covered graph if and only if* $n = m = 3$ *or* 4; $C_n \wedge C_m$ *is a very well covered graph if and only if* $n = m = 4$.

P r o o f. It is clear that if n and k are integers such that $n \geq 3$ and $\lceil n/3 \rceil \leq k \leq \lfloor n/2 \rfloor$, then in the cycle C_n there exists a maximal independent set of cardinality k. This implies that the cycle C_n is well covered if and only if $\lceil n/3 \rceil = \lfloor n/2 \rfloor$, that is, if and only if $n = 3, 4, 5$ or 7.

Certainly, C_4 is the only very well covered cycle. Therefore, by Theorem 3.2.5, $C_n \wedge C_m$ is very well covered if and only if $n = m = 4$. This proves the second part of the theorem. The well coveredness of $C_3 \wedge C_3$ follows from Proposition 3.2.8, since $C_3 = K_3$.

On the other hand, assume that the conjunction $C_n \wedge C_m$ is well covered. Then C_n and C_m are well covered by Theorem 3.2.4; hence, $n, m \in \{3, 4, 5, 7\}$. Again, by Theorem 3.2.4, none of the six graphs $C_3 \wedge C_4$, $C_3 \wedge C_5$, $C_3 \wedge C_7$, $C_4 \wedge C_5$, $C_4 \wedge C_7$, $C_5 \wedge C_7$ is well covered. One can verify that neither $C_5 \wedge C_5$ nor $C_7 \wedge C_7$ is well covered. Thus, $C_3 \wedge C_3$ and $C_4 \wedge C_4$ are the only well covered conjunctions of cycles. ∎

COROLLARY 3.2.8. *The conjunction* $C_n \wedge K_m$ *of a cycle* C_n $(n \geq 3)$ *and a complete graph* K_m $(m \geq 2)$ *is a well covered graph if and only if* $n = m = 3$ *or* $n = 4$ *and* $m = 2$; $C_n \wedge K_m$ *is very well covered if and only if* $n = 4$ *and* $m = 2$.

P r o o f. This follows at once from the above results. ∎

The cartesian product of graphs. The *cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph having vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G_1 \times G_2$ are adjacent if $[v_1u_1 \in E(G_1)$ and $v_2 = u_2]$ or $[v_1 = u_1$ and $v_2u_2 \in E(G_2)]$.

We are not able to give a complete description of the relationship between the well coveredness of graphs formed by the cartesian product and their factors. However, we consider some special cases which seem interesting. Since the cartesian product $nK_1 \times G$ is isomorphic to nG, we may only consider the cartesian product of graphs which are not totally disconnected. We begin by proving that for such graphs, the cycle C_4 ($= K_2 \times K_2$) is the only connected, bipartite, (very) well covered cartesian product of graphs.

THEOREM 3.2.6. *If G_1, G_2 are connected bipartite graphs and each of them is different from K_1, then $G_1 \times G_2$ is well covered if and only if $G_1 = G_2 = K_2$.*

Proof. If $G_1 = G_2 = K_2$, then $G_1 \times G_2 = C_4$ is well covered. Conversely, assume that $G_1 \times G_2$ is a well covered graph. Since G_1, G_2 are bipartite, $G_1 \times G_2$ is bipartite. Thus, according to Corollary 3.1.3, $G_1 \times G_2$ has a perfect matching M such that for every edge $(x,y)(x',y') \in M$, the subgraph induced by $N_{G_1 \times G_2}((x,y)) \cup N_{G_1 \times G_2}((x',y'))$ is a complete bipartite graph. We claim that $G_1 = G_2 = K_2$. For if not, without loss of generality, let G_1 be a counterexample and let v be a vertex of degree at least two in G_1. Then for any $v' \in N_{G_1}(v)$, $v'' \in N_{G_1}(v) - \{v'\}$, $u \in V(G_2)$ and $u' \in N_{G_2}(u)$, the vertices (v'',u) and (v',u') are not adjacent in $G_1 \times G_2$ but each of them is adjacent to exactly one of the vertices incident with the edge $(v,u)(v',u)$ (and $(v,u)(v,u')$). Therefore neither the subgraph induced by $N_{G_1 \times G_2}((v,u)) \cup N_{G_1 \times G_2}((v',u))$ (for any $v' \in N_{G_1}(v)$) nor the subgraph induced by $N_{G_1 \times G_2}((v,u)) \cup N_{G_1 \times G_2}((v,u'))$ (for any $u' \in N_{G_2}(u)$) is complete bipartite. This implies that no edge incident with the vertex (v,u) belongs to M, contrary to the hypothesis that M is a perfect matching in $G_1 \times G_2$. ∎

COROLLARY 3.2.9. *If G_1, G_2 are connected very well covered graphs, then $G_1 \times G_2$ is very well covered if and only if $G_1 = G_2 = K_2$.*

Proof. Assume that G_1, G_2, and $G_1 \times G_2$ are very well covered graphs. Let I_i be a maximum independent set in G_i ($i = 1,2$). It is then clear that the set $I_1 \times I_2$ is independent in $G_1 \times G_2$. Let I be a maximum independent superset of $I_1 \times I_2$ in $G_1 \times G_2$. Obviously, $|I| = |V(G_1 \times G_2)|/2 = |(I_1 \times I_2) \cup ((V(G_1) - I_1) \times (V(G_2) - I_2))|$. By the maximality of I_i, every vertex $v_i \in V(G_i) - I_i$ is adjacent to some vertex of I_i in G_i ($i = 1,2$). Thus every vertex $(v_1,v_2) \in ((V(G_1) - I_1) \times I_2) \cup (I_1 \times (V(G_2) - I_2))$ is adjacent to some vertex of $I_1 \times I_2$. Hence I is a subset of $(I_1 \times I_2) \cup ((V(G_1) - I_1) \times (V(G_2) - I_2))$ and so $I = (I_1 \times I_2) \cup ((V(G_1) - I_1) \times (V(G_2) - I_2))$. By the independence of $(V(G_1) - I_1) \times (V(G_2) - I_2)$ in $G_1 \times G_2$, the sets $V(G_1) - I_1$ and $V(G_2) - I_2$ are independent in G_1 and G_2, respectively. This implies the bipartition of G_1 and G_2. The rest follows from Theorem 3.2.6 and Proposition 3.1.4. ∎

For the cartesian product of complete graphs we have

PROPOSITION 3.2.10. *For all positive integers n and m, $K_n \times K_m$ is well covered.*

Proof. Assume $n \le m$ and $V(K_n \times K_m) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$. Let I be any maximal independent set in $K_n \times K_m$. In order to prove that $K_n \times K_m$ is well covered, we shall show that $\alpha(K_n \times K_m) = n$ and $|I| = n$. It is easy to see that $|I \cap (\{x_i\} \times V(K_m))| \le 1$ and $|I \cap (V(K_n) \times \{y_j\})| \le 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Hence $|I| \le n$ and therefore $\alpha(K_n \times K_m) \le n$. On the other hand, since the set $\{(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)\}$ has n elements and is independent in $K_n \times K_m$, $\alpha(K_n \times K_m) = n$. There remains only to show that $|I| = n$. Suppose indirectly that $|I| < n$. Then the sets $V(K_n) - \pi_{K_n}(I)$ and $V(K_m) - \pi_{K_m}(I)$ are nonempty, and for every $x \in V(K_n) - \pi_{K_n}(I)$ and $y \in V(K_m) - \pi_{K_m}(I)$, the proper superset $I \cup \{(x, y)\}$ of I is independent in $K_n \times K_m$, a contradiction. ■

We now study the well coveredness of the cartesian product of two cycles. Let C_n and C_m be two cycles with $V(C_n) = \{x_1, \ldots, x_n\}$, $V(C_m) = \{y_1, \ldots, y_m\}$, $E(C_n) = \{x_i x_{i+1} : i = 1, \ldots, n-1\} \cup \{x_1 x_n\}$, and $E(C_m) = \{y_j y_{j+1} : j = 1, \ldots, m-1\} \cup \{y_1 y_m\}$. For the cartesian product $C_n \times C_m$ of the cycles C_n and C_m, we define $I_{n,m}$ to be the set of those vertices (x_i, y_j) of $C_n \times C_m$ for which $i = 1, \ldots, 2\lfloor n/2 \rfloor$, $j = 1, \ldots, 2\lfloor m/2 \rfloor$ and $i + j$ is an even integer. Put $I^*_{n,m} = I_{n,m} \cup \{(x_n, y_m)\}$ if both n and m are odd, while $I^*_{n,m} = I_{n,m}$ in other cases. It is easy to check the following properties of the set $I^*_{n,m}$ in $C_n \times C_m$.

PROPOSITION 3.2.11. *For all integers $n, m \geq 3$, the set $I^*_{n,m}$ is a maximal independent set in $C_n \times C_m$; in addition, $|I^*_{n,m}| = 2\lfloor n/2 \rfloor \lfloor m/2 \rfloor + 1$ if both n and m are odd, whereas $|I^*_{n,m}| = 2\lfloor n/2 \rfloor \lfloor m/2 \rfloor$ in other cases.* ■

PROPOSITION 3.2.12. *For every integer $m \geq 3$, the cartesian product $C_3 \times C_m$ is well covered.*

Proof. Let I be any maximal independent set in $C_3 \times C_m$. As in the proof of Proposition 3.2.10, it is enough to show that $\alpha(C_3 \times C_m) = m$ and $|I| = m$. Since $|I \cap (V(C_3) \times \{y_j\})| \leq 1$ for $j = 1, \ldots, m$, so $|I| \leq m$ and $\alpha(C_3 \times C_m) \leq m$. On the other hand, by Proposition 3.2.11, the set $I^*_{3,m}$ is independent in $C_3 \times C_m$ and $|I^*_{3,m}| = m$. Hence $\alpha(C_3 \times C_m) = m$. We now claim that $|I| = m$. For if not, then $|I| < m$ and therefore $I \cap (V(C_3) \times \{y_j\}) = \emptyset$ for some $j \in \{1, \ldots, m\}$, say $j = 2$. The maximality of I implies that $N_{C_3 \times C_m}((x_i, y_2)) \cap I \neq \emptyset$ for each $i = 1, 2, 3$. From this and from the structure of $C_3 \times C_m$ it follows that the subset $\bigcup_{i=1}^{3} N_{C_3 \times C_m}((x_i, y_2)) \cap I$ of $V(C_3) \times \{y_1, y_3\}$ has at least three vertices. Hence, $I \cap (V(C_3) \times \{y_1\})$ or $I \cap (V(C_3) \times \{y_3\})$ has at least two vertices, a contradiction. ■

PROPOSITION 3.2.13. *For all integers $n, m \geq 4$, the cartesian product $C_n \times C_m$ is not well covered.*

Proof. The result follows from Theorem 3.2.6 if both n and m are even. Thus it suffices to show that $C_n \times C_m$ is not well covered if n or m is odd. We consider two cases.

Case 1: *n and m are odd.* By Proposition 3.2.11, the set $I^*_{n,m}$ is a maximal independent set in $C_n \times C_m$. On the other hand, it is easy to check that the set

$$J_{n,m} = (I^*_{n,m} - \{(x_1, y_1), (x_1, y_3), (x_2, y_2)\}) \cup \{(x_1, y_2), (x_n, y_3)\}$$

is also a maximal independent set in $C_n \times C_m$. Since $|I^*_{n,m}| \neq |J_{n,m}|$, $C_n \times C_m$ is not well covered.

Case 2: *Exactly one of n and m is odd.* Since $C_n \times C_m$ is isomorphic to $C_m \times C_n$, we may assume that m is odd. An easy verification shows that the set

$$N_{n,m} = (I^*_{n,m} - \{(x_1, y_1), (x_1, y_3), (x_2, y_2), (x_n, y_2)\}) \cup \{(x_1, y_2), (x_1, y_m)\}$$

is a maximal independent set in $C_n \times C_m$. Since $N_{n,m}$ is smaller than $I^*_{n,m}$, $C_n \times C_m$ is not a well covered graph. This completes the proof. ∎

We summarize the above results in the following corollary.

COROLLARY 3.2.10. *The cartesian product $C_n \times C_m$ of cycles C_n and C_m is well covered if and only if $n = 3$ or $m = 3$.* ∎

Staples [130, 131] has observed that $K_2 \times G$ is a W_{n-1} graph if G is a W_n graph ($n \geq 2$). This implies that the cartesian products $K_2 \times C_3$ and $K_2 \times C_5$ are well covered. We conclude this section with the observation that the cycles C_3, C_5, and the graph $K_1 + (K_2 \cup nK_1)$ (for $n \geq 1$) are the only unicyclic graphs G for which the cartesian product $K_2 \times G$ is well covered. We begin by proving the following useful proposition.

PROPOSITION 3.2.14. *Suppose that a connected graph G contains a bridge v_1v_2 such that v_1 is not an end vertex in G and the set $N_G(v_1)$ is independent. Then the cartesian product $K_2 \times G$ is not well covered.*

Proof. Let $V(K_2) = \{a, b\}$, $U = N_G(v_1) - \{v_2\}$, and let $G_i = G'_i - v_i$, where G'_i is the connected component of $G - v_1v_2$ that contains the vertex v_i ($i = 1, 2$). Let S be a maximal independent set in $G_1 - U$, let T be a maximal independent superset of U in $G_1 - S$, and let W be a maximal independent set in $K_2 \times G - N_{K_2\times G}(\{b\} \times (V(G'_1) \cup \{v_2\}))$ $(= K_2 \times G_2 - N_{K_2\times G}((b, v_2)))$. Notice that $W \cup (\{a\} \times S) \cup (\{b\} \times T) \cup \{(a, v_1), (b, v_2)\}$ and $W \cup (\{b\} \times S) \cup (\{a\} \times T) \cup \{(b, v_2)\}$ are maximal independent sets of different cardinality in $K_2 \times G$. Thus $K_2 \times G$ is not well covered. ∎

PROPOSITION 3.2.15. *If G is a connected unicyclic graph, then the cartesian product $K_2 \times G$ is well covered if and only if $G = C_3$, $G = C_5$ or $G = K_1 + (K_2 \cup nK_1)$ for some positive integer n.*

Proof. We consider two cases.

Case 1: *G is a cycle, $G = C_n$.* Since $K_2 \times C_n$ is a cubic, planar, 3-connected graph, it follows from Theorem 3.1.3 that $K_2 \times C_n$ is well covered if and only if $n = 3$ or $n = 5$.

Case 2: *G is not a cycle.* Let C be the unique cycle of G, $V(K_2) = \{a, b\}$, and assume that $K_2 \times G$ is a well covered graph. Then it easily follows from Proposition 3.2.14 that C is a cycle of length three and each end vertex of G is adjacent to a vertex of C. Let $V(C) = \{v_1, v_2, v_3\}$ be the vertex set of C and denote $p_i = |N_G(v_i) - V(C)|$ ($i = 1, 2, 3$). We may assume that $p_1 \geq p_2 \geq p_3$. We claim that $p_2 = p_3 = 0$. For if not, then $p_2 > 0$ and the sets $I = (\{a\} \times (V(G) - V(C))) \cup \{(b, v_3)\}$ and $I' = (\{a\} \times (N_G(\{v_1, v_2\}) - \{v_1, v_2\})) \cup (\{b\} \times (N_G(v_3) - \{v_2\}))$ are maximal independent sets of different cardinality in $K_2 \times G$, a contradiction. Hence, $G = K_1 + (K_2 \cup nK_1)$ (for $n = p_1$) and it is easy to check that $K_2 \times (K_1 + (K_2 \cup nK_1))$ is well covered. ∎

To conclude this section, let us observe that there are a number of questions raised by the results presented here. For example, the problem of finding a "nice" characterization of well covered graphs G_1 and G_2 for which $G_1 \wedge G_2$ is well covered has not been solved in this section. The results of the last subsection indicate the difficulty to find a characterization of graphs G_1 and G_2 for which $G_1 \times G_2$ is well covered. Finally, is it possible to find a pair of graphs, G_1 and G_2, for which $G_1 \times G_2$ is well covered but both G_1 and G_2 are not well covered?

3.3. Well covered simplicial and chordal graphs. It is quite easy to embed, as an induced subgraph, any graph G in a well covered supergraph. Indeed, for any graph G, the corona $H = G \circ K_1$ contains G and it is a well covered graph. The last graph is a simplicial graph as well. It also follows from other results presented in this chapter that well covered graphs are very often (but not always) simplicial graphs. In this section, we describe well covered and well dominated simplicial graphs. Next we characterize well covered and well dominated chordal graphs. Again it follows from this characterization that every well covered chordal graph is a simplicial graph. Finally, we discuss the concept of well coveredness for circular arc graphs and for $C_{(n)}$-trees. We begin with the following property of simplices in well covered graphs.

PROPOSITION 3.3.1. *If G is a well covered graph, then all its simplices are pairwise vertex-disjoint.*

Proof. Assume G is a well covered graph and suppose that S_1, S_2 are two distinct simplices of G containing a common vertex v. Let I be any maximal independent set of G containing the vertex v. Select two simplicial vertices v_1 and v_2 from S_1 and S_2, respectively. Since v_1 is not adjacent to v_2 and neither v_1 nor v_2 is adjacent to any vertex of $I - \{v\}$, the set $(I - \{v\}) \cup \{v_1, v_2\}$ is independent in G and contains one vertex more than I, a contradiction to the well coveredness of G. ∎

As a converse to Proposition 3.3.1 we now prove Proposition 3.3.2 below.

PROPOSITION 3.3.2. *If a graph G has n simplices and every vertex of G belongs to exactly one simplex of G, then $\gamma(G) = i(G) = \alpha(G) = \Gamma(G) = n$.*

Proof. Let $S_1, \ldots, S_n$ be the simplices of G and assume that every vertex of G belongs to exactly one of $S_1, \ldots, S_n$. Then the sets $V(S_1), \ldots, V(S_n)$ form a partition of $V(G)$. Proposition 2.1.3 implies that in order to prove the result, it suffices to show that every minimal dominating set of G has exactly n vertices. Let D be any minimal dominating set of G. First let us observe that $D \cap V(S_i) \neq \emptyset$ for $i = 1, \ldots, n$; otherwise $D \cap V(S_{i_0}) = \emptyset$ for some $i_0 \in \{1, \ldots, n\}$ and then D would not be dominating because $N_G[v] \cap D = V(S_{i_0}) \cap D = \emptyset$ for any simplicial vertex v of G belonging to S_{i_0}. Hence $|D \cap V(S_i)| \geq 1$ for $i = 1, \ldots, n$. On the other hand, it follows from the minimality of D that $|D \cap V(S_i)| \leq 1$ for $i = 1, \ldots, n$; otherwise $|D \cap V(S_{j_0})| \geq 2$ for some $j_0 \in \{1, \ldots, n\}$ and then, for any $u \in D \cap V(S_{j_0})$, $D - \{u\}$ would be a smaller dominating set of G. Consequently, $|D \cap V(S_i)| = 1$ for $i = 1, \ldots, n$ and therefore $|D| = n$. This completes the proof. ∎

The following theorem gives a simple characterization of the well covered simplicial graphs.

THEOREM 3.3.1. *A graph G is simplicial and well covered if and only if every vertex of G belongs to exactly one simplex of G.*

Proof. If G is a simplicial graph and $S_1 \ldots, S_n$ are the simplices of G, then $V(G) = \bigcup_{i=1}^{n} V(S_i)$. In addition, if G is well covered, then by Proposition 3.3.1, the sets $V(S_1), \ldots, V(S_n)$ are disjoint and therefore every vertex of G belongs to exactly one simplex of G. The converse implication follows from Proposition 3.3.2. ∎

The main result of this section is a characterization of well covered chordal graphs. This characterization is given in Theorem 3.3.2 and it follows from a more general characterization of well covered graphs without induced cycles of length four given in Proposition 3.3.4. The following property of C_4-free graphs is required for our proof of Proposition 3.3.4. This property is a simple generalization of a property established by Farber [56, Lemma 5].

PROPOSITION 3.3.3. *Let S and T be disjoint sets of vertices of a C_4-free graph G. If the subgraphs $G[S]$ and $G[T]$ are complete, then there exists a vertex s_0 in S such that $N_G(s_0) \cap T = N_G(S) \cap T$.*

Proof. The proof is by induction on $m = |S \cap N_G(T)|$. If $m \leq 1$, then the result is trivially true. Suppose $m > 1$ and that the result is valid for all $m' < m$. Take any $s \in S \cap N_G(T)$. By the induction hypothesis, there exists $s' \in S - \{s\}$ such that $N_G(S - \{s\}) \cap T = N_G(s') \cap T$. Certainly, if $N_G(s) \cap T \subseteq N_G(s') \cap T$ or $N_G(s') \cap T \subseteq N_G(s) \cap T$, then s' or s, respectively, is the desired vertex. Thus the proof will be complete if we can show that at least one of the two sets $N_G(s) \cap T$ and $N_G(s') \cap T$ contains the other one. Suppose to the contrary that neither $N_G(s) \cap T \subseteq N_G(s') \cap T$ nor $N_G(s') \cap T \subseteq N_G(s) \cap T$. Then, for every $t \in (N_G(s) - N_G(s')) \cap T$ and every $t' \in (N_G(s') - N_G(s)) \cap T$, the vertices s, s', t, t' form an induced cycle of length four in G, a contradiction. This completes the proof of the proposition. ∎

PROPOSITION 3.3.4. *If G is a C_4-free graph, then the following statements are equivalent:*

(1) *Every vertex of G belongs to exactly one simplex of G;*
(2) $i(G) = \alpha(G) = \theta(G)$.

Proof. Let $S_1, \ldots, S_n$ be the simplices of G. If every vertex of G belongs to exactly one of them, then $i(G) = \alpha(G) = n$ (by Proposition 3.3.2) and $\theta(G) \leq n$.

From this and from the obvious inequality $\alpha(G) \leq \theta(G)$, we also have $\alpha(G) = \theta(G)$. This proves the implication (1)⇒(2).

Let $S_1, \ldots, S_n$ be a clique covering of G, where $n = \theta(G) = \alpha(G) = i(G)$. To prove the implication (2)⇒(1), it suffices to prove the following two claims.

CLAIM 1. *$S_1, \ldots, S_n$ are mutually disjoint.*

CLAIM 2. *$S_1, \ldots, S_n$ are simplices of G.*

Proof of Claim 1. Suppose to the contrary that v is a common vertex of S_i and S_j ($i \neq j$), and let I be any maximal independent set of G containing v. Then, since $|I \cap (V(S_i) \cup V(S_j))| = 1$ and $|I \cap V(S_k)| \leq 1$ for $k = 1, \ldots, n$, we have $|I| \leq n - 1 < \alpha(G)$, a contradiction.

Proof of Claim 2. Suppose to the contrary that at least one of the cliques $S_1, \ldots, S_n$ is not a simplex of G, say S_n is not a simplex of G. Then $n \geq 2$ and every vertex of S_n is adjacent to some vertex of $V(G) - V(S_n)$. Let I be any minimal subset of $V(G) - V(S_n)$ such that $V(S_n) \subseteq N_G(I)$, say $|I| = k$. We claim that the set I is independent in G. Suppose not, and let v and u be adjacent vertices of I. Applying Proposition 3.3.3 to $\{v, u\}$ and $V(S_n)$, we have that $N_G(\{v, u\}) \cap V(S_n) = N_G(v) \cap V(S_n)$ or $N_G(\{v, u\}) \cap V(S_n) = N_G(u) \cap V(S_n)$. But then $V(S_n) \subseteq N_G(I - \{u\})$ or $V(S_n) \subseteq N_G(I - \{v\})$ and this contradicts the minimality of I. Thus I is independent and this implies that $|I \cap V(S_i)| \leq 1$ for $i = 1, \ldots, n - 1$. Hence, $k = |I| \leq n - 1$ and we may assume that $|I \cap V(S_i)| = 1$ for $i = 1, \ldots, k$. Let J be any (possibly empty) maximal independent set in the subgraph $G - N_G[I]$. Since $J \subseteq V(G) - N_G[I] \subseteq \bigcup_{j=k+1}^{n-1} V(S_j)$, it is immediate that $|J| \leq n-k-1$. Moreover, since $J \cap N_G[I] = \emptyset$, $I \cup J$ is a maximal independent set of G and $|I \cup J| \leq n - 1 < \alpha(G)$, a final contradiction. ■

The following simple characterization of well covered chordal graphs due to Prisner, Topp and Vestergaard [117] follows from Proposition 3.3.4.

THEOREM 3.3.2. *Let G be a chordal graph. Then G is well covered if and only if every vertex of G belongs to exactly one simplex of G.*

Proof. Let G be a chordal graph. Since $\alpha(G) = \theta(G)$ (by Proposition 2.3.3) and since every chordal graph is C_4-free, the result follows from Proposition 3.3.4. ■

COROLLARY 3.3.1. *If G is a chordal or simplicial graph, then the following statements are equivalent*:

(1) $\gamma(G) = \Gamma(G)$, *i.e. G is well dominated*;
(2) $\gamma(G) = \alpha(G)$;
(3) $i(G) = \Gamma(G)$;
(4) $i(G) = \alpha(G)$, *i.e. G is well covered*;
(5) *Every vertex of G belongs to exactly one simplex of G.*

Proof. The implications (1)⇒(2)⇒(4) and (1)⇒(3)⇒(4) are obvious from Proposition 2.1.3. The implication (4)⇒(5) follows from Theorem 3.3.1 if G is a simplicial graph and from Theorem 3.3.2 if G is a chordal graph. Finally, the implication (5)$\Rightarrow$(1) is the content of Proposition 3.3.2. ∎

Corollary 3.3.1 generalizes previously known equivalences for trees [59, 118, 131] and block graphs [149, 150]. Because of the equivalence (2)$\Leftrightarrow$(5), the chordal graphs in which every vertex belongs to exactly one simplex form a solution to the Szamkołowicz problem posed in [135] (and to Problem 1(c) of Laskar and Walikar [100]) for chordal graphs.

For a positive integer k, *k-trees* are defined recursively as follows. A complete graph on k vertices is the smallest k-tree, and a k-tree with $n+1>k$ vertices is obtained by adding to a k-tree with n vertices a new vertex adjacent to k mutually adjacent old vertices. Certainly, every 1-tree is a tree and vice versa. It is also easy to observe that every k-tree is a chordal graph and therefore Theorem 3.3.2 implies the following characterization of well covered k-trees.

COROLLARY 3.3.2. *A k-tree G is a well covered graph if and only if every vertex of G belongs to exactly one simplex of G.* ∎

The last corollary again generalizes previously known results for trees [59, 118, 131] and 2-trees [141] and completely solves the problem posed in [141]. Since every simplicial vertex of a k-tree of order at least $k+1$ is a vertex of degree k (and vice versa), as a consequence of the last corollary we have the following property of well covered k-trees.

COROLLARY 3.3.3. *If G is a well covered k-tree of order at least $k+1$, then G is a graph of order $(k+1)n$ for some positive integer n.* ∎

From Theorem 3.3.2 we can also deduce a characterization of well covered circular arc graphs. First we mention some definitions. A graph G is a *circular arc graph* if the vertices of G can be put in a one-to-one correspondence with a set of arcs on a circle such that two distinct vertices of G are adjacent if and only if their associated arcs intersect. Circular arc graphs were introduced as a generalization of *interval graphs* (similarly defined, except that intervals on a real line are used instead of arcs on a circle) and they have been extensively studied. The reader is referred to [75, Chapter 8.6] for more details. The class of circular arc graphs is not comparable to the class of chordal graphs. (See Figure 15.) However, it is easy to observe that if G is a circular arc graph, then for each vertex v, its subgraph $G-N_G[v]$ is an interval graph and therefore a chordal graph. One can verify that the complete bipartite graph $K_{2,3}$ is not an induced subgraph of any circular arc graph. It is obvious that a graph G with $i(G)=1$ is well covered if and only if G is a complete graph. Therefore in the next theorem we consider only circular arc graphs with $i(G)>1$.

THEOREM 3.3.3. *Let G be a circular arc graph with $i(G)>1$. Then the following statements are equivalent*:

(1) *G is a well covered graph*;

(2) *For each vertex v of G, $G-N_G[v]$ is a well covered graph*;

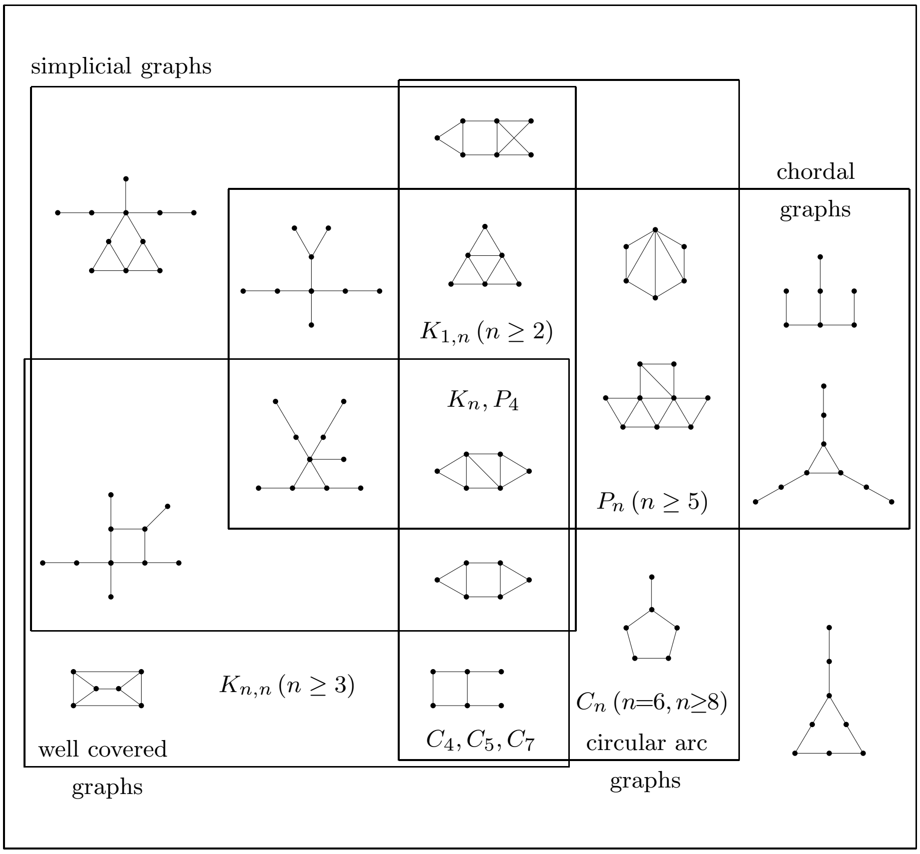

Fig. 15

(3) *For each vertex v of G, every vertex of $G - N_G[v]$ belongs to exactly one simplex of $G - N_G[v]$.*

P r o o f. The implication (1)⇒(2) follows from Proposition 3.1.5. If G is a circular arc graph, then for every vertex v of G, the subgraph $G - N_G[v]$ is a chordal graph and therefore the statements (2) and (3) are equivalent by Theorem 3.3.2. Finally, the implication (2)⇒(1) follows from Theorem 3.1.2 and the observation that every circular arc graph is a $K_{2,3}$-free graph. ∎

The classes of graphs which we have considered in this section are not comparable, i.e. no one of them is a subclass of any of the others. However, it follows from Theorem 3.3.2 that well covered chordal graphs are simplicial graphs. Figure 15 illustrates relationships between these classes of graphs.

A graph G is a $C_{(n)}$-*tree* if it can be constructed from a cycle of length n by a finite number of applications of the following operation: add a new cycle of length n and identify an edge of this cycle with an edge of the existing graph. Note that every 2-tree of order at least 3 is a $C_{(3)}$-tree and vice versa. A cycle of length n in a

$C_{(n)}$-tree is called an *elementary cycle*. Let $c(G)$ denote the number of elementary cycles in a $C_{(n)}$-tree G. An elementary cycle C of a $C_{(n)}$-tree G is called an *end cycle* of G if C has exactly two adjacent vertices of degree three or more in G. A simple induction on the number of elementary cycles shows that if n is even, then every $C_{(n)}$-tree is a bipartite graph and possesses a perfect matching. In the next proposition we show that there is no well covered $C_{(n)}$-tree G with $n \geq 4$ and $c(G) \geq 2$. Then we show that a $C_{(n)}$-tree is well covered if and only if it is a well covered 2-tree of order at least three or it is a cycle of length 4, 5, or 7.

PROPOSITION 3.3.5. *Let G be a $C_{(n)}$-tree with $n \geq 4$ and $c(G) \geq 2$. Then G is not a well covered graph.*

Proof. We distinguish two cases: $n = 4$, $n \geq 5$.

Case 1: $n = 4$. Let M be any perfect matching of G. By Corollary 3.1.3, it suffices to show that $G[N_G(\{v,u\})]$ is not a complete bipartite graph for some edge vu of M. Let C be an arbitrary end cycle in G, say $V(C) = \{a,b,c,d\}$, $E(C) = \{ab, bc, cd, da\}$, $d_G(a) \geq 3$, and $d_G(b) \geq 3$. Since M is a perfect matching in G, there exists a vertex x in $N_G(a)$ such that $ax \in M$. If $x \in V(C)$, then either $x = b$ or $x = d$. In both cases the vertex c belongs to $N_G(x)$ and it is not adjacent to all the vertices of $N_G(a)$ as $|N_G(c)| = 2$ and $|N_G(a)| \geq 3$. Therefore $G[N_G(\{a,x\})]$ is not a complete bipartite graph. If $x \notin V(C)$, then $c \notin N_G(x)$ and again $G[N_G(\{v,u\})]$ is not a complete bipartite graph because $d \in N_G(a)$ and d is not adjacent to all the neighbours of x as $N_G(d) \cap N_G(x) = \{x\}$ and $|N_G(x)| \geq 2$.

Case 2: $n \geq 5$. Since G has girth $n \geq 5$, by Theorem 3.1.4, it is sufficient to show that neither G belongs to the family $\mathcal{PC}$ nor G is one of the graphs given in Figures 12 and 13.

First, since $c(G) \geq 2$ and every end cycle in G has exactly $n-2$ vertices of degree 2 and two adjacent vertices of degree at least three, none of the five graphs in Figures 12 and 13 is a $C_{(n)}$-tree and therefore G is none of the graphs in Figures 12 and 13.

Furthermore, G does not belong to the family $\mathcal{PC}$; for if G were in $\mathcal{PC}$, then, since G does not have any end edge, every vertex of G would be in exactly one basic 5-cycle of G and therefore every end cycle of G would be a basic 5-cycle which is impossible as every end cycle of G contains exactly two adjacent vertices of degree three or more. ■

We now have the following characterization of well covered $C_{(n)}$-trees.

THEOREM 3.3.4 [141]. *Let G be a $C_{(n)}$-tree with $n \geq 3$. Then G is a well covered graph if and only if one of the following conditions is satisfied*:

(a) *G is a cycle of length* 3, 4, 5, *or* 7;

(b) *G is a $C_{(3)}$-tree in which every vertex belongs to exactly one end cycle of G.*

Proof. If G is a cycle of length n, then $i(G) = \lceil n/3 \rceil$, $\alpha(G) = \lfloor n/2 \rfloor$, and $\lceil n/3 \rceil = \lfloor n/2 \rfloor$ if and only if $n = 3$, 4, 5, or 7. Therefore G is well covered if and only if G is a cycle of length 3, 4, 5, or 7.

Assume now that G is a $C_{(n)}$-tree with $c(G) \geq 2$. If $n = 3$, then G is a 2-tree of order at least four and the result follows from Corollary 3.3.2. Finally, it follows from Proposition 3.3.5 that G is not well covered if $n \geq 4$. ∎

3.4. Well covered line and total graphs. In this section we shall consider edge and total versions of well covered graphs. A graph G is said to be *edge well covered* if every maximal independent set of edges of G is also maximum. Since there exists a one-to-one correspondence between independent sets of edges of a graph G and independent sets of vertices of the line graph $L(G)$ of G, G is edge well covered if and only if $L(G)$ is well covered. A graph G is *equimatchable* if every maximal matching of G is maximum, i.e., if all maximal matchings of G have the same cardinality. Our first theorem shows a relationship between equimatchable graphs, well covered graphs, and graphs for which the domination number equals the independence number.

THEOREM 3.4.1. *For a graph G and its line graph $L(G)$, the following statements are equivalent*:

(i) *G is equimatchable*;
(ii) *$L(G)$ is well covered*;
(iii) $\gamma(L(G)) = \alpha(L(G))$.

Proof. The equivalence of (i) and (ii) is obvious. Since every line graph is a $K_{1,3}$-free graph, it follows from Corollary 2.4.2 that $\gamma(L(G)) = i(L(G))$ for every graph G. This and Proposition 2.1.3 imply the equivalence of (ii) and (iii). ∎

The problem to determine which graphs are equimatchable (and therefore which line graphs are well covered) has been completely solved by Lewin [103] and Lesk, Plummer and Pulleyblank [102] (see also [61], [106], [132] and [148]). However, the application of their results to particular graphs is not easy. For this reason, in the next theorem we establish simple necessary and sufficient conditions for a tree to be equimatchable.

THEOREM 3.4.2. *Let T be a tree of order at least three and let Ω be the set of end vertices of T. Then the following statements are equivalent*:

(1) *T is equimatchable*;
(2) *$L(T)$ is well covered*;
(3) *For each interior edge uv of T, precisely one of u and v is incident with an end edge of T*;
(4) *The sets $N_T(\Omega)$ and $V(T) - N_T(\Omega)$ form a bipartition of T.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.4.1.

(2)⇔(3). Let $v_1, \ldots, v_n$ be the vertices of $N_T(\Omega)$. For $v_i \in N_T(\Omega)$, let E_i be the set of edges incident with v_i in T and let L_i be the subgraph of $L(T)$ induced by

E_i. Because every E_i contains an end edge of T and every end edge of T belongs to exactly one of the sets $E_1, \ldots, E_n$, the graphs $L_1, \ldots, L_n$ are the simplicies of $L(T)$. For the same reason, the sets $E_1, \ldots, E_n$ form a partition of the edge set of T if and only if every interior edge of T (if any) belongs to exactly one of the sets $E_1, \ldots, E_n$. The last condition is nothing but the statement that for each interior edge uv of T, exactly one of u and v is incident with an end edge of T. Since $L(T)$ is a chordal graph, it follows from Theorem 3.3.2 that $L(T)$ is well covered if and only if the vertex sets of $L_1, \ldots, L_n$ form a partition of the vertex set of $L(T)$. Equivalently, $L(T)$ is well covered if and only if the sets $E_1, \ldots, E_n$ form a partition of the edge set of T. Combining the above facts we obtain the equivalence of (2) and (3).

(3)$\Leftrightarrow$(4). Assume (3) holds. Since T has at least three vertices, we have $\Omega \cap N_T(\Omega) = \emptyset$ and therefore $\Omega \subseteq V(T) - N_T(\Omega)$. To prove (4), it suffices to show that every edge e of T joins a vertex of $V(T) - N_T(\Omega)$ to a vertex of $N_T(\Omega)$. This is clear if e is an end edge of T. If $e = uv$ is an interior edge of T, then $\{u, v\} \cap \Omega = \emptyset$ and, in addition, it follows from (3) that neither $\{u, v\} \subseteq N_T(\Omega)$ nor $\{u, v\} \subseteq V(T) - N_T[\Omega]$. Consequently, $e = uv$ has one vertex in $N_T(\Omega)$ and the other in $V(T) - N_T[\Omega] \subseteq V(T) - N_T(\Omega)$. This proves the implication (3)$\Rightarrow$(4). The converse implication (4)$\Rightarrow$(3) is obvious. ∎

We proceed now to the investigation of the total version of well covered graphs. For a graph G, a subset X of $V(G) \cup E(G)$ is a *totally independent set* of G if no two elements of X are adjacent or incident in G. A graph G is said to be *totally well covered* if all maximal totally independent sets of G have the same cardinality. Since a subset of vertices and edges of a graph G is totally independent in G if and only if it is an independent set of vertices of the total graph $T(G)$, G is totally well covered if and only if $T(G)$ is well covered. For this reason, in the next theorem we characterize well covered total graphs. First, we state some definitions and four propositions. A graph G is *factor-critical* if $G - v$ has a perfect matching for every vertex v of G. Certainly, every factor-critical graph is a connected graph of odd order. For a graph G, let D, A, C be the partition of $V(G)$, where D is the set of all vertices in G which are not covered by at least one maximum matching of G, $A = N_G(D) - D$, and $C = V(G) - A - D$. The Gallai–Edmonds structure theorem (see [105, p. 52] or [106, p. 94]) says that: (a) the components of $G[D]$ are factor-critical, (b) $G[C]$ has a perfect matching, and (c) if M is any maximum matching of G, then it contains a maximum matching of each component of $G[D]$, a perfect matching of each component of $G[C]$, and matches all vertices of A with vertices in distinct components of $G[D]$. It follows from this theorem that a connected graph G is factor-critical if and only if $D = V(G)$ (and so $A = C = \emptyset$). Thus, we have the following property of factor-critical graphs.

PROPOSITION 3.4.1. *A connected graph G is factor-critical if and only if every vertex of G is uncovered by at least one maximum matching of G.* ∎

In the proof of Theorem 3.4.3, we will use the following structural characterization of equimatchable factor-critical graphs with a cut vertex due to Favaron [61].

PROPOSITION 3.4.2. *A connected graph G with a cut vertex is equimatchable and factor-critical if and only if*:

(1) *G has exactly one cut vertex c, say*;

(2) *Every connected component G_i of $G-c$ is isomorphic to K_{2m_i} or to K_{m_i,m_i} for some positive integer m_i*;

(3) *c is adjacent to at least two adjacent vertices of each component G_i of $G-c$.* ∎

A graph G is *randomly matchable* if every maximal matching of G is perfect. Certainly, every randomly matchable graph is equimatchable. The next proposition (due to Sumner [132]) gives a characterization of randomly matchable graphs. The proof given here is due to Lesk, Plummer and Pulleyblank [102].

PROPOSITION 3.4.3. *A connected graph G is randomly matchable if and only if $G = K_{2n}$ or $G = K_{n,n}$ for some positive integer n.*

Proof. Clearly K_{2n} and $K_{n,n}$ are randomly matchable.

Conversely, suppose G is a connected randomly matchable graph. It is then easy to see that, in fact, either $G = K_2$ or G must be 2-connected.

Now suppose G is bipartite but not complete. Let V_1 and V_2 be partite sets of G and $u \in V_1$, $v \in V_2$ be nonadjacent. Since G is connected there is an odd path P joining u and v. Put the first, third, ..., and last edges of P into a matching and extend this matching to a perfect matching M for G. Then the symmetric difference $M \otimes P$ is a matching for G which leaves only points u and v exposed. Hence $M \otimes P$ cannot be extended to a perfect matching, a contradiction. Hence $G = K_{n,n}$ for some $n \geq 1$.

Now we consider the non-bipartite case.

CLAIM 1. *If G is any 2-connected non-bipartite graph, every vertex lies on an odd cycle.*

For let u be any vertex and let C be any odd cycle. If $u \in V(C)$ we are done, so suppose $u \notin V(C)$. Then by Menger's theorem there exist two openly disjoint paths from u to two different vertices of C and these two paths, together with one of the two parts of C intercepted, form an odd cycle.

CLAIM 2. *If G is any 2-connected non-bipartite graph, then every pair of vertices is joined by an odd path.*

Let u and v be any two vertices of G. If $uv \in E(G)$, there is nothing to prove, so suppose $uv \notin E(G)$. By Claim 1 we know there is an odd cycle C containing v. If $u \in V(C)$ again we are done, so suppose $u \notin V(C)$. Then there is a path P from u to the cycle C. If P meets C in a vertex different from v, we are done. Thus assume that all such paths P meet C at v. Then v is a cut vertex of G, a contradiction, and Claim 2 is proved.

Let u and v be any pair of nonadjacent vertices in G. By Claim 2, there is an odd path P joining u and v. Now proceed as in the bipartite case to get the same contradiction. Thus $G = K_{2n}$ for some $n \geq 1$. ∎

For a set of edges M of a graph G, we denote by $V(M)$ the set of vertices of G incident with at least one edge of M. The next proposition (due to Yannakakis and Gavril [160]) shows a connection between maximal matchings of a graph G and maximal independent sets of vertices of the total graph $T(G)$ of G.

PROPOSITION 3.4.4. *If M is a maximal matching of G, then $M \cup (V(G) - V(M))$ is a maximal independent set of vertices of $T(G)$. Also $|M \cup (V(G) - V(M))| = |V(G)| - |M|$.*

Proof. Since M is a maximal matching of G it follows that $V(G) - V(M)$ is an independent set of G, hence $M \cup (V(G) - V(M))$ is independent in $T(G)$. Since every vertex of $T(G)$ which is not in M is incident or adjacent to M or is in $V(G) - V(M)$, it follows that $M \cup (V(G) - V(M))$ is a maximal independent set in $T(G)$. Certainly, $|M \cup (V(G) - V(M))| = |M| + |V(G)| - 2|M| = |V(G)| - |M|$. ∎

With the above terminology and propositions, we now describe all connected graphs whose total graphs are well covered. The following theorem shows that the class of totally well covered graphs is quite restricted.

THEOREM 3.4.3 [148]. *If G is a connected graph, then the total graph $T(G)$ of G is well covered if and only if G is one of the graphs K_n, $K_{n,n}$ and $K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ for any positive integers n and $m_1, \ldots, m_n$.*

Proof. A trivial verification shows that each of the total graphs $T(K_n)$, $T(K_{n,n})$ and $T(K_1 + \bigcup_{i=1}^{n} K_{2m_i})$ is well covered.

Conversely, assume that G is a connected graph such that $T(G)$ is well covered. By Proposition 3.4.4, for any maximal matching M of G, $M \cup (V(G) - V(M))$ is a maximal (and therefore maximum) independent set of vertices in $T(G)$, say $p = |M \cup (V(G) - V(M))| = |V(G)| - |M|$. This observation implies that G is equimatchable. In addition, if G has a perfect matching, then every maximal matching of G is perfect and it follows from Proposition 3.4.3 that $G = K_{2n}$ or $G = K_{n,n}$ for $n \geq 1$. Thus assume that G is equimatchable but G has no perfect matching. Then it suffices to show that $G = K_{2n-1}$ or $G = K_1 + \bigcup_{i=1}^{n} K_{2m_i}$ for some positive integers $n, m_1, \ldots, m_n$. In the proof we frequently use the following claim.

CLAIM 1. *Let M be a maximum matching of G. Then for every $xy \in M$ and $t \in V(G) - V(M)$, either $\{x, y\} \subseteq N_G(t)$ or $\{x, y\} \cap N_G(t) = \emptyset$.*

Suppose that $x \in N_G(t)$ and $y \notin N_G(t)$. If $A = N_G(x) \cap N_G(y) \cap (V(G) - V(M)) = \emptyset$, then $I = (M - \{xy\}) \cup \{x\} \cup (V(G) - (V(M) \cup N_G(x)))$ is a maximal independent set of $T(G)$ and $|I| < p$, a contradiction. If $A \neq \emptyset$, then for any $s \in A$, $I \cup \{ys\}$ is a maximal independent set of $T(G)$ and $|I \cup \{ys\}| < p$, a contradiction.

Claim 2. *G is factor-critical.*

Let $D(G)$ be the set of vertices of G which are uncovered by at least one maximum matching of G. By Proposition 3.4.1, it suffices to prove that $D(G) = V(G)$. Since G is connected and $D(G) \neq \emptyset$ (as G has no perfect matching), it suffices to show that $N_G(t) \subseteq D(G)$ for every $t \in D(G)$. Take any $t \in D(G)$ and a maximum matching M of G that does not cover t. Then $t \notin V(M)$ and $N_G(t) \subseteq V(M)$. Take any $x \in N_G(t)$. Since $x \in V(M)$, there is $y \in V(M)$ such that $xy \in M$. By Claim 1, $\{x, y\} \subseteq N_G(t)$. Now $M' = (M - \{xy\}) \cup \{yt\}$ is a maximum matching avoiding x. Therefore $x \in D(G)$ and consequently $N_G(t) \subseteq D(G)$.

To complete the proof of the theorem, we consider two cases.

Case 1: *G contains a cut vertex c, say.* Since G is equimatchable and factor-critical, Proposition 3.4.2 implies that c is the only cut vertex of G. In addition, if G_i is a component of $G - c$, then $G_i = K_{2m_i}$ or $G_i = K_{m_i,m_i}$, and c is adjacent to at least two adjacent vertices of G_i. Let n be the number of components of $G - c$. For $i = 1, \ldots, n$, let $v_1^i, \ldots, v_{m_i}^i$, $u_1^i, \ldots, u_{m_i}^i$ be the vertices of G_i. We may assume that v_1^i and u_1^i are neighbours of c in G and every v_l^i is adjacent to every u_k^i, $l, k = 1, \ldots, m_i$. We shall prove that $G = K_1 + (K_{2m_1} \cup \ldots \cup K_{2m_n})$.

It is obvious that $M_i = \{v_k^i u_k^i : k = 1, \ldots, m_i\}$ is a perfect matching of G_i $(i = 1, \ldots, n)$ and $M = \bigcup_{i=1}^n M_i$ is a maximum matching of G. We shall prove that c is adjacent to every vertex of G_i, and that G_i is a complete graph, $i = 1, \ldots, n$. This is clear if $m_i = 1$. Thus assume that $m_i \geq 2$. For $k = 2, \ldots, m_i$, $M_{ik} = (M - \{v_1^i u_1^i, v_k^i u_k^i\}) \cup \{v_1^i u_k^i, v_k^i u_1^i\}$ is a maximum matching of G. Since $c \notin V(M_{ik})$ and c is adjacent to the vertex v_1^i (u_1^i, resp.) of the edge $v_1^i u_k^i$ ($v_k^i u_1^i$, resp.) which belongs to M_{ik}, we conclude from Claim 1 that c is adjacent to u_k^i (v_k^i, resp.). Thus c is adjacent to every vertex of G_i. Now for $k = 1, \ldots, m_i$, the set $M'_{ik} = (M - \{v_k^i u_k^i\}) \cup \{u_k^i c\}$ is a maximum matching of G which does not cover v_k^i. Since v_k^i is adjacent to every vertex u_l^i and $v_l^i u_l^i \in M'_{ik}$ if $l \neq k$, v_k^i is adjacent to every vertex v_l^i with $l \neq k$ (by Claim 1). Similarly, replacing M'_{ik} by $M''_{ik} = (M - \{v_k^i u_k^i\}) \cup \{v_k^i c\}$, we observe that u_k^i is adjacent to every vertex u_l^i, $l \neq k$. Thus G_i is a complete graph of order $2m_i$, $G_i = K_{2m_i}$. Finally, since the cut vertex c of G is adjacent to every vertex of G_i, $i = 1, \ldots, n$, we conclude that $G = K_1 + (K_{2m_1} \cup \ldots \cup K_{2m_n})$.

Case 2: *G has no cut vertex.* We claim that G is a complete graph (of odd order). Suppose this is not true. Then there exists a vertex p in G for which $N_G[p] \neq V(G)$. Consequently, since G is connected, the two sets $S = \{v \in N_G(p) : N_G(v) \not\subseteq N_G[p]\}$ and $R = \{x \in V(G) - N_G[p] : N_G(x) \cap N_G(p) \neq \emptyset\}$ are nonempty. Let M be a perfect matching of $G - p$. For a vertex w of $G - p$, let w^* denote the unique neighbour of w such that $ww^* \in M$. It is clear from Claim 1 that for every vertex w of $G - p$, either $\{w, w^*\} \subseteq N_G(p)$ or $\{w, w^*\} \subseteq V(G) - N_G[p]$. We make four additional observations.

(1) *For every* $v \in S$ *and* $x \in R$, *either* $\{x, x^*\} \subset N_G(v)$ *or* $\{x, x^*\} \cap N_G(v) = \emptyset$.

Assume $\{x, x^*\} \cap N_G(v) \neq \emptyset$. Because $M' = (M - \{vv^*\}) \cup \{v^*p\}$ is a maximum matching of G for which $v \notin V(M')$ and $xx^* \in M'$, we conclude from Claim 1 that $\{x, x^*\} \subset N_G(v)$.

(2) *For every* $v \in S$ *and* $x \in R$, *if* $\{x, x^*\} \subset N_G(v)$, *then* $\{x, x^*\} \cap N_G(v^*) = \emptyset$.

Assume $\{x, x^*\} \subset N_G(v)$ and suppose that $\{x, x^*\} \cap N_G(v^*) \neq \emptyset$. Then $\{x, x^*\} \subset N_G(v^*)$ by (1). But now $M' = (M - \{vv^*, xx^*\}) \cup \{vx, v^*x^*\}$ is a maximum matching of G. Because $p \notin V(M')$ and vx, $v^*x^* \in M'$ while $\{v, x\}$ and $\{v^*, x^*\}$ are contained neither in $N_G(p)$ nor in $V(G) - N_G[p]$, we get a contradiction to Claim 1.

(3) *For every* $v \in S$ *and* $x \in R$, *if* $x \in N_G(v)$, *then* $N_G(x) \cap S = \{v\}$.

Assume $x \in N_G(v)$ and suppose that there exists $u \in N_G(x) \cap S - \{v\}$. It follows from (2) that $u \neq v^*$. Then $M' = (M - \{xx^*, vv^*, uu^*\}) \cup \{vx, ux^*, pu^*\}$ is a maximum matching of G and it does not cover v^*. Since $vx \in M'$ and neither $\{v, x\} \subseteq N_G(v^*)$ nor $\{v, x\} \cap N_G(v^*) = \emptyset$, we reach a contradiction to Claim 1.

(4) *The set* S *has exactly one vertex* v, *say.*

Suppose $|S| \geq 2$ and $u \in S - \{v\}$. Let $x \in N_G(v) \cap R$ and $y \in N_G(u) \cap R$. It follows from (1) and (3) that $xy \notin M$; for otherwise (1) implies that $\{x, y\} \subset N_G(v)$ and $\{x, y\} \subset N_G(u)$ which contradicts (3). If $vu \in M$, then considering a maximum matching M' of G containing $(M - \{vu, xx^*, yy^*\}) \cup \{vx, uy\}$ (and then necessarily also x^*y^*), we get a contradiction just as in the proof of (2). If $vu \notin M$, then let M' be a maximum matching of G containing $(M - \{vv^*, uu^*, xx^*, yy^*\}) \cup \{vx, uy, pu^*\}$. Because the vertex v^* is adjacent neither to x (see (2)) nor to y^* (see (3)), $v^* \notin V(M')$. But now since $vx \in M'$ and neither $\{v, x\} \subseteq N_G(v^*)$ nor $\{v, x\} \cap N_G(v^*) = \emptyset$, we get a contradiction to Claim 1.

It is obvious from (4) and from definitions of S and R that v, the unique vertex of S, is a cut vertex of G. This, however, contradicts the assumption that G has no cut vertex and completes the proof of the theorem. ∎

3.5. Well covered generalized Petersen graphs. Let n and k be positive integers with $n \geq 3$ and $1 \leq k \leq n - 1$. The *generalized Petersen graph* $P_{n,k}$ is defined in the following way. It has $2n$ vertices $v_0, v_1, \ldots, v_{n-1}$, $u_0, u_1, \ldots, u_{n-1}$ and edges $v_i v_{i+1}$, $v_i u_i$, and $u_i u_{i+k}$ for all i satisfying $0 \leq i \leq n - 1$ with all subscripts taken modulo n. It is no problem to observe that each vertex v_i is of degree three in $P_{n,k}$. Similarly, each u_i is a vertex of degree three if $k \neq n/2$ but its degree is two if $k = n/2$. It is also easy to see that $P_{n,k}$ is isomorphic to $P_{n,n-k}$ and therefore we may always assume that $k \leq \lfloor n/2 \rfloor$. A simple analysis shows that $P_{n,k}$ is a graph of girth three if and only if $n = 3k$ for $k \geq 1$. Analogously, $P_{n,k}$ is a graph of girth four if and only if $k = 1$ and $n \geq 4$, $k = 2$ and $n = 4$ or $k \geq 1$ and $n = 4k$.

The following theorem characterizes well covered generalized Petersen graphs.

THEOREM 3.5.1 [148]. *There are exactly five well covered generalized Petersen graphs and they are shown in Figure* 16.

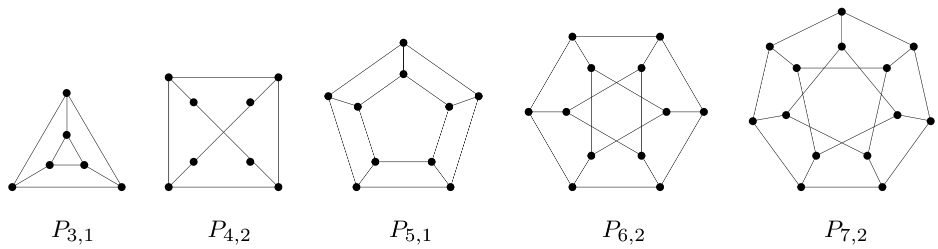

Fig. 16. The well covered generalized Petersen graphs

Proof. It is easy to check that the generalized Petersen graphs $P_{3,1}$, $P_{4,2}$, $P_{5,1}$, $P_{6,2}$, $P_{7,2}$ (given in Figure 16) are well covered.

Conversely, assume that a generalized Petersen graph $P_{n,k}$ is well covered. We first dispose of the case $k = 1$. It is easy to check that $P_{3,1}$ and $P_{5,1}$ are well covered while $P_{4,1}$ is not well covered. For $n \geq 6$, the set $I = \{u_1, u_5\}$ is independent in $P_{n,1}$ and $P_{n,1} - N_{P_{n,1}}[I]$ has a non-well covered component $K_{1,3}$ with vertices v_2, v_3, v_4 and u_4. Thus, by Proposition 3.1.5, $P_{n,1}$ is not well covered if $n \geq 6$.

We next dispose of the case that n is even and $k = n/2$. Certainly, $P_{4,2}$ is well covered. If $n \geq 6$, then $I = \{u_1, v_2, u_3, \ldots, u_{n/2}\}$ is an independent set in $P_{n,n/2}$. Since $P_{n,n/2} - N_{P_{n,n/2}}[I]$ is a non-well covered tree, it follows from Proposition 3.1.5 that $P_{n,n/2}$ is not well covered if $n \geq 6$.

For the remainder of the proof, we assume that $1 < k < n/2$. In this case $P_{n,k}$ is cubic and therefore it does not belong to the family $\mathcal{PC}$. Thus, if the girth of $P_{n,k}$ is at least 5, Theorem 3.1.4 forces that $P_{n,k}$ must be isomorphic to the graph $P_{14} = P_{7,2}$ in Figures 13 and 16. The proof of the theorem will be complete if we show that $P_{6,2}$ is the only well covered generalized Petersen graph $P_{n,k}$ of girth three or four with $1 < k < n/2$.

We first consider the case that $P_{n,k}$ is of girth three. The restriction $1 < k < n/2$ implies that every 3-cycle of $P_{n,k}$ consists of vertices u_i, u_{i+k}, u_{i+2k} for $i = 0, 1, \ldots, n-1$ and therefore it must be $n = 3k$. For $k = 2$, we get the well covered graph $P_{6,2}$. If $k \geq 3$, then $I = \{u_0, u_1, \ldots, u_{k-1}\}$ is an independent set in $P_{3k,k}$ and its subgraph $P_{3k,k} - N_{P_{3k,k}}[I]$ (shown in bold in Figure 17) is a non-well covered path of length $2k \geq 6$. This and Proposition 3.1.5 imply that $P_{3k,k}$ is not well covered if $k \geq 3$.

This now leaves us with the case that $P_{n,k}$ is of girth four. The restriction $1 < k < n/2$ implies now that every 4-cycle of $P_{n,k}$ consists of vertices $u_i, u_{i+k}, u_{i+2k}, u_{i+3k}$ for $i=0, 1, \ldots, n-1$ and $n=4k$. If $k=2$, then a simple verification shows that $P_{8,2}$ is not well covered. If $k \geq 3$, then $I = \{u_0, u_1, \ldots, u_{k-1}\}$ is an independent set in $P_{4k,k}$ and it is easy to check that its subgraph $P_{4k,k} - N_{P_{4k,k}}[I]$ (shown in bold in Figure 17) is a non-well covered tree of order $4k \geq 12$. This and Proposition 3.1.5 imply that $P_{4k,k}$ is not a well covered graph if $k \geq 3$.

This completes the proof of the theorem. ∎

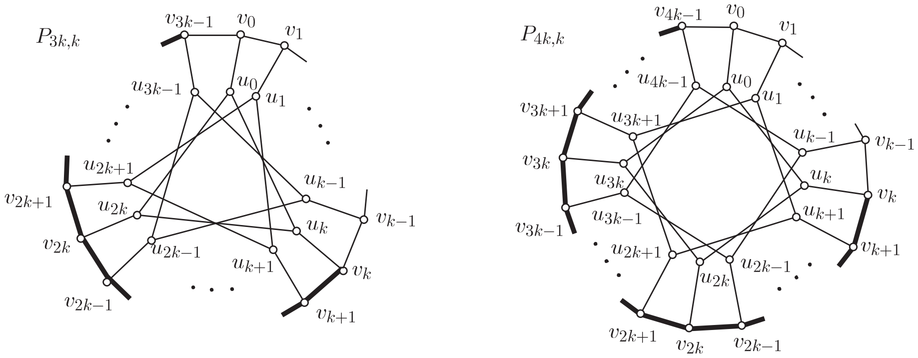

Fig. 17. The generalized Petersen graphs $P_{3k,k}$ and $P_{4k,k}$

In conclusion, let us observe that for the graphs of Figure 16 we have $\alpha(P_{3,1}) = 2 < \Gamma(P_{3,1}) = 3$, $\alpha(P_{4,2}) = 3 < \Gamma(P_{4,2}) = 4$, $\alpha(P_{5,1}) = 4 < \Gamma(P_{5,1}) = 5$, $\alpha(P_{6,2}) = 4 < \Gamma(P_{6,2}) = 6$ and $\alpha(P_{7,2}) = 5 < \Gamma(P_{7,2}) = 7$. This implies that none of the graphs of Figure 16 is well dominated (or well irredundant).

3.6. Well irredundant graphs. In this section, we focus our attention on well irredundant graphs. We characterize well irredundant graphs within the following three families: bipartite graphs, chordal graphs, graphs of girth at least five. It follows from Proposition 2.1.4 that for any graph G, the corona $G \circ K_1$ is a well irredundant graph. The next theorem, among other things, proves that the converse is true for connected bipartite graphs except for K_1 and C_4. Other proofs of the theorem can be found in [146, 149, 150]. The equivalence of (vi) and (vii) is also given in [66] but with a longer proof.

THEOREM 3.6.1. *Let G be a connected bipartite graph. Then the following statements are equivalent*:

(i) $\mathrm{ir}(G) = \mathrm{IR}(G)$, *i.e. G is well irredundant*;
(ii) $\mathrm{ir}(G) = \Gamma(G)$;
(iii) $\mathrm{ir}(G) = \alpha(G)$;
(iv) $\gamma(G) = \alpha(G)$;
(v) $\gamma(G) = \mathrm{IR}(G)$;
(vi) $\gamma(G) = \Gamma(G)$, *i.e. G is well dominated*;
(vii) $G \in \{K_1, C_4\}$ *or $G = H \circ K_1$ for some connected bipartite graph H.*

P r o o f. The statements (i)–(iii) ((iv)–(vi), resp.) are equivalent according to Theorem 2.4.6. The implication (iii)⇒(iv) follows from Proposition 2.1.3. The implication (vii)⇒(i) is obvious if $G \in \{K_1, C_4\}$ and follows from Proposition 2.1.4 if $G = H \circ K_1$ for some graph H. Finally, the equivalence of (iv) and (vii) is the content of Corollary 2.3.2. ∎

It follows from Theorem 3.6.1 that a bipartite graph G is well irredundant if and only if it is well dominated. Moreover, it follows from Proposition 2.1.3 and Theorem 3.6.1 that for a bipartite graph G, each of the equations (i)–(vi) of Theorem 3.6.1 implies the following nine equations: $i(G) = \alpha(G)$, $i(G) = \Gamma(G)$, $i(G) = \mathrm{IR}(G)$, $\alpha(G) = \Gamma(G)$, $\alpha(G) = \mathrm{IR}(G)$, $\Gamma(G) = \mathrm{IR}(G)$, $\mathrm{ir}(G) = \gamma(G)$, $\mathrm{ir}(G) = i(G)$, and $\gamma(G) = i(G)$. Each of the converse implications is false, as G_2 of Figure 9 and $K_{1,2}$ demonstrate.

The following two propositions are required for our proofs of characterizations of well irredundant chordal and block graphs.

PROPOSITION 3.6.1. *Let X be a set of vertices of a graph G. If every vertex of X belongs to at least one simplex of G but no two of them belong to the same simplex, then X is irredundant in G.*

Proof. For $x \in X$, let S be a simplex containing x and let s be a simplicial vertex from S. Since x is the only vertex of $X \cap V(S)$, $s \in I_G(x, X)$ and this implies the irredundance of X. ∎

PROPOSITION 3.6.2. *Let G be a graph of order n, and let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of nonempty graphs indexed by the vertices of G. Then* (i) $\mathrm{ir}(G \circ \mathcal{H}) = n$ *and* (ii) *$G \circ \mathcal{H}$ is a well irredundant graph if and only if $\mathcal{H}$ consists of complete graphs.*

Proof. The proposition is a direct consequence of the following four observations: (1) $V(G)$ is a maximal irredundant set in $G \circ \mathcal{H}$, (2) a subset J of $V(G \circ \mathcal{H})$ is a maximal irredundant set in $G \circ \mathcal{H}$ if and only if for each $v \in V(G)$, either $v \in J$ and $J \cap V(H_v) = \emptyset$ or $v \notin J$ and $J \cap V(H_v)$ is a maximal irredundant set of H_v, (3) $\mathrm{IR}(G \circ \mathcal{H}) = \sum_{v \in V(G)} \mathrm{IR}(H_v)$, and (4) for each $v \in V(G)$, $\mathrm{IR}(H_v) = 1$ if and only if H_v is a complete graph. ∎

The next result due to Topp and Vestergaard [146] characterizes well irredundant chordal graphs.

THEOREM 3.6.2. *A chordal graph G is well irredundant if and only if*

(1) *every vertex of G belongs to exactly one simplex of G and*

(2) *if G has an induced subgraph A given in Figure* 18, *then the unique vertex of degree two in A is not a simplicial vertex of G.*

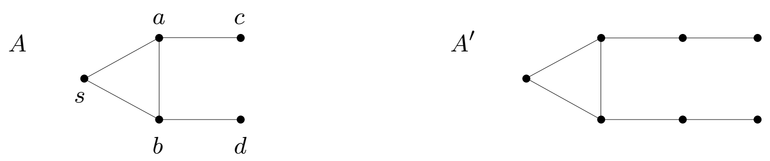

Fig. 18

Proof. Let G be a chordal graph. Let $S_1, \ldots, S_n$ be the simplices of G and S a set of n vertices containing exactly one simplicial vertex s_i, say, from each S_i.

Assume G is well irredundant. Then G is well covered and by Theorem 3.3.2, every vertex of G belongs to exactly one of the simplices $S_1, \ldots, S_n$. In addition, S is a maximal irredundant set in G. Suppose G has an induced subgraph A (see Figure 18) whose unique vertex s of degree two is a simplicial vertex of G, say $s = s_1$. Then the neighbours a and b of s belong to S_1 but their neighbours c and d belong to two other simplices of G, say c is in S_2 and d is in S_3. Now $S' = \{a, b\} \cup \{s_4, \ldots, s_n\}$ is another maximal irredundant set in G and $|S'| < |S|$ which contradicts the well irredundance of G.

Conversely, assume G has properties (1) and (2). It is obvious from (1) that every minimal dominating set of G contains exactly one vertex from each simplex of G. Therefore $\gamma(G) = \Gamma(G) = n$. Let J be a maximal irredundant set of G. The proof will be complete if we show that $|V(S_i) \cap J| = 1$ $(i = 1, \ldots, n)$, which, in turn, implies that $|J| = n$. First we show that $|V(S_i) \cap J| \leq 1$. If this is not the case, let a and b be distinct vertices from $V(S_i) \cap J$. Note that neither a nor b can be a simplicial vertex, else J would not be irredundant. On the other hand, if a and b are nonsimplicial vertices from S_i, then for any $c \in I_G(a, J)$ and $d \in I_G(b, J)$, $G[\{s_i, a, b, c, d\}]$ is isomorphic to A, a contradiction to (2). Hence, $|V(S_i) \cap J| \leq 1$. Finally, $V(S_i) \cap J \neq \emptyset$, for otherwise $J \cup \{s_i\}$ is irredundant (by Proposition 3.6.1) and this contradicts the maximality of J. This completes the proof. ∎

It is easy to observe that in the characterization of well irredundant chordal graphs, condition (2) of Theorem 3.6.2 may be replaced by each of the following conditions: (2′) if G has an induced subgraph A' given in Figure 18, then at least one simplicial vertex of A' is a nonsimplicial vertex of G; (2″) if vertices x and y belong to the same simplex of G, then at least one of the sets $N_G[x] - N_G[y]$ and $N_G[y] - N_G[x]$ is empty.

Corollary 3.6.1. *If G is a connected block graph, then the following statements are equivalent*:

(1) *G is well irredundant*;

(2) *Every vertex of G belongs to exactly one end block of G*;

(3) *$G = K_1$ or $G = H \circ \{H_v : v \in V(H)\}$ where H is a connected block graph and every graph of the family $\{H_v : v \in V(H)\}$ is complete.*

Proof. The result is obvious if $G = K_1$ or $G = K_1 \circ K_{n-1} = K_n$ for $n \geq 2$. Thus assume that G is a connected noncomplete block graph and let C be the set of all cut vertices of G.

Assume that every vertex of G belongs to exactly one end block of G. For $v \in C$, let B_v be the end block of G that contains v. Certainly, the subgraphs $H_v = B_v - v$ are nonempty and complete. In addition, $G[C] = G - \bigcup_{v \in C} V(H_v)$ is a connected block graph. Further, the corona $G[C] \circ \{H_v : v \in C\}$ is well irredundant by Proposition 3.6.2. Thus, G is well irredundant since G is isomorphic to $G[C] \circ \{H_v : v \in C\}$. This proves the implications (2)⇒(3)⇒(1).

Assume G is a well irredundant block graph. Then, by Theorem 3.6.2, every vertex of G belongs to exactly one simplex and it remains to show that every simplex of G is an end block. Suppose G has a simplex S which is not an end block. Then S has a simplicial vertex s, say, and distinct cut vertices c_1, c_2 of G. Further, since c_i has a neighbour d_i such that d_i and c_{3-i} belong to different components of $G - c_i$ ($i = 1, 2$), the set $\{s, c_1, c_2, d_1, d_2\}$ induces in G a graph A which contradicts Theorem 3.6.2. This proves the implication (1)⇒(2) and completes the proof. ∎

We now turn our attention to well irredundant graphs of girth at least five. The following theorem due to Topp and Vestergaard [146] is a counterpart of Theorem 3.1.5 for well irredundant graphs.

THEOREM 3.6.3. *If a graph G belongs to the family $\mathcal{PC}$, then G is well irredundant if and only if for every pair of basic* 5*-cycles there is either no edge joining them, exactly two edges and they are vertex disjoint, or four edges.*

Proof. If $G \in \mathcal{PC}$ and G is well irredundant, then G is well dominated and the "only if" part of the theorem follows from Theorem 3.1.5.

Conversely, assume $G \in \mathcal{PC}$ and for every pair of basic 5-cycles of G there is either no edge joining them, exactly two edges and they are vertex disjoint, or four edges. Let J be a maximal irredundant set in G. To prove that G is well irredundant, it suffices to show that $|J| = |E_e| + 2|\mathcal{C}|$ where E_e is the set of end edges of G and $\mathcal{C}$ is the set of basic 5-cycles of G, respectively. Since J is irredundant, every end edge of G has at most one vertex in J and every basic 5-cycle has at most three vertices in J. Thus, E_e can be partitioned into two subsets $E_e^i = \{vu \in E_e : |\{v, u\} \cap J| = i\}$, $i = 0, 1$. Similarly, $\mathcal{C}$ can be partitioned into four subsets $\mathcal{C}_i = \{C \in \mathcal{C} : |V(C) \cap J| = i\}$, $i = 0, 1, 2, 3$. Certainly, $|J| = |E_e^1| + |\mathcal{C}_1| + 2|\mathcal{C}_2| + 3|\mathcal{C}_3| = |E_e| + 2|\mathcal{C}| + (|\mathcal{C}_3| - |E_e^0| - 2|\mathcal{C}_0| - |\mathcal{C}_1|)$ and it suffices to prove that $|\mathcal{C}_3| = |E_e^0| + 2|\mathcal{C}_0| + |\mathcal{C}_1|$.

We now give a few remarks needed for the rest of proof. We omit simple proofs of the first four properties.

(1) *If $C \in \mathcal{C}_0$, then C has two vertices of degree three or more.*

(2) *If $C \in \mathcal{C}_1$, then C has two vertices of degree three or more and the unique vertex of $V(G) \cap J$ is adjacent to exactly one of them.*

(3) *If $C \in \mathcal{C}_3$, then one vertex of $V(C) \cap J$ is of degree at least three, we denote it by $t(C)$, and the other two are of degree two and adjacent to $t(C)$.*

(4) *$I_G(x, J) \cap \Omega \neq \emptyset$ if $x \in J \cap V_p$, where Ω is the set of end vertices of G.*

(5) *Let x and v be vertices such that $x \in J$, $v \in V(G) - (J \cup \Omega)$ and $I_G(x, J) = \{v\}$. If x and v do not belong to the same basic* 5*-cycle, then $x = t(D)$ for a cycle $D \in \mathcal{C}_3$.*

The assumptions and (4) imply that x belongs to some basic 5-cycle $D = (x_1, \ldots, x_5)$, say $x = x_1$. Since $I_G(x_1, J) = \{v\}$ is disjoint with $V(D)$, it follows that $V(D)\cap J=\{x_1,x_2,x_5\}$; otherwise $\{x_2,x_5\}\cap I_G(x_1,J)\neq\emptyset$ or J is redundant. Thus, $D\in\mathcal{C}_3$ and $x=t(D)$.

(6) *For any $C\in\mathcal{C}_3$, no vertex of $I_G(t(C),J)$ belongs to Ω or to a cycle from $\mathcal{C}_2\cup\mathcal{C}_3$.*

Since $G\in\mathcal{PC}$, no vertex of V_c is adjacent to an end vertex of G and so $I_G(t(C),J)\cap\Omega=\emptyset$. Suppose $C=(x_1,\ldots,x_5)\in\mathcal{C}_3$, $V(C)\cap J=\{x_1=t(C),x_2,x_5\}$ (see (3)), and a vertex x of $I_G(x_1,J)$ belongs to a basic 5-cycle $D=(y_1,\ldots,y_5)$. We may assume that $x=y_1$ and x_3 is adjacent to y_3 (and possibly to y_1 but then x_1 is also adjacent to y_3). Since $y_1\in I_G(x_1,J)$, no vertex of $N_G[y_1]-\{x_1\}$ belongs to J. In particular, $\{y_1,y_2,y_5\}\cap J=\emptyset$. Similarly, since $x_3\in I_G(x_2,J)$, no vertex of $N_G[x_3]-\{x_2\}$ belongs to J and so $y_3\notin J$. Thus, $|V(D)\cap J|\leq 1$ and so $D\notin\mathcal{C}_2\cup\mathcal{C}_3$.

Let $S=P_0\cup P_1\cup P_2$, where P_0, P_1 and P_2 are vertex sets defined by

$P_0=\{v\in V_p-\Omega : v$ is incident with an end edge from $E_e^0\}$,
$P_1=\{v\in V_c : d_G(v)\geq 3$ and $v\in V(C)-N_G[V(C)\cap J]$ for some $C\in\mathcal{C}_1\}$,
$P_2=\{v\in V_c : d_G(v)\geq 3$ and $v\in V(C)$ for some $C\in\mathcal{C}_0\}$.

Certainly, $|P_0|=|E_e^0|$. Similarly, it follows from (1) and (2) that $|P_2|=2|\mathcal{C}_0|$ and $|P_1|=|\mathcal{C}_1|$, respectively. Hence, $|S|=|E_e^0|+2|\mathcal{C}_0|+|\mathcal{C}_1|$. From (3), (6) and the definition of private neighbourhood it follows that $\{I_G(t(C),J) : C\in\mathcal{C}_3\}$ is a family of nonempty disjoint subsets of S. Thus,

$$|S|\geq\Big|\bigcup_{C\in\mathcal{C}_3} I_G(t(C),J)\Big|=\sum_{C\in\mathcal{C}_3}|I_G(t(C),J)|\geq|\mathcal{C}_3|.$$

The proof will be complete if we show that for every $v\in S$ there is $D\in\mathcal{C}_3$ such that $I_G(t(D),J)=\{v\}$, which, in turn, implies that $|S|\leq|\mathcal{C}_3|$ and consequently $|\mathcal{C}_3|=|S|=|E_e^0|+2|\mathcal{C}_0|+|\mathcal{C}_1|$. To prove this, we consider three cases.

Case 1: $v\in P_0$. Let $u\in\Omega$ be such that $vu\in E_e^0$. Since $u\notin N_G[J]$ and $N_G(u)=\{v\}$, there exists x in J such that $I_G(x,J)=\{v\}$. This and (5) imply that $x=t(D)$ for some $D\in\mathcal{C}_3$.

Case 2: $v\in P_1$. Let $C=(a_1,\ldots,a_5)\in\mathcal{C}_1$ be the cycle containing v. By (2) we may assume that $d_G(a_1)\geq 3$, $d_G(a_3)\geq 3$, $V(C)\cap J=\{a_4\}$ and $v=a_1$. Now $a_2\notin N_G[J]$, so there is $x\in J$ such that $I_G(x,J)\subseteq N_G(a_2)=\{a_1,a_3\}$. Since $a_4\in I_G(a_4,J)$ and $a_3\in N_G(a_4)$, we have $x\neq a_4$ and $I_G(x,J)=\{a_1\}$. By (5), there is $D\in\mathcal{C}_3$ such that $I_G(t(D),J)=\{a_1\}$.

Case 3: $v\in P_2$. Let $C=(a_1,\ldots,a_5)\in\mathcal{C}_0$ be the cycle containing v. By (1) we may assume that $d_G(a_1)\geq 3$ and $d_G(a_3)\geq 3$, so $v\in\{a_1,a_3\}$. Since $\{a_2,a_4,a_5\}\cap N_G[J]=\emptyset$, there are $x,y\in J$ such that $I_G(x,J)\subseteq N_G(a_5)=\{a_1,a_4\}$ and $I_G(y,J)\subseteq N_G(a_4)=\{a_3,a_5\}$. Consequently, $I_G(x,J)=\{a_1\}$, $I_G(y,J)=\{a_3\}$ (and $I_G(x,J)\cup I_G(y,J)=N_G(a_2)$) since $a_4,a_5\notin N_G[J]$. Now it follows from (5) that there are cycles D and D' in $\mathcal{C}_3$ such that $I_G(t(D),J)=\{a_1\}$ and $I_G(t(D'),J)=\{a_3\}$, respectively.

This completes the proof. ■

It is easy to verify that the graphs C_7 and P_{10} of Figure 13, together with K_1, are well irredundant, whereas P_{14} is not well irredundant because $\{1, 2, 3, 4, 5, 6, 7\}$ and $\{a, 2, 4, 5, 7\}$ are both maximal irredundant sets in P_{14}. This observation, Corollaries 3.1.5 and 3.1.6, and Theorem 3.6.3 immediately imply the following corollary.

COROLLARY 3.6.2. (i) *Let G be a connected graph of girth at least five. Then G is well irredundant if and only if either $G = K_1$, or G is one of the graphs C_7 and P_{10} of Figure* 13, *or G belongs to the family $\mathcal{PC}$ and for every pair of basic* 5-*cycles there is either no edge joining them or exactly two edges and they are vertex disjoint.*

(ii) *If G is a connected graph of girth at least six*, *then the following statements are equivalent*: (a) *G is well irredundant*; (b) *G is well dominated*; (c) *G is well covered*; (d) $G \in \{K_1, C_7\} \cup \{H \circ K_1 : H$ *is a connected graph of girth* $\geq 6\}$. ■

Note that all well dominated graphs of the family $\mathcal{PC}$ are well irredundant and, certainly, vice versa. Moreover, the graph P_{14} (shown in Figure 13) is the generalized Petersen graph $P_{7,2}$ and it is the only connected well dominated graph of girth at least five which is not well irredundant.

We conclude this section with a characterization of well irredundant unicyclic graphs. A graph is *unicyclic* if it is connected and has exactly one cycle. Let $\mathcal{U}$ be the set of all unicyclic graphs, and we let $\mathcal{KU} = \{H \circ K_1 : H \in \mathcal{U}\}$. We say that a graph G is in the family $\mathcal{S}_5$ if $G \in \mathcal{U} \cap \mathcal{PC}$ and it has a basic 5-cycle. Finally, a unicyclic graph G is in the family $\mathcal{S}_3^1$ if $G = T \circ \mathcal{H}$ where T is a tree and the family $\mathcal{H} = \{H_v : v \in V(T)\}$ consists of K_2 and $|V(T)| - 1$ copies of K_1. The next corollary may be obtained by routine arguments from Proposition 3.6.2, Theorem 3.6.1, and Corollaries 3.6.1 and 3.6.2.

COROLLARY 3.6.3. *A unicyclic graph G is well irredundant if and only if* $G \in \{C_4, C_7\} \cup \mathcal{S}_3^1 \cup \mathcal{S}_5 \cup \mathcal{KU}$. ■

4. Graphical sequences and sets of integers

The literature of graph theory contains many graphical sequences and sets of integers that concern graphical invariants (see Buckley and Harary [25]). For a given graph G and a given graphical invariant π, such sequences and sets of integers are usually lists of π-values of all (or some) vertices or subgraphs of G. An advantage of studying and using such sequences and sets of integers is that they are often nearly as easy to calculate as single numerical invariants yet they carry far more information about graphs they represent and about invariants for which they are formed. In §4.1 of this chapter we discuss some sequences concerning the irredundance, domination and independence numbers. In §4.2, we study interpolation properties of the independence, domination and irredundance numbers.

4.1. Domination-feasible sequences. Let π be an integer-valued graphical invariant. A sequence $(a_0, a_1, \dots, a_n)$ of positive integers is said to be a π-*feasible sequence* if there exists a graph G with distinguished vertices $v_1, v_2, \dots, v_n$ such that $\pi(G) = a_0$ and $\pi(G - v_1 - v_2 - \dots - v_i) = a_i$ for $i = 1, 2, \dots, n$. π-feasible sequences describe possible behaviors of the invariant π in successive vertex-deleted subgraphs and they have been studied by Harary and Kabell [78] for π being the connectivity κ, the line connectivity λ, the chromatic index χ', the diameter d, the number of edges q, the minimum degree δ, and the maximum degree Δ of a graph. In this section, we characterize π-feasible sequences for the parameter π being the upper irredundance number IR, the lower (upper) independence number i (α), and the lower domination number γ. Since the deletion of a vertex from a graph can change dramatically the lower irredundance number and the upper domination number (see Theorem 2.2.2 and Proposition 2.2.1), it is not easy to find a complete characterization of all ir- and Γ-feasible sequences. For this reason, for ir- and Γ-feasible sequences we only have partial results. The next two theorems due to Topp [142] characterize γ-, i-, α-, and IR-feasible sequences.

THEOREM 4.1.1. *Let $\eta = (a_0, a_1, \dots, a_n)$ be a sequence of positive integers. Then the following three statements are equivalent*:

(1) *η is a γ-feasible sequence*;
(2) *η is an i-feasible sequence*;
(3) *$a_l \geq a_{l-1} - 1$ for $l = 1, 2, \dots, n$.*

Moreover, *each of the statements* (1)–(3) *implies the statement*

(4) *η is an* ir-*feasible sequence.*

Proof. The implications (1)⇒(3) and (2)⇒(3) easily follow from Theorem 2.2.1. We shall prove that (3) implies (1), (2), and (4). Assume that $\eta = (a_0, a_1, \dots, a_n)$ is a sequence of positive integers with $a_l \geq a_{l-1} - 1$ for $l = 1, \dots, n$. Let p be any integer greater than $\max\{n + a_i : i = 0, 1, \dots, n\}$, and let Y and W be two disjoint sets of cardinality p and $a_n + n$, respectively, say $Y = \{y_1, y_2, \dots, y_p\}$ and $W = \{w_1, w_2, \dots, w_{a_n+n}\}$. For the sake of convenience, we order the set $\{y_1\} \cup W$ by stipulating that $y_1 < w_1 < w_2 < \dots < w_{a_n+n}$. Taking elements of the set W and the complete graph $K[Y]$ on the vertex set Y, we construct successively graphs $G_n, G_{n-1}, \dots, G_1$, and G_0. First, let G_n be the graph with vertex set $Y \cup \{w_1, w_2, \dots, w_{a_n}\}$ and edge set $E(K[Y]) \cup \{y_i w_i : i = 1, 2, \dots, a_n\}$. It is easy to observe that $J_n = \{y_1\} \cup \{w_i : 2 \leq i \leq a_n\}$ ($J_n = \{y_1\}$ if $a_n = 1$) is a smallest maximal irredundant set and a maximal independent set in G_n. From this and from Proposition 2.1.3 we have $\mathrm{ir}(G_n) = \gamma(G_n) = i(G_n) = |J_n| = a_n$. Suppose now that for some integer m, $n \geq m \geq 1$, the graphs $G_n, G_{n-1}, \dots, G_m$ are already constructed. In addition, assume that for every integer i, $n \geq i \geq m$, there exists a subset J_i of $\{y_1\} \cup \{w_2, w_3, \dots, w_{a_n+n-i}\}$ which is a smallest maximal irredundant set and a maximal independent set of cardinality a_i in G_i. Then we construct G_{m-1} by taking G_m and the vertex $t = w_{a_n+n-m+1}$, and joining t to the vertex

$y_{a_n+n-m+1}$ of G_m if $a_{m-1} = a_m + 1$, or to all the vertices of G_m if $a_m = 1$. If $1 < a_{m-1} \le a_m$ and $J_m = \{x_1, x_2, \ldots, x_{a_m}\}$, where the elements of J_m are arranged increasingly, then we join the vertex t to the vertices of $N_{G_m}[\{x_i : a_{m-1} \le i \le a_m\}]$. Let J_{m-1} be the set defined by $J_{m-1} = J_m \cup \{t\}$ if $a_{m-1} = a_m + 1$, or $J_{m-1} = \{t\}$ if $a_{m-1} = 1$, or $J_{m-1} = (J_m - N_{G_{m-1}}(t)) \cup \{t\}$ if $1 < a_{m-1} \le a_m$, respectively. One sees immediately that J_{m-1} is a maximal independent set of cardinality a_{m-1} in G_{m-1}. Thus $\mathrm{ir}(G_{m-1}) \le \gamma(G_{m-1}) \le i(G_{m-1}) \le |J_{m-1}| = a_{m-1}$. We now claim that J_{m-1} is a smallest maximal irredundant set in G_{m-1}. This claim is trivial if $a_{m-1} = 1$. Thus assume that $a_{m-1} \ge 2$ and suppose to the contrary that $\mathrm{ir}(G_{m-1}) < a_{m-1} = |J_{m-1}|$. Let J be a smallest maximal irredundant set in G_{m-1}. It is no problem to observe that if $J_{m-1} = \{u_1, u_2, \ldots, u_{a_{m-1}}\}$, where the elements of J_{m-1} are again written in the increasing order, then the sets $V_1 = N_{G_{m-1}}[u_1] - \bigcup_{i=2}^{a_{m-1}} N_{G_{m-1}}[u_i]$, $V_2 = N_{G_{m-1}}[u_2], \ldots, V_{a_{m-1}} = N_{G_{m-1}}[u_{a_{m-1}}]$ are nonempty and form a partition of the vertex set of G_{m-1}. Since the cardinality of J is smaller than a_{m-1}, at least one of the sets $J \cap V_i$, $1 \le i \le a_{m-1}$, is empty. Let i_0 be the smallest integer i, $1 \le i \le a_{m-1}$, such that $J \cap V_i = \emptyset$. There are two cases to be considered: $i_0 = 1$, $i_0 > 1$.

Case 1: $i_0 = 1$. In this case it follows from the construction of the graphs $G_n, G_{n-1}, \ldots, G_{m-1}$ that $u_1 = y_1$ if $a_j \ge 2$ for each $j \in \{m, m+1, \ldots, n-1\}$ or $u_1 = w_{a_n+n-j_0}$, where j_0 is the smallest integer $j \in \{m, m+1, \ldots, n-1\}$ such that $a_j = 1$. Since $J \cap V_1 = \emptyset$ and $N_{G_{m-1}}[w_1] = \{y_1, w_1\} \subset V_1$ if $u_1 = y_1$ or $N_{G_{m-1}}[w_1] \subset \{y_1, w_1, w_2, \ldots, w_{a_n+n-j_0}\} \subset V_1$ if $u_1 = w_{a_n+n-j_0}$, w_1 is an isolated vertex in $N_{G_{m-1}}[J \cup \{w_1\}]$ and the maximality of J implies that $J \cup \{w_1\}$ is not an irredundant set in G_{m-1}. Thus, there exists a vertex x in J such that $I_{G_{m-1}}(x, J \cup \{w_1\}) = \emptyset$, while $I_{G_{m-1}}(x, J) \ne \emptyset$. Consequently, the set $I_{G_{m-1}}(x, J)$ is a subset of $N_{G_{m-1}}[w_1]$. This forces that x belongs to the set Y. In addition, x is the unique vertex which belongs to $J \cap Y$; for if there were another x' in $J \cap Y$, then since $N_{G_{m-1}}[x'] \cap N_{G_{m-1}}[w_1] = N_{G_{m-1}}[x] \cap N_{G_{m-1}}[w_1]$, the set $I_{G_{m-1}}(x, J)$ would be empty and this would contradict the irredundance of J in G_{m-1}. Since $J \cap Y = \{x\}$ and $p > \max\{a_i + n : 0 \le i \le n\}$, it is easy to observe that $\{y_1\} \cup \{y_i : a_n + n - m + 1 < i \le p\}$ is a subset of $I_{G_{m-1}}(x, J)$. Hence $|Y \cap I_{G_{m-1}}(x, J)| \ge 2$ and this contradicts the fact that $I_{G_{m-1}}(x, J)$ is a subset of $N_{G_{m-1}}[w_1]$ since $|Y \cap N_{G_{m-1}}[w_1]| = 1$.

Case 2: $i_0 > 1$. Since J is a maximal irredundant set in G_{m-1} and J is disjoint to $V_{i_0} = N_{G_{m-1}}[u_{i_0}]$, $J \cup \{u_{i_0}\}$ is not an irredundant set in G_{m-1}. Therefore, as in the first case, there exists $x \in J$ such that $I_{G_{m-1}}(x, J \cup \{w_{i_0}\}) = \emptyset$ and $I_{G_{m-1}}(x, J) \ne \emptyset$. Thus $I_{G_{m-1}}(x, J)$ is a subset of $N_{G_{m-1}}[u_{i_0}]$. Moreover, in this case the structure of G_{m-1} forces that $I_{G_{m-1}}(x, J)$ is a subset of $Y \cap N_{G_{m-1}}[u_{i_0}]$, and the vertex y_p does not belong to $N_{G_{m-1}}[u_{i_0}]$. On the other hand, a simple verification shows that Y is a subset of $N_{G_{m-1}}[x]$ and, in particular, $y_p \in N_{G_{m-1}}[x]$. In addition, $y_p \notin N_{G_{m-1}}[J - \{x\}]$, as otherwise, if $y_p \in N_{G_{m-1}}[J - \{x\}]$, then $Y \subset N_{G_{m-1}}[x']$ for some $x' \in J - \{x\}$ and the set $I_{G_{m-1}}(x, J)$ would be empty, contrary to the assumption that J is an irredundant set in G_{m-1}. Hence, $y_p \in I_{G_{m-1}}(x, J)$

and this contradicts the fact that $I_{G_{m-1}}(x, J)$ is a subset of $Y \cap N_{G_{m-1}}[u_{i_0}]$ (since $y_p \notin N_{G_{m-1}}[u_{i_0}]$).

Since both the cases lead to contradictions, we must reject the assumption that $\mathrm{ir}(G_{m-1}) < a_{m-1}$. Consequently, $\mathrm{ir}(G_{m-1}) = \gamma(G_{m-1}) = i(G_{m-1}) = a_{m-1}$. Finally, for $m = 1$, we have $\mathrm{ir}(G_0) = \gamma(G_0) = i(G_0) = a_0$. Moreover, since $G_0 - w_{a_n+n} - w_{a_n+n-1} - \ldots - w_{a_n+n-i+1} = G_i$ and $\mathrm{ir}(G_i) = \gamma(G_i) = i(G_i) = a_i$ for $i = 1, 2, \ldots, n$, the sequence η is ir-feasible, γ-feasible, and i-feasible. This proves the implications (3)⇒(1), (3)⇒(2), (3)⇒(4) and completes the proof. ∎

Figure 19 illustrates the proof of Theorem 4.1.1 for $\eta = (3, 2, 3, 2, 1, 2)$.

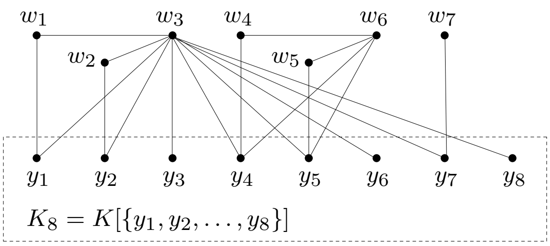

Fig. 19. A graph to illustrate the proof of Theorem 4.1.1

For the upper independence, irredundance and domination numbers, we have the following counterpart of Theorem 4.1.1.

THEOREM 4.1.2. *If* $\eta = (a_0, a_1, \ldots, a_n)$ *is a sequence of positive integers, then the following three statements are equivalent*:

(1) η *is an* α*-feasible sequence*;
(2) η *is an* IR*-feasible sequence*;
(3) $a_{l-1} \geq a_l \geq a_{l-1} - 1$ *for* $l = 1, 2, \ldots, n$.

Moreover, *each of the statements* (1)–(3) *implies the statement*

(4) η *is an* Γ*-feasible sequence.*

Proof. The implications (1)⇒(3) and (2)⇒(3) follow from Theorem 2.2.1. Thus it suffices to prove that (3) implies (1), (2), and (4). Assume that $\eta = (a_0, a_1, \ldots, a_n)$ is a sequence of positive integers with $a_{l-1} \geq a_l \geq a_{l-1} - 1$ for $l = 1, 2, \ldots, n$. Let K_{n+1} and K_{a_0} be disjoint complete graphs with vertex sets $\{x_1, v_2, \ldots, v_{n+1}\}$ and $\{y_1, \ldots, y_{a_0}\}$, respectively, and define the graph G to be the join $K_{n+1} + \overline{K}_{a_0}$ where $\overline{K}_{a_0}$ is the complement of K_{a_0}. Then $\alpha(G) = \Gamma(G) = \mathrm{IR}(G) = a_0$. Now let $v_1, v_2, \ldots, v_n$ be vertices of G such that

$$v_i = \begin{cases} x_{i-k} & \text{if } a_i = a_{i-1} \text{ and } a_0 - a_i = k \\ y_k & \text{if } a_i = a_{i-1} - 1 \text{ and } a_0 - a_i = k \end{cases}$$

for $i = 1, \ldots, n$. Then for $i = 1, \ldots, n$, if $a_i = a_0 - k$ for some nonnegative integer k, the graph $G - v_1 - \ldots - v_i$ is obtained from $K_{n+1} + \overline{K}_{a_0}$ by the removal of k vertices belonging to the subgraph $\overline{K}_{a_0}$ and of $i - k$ vertices belonging to the subgraph K_{n+1} and so $G - v_1 - \ldots - v_i$ is isomorphic to $K_{n+1-(i-k)} + \overline{K}_{a_0-k}$ which, in turn, implies that $\alpha(G - v_1 - \ldots - v_i) = \Gamma(G - v_1 - \ldots - v_i) = \mathrm{IR}(G - v_1 - \ldots - v_i) = a_0 - k = a_i$. Thus the sequence η is α-, Γ- and IR-feasible and the proof is complete. ∎

Proposition 2.2.1 and the remarks following it indicate some difficulties in finding a characterization of Γ-feasible sequences. In fact, we do not know a complete characterization of Γ-feasible sequences. Instead, we give a characterization of Γ-feasible sequences $(a_0, a_1, \ldots, a_n)$ with $a_n \geq 2$. In the proof, we will use graphs A_n and D_n defined in §2.2 and the following two propositions.

PROPOSITION 4.1.1. *For any two graphs F and H, we have $\Gamma(F + H) = \max\{\Gamma(F), \Gamma(H)\}$, where $F + H$ is the join of F and H.*

Proof. If both F and H are complete graphs, then $F + H$ is a complete graph and the result is obvious; so we assume that F or H is not complete. Then $\Gamma(F + H) \geq 2$ and $\max\{\Gamma(F), \Gamma(H)\} \geq 2$. It is easy to observe that if D is a minimal dominating set of one of the graphs F and H, then D is a minimal dominating set of $F + H$ and therefore $\Gamma(F + H) \geq \max\{\Gamma(F), \Gamma(H)\}$. To prove that $\Gamma(F + H) \leq \max\{\Gamma(F), \Gamma(H)\}$, let D be a largest minimal dominating set of $F + H$. First, if both $D \cap V(F)$ and $D \cap V(H)$ are nonempty sets, then it follows from the minimality of D that $|D \cap V(F)| = |D \cap V(H)| = 1$ and therefore $\Gamma(F + H) = 2 \leq \max\{\Gamma(F), \Gamma(H)\}$. Finally, if exactly one of the sets $D \cap V(F)$ and $D \cap V(H)$ is nonempty, say $D \cap V(F) \neq \emptyset$, then D is a minimal dominating set of F and so $\Gamma(F + H) = |D| \leq \Gamma(F) \leq \max\{\Gamma(F), \Gamma(H)\}$. This completes the proof. ∎

If G_1 and G_2 are graphs having exactly one common vertex c, say, then let $G_1 * G_2$ be the graph in which $V(G_1 * G_2) = V(G_1) \cup V(G_2)$, $N_{G_1 * G_2}(c) = N_{G_1}(c) \cup N_{G_2}(c)$ and $G_1 * G_2 - c = (G_1 - c) + (G_2 - c)$.

PROPOSITION 4.1.2. *If graphs F and H have exactly one common vertex c, say, then $\Gamma(F * H) = \max\{\Gamma(F), \Gamma(H)\}$ and $\Gamma(F * H - c) = \max\{\Gamma(F - c), \Gamma(H - c)\}$.*

Proof. The equality $\Gamma(F * H - c) = \max\{\Gamma(F - c), \Gamma(H - c)\}$ follows from Proposition 4.1.1 and from the fact that $F * H - c = (F - c) + (H - c)$. Thus it remains only to verify that $\Gamma(F * H) = \max\{\Gamma(F), \Gamma(H)\}$. The last equality is obvious if F and H are complete graphs. Therefore we assume that F or H is not a complete graph. Then $\Gamma(F * H) \geq 2$ and $\max\{\Gamma(F), \Gamma(H)\} \geq 2$. It is easy to observe that every minimal dominating set of F or of H is a minimal dominating set of $F * H$, which, in turn, implies the inequality $\Gamma(F * H) \geq \max\{\Gamma(F), \Gamma(H)\}$. To prove the inequality $\Gamma(F * H) \leq \max\{\Gamma(F), \Gamma(H)\}$, let D be a largest minimal dominating set of $F * H$. We consider two cases.

Case 1: $c \notin D$. If both $D \cap V(F)$ and $D \cap V(H)$ are nonempty sets, then it follows from the minimality of D that $|D| = 2$ and so $\Gamma(F * H) = 2 \leq \max\{\Gamma(F), \Gamma(H)\}$. If exactly one of the sets $D \cap V(F)$ and $D \cap V(H)$ is nonempty, say $D \cap V(F) \neq \emptyset$, then D is a minimal dominating set of F and therefore $\Gamma(F * H) = |D| \leq \Gamma(F) \leq \max\{\Gamma(F), \Gamma(H)\}$.

Case 2: $c \in D$. In this case $D \cap N_{F*H}(c) = \emptyset$; otherwise $I_{F*H}(c, D) = \emptyset$ and D would not be a minimal dominating set of $F * H$. Assume first that both $D \cap V(F - c)$ and $D \cap V(H - c)$ are nonempty sets. Then, since D is a minimal dominating set, $|D \cap V(F-c)| = 1 = |D \cap V(H-c)|$ and therefore $\Gamma(F*H) = 3$. Let x and y be the unique vertices of $D \cap (F-c)$ and $D \cap V(H-c)$, respectively. Since x is adjacent to y and to every other vertex of $H-c$, $I_{F*H}(y, D)$ is a nonempty subset of $V(F-c)$. Now, for any $z \in I_{F*H}(y, D)$, the set $\{x, z, c\}$ is independent in F and so $3 \leq \alpha(F) \leq \Gamma(F)$. Consequently, $\Gamma(F * H) = 3 \leq \Gamma(F) \leq \max\{\Gamma(F), \Gamma(H)\}$. Finally, assume that either $D \cap V(F - c)$ or $D \cap V(H - c)$ is nonempty, say $D \cap V(F - c) \neq \emptyset$. Then D is a minimal dominating set of F and $\Gamma(F * H) = |D| \leq \Gamma(F) \leq \max\{\Gamma(F), \Gamma(H)\}$. This completes the proof. ■

THEOREM 4.1.3. *Let $\eta = (a_0, a_1, \ldots, a_n)$ be a sequence of positive integers with $a_n \geq 2$. Then η is a Γ-feasible sequence if and only if $a_i \geq 2$ for $i = 0, 1, \ldots, n-1$.*

Proof. Assume first that $(a_0, a_1, \ldots, a_n)$ is a Γ-feasible sequence with $a_n \geq 2$. Then there exists a graph G with distinguished vertices $v_1, \ldots, v_n$ such that $\Gamma(G) = a_0$ and $\Gamma(G - v_1 - \ldots - v_i) = a_i$ for $i = 1, \ldots, n$. Now, if there were $a_i = 1$ for some $i < n$, then $G - v_1 - \ldots - v_i$ (or G if $i = 0$) and each induced subgraph of $G - v_1 - \ldots - v_i$ (of G if $i = 0$) would be a complete graph. In particular, $G - v_1 - \ldots - v_n$ would be a complete graph and so $a_n = \Gamma(G - v_1 - \ldots - v_n) = 1$, contradicting the assumption that $a_n \geq 2$.

To prove the "only if" part of the theorem, we assume that $(a_0, a_1, \ldots, a_n)$ is a sequence of positive integers with $a_i \geq 2$ for $i = 0, 1, \ldots, n$. Take $n + 1$ disjoint graphs D_{a_0}, $D_{a_1}, \ldots, D_{a_n}$. Since $a_i \geq 2$, D_{a_i} contains exactly one vertex u_i, say, adjacent to every vertex of maximum degree in D_{a_i}, $i = 1, \ldots, n$. Now, for $i = 1, \ldots, n$, let v_i be a fixed vertex of maximum degree in $D_{a_{i-1}}$ and consider the graph A_{a_i} obtained from D_{a_i} by adding the vertex v_i and joining it to all vertices of $D_{a_i} - N_{D_{a_i}}[u_i]$. Note that v_1 is the only common vertex of D_{a_0} and A_{a_1}. For $i = 2, \ldots, n$, v_i is the only common vertex of $(\ldots(D_{a_0} * A_{a_1}) * \ldots * A_{a_{i-2}}) * A_{a_{i-1}}$ and A_{a_i}. Thus, the graph

$$G = (\ldots((D_{a_0} * A_{a_1}) * A_{a_2}) * \ldots * A_{a_{n-1}}) * A_{a_n}$$

is well-defined and it easily follows from Propositions 2.2.1 and 4.1.2 that $\Gamma(G) = a_0$. Moreover,

$$G - v_1 = (\ldots(((D_{a_0} - v_1) + (A_{a_1} - v_1)) * A_{a_2}) * \ldots * A_{a_{n-1}}) * A_{a_n}$$

is isomorphic to $(\ldots((A_{a_0 - 1} + D_{a_1}) * A_{a_2}) * \ldots * A_{a_{n-1}}) * A_{a_n}$ and therefore it follows from Propositions 2.2.1, 4.1.1 and 4.1.2 that $\Gamma(G - v_1) = a_1$. Finally, for $i = 2, \ldots, n$, the graph $G - v_1 - \ldots - v_i = (\ldots((D_{a_0} - v_1) + (A_{a_1} - v_1 - v_2) + \ldots + (A_{a_{i-1}} - v_{i-1} - v_i) + (A_{a_i} - v_i)) * A_{a_{i+1}} * \ldots * A_{a_{n-1}}) * A_{a_n}$ is isomorphic to $(\ldots(A_{a_0 - 1} + A_{a_1 - 1} + \ldots + A_{a_{i-1} - 1} + D_{a_i}) * A_{a_{i+1}} * \ldots * A_{a_{n-1}}) * A_{a_n}$ and again it follows from Propositions 2.2.1, 4.1.1 and 4.1.2 that $\Gamma(G - v_1 - \ldots - v_i) = a_i$. This proves that $(a_0, a_1, \ldots, a_n)$ is a Γ-feasible sequence. ∎

4.2. Interpolation properties of domination parameters. In 1980, G. Chartrand [32] raised the following problem: If a graph G possesses a spanning tree having m end vertices and another having M end vertices, where $M > m$, does G possess a spanning tree having k end vertices for every k between m and M? This question was answered affirmatively in [125] and it led to a number of papers studying the interpolation properties of parameters of spanning trees of a given graph. In [82], the various known interpolation results are examined and classified on the basis of the proof techniques used in establishing them. Motivated by results of the papers [82] and [83], we investigate the interpolation properties of the irredundance, domination, and independence numbers of a graph. For the sake of completeness we give a few definitions here. For a connected graph G, let $\mathcal{T}(G)$ be the set of all spanning trees of G. Let T be a spanning tree of G and let e be an edge of G which is not in T. If f is an edge which is in the unique cycle of $T + e$, then $T + e - f$ is a spanning tree of G and the transformation $T \to T + e - f$ is called a fundamental exchange. If e and f are adjacent edges of G, then the transformation $T \to T + e - f$ is called a neighbour exchange. A neighbour exchange $T \to T + e - f$ is called an end-edge exchange if f is an end edge in T. It is known that any spanning tree $T \in \mathcal{T}(G)$ can be transformed into a spanning tree $T^* \in \mathcal{T}(G)$ by a sequence of neighbour exchanges. Lovász [105, p. 269] and Harary, Mokken and Plantholt [81] have proved that if G is a 2-connected graph, then any $T \in \mathcal{T}(G)$ can be transformed into any $T^* \in \mathcal{T}(G)$ by a sequence of end-edge exchanges.

An integer-valued graph function π is said to *interpolate over a connected graph* G if the set $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$, listed in increasing order, is a set of consecutive integers. A function π *interpolates over a family* $\mathcal{F}$ of graphs, if π interpolates over each graph of the family $\mathcal{F}$. Finally, we shall say that π is an *interpolating function* if π interpolates over each connected graph.

Our first theorem indicates that unicyclic graphs play a significant role in investigating of the interpolation properties of integer-valued graph functions. Among other things, it follows from Theorem 4.2.1 that if an integer-valued graph function π is not an interpolating function, then there exists a unicyclic graph G such that π does not interpolate over G.

THEOREM 4.2.1. *An integer-valued graph function π is an interpolating function if and only if π interpolates over the family of all unicyclic graphs.*

Proof. The necessity of the condition is clear. To prove the sufficiency, assume that π interpolates over the family of all unicyclic graphs and let G be any connected graph. Then it suffices to show that $\pi(\mathcal{T}(G))$ is a set of consecutive integers if G has at least two cycles and $|\pi(\mathcal{T}(G))| \geq 2$. Let m and M be the smallest and the largest integer of $\pi(\mathcal{T}(G))$, respectively. Let T_0, $T^* \in \mathcal{T}(G)$

be such that $\pi(T_0) = m$ and $\pi(T^*) = M$, and let $T_0, T_1, \ldots, T_n = T^*$ be a sequence of neighbour exchanges transforming T_0 into T^*. For $i = 0, 1, \ldots, n-1$, let e_i and f_i be the edges of G such that $T_{i+1} = T_i + e_i - f_i$. Since $T_i + e_i$ is a unicyclic graph, according to our hypothesis $\pi(\mathcal{T}(T_i + e_i))$ is a set of consecutive integers for $0 \le i \le n-1$. Moreover, since $T_i, T_{i+1} \in \mathcal{T}(T_i + e_i)$, the sets $\pi(\mathcal{T}(T_i + e_i))$ and $\pi(\mathcal{T}(T_{i+1} + e_{i+1}))$ are not disjoint and therefore their union $\pi(\mathcal{T}(T_i + e_i)) \cup \pi(\mathcal{T}(T_{i+1} + e_{i+1}))$ is a set of consecutive integers. Consequently, the union $\bigcup_{i=0}^{n-1} \pi(\mathcal{T}(T_i + e_i))$ is a set of consecutive integers. Finally, we have $\{m, m+1, \ldots, M\} \subseteq \bigcup_{i=0}^{n-1} \pi(\mathcal{T}(T_i + e_i)) \subseteq \pi(\mathcal{T}(G)) \subseteq \{m, m+1, \ldots, M\}$ and therefore $\pi(\mathcal{T}(G)) = \{m, m+1, \ldots, M\}$ is a set of consecutive integers. ∎

The following corollary gives a useful sufficient condition for an integer-valued graph function to be an interpolating function. This corollary was first observed by Harary and Plantholt [82] and it follows immediately from Theorem 4.2.1.

COROLLARY 4.2.1. *An integer-valued graph function π is an interpolating function if one of the conditions is satisfied*:

(1) *For every graph H and every edge vu of H, $\pi(H) \le \pi(H - vu) \le \pi(H) + 1$;*
(2) *For every graph H and every edge vu of H, $\pi(H) - 1 \le \pi(H - vu) \le \pi(H)$.* ∎

COROLLARY 4.2.2. *For any positive integer k, the k-packing number α_k and the k-covering number γ_k are interpolating functions.*

Proof. The result follows from Theorem 2.2.3 and Corollary 4.2.1. ∎

COROLLARY 4.2.3 [83]. *The independence number α and the domination number γ are interpolating functions.*

Proof. This follows from Corollary 4.2.2 and the observation that for any graph H, $\alpha(H) = \alpha_1(H)$ and $\gamma(H) = \gamma_1(H)$. ∎

COROLLARY 4.2.4. *The upper domination number Γ and the upper irredundance number* IR *are interpolating functions.*

Proof. Let G be any connected graph. It follows from Theorem 2.4.6 that $\alpha(T) = \Gamma(T) = \mathrm{IR}(T)$ for every tree $T \in \mathcal{T}(G)$. Thus, $\Gamma(\mathcal{T}(G)) = \mathrm{IR}(\mathcal{T}(G)) = \alpha(\mathcal{T}(G))$ and the result follows from Corollary 4.2.3. ∎

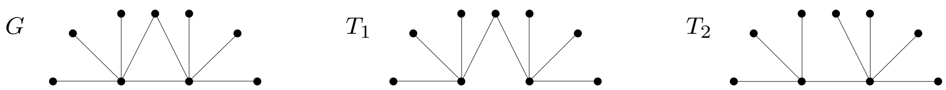

Fig. 20. A graph G and its nonisomorphic spanning trees T_1 and T_2 with $i(T_1) = 2$ and $i(T_2) = 4$

Harary and Schuster [83] have observed that the lower independence number i is not an interpolating function. This follows from the simple counter-example shown in Figure 20, in which the graph G has only two nonisomorphic spanning trees T_1 and T_2 with $i(T_1) = 2$ and $i(T_2) = 4$. Next the following theorem was shown by Harary and Plantholt [82].

THEOREM 4.2.2. *The lower independence number i interpolates over any 2-connected graph.*

Proof. Assume G is a 2-connected graph such that $|i(\mathcal{T}(G))| \geq 2$. Let m and M be the smallest and the largest integer of the set $i(\mathcal{T}(G))$, respectively, and let $T_0, T^* \in \mathcal{T}(G)$ be such that $i(T_0) = m$ and $i(T^*) = M$. As it was shown in [105, p. 269] and [81], there exists a sequence of end-edge exchanges $T_0, T_1, \ldots, T_n = T^*$ transforming T_0 into T^*.

We claim that $i(T_{k+1}) \leq i(T_k)+1$ for $0 \leq k \leq n-1$. To prove this, let I be any minimum maximal independent set in T_k and suppose that $T_{k+1} = T_k + wv - vu$, where v is an end vertex of T_k (and T_{k+1}). We consider four cases.

Case 1: $v \in I$, $w \notin I$. If $u \in N_{T_k}(I - \{v\})$ ($u \notin N_{T_k}(I - \{v\})$, resp.), then I ($I \cup \{u\}$, resp.) is a maximal independent set in T_{k+1}.

Case 2: $v \in I$, $w \in I$. If $u \in N_{T_k}(I - \{v\})$ ($u \notin N_{T_k}(I - \{v\})$, resp.), then $I - \{v\}$ ($(I - \{v\}) \cup \{u\}$, resp.) is a maximal independent set in T_{k+1}.

Case 3: $v \notin I$, $w \notin I$. Here $u \in I$ and $I \cup \{v\}$ is a maximal independent set in T_{k+1}.

Case 4: $v \notin I$, $w \in I$. Again $u \in I$ and it is easy to observe that I is a maximal independent set in T_{k+1}.

In each case the tree T_{k+1} has a maximal independent set of cardinality at most $|I| + 1$. Thus, $i(T_{k+1}) \leq i(T_k) + 1$. The last property implies that the sequence $(i(T_0), i(T_1), \ldots, i(T_n))$ contains $(m, m+1, \ldots, M)$ as a subsequence. Hence, $i(\mathcal{T}(G)) = \{m, m+1, \ldots, M\}$, so i interpolates over G. ∎

It follows from Corollary 2.2.2 and Theorem 2.2.6 that adding a new edge to a graph G or removing an edge from G may cause an increase or decrease of the lower irredundance number ir and that the extent to which the lower irredundance number can vary may be arbitrarily large. Therefore the analysis used for establishing Corollary 4.2.2 or Theorem 4.2.2 fails to yield any knowledge of the interpolating character of the lower irredundance number ir. Our preliminary observations that have been made so far convince us to formulate the following conjecture: The lower irredundance number ir is an interpolating function. Certainly, according to Theorem 4.2.1 it is enough to check whether the lower irredundance number ir interpolates or not over the unicyclic graphs.

Some other results concerning the interpolation properties of covering and domination numbers of a graph can be found in [84] and [143, 144, 145, 147].

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