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GRZEGORZ ANDRZEJCZAK

A semi-simplicial approach to foliations and their transverse structure

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CONTENTS

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Introduction

Since the Colloquium in Toulouse: "Structure Transverse des Feuilletages" in February 1982 it has been accepted that the structure of the space of leaves of a foliation is an equivalence class of its holonomy pseudogroups or, more generally, groupoids. In fact both the pseudogroups and the associated groupoids are involved because manifolds themselves have turned out to be too restrictive. On the other hand, classification problems for various types of foliations lead to immense topological spaces which are—as a rule—realizations of some semi-simplicial manifolds. The aim of the present paper is to reconcile these two aspects of foliation theory: both the transverse structure and the classifying objects can be identified within an appropriate category of foliated semi-simplicial manifolds.

The paper arose from the author's attempts to find universal characteristic classes for some elementary factors of the Bott characteristic classes. In particular, it turned out necessary to describe a geometrical setting in which the semi-simplicial nerve of Γ_q could play the role of the classifying space. Our solution to this problem refers to J. L. Dupont's lecture notes on semi-simplicial manifolds [10], R. Bott's papers on characteristic classes of foliations [6], [7], and A. Haefliger's [16] and W. T. van Est's [12] theory of groupoids and pseudogroups. It was first presented, in brief, in a series of preprints [2]–[4] prepared at the Max Planck Institut für Mathematik (under SFB 40—Univ. Bonn), to which the author expresses his gratitude. The present paper is a revised, completely rearranged and extended version of those preprints.

The paper consists of two parts. Part I is a self-contained introduction to ss-manifolds, groupoids, and principial Γ-bundles. Semi-simplicial manifolds provide both an elegant and visual way of representing manifolds equipped with groupoid-like structures, e.g. manifolds with open coverings, Lie groups, and pseudogroups of diffeomorphisms; via the appropriate smooth classification theorem, principal G-bundles as well as Γ-structures are identified with morphisms of our category (ss-morphisms). A localization procedure, deeply involved in the semi-simplicial approach, keeps control over those global concepts and constructions associated with ss-manifolds which admit a local description. In fact, the procedure automatically translates the topological complexity of a manifold, for example, into the combinatorial complexity of any of its coverings. Thus the semi-simplicial way of finding geometric-algebraic objects like fundamental groups for manifolds, holonomy pseudogroups for foliated manifolds, or characteristic homomorphisms for foliated G-bundles is applicable to any set of local data characterizing the (foliated) manifold or bundle.

Part II deals with foliations, and requires some elementary experience in this subject (cf. e.g. [8], [18], [20]) as we do not consider any concrete foliated manifold, and examples refer—as a rule—to specific particular cases of the theory. We have already mentioned that it is the holonomy pseudogroup which mirrors the transverse geometry of any foliation. One writes *the* holonomy pseudogroup because for any complete transversal T the associated pseudogroup of all the holonomy translations of portions of T is—up to canonical equivalences of pseudogroups—independent of the transversal. For foliations of ss-manifolds one has to answer the following fundamental question: which pseudogroups, and why, are to be *the* holonomy pseudogroups? In order to answer this question we introduce the notion of a transverse projection for a foliated ss-manifold (X, F) and define the *holonomy groupoid* $\mathit{\Gamma}_F$ together with an associated *minimal transverse projection* $\mathit{\Pi}_F : X \rightharpoonup \mathcal{N}\mathit{\Gamma}_F$ as any initial object in the appropriate category. This ensures the uniqueness of the holonomy groupoid (up to equivalence), and characterizes various canonical morphisms and equivalences between holonomy groupoids as the morphisms (equivalences) which transfer one minimal transverse projection to another. That abstract presentation is completed with three explicit models of the holonomy groupoid (pseudogroup). In this general setting, we reprove van Est's epimorphism theorem for the morphism of fundamental groupoids (groups) induced by a minimal transverse projection $\mathit{\Pi}_F$. Furthermore, it is shown that after passage to the leaves $\mathit{\Pi}_F$ splits into holonomy homomorphisms of the fundamental groups of the leaves of F onto the holonomy groups. The last section of Part II deals with G-structures and—more generally—foliated G-bundles on foliated ss-manifolds. Geometric G-structures provide numerous examples of $\mathit{\Gamma}$-foliations for specific $\mathit{\Gamma}'$s associated with closed subgroups of the linear groups of any order. The paper is concluded with a sample of smooth classification theorems—for foliated G-bundles and for so-called $\overline{G}$-integrable G-foliations.

The reader interested in further topological aspects of the theory is referred to the author's thesis [5] where some admissible (i.e. natural with respect to ss-morphisms) sheaf cohomology functors are considered as well. Foliated ss-manifolds admit also a Čech–de Rham cohomology functor for which the classical constructions of characteristic classes for foliations hold true (after some minor modification). A discussion of the respective cohomology classes—including the elementary classes [1], [2], and vanishing theorems—will be presented elsewhere.

Throughout the paper we adopt a general convention that the brackets [ ]—if they do not refer to the bibliography—indicate the equivalence class of an equivalence relation clear from the context. Thus the brackets are used to denote: germs of functions, homotopy classes of paths, elements of the fat realization of an ss-manifold, etc.; evidently, the Lie bracket of vector fields does not lead to any confusion.

I. The category of semi-simplicial manifolds

We present a category whose objects are smooth semi-simplicial manifolds and whose morphisms (ss-morphisms) are equivalence classes of some ss-maps. This modification of the classical theory [10] makes semi-simpicial manifolds more flexible without losing their differentiable structure. The main result here is a smooth classification theorem for principal Γ-bundles, Γ being any differentiable groupoid (Thm. 2.12). Via the classification theorem, ss-morphisms between nerves of groupoids can be identified with generalized homomorphisms of the groupoids in the sense of Haefliger [16]; we prove Haefliger's invertibility criterion for such ss-morphisms (Thm. 3.2). In Section 4 we introduce a (combinatorial) fundamental group for connected ss-manifolds and formulate a universal property which characterizes both the group and the associated simply connected covering ss-manifold (Thm. 4.11).

I.1. Semi-simplicial manifolds and semi-simplicial morphisms. Throughout the paper, by a *semi-simplicial manifold* (*ss-manifold*) we mean any semi-simplicial object [21] in the category of smooth not necessarily Hausdorff manifolds, i.e. any sequence $X = (X_n)_{n\geq 0}$ of manifolds (*levels* of X) together with a collection of (smooth) *face operators* $\varepsilon_i : X_n \to X_{n-1}$, $i \leq n$, and *degeneracy operators* $\eta_i : X_n \to X_{n+1}$, $i \leq n$. The *structure operators* ε_i, η_j are assumed to satisfy the following commutation relations:

$$
\tag{1.1}
\begin{aligned}
\varepsilon_i\varepsilon_j &= \varepsilon_{j-1}\varepsilon_i && \text{if } i < j\,,\\
\varepsilon_i\eta_j &= \begin{cases} \eta_{j-1}\varepsilon_i & \text{if } i < j\,,\\ \mathrm{id} & \text{if } i = j, j+1\,,\\ \eta_j\varepsilon_{i-1} & \text{if } i > j+1\,,\end{cases}\\
\eta_i\eta_j &= \eta_{j+1}\eta_i && \text{if } i \leq j\,.
\end{aligned}
$$

An *ss-map* $f : X \to Y$ of $X = (X_n)$ to $Y = (Y_n)$ is a sequence $f = (f_n)$ of smooth maps $f_n : X_n \to Y_n$ commuting with the structure operators. Isomorphic ss-manifolds will be denoted by $X \cong Y$.

EXAMPLE 1.1. By associating to any manifold M the ss-manifold $\mathcal{N}M = (M)_{n\geq 0}$ with $\varepsilon_i = \eta_i = \mathrm{id}$ for all n, i, one identifies manifolds with a full subcategory of ss-manifolds.

EXAMPLE 1.2 ([10]). Any open covering $\mathcal{U} = (U_a)_{a\in A}$ of a manifold M gives rise to an ss-manifold $\mathcal{N}\mathcal{U} = (\mathcal{N}_n\mathcal{U})$, the *nerve* of $\mathcal{U}$, such that

$$\mathcal{N}_n\mathcal{U} = \coprod_{(a_0,\dots,a_n)\in A^{n+1}} U_{a_0} \cap \dots \cap U_{a_n} \qquad \text{(disjoint union)}$$

and the structure operators are the inclusions

$$U_{a_0} \cap \dots \cap U_{a_n} \ni x \quad \begin{matrix} \overset{\varepsilon_i}{\nearrow} \; x \in U_{a_0} \cap \dots \cap U_{a_{i-1}} \cap U_{a_{i+1}} \cap \dots \cap U_{a_n} \\ \underset{\eta_i}{\searrow} \; x \in U_{a_0} \cap \dots \cap U_{a_i} \cap U_{a_i} \cap \dots \cap U_{a_n}\,. \end{matrix}$$

Roughly speaking, $\mathcal{N}\mathcal{U}$ has the same differentiable structure as M, while some part of the topological complexity of M is expressed in combinatorial language.

EXAMPLE 1.3. We recall that a (differentiable) *groupoid* Γ over a manifold N is a small category with only invertible morphisms, having N as the set of objects, and endowed with a differentiable structure such that:
— the *source* and *target* maps $\alpha, \beta : \Gamma \to N$ are submersions, and
— the composition and the inverse mapping are smooth.
One identifies N with the submanifold of units of Γ.

Every groupoid Γ gives rise to an ss-manifold $\mathcal{N}\Gamma = (\mathcal{N}_n\Gamma)$, the *nerve* of Γ ([21], [10]), such that

$$\mathcal{N}_0\Gamma = N\,,$$
$$\mathcal{N}_n\Gamma = \{(g_1,\dots,g_n) \in \Gamma^n; \alpha g_1 = \beta g_2, \dots, \alpha g_{n-1} = \beta g_n\}$$

for $n \geq 1$, and the structure operators are defined as follows:

$$\varepsilon_0 = \alpha,\ \varepsilon_1 = \beta \quad \text{on } \mathcal{N}_1\Gamma = \Gamma,$$

$$\varepsilon_i(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & \text{for } i = 0\,, \\ (\dots, g_i g_{i+1}, \dots) & \text{for } i = 1, \dots n-1\,, \\ (g_1,\dots,g_{n-1}) & \text{for } i = n\,, \end{cases}$$

$$\eta_0 : N \hookrightarrow \Gamma\,,$$

$$\eta_i(g_1,\dots,g_n) = \begin{cases} (\beta g_1, g_1, \dots, g_n) & \text{for } i = 0\,, \\ (\dots, g_i, \alpha g_i, g_{i+1}, \dots) & \text{for } i > 0\,. \end{cases}$$

For any two groupoids Γ and Γ' the restriction to $\mathcal{N}_1$ yields a bijective correspondence between ss-maps $\mathcal{N}\Gamma \to \mathcal{N}\Gamma'$ and homomorphisms (smooth functors) $\Gamma \to \Gamma'$; the extension of $h : \Gamma \to \Gamma'$ to an ss-map will be denoted by $\mathcal{N}h$.

Definition 1.4. The *localization* of an ss-manifold $X = (X_n)$ to an open covering $\mathcal{U} = (U_a)_{a\in A}$ of X_0 is an ss-manifold $X_{\mathcal{U}}$ such that

$$X_{\mathcal{U}}(n) = \coprod_{(a_0,\dots,a_n)\in A^{n+1}} \bigcap_{i=0}^{n} ({\varepsilon_1}^{n-i}{\varepsilon_0}^{i})^{-1} U_{a_i}$$

(${\varepsilon_i}^j := \varepsilon_i \circ \dots \circ \varepsilon_i$, j times) and the structure operators are

$$(a_0,\dots,a_n;x) \xrightarrow{\varepsilon_i} (\dots,a_{i-1},a_{i+1},\dots;\varepsilon_i x), \qquad (a_0,\dots,a_n;x) \xrightarrow{\eta_i} (\dots,a_i,a_i,\dots;\eta_i x)$$

where $(a_0,\dots,a_n;x) := ((a_0,\dots,a_n),x)$. A careful application of the axioms ensures that the new maps ε_i, η_i are well defined and satisfy (1.1).

Example 1.5. For any open covering $\mathcal{U}$ of a manifold M, one has $\mathcal{N}\mathcal{U} = (\mathcal{N}M)_{\mathcal{U}} \cong \mathcal{N}(M_{\mathcal{U}})$ where $M_{\mathcal{U}}$ stands for the manifold $\mathcal{N}_1\mathcal{U}$ equipped with an appropriate groupoid structure.

Example 1.6. Let $X = (X_n)$ be an ss-manifold, $\mathcal{U} = (U_a)_{a\in A}$ an open covering of X_0, and Γ a groupoid. For any ss-map $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$ let $\gamma_{ab} : \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b \to \Gamma$, $a,b \in A$, be the components of f_1. Then the assingment $f \rightsquigarrow (\gamma_{ab})_{a,b\in A}$ establishes a bijective correspondence between ss-maps $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$ and collections of maps $\gamma_{ab}, a, b \in A$, such that

$$(\gamma_{ab}\varepsilon_2)(\gamma_{bc}\varepsilon_0) = \gamma_{ac}\varepsilon_1 \tag{1.2}$$

on $(\varepsilon_1\varepsilon_1)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0\varepsilon_0)^{-1}U_c \subset X_2$. Generalizing the classical notion of Γ-*cocycle* ([14]) we shall call any collection of maps satisfying (1.2) a Γ-*cocycle* on X with respect to the covering $\mathcal{U}$.

In order to extend a Γ-cocycle (γ_{ab}) to an ss-map $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$, one has to set

$$\begin{aligned} f_0(a,x) &= \gamma_{aa}(\eta_0 x)\,, \\ f_n(a_0,\dots,a_n;x) &= (\gamma_{a_0a_1}({\varepsilon_2}^{n-1}x), \gamma_{a_1a_2}({\varepsilon_2}^{n-2}\varepsilon_0 x), \dots \\ &\quad \dots, \gamma_{a_{n-1}a_n}({\varepsilon_0}^{n-1}x)) \end{aligned} \tag{1.3}$$

for $n \geq 1$.

For any open covering $\mathcal{U} = (U_a)_{a\in A}$ of the 0-th level X_0 of an ss-manifold X there is a canonical *gluing projection* $\lambda = \lambda^{(\mathcal{U})} : X_{\mathcal{U}} \to X$,

$$(a_0,\dots,a_n;x) \to x\,.$$

Furthermore, if $\mathcal{V} = (V_i)_{i\in I}$ is any refinement of U, then each *refinement map* $\varrho : I \to A$ ($V_i \subset U_{\varrho(i)}$) gives rise to an ss-map $\varrho_\# : X_\mathcal{V} \to X_\mathcal{U}$,

$$(i_0, \ldots, i_n; x) \to (\varrho(i_0), \ldots, \varrho(i_n); x)\,,$$

which is evidently compatible with the gluing projections.

Given ss-manifolds X and Y, two ss-maps $f : X_\mathcal{U} \to Y$ and $g : X_\mathcal{V} \to Y$ will be called *elementarily equivalent* iff one of the coverings is a refinement of the other, e.g. $\mathcal{U}$ of $\mathcal{V}$, and there is a refinement map ϱ such that the triangle

$$\begin{array}{ccc} X_\mathcal{U} & & \\ \varrho_\# \big\downarrow & \searrow^{f} & \\ & & Y \\ & \nearrow_{g} & \\ X_\mathcal{V} & & \end{array} \tag{1.4}$$

commutes (cf. [2]). The elementary equivalences generate an equivalence relation in the family of all ss-maps of the form $X_\mathcal{W} \to Y$, where $\mathcal{W}$ ranges over (open) coverings of X_0.

DEFINITION 1.7. An *ss-morphism* $\mathbf{f} : X \rightharpoonup Y$ (of X to Y) is any equivalence class of the relation generated by elementary equivalences.

Since every ss-morphism of X in Y has representatives $X_\mathcal{W} \to Y$ such that $\mathcal{W}$ is a non-indexed covering (indexed by itself) and these representatives form a set, it is possible to organize in a set all ss-morphisms of X in Y.

EXAMPLE 1.8. All the gluing projections $X_\mathcal{U} \to X$ represent the same *identity ss-morphism* $\mathbf{1}_X : X \rightharpoonup X$.

EXAMPLE 1.9. Any ss-map $\mathcal{N}M_\mathcal{U} \to \mathcal{N}M'$, M and M' being manifolds, comes from a uniquely defined map $M \to M'$. Thus the notions of a map $M \to M'$, an ss-map $\mathcal{N}M \to \mathcal{N}M'$, and an ss-morphism $\mathcal{N}M \rightharpoonup \mathcal{N}M'$ mean the same.

For any ss-manifolds X, Y, and Z, let $\mathbf{f} : X \rightharpoonup Y$ and $\mathbf{g} : Y \rightharpoonup Z$ be ss-morphisms; we take any representatives $f : X_\mathcal{U} \to Y$ and $g : Y_\mathcal{V} \to Z$ of $\mathbf{f}$ and $\mathbf{g}$, respectively, where $\mathcal{U} = (U_a)_{a\in A}$ and $\mathcal{V} = (V_i)_{i\in I}$. Let

$$f^{-1}\mathcal{V} := (f_{0a}^{-1} V_i)_{(a,i)\in A\times I} \tag{1.5}$$

where $f_0 = \sum f_{0a} : \coprod U_a \to Y_0$. In order to compose f and g, we localize f to $\mathcal{V}$ and define an ss-map $f_\mathcal{V} : X_{f^{-1}\mathcal{V}} \to Y_\mathcal{V}$,

$$((a_0, i_0), \ldots, (a_n, i_n); x) \to (i_0, \ldots, i_n; f_n(a_0, \ldots, a_n; x))\,. \tag{1.6}$$

DEFINITION 1.10. The *composition* $\mathbf{g} \circ \mathbf{f} : X \rightharpoonup Z$ of $\mathbf{f} : X \rightharpoonup Y$ and $\mathbf{g} : Y \rightharpoonup Z$ (also written $\mathbf{gf}$) is the ss-morphism represented by $g \circ f_\mathcal{V} : X_{f^{-1}\mathcal{V}} \to Z$.

PROPOSITION 1.11. *Semi-simplicial manifolds and their ss-morphisms form a category.*

P r o o f. We first show that the composition of ss-morphisms is well defined. Clearly, any commuting triangle of the form

$$\begin{array}{ccc} X_{\mathcal{U}} & & \\ & \searrow^{f} & \\ \varrho_{\#}\big\downarrow & & Y \\ & \nearrow_{\bar{f}} & \\ X_{\overline{\mathcal{U}}} & & \end{array}$$

gives rise to another one,

$$\begin{array}{ccc} X_{f^{-1}\mathcal{V}} & & \\ & \searrow^{f_{\mathcal{V}}} & \\ (\varrho\times\mathrm{id})_{\#}\big\downarrow & & Y_{\mathcal{V}} \\ & \nearrow_{\bar{f}_{\mathcal{V}}} & \\ X_{\bar{f}^{-1}\mathcal{V}} & & \end{array} \tag{1.7}$$

whose commutativity ensures that for any $g : Y_{\mathcal{V}} \to Z$, $g \circ f_{\mathcal{V}}$ and $g \circ \overline{f}_{\mathcal{V}}$ represent the same ss-morphism $X \rightharpoonup Z$.

On the other hand, if $g : Y_{\mathcal{V}} \to Z$ is elementarily equivalent to $\overline{g} : Y_{\overline{\mathcal{V}}} \to Z$ where $\mathcal{V}$ is a refinement of $\overline{\mathcal{V}}$, then for every ss-map $f : X_{\mathcal{U}} \to Y$ the refinement map ϱ induces a commuting square

$$\begin{array}{ccc} X_{f^{-2}\mathcal{V}} & \xrightarrow{f_{\mathcal{V}}} & Y_{\mathcal{V}} \\ (\mathrm{id}\times\varrho)_{\#}\big\downarrow & & \big\downarrow\varrho_{\#} \\ X_{f^{-1}\bar{\mathcal{V}}} & \xrightarrow[f_{\bar{\mathcal{V}}}]{} & Y_{\bar{\mathcal{V}}} \end{array} \tag{1.8}$$

and thus $g \circ f_{\mathcal{V}}$ is elementarily equivalent to $\overline{g} \circ f_{\bar{\mathcal{V}}}$.

For any ss-manifold X the identity ss-morphism $\mathbf{1}_X : X \rightharpoonup X$ is represented by the identity $X_{\{X_0\}} \to X$ and is therefore a unit of the composition.

In order to verify the associativity of the composition, we consider any three representatives $f : X_{\mathcal{U}} \to Y$, $g : Y_{\mathcal{V}} \to Z$, and $h : Z_{\mathcal{W}} \to T$ of ss-morphisms $\mathbf{f} : X \rightharpoonup Y$, $\mathbf{g} : Y \rightharpoonup Z$, and $\mathbf{h} : Z \rightharpoonup T$, respectively. Then $(\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$ is represented by

$$(h \circ g_{\mathcal{W}}) \circ f_{g^{-1}\mathcal{W}} : X_{f^{-1}g^{-1}\mathcal{W}} \to T$$

while $\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f})$ by

$$h \circ (g \circ f_{\mathcal{V}})_{\mathcal{W}} : X_{(g\circ f_{\mathcal{V}})^{-1}\mathcal{W}} \to T\,.$$

Fortunately, despite a relative complexity of the formulas, the coverings $f^{-1}g^{-1}\mathcal{W}$ and $(g \circ f_{\mathcal{V}})^{-1}\mathcal{W}$ of X_0 are essentially the same, and the two compositions of ss-maps are equal. ■

Clearly, there is a canonical functor carrying any ss-map $f : X \to Y$ to the ss-morphism

$$[f] : X \rightharpoonup Y \tag{1.9}$$

represented by $\widetilde{f} : X_{\{X_0\}} \cong X \xrightarrow{f} Y$.

The relationship between ss-morphisms and ss-maps is given by the following:

THEOREM 1.12 ([2]). *Let X be an ss-manifold and $\mathcal{U}$ an open covering of X_0.*

(i) *The ss-morphism $[\lambda] : X_{\mathcal{U}} \rightharpoonup X$ associated with the gluing projection $\lambda : X_{\mathcal{U}} \to X$ is invertible.*

(ii) *If $\mathbf{f} : X \rightharpoonup Y$ is an ss-morphism and $f : X_{\mathcal{U}} \to Y$ an ss-map representing $\mathbf{f}$, then $\mathbf{f} = [f][\lambda]^{-1}$.*

Proof. (i) Let $\mathbf{h} : X \rightharpoonup X_{\mathcal{U}}$ be the ss-morphism represented by $\mathrm{id} : X_{\mathcal{U}} \to X_{\mathcal{U}}$; the composition $[\lambda] \circ \mathbf{h}$ is represented by λ and thus equal to $\mathbf{1}_X$. In order to show that $\mathbf{h} \circ [\lambda] = \mathbf{1}_{X_{\mathcal{U}}}$, we have to compare the ss-map $\widetilde{\lambda}_{\mathcal{U}} : (X_{\mathcal{U}})_{\widetilde{\lambda}^{-1}\mathcal{U}} \to X_{\mathcal{U}}$,

$$(a_0, \ldots, a_n; (b_0, \ldots, b_n; x)) \to (a_0, \ldots, a_n; x)$$

where $\mathcal{U} = (U_a)_{a \in A}$ as usual (this is not a gluing projection!), with a representative of $\mathbf{1}_{X_{\mathcal{U}}}$. It suffices to connect $\widetilde{\lambda}_{\mathcal{U}}$ and $(\mathrm{id}_{X_{\mathcal{U}}})\widetilde{\ }$ by a chain of two elementary equivalences. So let $\mathcal{W} := \widetilde{\lambda}^{-1}\mathcal{U} \amalg \{X_{\mathcal{U}}(0)\}$ with an extra index $*$ for the attached set. The ss-map $\mu : (X_{\mathcal{U}})_{\mathcal{W}} \to X_{\mathcal{U}}$,

$$(\ldots, a_i, \ldots, \underset{j}{*}, \ldots; (b_0, \ldots, b_n; x)) \to (\ldots, a_i, \ldots, b_j, \ldots; x)$$

(we replace all the $*$'s with the respective b_j's) makes the following diagram commutative:

$$\begin{array}{ccc} (X_{\mathcal{U}})_{\widetilde{\lambda}^{-1}\mathcal{U}} & & \\ \cap & \searrow^{\widetilde{\lambda}_{\mathcal{U}}} & \\ (X_{\mathcal{U}})_{\mathcal{W}} & \xrightarrow{\mu} & X_{\mathcal{U}} \\ \uparrow & \nearrow_{\mathrm{id}} & \\ X_{\mathcal{U}} & & \end{array}$$

(ii) As follows from the proof of (i), the composition $[f][\lambda]^{-1}$ is represented by $f \circ \mathrm{id} = f$. ■

Two ss-manifolds X and Y will be called *equivalent* (notation $X \approx Y$) if there exists an invertible ss-morphism (an *equivalence*) $X \rightharpoonup Y$.

EXAMPLE 1.13. For any surmersion $\varphi : M \to Q$ the fibre product $M \times_Q M \subset M \times M$ equipped with the composition rule

$$(x, y) \cdot (x', y') = (x, y') \quad \text{iff} \quad y = x'$$

is a groupoid over Q. Its nerve is canonically isomorphic to an ss-manifold $\overline{\mathcal{N}} M^{\varphi}$ ($\overline{\mathcal{N}} M$ if φ is known from the context) such that

$$\overline{\mathcal{N}}_n M^{\varphi} = \{(x_0, \ldots, x_n) \in M^{n+1}; \varphi(x_0) = \ldots = \varphi(x_n)\}$$

for $n \geq 0$, and

$$\overline{\mathcal{N}}_n M^{\varphi} \ni (x_0, \ldots, x_n) \begin{array}{l} \overset{\varepsilon_i}{\nearrow} (\ldots, x_{i-1}, x_{i+1}, \ldots) \in \overline{\mathcal{N}}_{n-1} M^{\varphi} \\ \underset{\eta_i}{\searrow} (\ldots, x_i, x_i, \ldots) \in \overline{\mathcal{N}}_{n+1} M^{\varphi} \end{array}$$

We shall show that the ss-map $\Phi : \overline{\mathcal{N}} M^{\varphi} \to \mathcal{N} Q, (x_0, \ldots, x_n) \to \varphi(x_0)$, generates an equivalence of ss-manifolds,

$$[\Phi] : \overline{\mathcal{N}} M^{\varphi} \overset{\approx}{\rightharpoonup} \mathcal{N} Q .$$

Indeed, by choosing a collection of local sections of $\varphi, Q \supset V_i \overset{s_i}{\to} M$, over a covering $\mathcal{V} = (V_i)_{i \in I}$ of Q we get an ss-map $S : \mathcal{N}\mathcal{V} \to \overline{\mathcal{N}} M^{\varphi}$,

$$(i_0, \ldots, i_n; u) \to (s_{i_0} u, \ldots, s_{i_n} u) ,$$

such that $\Phi \circ S : \mathcal{N}\mathcal{V} \to \mathcal{N} Q$ is the gluing projection. On the other hand, the composition $S \circ \Phi_{\mathcal{V}} : \overline{\mathcal{N}} M^{\varphi}_{\varphi^{-1}\mathcal{V}} \to \overline{\mathcal{N}} M^{\varphi}$,

$$(i_0, \ldots, i_n; (x_0, \ldots, x_n)) \to (s_{i_0} \varphi x_0, \ldots, s_{i_n} \varphi x_n) ,$$

together with the identity on $\overline{\mathcal{N}} M^{\varphi}$ are both elementarily equivalent to an ss-map $h : \overline{\mathcal{N}} M^{\varphi}_{\mathcal{W}} \to \overline{\mathcal{N}} M^{\varphi}$,

$$(\ldots, i_k, \ldots, \underset{l}{*}, \ldots; (x_0, \ldots, x_n)) \to (\ldots, s_{i_k} \varphi x_k, \ldots, x_l, \ldots) ,$$

where $\mathcal{W}$ is the disjoint union of $\varphi^{-1}\mathcal{V}$ and the trivial covering $\{M\}$ indexed by $*$.

For any collection of ss-manifolds $X^{\alpha} = (X^{\alpha}_n)_{n \geq 0}$, $\alpha \in A$, the disjoint unions $X_n := \coprod_{\alpha} X^{\alpha}_n$, $n \geq 0$, form an ss-manifold X provided for each n the manifolds X^{α}_n, $\alpha \in A$, are of the same dimension. We shall denote the resulting ss-manifold by $\coprod_{\alpha} X^{\alpha}$ and call it the *union* of the collection. Note that for every β there is a canonical ss-map $X^{\beta} \to \coprod_{\alpha} X^{\alpha}$ composed of inclusions.

DEFINITION 1.14. An ss-manifold X is *connected* if it is not isomorphic to the union of any non-trivial collection of non-empty ss-manifolds.

PROPOSITION 1.15. *For every non-empty ss-manifold X there is a unique* (*up to ordering*) *family of connected ss-manifolds X^{α}, $\alpha \in A$, such that all the X^{α}_n's are open in X_n, and*

$$X_n = \bigcup_{\alpha \in A} X^{\alpha}_n , \qquad X^{\alpha}_n \cap X^{\beta}_n = \emptyset \quad \textit{iff} \quad \alpha \neq \beta \tag{1.10}$$

for $n = 0, 1, \ldots$ The decomposition (1.10) *of X identifies it with the union $\coprod X^{\alpha}$.*

DEFINITION 1.16. The ss-manifolds X^α, $\alpha \in A$, are the *connected components* of X.

Proof of Proposition 1.15. We consider the smallest equivalence relation $\sim$ in X_0 generated by the following one: for $x, y \in X_0$, $x \sim y$ if the maximal connected components U_x and U_y of X_0 which contain, respectively, x and y satisfy

$$(1.11) \qquad (\varepsilon_1^{-1} U_x \cap \varepsilon_0^{-1} U_y) \cup (\varepsilon_1^{-1} U_y \cap \varepsilon_0^{-1} U_x) \neq \emptyset .$$

Let X_0^α, $\alpha \in A$, be the equivalence classes of $\sim$, so that (1.10) holds true for $n = 0$, and let

$$(1.12) \qquad X_n^\alpha := \bigcup_{i=0}^{n} ({\varepsilon_1}^{n-i} {\varepsilon_0}^{i})^{-1} X_0^\alpha$$

for $n > 0$. Since it is evident that $X^\alpha = (X_n^\alpha)$ are connected ss-manifolds such that $X_n = \bigcup_\alpha X_n^\alpha$, it remains to check that $X_n^\alpha \cap X_n^\beta = \emptyset$ unless $\alpha = \beta$. If $x \in X_n^\alpha \cap X_n^\beta$ is any element such that ${\varepsilon_1}^{n-i} {\varepsilon_0}^{i} x \in X_0^\alpha$ and ${\varepsilon_1}^{n-j} {\varepsilon_0}^{j} x \in X_0^\beta$ then we may assume $i < j$ and consider the sequence ${\varepsilon_2}^{n-i-h} {\varepsilon_0}^{i+h-1} x \in X_1$, $h = 1, \ldots, j - i$. The identities

$$\varepsilon_0({\varepsilon_2}^{n-i-h} {\varepsilon_0}^{i+h-1} x) = {\varepsilon_1}^{n-i-h} {\varepsilon_0}^{i+h} x ,$$
$$\varepsilon_1({\varepsilon_2}^{n-i-h} {\varepsilon_0}^{i+h-1} x) = {\varepsilon_1}^{n-i-h+1} {\varepsilon_0}^{i+h-1} x$$

ensure that all the elements ${\varepsilon_1}^{n-i-h} {\varepsilon_0}^{i+h} x \in X_0$, $h = 0, 1, \ldots, j - i$, are in the same equivalence class. ∎

EXAMPLE 1.17. For every groupoid Γ the connected components of the nerve $\mathcal{N}\Gamma$ are the nerves of some open subgroupoids of Γ.

In order to establish a relationship between connectedness and ss-morphisms we need the following

LEMMA 1.18. *If X is a connected ss-manifold then all its localizations are connected.*

Proof. Let $\mathcal{U} = (U_a)_{a \in A}$ be an open covering of X_0; we wish to show that if $x, y \in X_0$ and $x \sim y$ then $(a, x) \sim (b, y)$ for any $a, b \in A$ such that $x \in U_a$, $y \in U_b$. Here $\sim$ stands for the natural equivalence relation in X_0 and, respectively, in $X_{\mathcal{U}}(0)$ generated by (1.11). So consider $x, y \in X_0$ and assume that there is a $z \in X_1$ such that x and $\varepsilon_1 z$ as well as y and $\varepsilon_0 z$ are in the same connected components of X_0. If $x \in U_a$ and $y \in U_b$, then there are indices $a = a_0, a_1, \ldots, a_r$ and $b = b_0, b_1, \ldots, b_s$ in A and points $x_i \in U_{a_{i-1}} \cap U_{a_i}$, $y_i \in U_{b_{j-1}} \cap U_{b_j}$ for $i \leq r$, $j \leq s$, such that $\varepsilon_1 z \in U_{a_r}$, $\varepsilon_0 z \in U_{b_s}$, and x_i and x_{i+1} (y_j and y_{j+1}) are in the same connected component of U_{a_i} (of U_{b_j}) for $i = 0, 1, \ldots, r$ ($j = 0, 1, \ldots, s$);

here we set $x_0 = x$, $x_{r+1} = \varepsilon_1 z$ (respectively, $y_0 = y$ and $y_{s+1} = \varepsilon_0 z$). By construction, in the sequence

$$(a, x), (a_1, x_1), \dots, (a_r, x_r), (b_s, y_s), \dots, (b_1, y_1), (b, y)$$

of elements of $X_{\mathcal{U}}(0)$, the neighbouring elements satisfy (1.11). ∎

Proposition 1.19. *Let* $\mathbf{f} : X \rightharpoonup Y$ *be an ss-morphism. For every connected component* X' *of* X *there is a* unique *connected component* Y' *of* Y *and a* unique *ss-morphism* $\mathbf{f}' : X' \rightharpoonup Y'$ *such that the square*

$$\begin{array}{ccc} X' & \xrightharpoonup{\mathbf{f}'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightharpoonup{\mathbf{f}} & Y \end{array}$$

commutes (*after passing to ss-morphisms*).

Proof. We assume, without loosing generality, that $X = X'$ is connected. If Y' and $\mathbf{f}'$ exist, then every representative of $\mathbf{f}'$ is a representative of $\mathbf{f}$; thus $\mathbf{f}$ has representatives with values in Y'. In order to find the required component of Y we consider any representative $f : X_{\mathcal{U}} \to Y$ of $\mathbf{f}$. Since $X_{\mathcal{U}}$ is connected, the image $f_0(X_{\mathcal{U}}(0)) \subset Y_0$ is contained in an equivalence class of the relation in Y_0; let Y' be the corresponding connected component of Y. As (1.12) ensures that image $(f_n) \subset Y'_n$ for $n \geq 0$, f descends to an ss-map $f : X_{\mathcal{U}} \to Y'$ which represents the desired ss-morphism. It is readily seen that any representative of $\mathbf{f}$ elementarily equivalent to f gives rise to the same component of Y and the same ss-morphism. ∎

Corollary 1.20. *If* $X \approx Y$ *and* X *is connected*, *then so is* Y.

Proof. Let $\mathbf{f} : X \rightharpoonup Y$ be any equivalence, and Y' the component of Y through which $\mathbf{f}$ can be factorized. The commuting diagram

$$\begin{array}{ccccc} Y & \xrightharpoonup{\mathbf{f}^{-1}} & X & \xrightharpoonup{\mathbf{f}'} & Y' \\ & \mathbf{1}_X \searrow & & & \downarrow \\ & & & & Y \end{array}$$

factorizes $\mathbf{1}_Y$ through Y', so that Y cannot have connected components other than Y'. ∎

Connectedness of an ss-manifold X is an example of a topological property—it depends on the topology of the associated topological space $\|X\|$. We recall that for any ss-manifold $X = (X_n)$ its *fat realization* $\|X\|$ is defined by

$$\|X\| = \coprod_{n \geq 0} \Delta^n \times X_n / (\varepsilon^i t, x) \sim (t, \varepsilon_i x)$$

where Δ^n stands for the standard affine n-simplex in $\mathbb{R}^{n+1}$, $n \geq 0$, and $\varepsilon^i : \Delta^n \to \Delta^{n+1}$ are the face maps (cf. [25]). An ss-map $f : X \to Y$ gives rise to a continuous map

$$\|f\| : \|X\| \to \|Y\|, \quad [t, x] \to [t, f_n(x)] .$$

The relationship between ss-morphisms and topology is explained by the following theorem (cf. [5]).

THEOREM 1.21. (i) *For every ss-manifold X and any coverings $\mathcal{U}$ and $\mathcal{V}$ of X_0 such that $\mathcal{U}$ is a refinement of $\mathcal{V}$ the homotopy class*

$$i_{\mathcal{V}}^{\mathcal{U}} := [\|\varrho_{\#}\|] \in [\|X_{\mathcal{U}}\|, \|X_{\mathcal{V}}\|]$$

is independent of the refinement map ϱ. The resulting inverse system converges to any $\|X_{\mathcal{W}}\|$ with $\mathcal{W}$ consisting of paracompact sets.

(ii) *There exists a unique covariant functor $\|\cdot\|$ from the category of ss-manifolds with paracompact* 0*-level and ss-morphisms to topological spaces and homotopy classes of maps such that*

— *for every X, $\|X\|$ is the fat realization, and*
— *for every ss-map $f : X \to Y$*

$$\|[f]\| = [\|f\|] \in [\|X\|, \|Y\|] ,$$

i.e. $\|[f]\|$ is the homotopy class of $\|f\|$ (cf. (1.9)).

Proof. See [5], Thm. 1.1, Prop. 2.1. The proof relies upon Theorem 1.12. ∎

I.2. Γ-bundles over ss-manifolds. In order to visualize the abstract notion of an ss-morphism we now give an important geometric interpretation of ss-morphisms $X \rightharpoonup \mathcal{N}\Gamma$ of an arbitrary ss-manifold X to the nerve of a groupoid Γ. Anticipating the definitions, one can say that such ss-morphisms are nothing but (isomorphy classes of) principal Γ-bundles over X.

Let Γ be a groupoid over a manifold N. A (right) *principal Γ-bundle* over a manifold M ([13], [16]) is a manifold E endowed with two maps, the *bundle projection* $\pi : E \to M$ and the *source map* $\alpha : E \to N$, and with a right Γ-action $E \times_{(\alpha,\beta)} \Gamma \to E$ in the fibres of π (i.e. $zg = z$ if $g \in N$, $(zg_1)g_2 = z(g_1g_2)$, and $\pi(zg) = \pi z$; left principal Γ-bundles are defined analogously and are endowed with a *target map* β). One requires a *local triviality condition*: on a neighbourhood of each point $x \in M$ there is a section $s : M \supset V \to E$ of π such that the map

$$V \times_{(\alpha s,\beta)} \Gamma \ni (y, g) \to s(y)g \in \pi^{-1}(V) \subset E$$

is a diffeomorphism. The following elementary lemma is standard:

LEMMA 2.1. *For any homomorphism $E' \to E$ of principal Γ-bundles, i.e. any commuting square*

$$\begin{array}{ccc} E' & \xrightarrow{\bar f} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M' & \xrightarrow{f} & M \end{array}$$

such that $\bar f$ is Γ-equivariant, the mapping

$$(\pi, \bar f) : E' \to f^*E := M' \times_{(f,\pi)} E$$

is an isomorphism. ■

DEFINITION 2.2. A *principal Γ-bundle over an ss-manifold $X = (X_n)$* is an ss-manifold $E = (E_n)$ together with an ss-map $\pi : E \to X$ (the *bundle projection*) such that

(i) for every $n \geq 0$, $\pi_n : E_n \to X_n$ is a principal Γ-bundle,
(ii) the structure operators of E are Γ-equivariant.

A *homomorphism* $h : E \to E'$ of principal Γ-bundles over ss-manifolds is an ss-map such that each $h_n : E_n \to E'_n, n \geq 0$, is Γ-equivariant.

EXAMPLE 2.3. The *universal principal Γ-bundle* [21], [10]. For any groupoid Γ the ss-manifold $\overline{\mathcal{N}}\Gamma = \overline{\mathcal{N}}\Gamma^\alpha$ (cf. Example 1.12) carries a canonical structure of a principal Γ-bundle over $\mathcal{N}\Gamma$ such that

$$(g_0, \dots, g_n) \cdot g = (g_0 g, \dots, g_n g)$$

and

$$\overline{\mathcal{N}}_n\Gamma \ni (g_0, \dots, g_n) \xrightarrow{\pi} (g_0 g_1^{-1}, \dots, g_{n-1} g_n^{-1}) \in \mathcal{N}_n\Gamma\,.$$

The universal properties of this Γ-bundle will be clarified in Theorem 2.12.

EXAMPLE 2.4. For any principal Γ-bundle $\pi : E \to X$ and any open covering $\mathcal{U} = (U_a)_{a\in A}$ of X_0 the localization $E_{\pi^{-1}\mathcal{U}}$ of E to $\pi^{-1}\mathcal{U} := (\pi_0^{-1} U_a)_{a\in A}$ is a principal Γ-bundle over $X_{\mathcal{U}}$. We shall see that converse statement is also true.

PROPOSITION 2.5. *For every principal Γ-bundle $E \to X_{\mathcal{U}}$ over the localization of an ss-manifold X to $\mathcal{U} = (U_a)_{a\in A}$ the isomorphisms*

$$\begin{array}{ccc} E_n|_{(b_0,\dots,b_n;x)} & \xrightarrow{(\varepsilon_n \dots \varepsilon_1 \varepsilon_0)^{-1}} & E_{2n+1}|_{(a_0,b_0,\dots,a_n,b_n;\eta_0\eta_1\dots\eta_n x)} \\ & {\scriptstyle I_{ab}}\searrow & \downarrow{\scriptstyle \varepsilon_{n+1}\varepsilon_n\dots\varepsilon_1} \\ & & E_n|_{(a_0,\dots,a_n;x)} \end{array} \tag{2.1}$$

define a Γ-equivariant equivalence relation in the fibres of $E_n, n \geq 0$; the quotients $\widehat{E}_n := E_n/\sim$ constitute a principal Γ-bundle $\widehat{E}$ over X such that E is canonically isomorphic to the localization $\widehat{E}_{\pi^{-1}\mathcal{U}}$ of $\widehat{E}$.

(ii) *If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then every homomorphism $E' \to E$ of principal Γ-bundles over $X_{\mathcal{V}}$ and $X_{\mathcal{U}}$, respectively, which projects to a $\varrho_{\#} : X_{\mathcal{V}} \to X_{\mathcal{U}}$ descends to an isomorphism $\widehat{E}' \overset{\cong}{\to} \widehat{E}$. In particular, any isomorphism $E''_{\pi^{-1}\mathcal{U}} \overset{\cong}{\to} E$ (E'' a Γ-bundle over X) induces $E'' \overset{\cong}{\to} \widehat{E}$.*

Proof. (i) Assume that $x \in X_n$ belongs to three components of $X_{\mathcal{U}}(n)$ so that there are indices $a_i, b_i, c_i,\ i \leq n$, such that ${\varepsilon_1}^{n-i}{\varepsilon_0}^{i}x \in U_{a_i} \cap U_{b_i} \cap U_{c_i}$ for $i = 0, \ldots, n$. Then the triangle

$$\begin{array}{ccc} E_n|_{(c_0,\ldots,c_n;x)} & \xrightarrow{I_{bc}} & E_n|_{(b_0,\ldots,b_n;x)} \\ & {}_{I_{ac}}\searrow \quad \swarrow_{I_{ab}} & \\ & E_n|_{(a_0,\ldots,a_n;x)} & \end{array}$$

commutes. Indeed, for any $u \in E_n$ with $\pi_n u = (c_0, \ldots, c_n; x)$, there is a unique $t \in E_{3n+2}$ such that $\pi_{3n+2}t = (a_0, b_0, \ldots, b_n, c_n; \eta_0^2\eta_1^2 \ldots \eta_n^2 x)$ and $\varepsilon_n^2 \ldots \varepsilon_1^2\varepsilon_0^2 t = u$ (cf. Lemma 2.1). This implies

$$u = \varepsilon_n \ldots \varepsilon_1\varepsilon_0(\varepsilon_{2n} \ldots \varepsilon_2\varepsilon_0 t)\,,$$
$$\pi_{2n+1}(\varepsilon_{2n} \ldots \varepsilon_2\varepsilon_0 t) = (b_0, c_0, \ldots, b_n, c_n; \eta_0 \ldots \eta_n x)\,,$$

which means that $w' = \varepsilon_{2n} \ldots \varepsilon_2\varepsilon_0 t$ is exactly the image of u in $E_{2n+1}|_{(b_0,c_0,\ldots,b_n,c_n;\eta_0\ldots\eta_n x)}$; therefore

$$I_{bc}(u) = \varepsilon_{n+1} \ldots \varepsilon_1 w' = \varepsilon_n \ldots \varepsilon_1\varepsilon_0(\varepsilon_{2n+2} \ldots \varepsilon_4\varepsilon_2 t)\,.$$

Since $\pi_{2n+1}(\varepsilon_{2n+2} \ldots \varepsilon_4\varepsilon_2 t) = (a_0, b_0, \ldots, a_n, b_n; \eta_0 \ldots \eta_n x)$, we conclude that

$$I_{ab}I_{bc}(u) = \varepsilon_{n+1} \ldots \varepsilon_1(\varepsilon_{2n+2} \ldots \varepsilon_4\varepsilon_2 t) = \varepsilon_{n+1}^2 \ldots \varepsilon_1^2 t\,.$$

On the other hand, the identities

$$u = \varepsilon_n \ldots \varepsilon_1\varepsilon_0(\varepsilon_{2n+1} \ldots \varepsilon_3\varepsilon_1 t)\,,$$
$$\pi_{2n+1}(\varepsilon_{2n+1} \ldots \varepsilon_3\varepsilon_1 t) = (a_0, c_0, \ldots, a_n, c_n; \eta_0 \ldots \eta_n x)$$

imply

$$I_{ac}(u) = \varepsilon_{n+1} \ldots \varepsilon_1(\varepsilon_{2n+1} \ldots \varepsilon_3\varepsilon_1 t) = \varepsilon_{n+1}^2 \ldots \varepsilon_1^2 t = I_{ab}I_{bc}(u)\,.$$

This cocycle condition $I_{ab}I_{bc} = I_{ac}$ implies $I_{aa} = \mathrm{id}$ and $I_{ba} = I_{ab}^{-1}$, so that the equivalence relation is correctly defined and leads to a sequence of quotient Γ-bundles $\widehat{E}_n$, $n \geq 0$. It remains to check that the structure operators of E descend to the $\widehat{E}_n$'s. So let $x \in X_n$ satisfy ${\varepsilon_1}^{n-i}{\varepsilon_0}^{i}x \in U_{a_i} \cap U_{b_i}$ for $i \leq n$, and let $u \in E_n$ be any element of the fibre over $(b_0, \ldots, b_n; x)$. We denote by w the image of u in E_{2n+1} over $(a_0, b_0, \ldots, a_n, b_n; \eta_0 \ldots \eta_n x)$, so that $\varepsilon_n \dots \varepsilon_1\varepsilon_0 w = u$ implies

$$\varepsilon_{n-1} \dots \varepsilon_1\varepsilon_0(\varepsilon_{2i}^2 w) = \varepsilon_i u, \qquad \varepsilon_{n+1} \dots \varepsilon_1\varepsilon_0(\eta_{2i+1}^2 w) = \eta_i u$$

for every $i \leq n$. Thus the element $\varepsilon_i u \in E_{n-1}|_{(b_0,\dots,\widehat{b_i},\dots,b_n;\varepsilon_i x)}$ is equivalent to

$$\varepsilon_n \dots \varepsilon_1(\varepsilon_{2i}^2 w) = \varepsilon_i(\varepsilon_{n+1}\varepsilon_n \dots \varepsilon_1 w) = \varepsilon_i I_{ab}(u)\,,$$

whereas the element $\eta_i u \in E_{n+1}|_{(b_0,\dots,b_i,b_i,\dots,b_n;\eta_i x)}$ is equivalent to $v \in E_{n+1}|_{(a_0,\dots,a_i,b_i,\dots,a_n;\eta_i x)}$ such that

$$v = \varepsilon_{n+2} \dots \varepsilon_1(\eta_{2i+1}^2 w) = \varepsilon_{n+2} \dots \varepsilon_{i+3}\varepsilon_i \dots \varepsilon_1 w\,.$$

In order to compare the last element with $\eta_i I_{ab}(u)$ we need a $w' \in E_{2n+5}$ such that

$$\begin{aligned} \varepsilon_{n+1}\varepsilon_n \dots \varepsilon_0 w' &= v\,, \\ \pi_{2n+5} w' &= (\dots, a_i, a_i, a_i, b_i, a_{i+1}, a_{i+1}, \dots; \eta_0 \dots \eta_{n+1}(\eta_i x)) \\ &= \eta_0\eta_1 \dots \eta_i^2 \eta_{i+2} \dots \eta_{n+1}\pi_{n+1} v\,. \end{aligned}$$

The only possibility is $w' = \eta_0\eta_1 \dots \eta_i^2\eta_{i+2} \dots \eta_{n+1} v$, which proves that v (as well as $\eta_i u$) is equivalent to

$$\varepsilon_{n+2} \dots \varepsilon_1 w' = \eta_i \varepsilon_{i+1} v = \eta_i I_{ab}(u)\,.$$

(ii) The assertion amounts to the naturality of the equivalence relation and is an immediate consequence of Lemma 2.1. ■

Let $E = (E_n)$ be a principal Γ-bundle over an ss-manifold Y. For any ss-map $f : X \to Y$

(2.2) $$f^*E = (f_n^* E_n)$$

is a well-defined pull-back Γ-bundle over X, and there is a canonical lift $f^*E \to E$ of f, $f_n^*E_n \ni (x, u) \to u \in E_n$. In order to extend this pull-back operation to an arbitrary ss-morphism $\mathbf{f} : X \rightharpoonup Y$ we consider any representative $f : X_{\mathcal{U}} \to Y$ of $\mathbf{f}$ and set $E^f := (f^*E)^\wedge$, so that there is a canonical homomorphism of Γ-bundles $E^f_{\pi^{-1}\mathcal{U}} \cong f^*E \to E$

(2.3) $$(a_0, \dots, a_n; [(a_0, \dots, a_n; x), u]) \to u\,,$$

where the brackets [ ] indicate the equivalence class in E^f.

The following lemma is a straightforward consequence of Proposition 2.5.

Lemma 2.6. *For any elementary equivalence* (1.4) *the homomorphism* $\widetilde{\varrho} : f^*E \to g^*E$, $((a_0, \dots, a_n; x), u) \to ((\varrho(a_0), \dots, \varrho(a_n); x), u)$ *descends to an isomorphism of the globalized Γ-bundles* $\varrho_* : E^f \xrightarrow{\cong} E^g$ *such that the* *following square commutes*:

$$\begin{array}{ccc} E^f_{\pi^{-1}\mathcal{U}} & \longrightarrow & E \\ {\scriptstyle \varrho_\#}\downarrow & & \uparrow \\ E^f_{\pi^{-1}\mathcal{V}} & \xrightarrow{\varrho_*} & E^g_{\pi^{-1}\mathcal{V}} \end{array} \qquad \blacksquare$$

PROPOSITION 2.7. *Let E' and E be principal Γ-bundles over ss-manifolds X and Y, respectively.*

(i) *If an ss-map $f : X_{\mathcal{U}} \to Y$ is covered by a homomorphism $\overline{f} : E'_{\pi^{-1}\mathcal{U}} \to E$ then there is an isomorphism $E' \overset{\cong}{\to} E^f$ such that $\overline{f}$ becomes the composition of $E'_{\pi^{-1}\mathcal{U}} \overset{\cong}{\to} E^f_{\pi^{-1}\mathcal{U}}$ and the canonical homomorphism* (2.3).

(ii) *For every homomorphism $\overline{f} : E'_{\pi^{-1}\mathcal{U}} \to E$ which projects to f and any elementary equivalence* (1.4) *there is a unique homomorphism $\overline{g} : E'_{\pi^{-1}\mathcal{V}} \to E$ which projects to g, such that $\overline{f} = \overline{g} \circ \varrho_\#$.*

COROLLARY 2.8. *Every chain of elementary equivalences between $f : X_{\mathcal{U}} \to Y$ and $h : X_{\mathcal{W}} \to Y$ admits a unique lift to elementary equivalences connecting $\overline{f} : E'_{\pi^{-1}\mathcal{U}} \to E$ and a homomorphism $\overline{h} : E'_{\pi^{-1}\mathcal{W}} \to E$ which projects to h.* ■

Proof of Proposition 2.7. (i) By Lemma 2.1, the sequence of maps $(\pi_n, \overline{f}_n) : E'_{\pi^{-1}\mathcal{U}}(n) \to f^*_n E_n$, $n \geq 0$, constitutes an isomorphism of Γ-bundles $(\pi, \overline{f}) : E'_{\pi^{-1}\mathcal{U}} \to f^*E$ such that $\overline{f}$ is the composition $E'_{\pi^{-1}\mathcal{U}} \to f^*E \to E$. Passing to the globalized Γ-bundles, we get $[\pi, \overline{f}] : E' \overset{\cong}{\to} E^f$.

(ii) If $f = g \circ \varrho_\#$ and $\overline{g}$ is a lift of g such that $\overline{f} = \overline{g} \circ \varrho_\#$, then there is a commuting diagram

$$\begin{array}{ccccc} E'_{\pi^{-1}\mathcal{U}} & & \xrightarrow{(\pi,\bar{f})} & & f^*E \\ & {\scriptstyle \bar{f}}\searrow & & \swarrow & \\ {\scriptstyle \varrho_\#}\downarrow & & E & & \downarrow{\scriptstyle \tilde{\varrho}} \\ & \nearrow{\scriptstyle \bar{g}} & & \nwarrow & \\ E'_{\pi^{-1}\mathcal{V}} & & \xrightarrow[(\pi,\bar{g})]{} & & g^*E \end{array}$$

Since the outer square of the diagram yields (after globalization)

$$\begin{array}{ccc} & & E^f \\ & {\scriptstyle [\pi,\bar{f}]}\nearrow & \\ E' & & \downarrow{\scriptstyle \varrho_*} \\ & {\scriptstyle [\pi,\bar{g}]}\searrow & \\ & & E^g \end{array}$$

the decomposition (i) of $\overline{g}$ reads

$$\overline{g} : \quad E'_{\pi^{-1}\mathcal{V}} \xrightarrow{[\pi,\bar{f}]} E^f_{\pi^{-1}\mathcal{V}} \xrightarrow{\varrho_*} E^g_{\pi^{-1}\mathcal{V}} \to E\,.$$

Such a $\overline{g}$ does project to g, and the required equality $\bar{f} = \overline{g} \circ \varrho_{\#}$ follows from the decomposition (i) of $\bar{f}$ and from Lemma 2.6. ■

Motivated by Corollary 2.8, we propose the following

DEFINITION 2.9. For any ss-morphism $\mathbf{f} : X \rightharpoonup Y$ and any principal Γ-bundle E over Y, a pair $(E', \overline{\mathbf{f}})$ is a *pull-back of E by $\mathbf{f}$* (to be denoted also by $\mathbf{f}^*E$ if this causes no confusion) if E' is a principal Γ-bundle over X and $\overline{\mathbf{f}}$ is an *ss-morphism of Γ-bundles* over $\mathbf{f}$, i.e. a maximal connected (with respect to elementary equivalences) family of homomorphisms $E'_{\pi^{-1}\mathcal{U}} \to E$ which project to representatives $X_{\mathcal{U}} \to Y$ of $\mathbf{f}$. Elements of $\overline{\mathbf{f}}$ are *distinguished lifts* of the representatives of $\mathbf{f}$.

By Proposition 2.7, the pull-back $\mathbf{f}^*E$ always exists, is unique up to isomorphism of Γ-bundles, and is completely characterized (generated) by any one of the distinguished lifts.

PROPOSITION 2.10. *For any ss-morphisms $\mathbf{f} : X \rightharpoonup Y$ and $\mathbf{g} : Y \rightharpoonup Z$ and every principal Γ-bundle E over Z,*

$$\mathbf{f}^*\mathbf{g}^*E = (\mathbf{gf})^*E\,.$$

More precisely, if $(E', \overline{\mathbf{g}})$ is a pull-back of E by $\mathbf{g}$, and $(E'', \overline{\mathbf{f}})$ is a pull-back of E' by $\mathbf{f}$, then $(E'', \overline{\mathbf{g}} \circ \overline{\mathbf{f}})$ is a pull-back of E by $\mathbf{gf}$, the composition $\overline{\mathbf{g}} \circ \overline{\mathbf{f}}$ being defined as that of ss-morphisms. ■

Remark 2.11. An ss-morphism of Γ-bundles $\overline{\mathbf{f}}$ generates an ss-morphism $\overline{\mathbf{f}} : E' \rightharpoonup E$ of the underlying ss-manifolds. More generally:

If Γ, a groupoid over N, acts on the left on a manifold Q with respect to a submersion $p : Q \to N$ (i.e. each $g \in \Gamma$ sends $p^{-1}(\alpha g)$ to $p^{-1}(\beta g)$), then for any principal Γ-bundle $E \to Y$ the manifolds

$$E_n \times_\Gamma Q = E_n \times_{(\alpha,p)} Q/(ug, z) \sim (u, gz) \quad \text{for} \quad g \in \Gamma$$

form the *associated Q-bundle* over Y,

$$E(Q) := (E_n \times_\Gamma Q)_{n \geq 0}\,, \tag{2.4}$$

which inherits from E both the structure operators and the projection on Y. Given any pull-pack $(E', \overline{\mathbf{f}})$ of E by an ss-morphism $\mathbf{f} : X \rightharpoonup Y$, $\overline{\mathbf{f}}$ generates a well defined ss-morphism $\overline{\mathbf{f}} : E'(Q) \rightharpoonup E(Q)$, the *lift* of $\mathbf{f}$.

The main result of the present section is

THEOREM 2.12. *For each Γ-bundle E over an ss-manifold X there exists* exactly one *ss-morphism $\mathbf{f}_E : X \rightharpoonup \mathcal{N}\Gamma$ such that E (together with some ss-morphism of Γ-bundles) is a pull-back $\mathbf{f}_E^*\overline{\mathcal{N}\Gamma}$ of the universal principal Γ-bundle over $\mathcal{N}\Gamma$.*

The ss-morphism $\mathbf{f}_E$ will be referred to as the *classifying ss-morphism* for the principal Γ-bundle E.

Before proving the theorem, let us simplify the structure of an arbitrary principal Γ-bundle over an ss-manifold.

PROPOSITION 2.13. *Let $X = (X_n)$ be an ss-manifold, Γ a groupoid, and $\pi : E_0 \to X_0$ a principal Γ-bundle over X_0. Assume that a map $\overline{\varepsilon}_1 : \varepsilon_0^* E_0 \to E_0$ satisfies the following conditions*:

(i) *$\overline{\varepsilon}_1$ is Γ-equivariant and induces $\varepsilon_1 : X_1 \to X_0$ on the base manifolds*,
(ii) *$\overline{\varepsilon}_1(\eta_0 \pi u, u) = u$ for $u \in E_0$, and*
(iii) *$\overline{\varepsilon}_1(\varepsilon_2 x, \overline{\varepsilon}_1(\varepsilon_0 x, u)) = \overline{\varepsilon}_1(\varepsilon_1 x, u)$ for $(x,u) \in ({\varepsilon_0}^2)^* E_0$.*

Then the projection $\overline{\varepsilon}_0 : \varepsilon_0^* E_0 \to E_0$ together with $\overline{\varepsilon}_1$ and the maps

$$({\varepsilon_0}^n)^* E_0 \ni (x,u) \xrightarrow{\overline{\varepsilon}_i} \begin{cases} (\varepsilon_i x, u) \in ({\varepsilon_0}^{n-1})^* E_0 & \text{for } i < n\,, \\ (\varepsilon_n x, \overline{\varepsilon}_1({\varepsilon_0}^{n-1} x, u)) \in ({\varepsilon_0}^{n-1})^* E_0 & \text{for } i = n\,, \end{cases}$$

$$E_0 \ni \xrightarrow{\overline{\eta}_0} (\eta_0 \pi u, u) \in \varepsilon_0^* E_0\,,$$

$$({\varepsilon_0}^n)^* E_0 \ni (x,u) \xrightarrow{\overline{\eta}_i} (\eta_i x, u) \in ({\varepsilon_0}^{n+1})^* E_0 \qquad \text{for } i \le n,\ n > 0\,,$$

make the sequence $E = (({\varepsilon_0}^n)^* E_0)_{n \ge 0}$ of principal Γ-bundles a principal Γ-bundle over X.

Conversely, if $E = (E_n)$ is any principal Γ-bundle over X, then the map

$$\overline{\varepsilon}_1 : \varepsilon_0^* E_0 \xrightarrow{(\pi_1, \varepsilon_0)^{-1}} E_1 \xrightarrow{\varepsilon_1} E_0$$

satisfies conditions (i)–(iii), *and the homomorphisms*

$$E_n \xrightarrow{(\pi_n, {\varepsilon_0}^n)} X_n \times_{({\varepsilon_0}^n, \pi_0)} E_0 = ({\varepsilon_0}^n)^* E_0$$

constitute an isomorphism of E and the Γ-bundle reconstructed from E_0 and $\overline{\varepsilon}_1$.

P r o o f. The first part of the proposition requires only a careful verification of the axioms (1.1). In order to demonstrate the second part, observe that for $u \in E_0$,

$$\overline{\varepsilon}_1(\eta_0 \pi_0 u, u) = \overline{\varepsilon}_1(\pi_1, \varepsilon_0)\eta_0 u = \varepsilon_1 \eta_0 u = u\,.$$

Furthermore, if $(x,u) \in ({\varepsilon_0}^2)^* E_0$ then there is a $v \in E_2$ such that $\pi_2 v = x$ and $\varepsilon_0 \varepsilon_0 v = u$ (cf. Lemma 2.1) and thus

$$\begin{aligned} \overline{\varepsilon}_1(\varepsilon_2 x, \overline{\varepsilon}_1(\varepsilon_0 x, u)) &= \overline{\varepsilon}_1(\varepsilon_2 x, \varepsilon_1 \varepsilon_0 v) = \overline{\varepsilon}_1(\varepsilon_2 x, \varepsilon_0 \varepsilon_2 v) \\ &= \varepsilon_1 \varepsilon_2 v = \varepsilon_1 \varepsilon_1 v = \overline{\varepsilon}_1(\varepsilon_1 x, u)\,. \end{aligned}$$

It remains to show that the isomorphisms $E_n \to ({\varepsilon_0}^n)^* E_0$ (cf. Lemma 2.1) commute with the structure operators. For $u \in E_n$, the only non-trivial relation is

$$\begin{aligned}\overline{\varepsilon}_n(\pi_n, {\varepsilon_0}^n)u &= (\varepsilon_n\pi_n u, \overline{\varepsilon}_1(\pi_1{\varepsilon_0}^{n-1}u, {\varepsilon_0}^n u)) \\ &= (\varepsilon_n\pi_n u, \varepsilon_1{\varepsilon_0}^{n-1}u) = (\pi_{n-1}, {\varepsilon_0}^{n-1})\varepsilon_n u\,. \quad \blacksquare\end{aligned}$$

COROLLARY 2.14. *Let $f : X \to Y$ be an ss-map, and E, E' principal Γ-bundles over X and Y, respectively. Assume that $I_0 : E_0 \to E'_0$ is a Γ-equivariant map inducing f_0 on the base manifolds. Then the maps*

$$I_n : \quad E_n \xrightarrow{(\pi_n, {\varepsilon_0}^n)} (\varepsilon_0^n)^* E_0 \xrightarrow{f_n \times I_0} (\varepsilon_0^n)^* E'_0 \xrightarrow{(\pi_n, {\varepsilon_0}^n)^{-1}} E'_n$$

constitute a homomorphism of principal Γ-bundles $I = (I_n)$ iff

$$\varepsilon_1 I_1 = I_0 \varepsilon_1\,.$$

I is an isomorphism iff so is I_0. $\blacksquare$

Proof of Theorem 2.12. Let $\pi : E \to X$ be a principal Γ-bundle over X. We consider any collection of local sections $s_a : X_0 \supset U_a \to E_0$ over a covering $\mathcal{U} = (U_a)_{a\in A}$ of X_0. Since there are canonical isomorphisms of principal Γ-bundles (cf. Lemma 2.1),

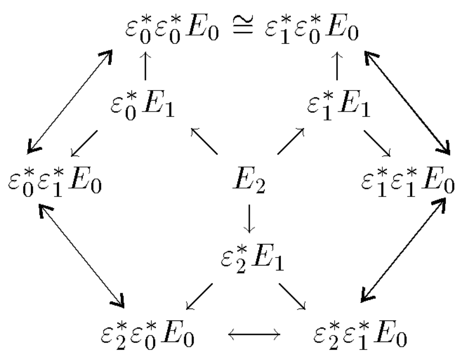
$$\varepsilon_1^* E_0 \xleftarrow{(\pi_1, \varepsilon_1)} E_1 \xrightarrow{(\pi_1, \varepsilon_0)} \varepsilon_0^* E_0\,,$$

the sections induce *two* collections of sections of E_1. By comparison, one has

$$(2.5) \qquad (\pi_1, \varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x)) = (\pi_1, \varepsilon_1)^{-1}(x, s_a(\varepsilon_1 x)) \cdot \gamma_{ab}(x)$$

for a $\gamma_{ab}(x) \in \Gamma$, $x \in \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b \subset X_1$. We claim that the resulting maps $\gamma_{ab}, a, b \in A$, form a Γ-cocycle (cf. Example 1.6). Indeed, any x in $(\varepsilon_1\varepsilon_1)^{-1}U_a \cap (\varepsilon_1\varepsilon_0)^{-1}U_b \cap (\varepsilon_0\varepsilon_0)^{-1}U_c, a, b, c \in A$, gives rise to three elements of E_0 : $s_a(\varepsilon_1\varepsilon_1 x), s_b(\varepsilon_1\varepsilon_0 x)$, and $s_c(\varepsilon_0\varepsilon_0 x)$, six elements of E_1 : $(\pi_1, \varepsilon_1)^{-1}(\varepsilon_2 x, s_a(\varepsilon_1\varepsilon_2 x)), (\pi_1, \varepsilon_1)^{-1}(\varepsilon_1 x, s_a(\varepsilon_1\varepsilon_1 x)), (\pi_1, \varepsilon_0)^{-1}(\varepsilon_2 x, s_b(\varepsilon_0\varepsilon_2 x))$, etc., and three of E_2 : $(\pi_2, \varepsilon_1\varepsilon_1)^{-1}(x, s_a(\varepsilon_1\varepsilon_1 x))$, etc. It is instructive to locate all these elements in the following commuting diagram of isomorphisms of principal Γ-bundles:

(2.6)



Starting from the equality (in E_1)

$$(\pi_1,\varepsilon_1)^{-1}(\varepsilon_1 x, s_a(\varepsilon_1\varepsilon_1 x))\gamma_{ac}(\varepsilon_1 x) = (\pi_1,\varepsilon_0)^{-1}(\varepsilon_1 x, s_c(\varepsilon_0\varepsilon_1 x))$$

we get in E_2

$$\begin{aligned}(\pi_2,\varepsilon_1\varepsilon_1)^{-1}&(x, s_a(\varepsilon_1\varepsilon_1 x))\gamma_{ac}(\varepsilon_1 x)\\ &= (\pi_2,\varepsilon_1)^{-1}(x,(\pi_1,\varepsilon_1)^{-1}(\varepsilon_1 x, s_a(\varepsilon_1\varepsilon_1 x))\gamma_{ac}(\varepsilon_1 x))\\ &= (\pi_2,\varepsilon_1)^{-1}(x,(\pi_1,\varepsilon_0)^{-1}(\varepsilon_1 x, s_c(\varepsilon_0\varepsilon_1 x)))\\ &= (\pi_2,\varepsilon_0\varepsilon_1)^{-1}(x, s_c(\varepsilon_0\varepsilon_1 x))\,.\end{aligned}$$

In a similar vein, we check the equalities

$$(\pi_2,\varepsilon_1\varepsilon_2)^{-1}(x, s_a(\varepsilon_1\varepsilon_2 x))\gamma_{ab}(\varepsilon_2 x) = (\pi_2,\varepsilon_0\varepsilon_2)^{-1}(x, s_b(\varepsilon_0\varepsilon_2 x))\,,$$
$$(\pi_2,\varepsilon_1\varepsilon_0)^{-1}(x, s_b(\varepsilon_1\varepsilon_0 x))\gamma_{bc}(\varepsilon_0 x) = (\pi_2,\varepsilon_0\varepsilon_0)^{-1}(x, s_c(\varepsilon_0\varepsilon_0 x))\,.$$

Since $\varepsilon_0\varepsilon_0 = \varepsilon_0\varepsilon_1$, $\varepsilon_1\varepsilon_0 = \varepsilon_0\varepsilon_2$, and $\varepsilon_1\varepsilon_1 = \varepsilon_1\varepsilon_2$, this implies the cocycle condition (1.2), and thus formulas (1.3) of Example 1.6 define an ss-map $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$.

If an ss-map $g : X_{\mathcal{V}} \to \mathcal{N}\Gamma$ is determined by another collection of local sections of E_0, then both $\mathcal{U}$ and $\mathcal{V}$ are refinements of $\mathcal{W} = \mathcal{U} \amalg \mathcal{V}$, and all the sections together give rise to a Γ-cocycle such that the corresponding ss-map $X_{\mathcal{W}} \to \mathcal{N}\Gamma$ is elementarily equivalent to both f and g. We have thus associated with E a well defined ss-morphism $\mathbf{f}_E : X \rightharpoonup \mathcal{N}\Gamma$.

In order to identify E with a pull-back $\mathbf{f}_E^*\overline{\mathcal{N}}\Gamma$ it suffices to indicate an isomorphism betwen E and $(\overline{\mathcal{N}}\Gamma)^f$. Clearly, when applied to $\eta_0 x$ with $x \in U_a \cap U_b$, (2.5) reduces to

$$s_b(x) = s_a(x)\gamma_{ab}(\eta_0 x)$$

so that $(\gamma_{ab}\eta_0)_{a,b\in A}$ is a Γ-cocycle describing E_0. Since the gluing equivalence relation in

$$f_0^*\overline{\mathcal{N}}_0\Gamma \cong \coprod U_a \times_{(\gamma_{aa}\eta_0,\beta)} \Gamma$$

is (cf. (2.1))

$$((a,x),g) \sim ((b,x),g') \quad \text{iff} \quad g = \gamma_{ab}(\eta_0 x)g'$$

the map

$$(\overline{\mathcal{N}}\Gamma)^f(0) \ni [(a,x),g] \xrightarrow{I_0} s_a(x)g \in E_0$$

is a well defined isomorphism. According to Corollary 2.14, the only extension of I_0 to the first level is

$$(\overline{\mathcal{N}}\Gamma)^f(1) \ni [(a,b;x),(\gamma_{ab}(x)g,g)] \xrightarrow{I_1} (\pi_1,\varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x)g)$$

and the criterion $\varepsilon_1 I_1 = I_0\varepsilon_1$ of extendability of I_0 to an isomorphism $I : (\overline{\mathcal{N}}\Gamma)^f \to E$ is exactly (2.5).

A canonical distinguished lift $\bar{f} : E_{\pi^{-1}\mathcal{U}} \to \overline{\mathcal{N}\Gamma}$ of f is given by

$$\text{(2.7)} \qquad \bar{f}_n(a_0, \dots, a_n; u) = (g_0, \dots, g_n) \quad \text{if} \quad u = s_{a_i}^{(i)}(\pi_n u) g_i,\ i \le n\,,$$

where the $s_a^{(i)}$'s are the induced sections of E_n,

$$s_a^{(i)}(x) = (\pi_n, {\varepsilon_1}^{n-1}{\varepsilon_0}^i)^{-1}(x, s_a({\varepsilon_1}^{n-i}{\varepsilon_0}^i x))\,.$$

In terms of the Γ-cocycle,

$$\text{(2.8)} \qquad \bar{f}_n(a_0, \dots, a_n; s_{a_n}({\varepsilon_0}^n x) g) \\ = (\gamma_{a_0 a_n}({\varepsilon_1}^{n-1}x) g, \gamma_{a_1 a_n}({\varepsilon_1}^{n-2}\varepsilon_0 x) g, \dots, \gamma_{a_{n-1} a_n}({\varepsilon_0}^{n-1}x) g, g)\,.$$

By (2.7), the ss-morphism of Γ-bundles $\bar{\mathbf{f}}_E$ generated by $\bar{f}$ is independent of the chosen collection of local sections of E_0.

It remains to prove that any ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ such that $\mathbf{f}^*\overline{\mathcal{N}\Gamma} = E$ is equal to $\mathbf{f}_E$. If $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$, with $\mathcal{U} = (U_a)_{a\in A}$, is any ss-map covered by a homomorphism $\bar{f} : E_{\pi^{-1}\mathcal{U}} \to \overline{\mathcal{N}\Gamma}$ then there is an isomorphism $(\pi, \bar{f}) : E_{\pi^{-1}\mathcal{U}} \to f^*\overline{\mathcal{N}\Gamma}$. By transfer of structure, the maps

$$U_a \ni x \to (x, f_0(a, x)) \in U_a \times_{(f_0|U_a, \beta)} \Gamma$$

define local sections $s_a : U_a \to E_0$. For $x \in \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$, there exist $g_0, g_1 \in \Gamma$ such that $g_0 g_1^{-1} = f_1(a, b; x)$ and

$$(\pi_1, \bar{f}_1)(a, b; (\pi_1, \varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x))) = ((a, b; x), (g_0, g_1))\,.$$

On applying ε_0 to the above equality, we find that $g_1 = f_0(b, \varepsilon_0 x)$ (a unit) and $g_0 = f_1(a, b; x)$. Hence

$$(\pi_1, \varepsilon_1)^{-1}(x, s_a(\varepsilon_1 x)) f_1(a, b; x) = (\pi_1, \varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x))$$

so that f is the ss-map determined by our collection of local sections of E_0, i.e. a representative of $\mathbf{f}_E : X \rightharpoonup \mathcal{N}\Gamma$. ∎

Remark 2.15. By considering *all* local sections of E_0 we get a *maximal* representative of $\mathbf{f}_E$, to which every representative is elementarily equivalent.

Remark 2.16. In the proof of Theorem 2.12 we have actually obtained an explicit form of the correspondence between Γ-cocycles and principal Γ-bundles classified by the associated ss-morphisms. Namely, if $\gamma = (\gamma_{ab})$ is any Γ-cocycle on X with respect to a covering $\mathcal{U} = (U_a)_{a\in A}$ of X_0 then—in terms of Proposition 2.13—there is a corresponding Γ-bundle E over X such that

$$E_0 = \coprod U_a \times_{(\gamma_{aa}\eta_0, \beta)} \Gamma / \sim$$

where $(b, x, g) \sim (a, x, \gamma_{ab}(\eta_0 x) g)$ for $x \in U_a \cap U_b$, and the crucial map $\varepsilon_1 : \varepsilon_0^* E_0 = E_1 \to E_0$ is given by

$$(y, [b, \varepsilon_0 y, g]) \to [a, \varepsilon_1 y, \gamma_{ab}(y) g]$$

for $y \in \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b$, $a, b \in A$.

I.3. Morphisms of groupoids. Let Γ and Γ' be groupoids (over manifolds N and N', respectively). For any Γ'-bundle E over the nerve $\mathcal{N}\Gamma$ consider the product map

$$\Gamma \times_{(\alpha,\pi_0)} E_0 = \varepsilon_0^* E_0 \cong E_1 \xrightarrow{\varepsilon_1} E_0 \,.$$

We call it a product and write $(g, u) \to gu$, for conditions (i)–(iii) of Proposition 2.13, mean exactly

(i) $g(ug') = (gu)g'$ iff the triple is composable,
(ii) $eu = u$ if $e = \pi_0 u$ is the unit, and
(iii) $g_1(g_2 u) = (g_1 g_2)u$.

In view of Theorem 2.12 and Proposition 2.13, we get canonical bijective correspondences:

ss-morphisms $\mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma'$
$\updownarrow$
isomorphy classes of principal Γ'-bundles over $\mathcal{N}\Gamma$
$\updownarrow$
isomorphy classes of principal Γ'-bundles over N equipped with a (left) Γ'-equivariant Γ-action

By transfer of structure, it follows that any principal Γ'-bundle Σ over N equipped with a Γ'-equivariant left Γ-action represents a (smooth) *morphism* $\Sigma : \Gamma \rightharpoonup \Gamma'$, of Γ to Γ' (a generalized homomorphism in the sense of [16]). One can readily check that

1° for any three groupoids Γ, Γ', and Γ'', and any morphisms $\Sigma : \Gamma \rightharpoonup \Gamma'$ and $\Sigma' : \Gamma' \rightharpoonup \Gamma''$, the *composition* $\Gamma \overset{\Sigma}{\rightharpoonup} \Gamma' \overset{\Sigma'}{\rightharpoonup} \Gamma''$ is represented by the principal Γ''-bundle

$$\Sigma' \circ \Sigma := \Sigma \times_{\Gamma'} \Sigma' = \Sigma \times_{(\alpha,\pi)} \Sigma'/(ug', u') \sim (u, g'u') \quad \text{for} \quad g' \in \Gamma'$$

where both the bundle projection and the Γ-action are inherited from Σ;

2° for any Γ, the *identity morphism* $\Gamma \rightharpoonup \Gamma$ is represented by Γ itself.

The above reduction of ss-morphisms of nerves to morphisms of groupoids can be summarized in

PROPOSITION 3.1. *Groupoids and their morphisms constitute a full subcategory of the category of ss-manifolds and ss-morphisms.* ∎

We shall denote by $\mathcal{N}$ the identifying functor

$$\Sigma : \Gamma \rightharpoonup \Gamma' \rightsquigarrow \mathcal{N}\Sigma : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma' .$$

Clearly, there is another functor which sends any homomorphism of groupoids $h : \Gamma \to \Gamma'$ to a morphism $[h] : \Gamma \rightharpoonup \Gamma'$ represented by

(3.1) $$[h] := N \times_{(h,\beta)} \Gamma', \quad g(\alpha g, g') := (\beta g, h(g)g') \quad \text{for } g \in \Gamma ;$$

one has $\mathcal{N}[h] = [\mathcal{N}h] : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma'$. By triviality of the units, every morphism $G' \rightharpoonup G$ of Lie groups is of the form $[h]$ for some homomorphism $h : G' \to G$ determined up to an inner automorphism of G.

The following theorem was announced in [16]:

THEOREM 3.2. *For any groupoids Γ and Γ' and every principal Γ'-bundle Σ over the units of Γ equipped with a Γ'-equivariant left Γ-action, the morphism $\Sigma : \Gamma \rightharpoonup \Gamma'$ is invertible iff Σ is also a (left) principal Γ-bundle with respect to the Γ-action. If this is the case then ${}^t\Sigma$, i.e. Σ considered with the transposed actions of the groupoids, represents the inverse morphism $\Gamma' \rightharpoonup \Gamma$.*

Proof. $\Leftarrow$ If ${}^t\Sigma$ is a principal Γ-bundle with respect to the right action $u \cdot g := g^{-1}u$ for $g \in \Gamma$, $u \in \Sigma$ then there are canonical isomorphisms

$$\Sigma \times_{\Gamma'} {}^t\Sigma = \Sigma \times_{(\alpha,\alpha)} \Sigma/\Gamma' \overset{\cong}{\to} \Gamma, \quad (gu, u)\Gamma' \to g ,$$

and

$${}^t\Sigma \times_{\Gamma} \Sigma = \Gamma \backslash (\Sigma \times_{(\pi,\pi)} \Sigma) \overset{\cong}{\to} \Gamma', \quad \Gamma(u, ug') \to g' ,$$

which show that ${}^t\Sigma$ represents $(\Sigma : \Gamma \rightharpoonup \Gamma')^{-1}$.

$\Rightarrow$ Let $\Sigma' : \Gamma' \rightharpoonup \Gamma$ be an inverse to $\Sigma : \Gamma \rightharpoonup \Gamma'$, so that Σ' is a principal Γ-bundle over the units N' of Γ' endowed with a Γ'-action and there are isomorphisms $\Sigma \times_{\Gamma'} \Sigma' \overset{J}{\to} \Gamma$ and $\Sigma' \times_{\Gamma} \Sigma \overset{J'}{\to} \Gamma'$. Since the projection $\Sigma' \times_{(\alpha,\pi)} \Sigma \to \Sigma' \times_{\Gamma} \Sigma$ is a submersion, for every $x \in N'$ there exists a local section

$$\Gamma' \supset N' \supset U \overset{(\varphi,\psi)}{\longrightarrow} \Sigma' \times_{(\alpha,\pi)} \Sigma$$

over a neighbourhood $U \ni x$. This means $\alpha\varphi = \pi\psi$, $J'(\varphi(y)\psi(y)) = y$ for $y \in U$ and, in particular, $\alpha\psi = \mathrm{id}_U$ so that $\psi : U \to \Sigma$ is a *section* of $\alpha : \Sigma \to N'$. It remains to prove that the mapping

$$\Gamma \times_{(\alpha,\pi\psi)} U \ni (g, y) \to g\psi(y) \in \Sigma$$

is a diffeomorphism onto $\alpha^{-1}U$. If there is a map $g : \alpha^{-1}u \to \Gamma$ such that $g(w)\psi(\alpha w) = w$ for $w \in \alpha^{-1}U$, then

$$J(w\varphi(\alpha w)) = g(w)J(\psi(\alpha w)\varphi(\alpha w))$$

and thus

$$g(w) = J(w\varphi(\alpha w))J(\psi(\alpha w)\varphi(\alpha w))^{-1} .$$

The $g(w)$ thus obtained is a smooth function of $w \in \alpha^{-1}U$ and satisfies $w\varphi(\alpha w) = g(w)\psi(\alpha w)\varphi(\alpha w)$ in $\Sigma \times_{\Gamma'} \Sigma'$; hence, there is a $g' \in \Gamma'$ such that $wg' = g(w)\psi(\alpha w)$, and $\varphi(\alpha w) = g'\varphi(\alpha w)$. The second equality immediately implies

$$J'(\varphi(\alpha w)\psi(\alpha w)) = g'J'(\varphi(\alpha w)\psi(\alpha w)),$$

i.e. $g' = \alpha w \in N'$, and thus $w = g(w)\psi(\alpha w)$ as was to be shown. ∎

EXAMPLE 3.3. For any surmersion $\varphi : M \to Q$ (cf. Example 1.13) the equivalence relation $R \subset M \times M$ which defines Q as a quotient of M is a groupoid over M such that $\mathcal{N}R \cong \overline{\mathcal{N}}M^{\varphi}$. When endowed with the R-action $(x, y) \cdot y = x$ and with the projections

$$\begin{array}{ccc} & M & \\ {}^{\mathrm{id}}\swarrow & & \searrow^{\varphi} \\ M & & Q \end{array}$$

M represents an invertible morphism $R \rightharpoonup Q$. The associated equivalence $\overline{\mathcal{N}}M^{\varphi} \cong \mathcal{N}R \rightharpoonup \mathcal{N}Q$ is exactly the ss-morphism considered in Example 1.13.

For arbitrary groupoids Γ and Γ' let Σ represent a morphism of Γ to Γ'. According to (2.4), every principal Γ-bundle E over an ss-manifold X gives rise to an associated Σ-bundle $E(\Sigma)$, and the Γ'-action inherited from Σ makes the associated bundle a principal Γ'-bundle over X, to be denoted by Σ_*E. In this way, Σ *transfers* Γ-bundles to Γ'-bundles. The transfer is natural with respect to homomorphisms of principal Γ-bundles.

EXAMPLE 3.4. If $\Sigma = [h]$ comes from a homomorphism $h : \Gamma \to \Gamma'$ then for every Γ-bundle E the induced Γ'-bundle $[h]_*E$ is canonically isomorphic to $h_*E := (E_n \times_h \Gamma')_{n\geq 0}$ where

$$E_n \times_h \Gamma' = E_n \times_{(h\alpha,\beta)} \Gamma'/(ug, g') \sim (u, h(g)g'), \qquad n \geq 0.$$

EXAMPLE 3.5. For any Σ the induced principal Γ'-bundle $\Sigma_*\overline{\mathcal{N}}\Gamma$ over $\mathcal{N}\Gamma$ is (up to isomorphism) the Γ'-bundle classified by the ss-morphism $\mathcal{N}\Sigma : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma'$, i.e. $\Sigma_*\overline{\mathcal{N}}\Gamma = (\mathcal{N}\Sigma)^*\overline{\mathcal{N}}\Gamma'$.

The next property completes Propositions 2.10 and 3.1.

PROPOSITION 3.6. *Let $E \to X$ be any principal Γ-bundle*, $\mathbf{f} : Y \rightharpoonup X$ *an ss-morphism*, *and $\Sigma : \Gamma \rightharpoonup \Gamma'$ a morphism of groupoids. Then* $\mathbf{f}^*(\Sigma_*E) = \Sigma_*(\mathbf{f}^*E)$. *More precisely*, *for any pull-back* $(E', \overline{\mathbf{f}})$ *of E by* $\mathbf{f}$ *there exists a pull-back of Σ_*E of the form* $(\Sigma_*E', \Sigma_*\overline{\mathbf{f}})$.

P r o o f. Every homomorphism $\overline{f} : E'_{\pi^{-1}\mathcal{U}} \to E$ generates a $\Sigma_*\overline{f}$: $(\Sigma_*E')_{\pi^{-1}\mathcal{U}} \cong \Sigma_*(E'_{\pi^{-1}\mathcal{U}}) \to \Sigma_*E$; the connectedness of the collection $\{\Sigma_*\overline{f}; \overline{f} \in \overline{\mathbf{f}}\}$ amounts to the functoriality of Σ_*. ∎

COROLLARY 3.7. *The classifying ss-morphisms* $\mathbf{f}_E$ *and* $\mathbf{f}_{\Sigma_* E}$ *yield a commuting triangle*

$$\begin{array}{ccc} & & \mathcal{N}\Gamma \\ & {}^{\mathbf{f}_E}\nearrow & \\ X & & \downarrow \mathcal{N}\Sigma \\ & {}_{\mathbf{f}_{\Sigma_* E}}\searrow & \\ & & \mathcal{N}\Gamma' \end{array}$$

P r o o f. $\Sigma_* E = \Sigma_*(\mathbf{f}_E^* \overline{\mathcal{N}}\Gamma) = \mathbf{f}_E^*(\mathcal{N}\Sigma)^* \overline{\mathcal{N}}\Gamma'$, by Example 3.5. ■

According to Example 3.5, every principal Γ-bundle F over $\mathcal{N}\Gamma$ is isomorphic to $\Sigma_* \overline{\mathcal{N}}\Gamma$, $\Sigma = F_0$. If we consider the manifold

$$\Gamma * \Sigma = \Gamma \times_{(\alpha,\pi)} \Sigma$$

and regard it as a groupoid over Σ such that

$$(g, u) \cdot (g', u') = (gg', u') \quad \text{iff} \quad u = g'u'$$

then there is a canonical isomorphism of ss-manifolds

$$F \cong \Sigma_* \overline{\mathcal{N}}\Gamma \cong \mathcal{N}(\Gamma * \Sigma)\,. \tag{3.2}$$

Using this particular form of the Γ'-bundle F we can reformulate and reinforce Corollary 3.7.

PROPOSITION 3.8. *Let* $\Sigma : \Gamma \rightharpoonup \Gamma'$ *be a morphism of groupoids,* $F = \mathcal{N}(\Gamma * \Sigma)$ *a* Γ'*-bundle over* $\mathcal{N}\Gamma$ *classified by* $\mathcal{N}\Sigma$*, and* $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ *an ss-morphism. If* E *is a principal* Γ*-bundle over* X *classified by* $\mathbf{f}$*, then there exists a pull-back* $(F', \overline{\mathbf{f}})$ *of* F *by* $\mathbf{f}$ *such that*

(i) $F' = \Sigma_* E = (E_n \times_\Gamma \Sigma)_{n \geq 0}$, *and*

(ii) *the lift* $\overline{\mathbf{f}} : F' \rightharpoonup F = \mathcal{N}(\Gamma * \Sigma)$ *is the classifying ss-morphism for the principal* $(\Gamma * \Sigma)$*-bundle*

$$\overline{E} := (E_n \times_{(\alpha,\pi)} \Sigma) \to (E_n \times_\Gamma \Sigma)$$

where $\Gamma * \Sigma$ *acts on* $\overline{E}_n$ *by the formula*

$$(w, u) \cdot (g, v) = (wg, v) \quad \textit{iff} \quad u = gv\,.$$

P r o o f. Fix a collection of local sections $s_a : U_a \to E_0$, $a \in A$, over a covering $\mathcal{U} = (U_a)_{a \in A}$ of X_0; the corresponding Γ-cocycle $(\gamma_{ab})_{a,b \in A}$ on X is given by (2.5). Each s_a gives rise to a section

$$\overline{s}_a : E_0 \times_\Gamma \Sigma \supset \pi_0^{-1} U_a \to E_0 \times_{(\alpha,\pi)} \Sigma\,, \qquad (s_a(x)g) \cdot u \to (s_a(x), gu)$$

(in multiplicative notation), and the corresponding transition $(\Gamma * \Sigma)$-cocycle $(\overline{\gamma}_{ab})_{a,b \in A}$ on F' is characterized by the equality

$$(\pi_1, \varepsilon_0)^{-1}(w \cdot u, \overline{s}_b(\varepsilon_0 w \cdot u)) = (\pi_1, \varepsilon_1)^{-1}(w \cdot u, \overline{s}_a(\varepsilon_1 w \cdot u))\overline{\gamma}_{ab}(w \cdot u)$$

for $w \cdot u \in \pi_1^{-1}(\varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b) \subset E_1 \times_\Gamma \Sigma$. If we take $w = (\pi_1, \varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x))$, $x = \pi_1(w \cdot u)$, then

$$(w, u) = (w\gamma_{ab}(x)^{-1}, \gamma_{ab}(x)u)\overline{\gamma}_{ab}(w \cdot u) \quad \text{in } E_1 \times_{(\alpha,\pi)} \Sigma$$

implies

$$\overline{\gamma}_{ab}((\pi_1, \varepsilon_0)^{-1}(x, s_b(\varepsilon_0 x)) \cdot u) = (\gamma_{ab}(x), u) \in \Gamma * \Sigma\,.$$

According to the last formula, the ss-map $\overline{f} : E'_{\pi^{-1}\mathcal{U}} \to \mathcal{N}(\Gamma * \Sigma)$ determined by the cocycle (cf. Example 1.5) is a Γ'-equivariant lift of the representative $f \in \mathbf{f}$ associated with (γ_{ab}). Evidently, the collection of all such lifts is connected with respect to elementary equivalences. ■

I.4. The fundamental groupoid of an ss-manifold. It is a well-known fact that the Poincaré group of a simplicial complex is a combinatorial construction. Stimulated by van Est [12] and Ver Eecke [24], we shall show that a Poincaré groupoid is naturally associated with any ss-manifold X.

Definition 4.1. A (continuous) *path* in an ss-manifold X is any sequence $C = (c_0, y_1^{e_1}, c_1, \ldots, c_{r-1}, y_r^{e_r}, c_r)$, $r \geq 0$, such that

1° $c_i : [0,1] \to X_0$, $i = 0, \ldots, r$, are continuous paths in X_0,

2° $y_1 \in X_1$ for $i = 1, \ldots, r$, and

3° the exponent $e_i = \pm 1$ indicates the direction of y_i, in the sense that

$$\varepsilon_1 y_i = c_{i-1}(1), \quad \varepsilon_0 y_i = c_i(0) \quad \text{if } e_i = +1\,,$$
$$\varepsilon_0 y_i = c_{i-1}(1), \quad \varepsilon_1 y_i = c_i(0) \quad \text{if } e_i = -1\,,$$

for $i = 1, \ldots, r$.

$c_0(0)$ and $c_r(1)$ are, respectively, the *initial point* and the *endpoint* of the path.

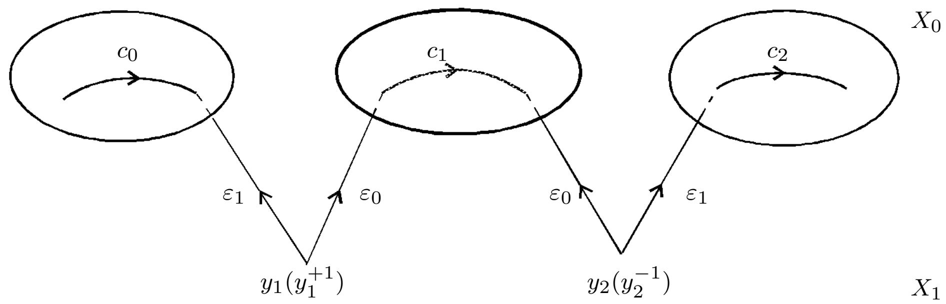


Fig. 1. A path in X

For brevity, we shall suppress the exponents $e = 1$ and say that the exponent $e = -1$ interchanges the roles of the face operators so that $\varepsilon_0 y^{-1} = \varepsilon_1 y$ and $\varepsilon_1 y^{-1} = \varepsilon_0 y$ for $y \in X_1$. Then condition 3° above reads

3°° $c_{i-1}(1) = \varepsilon_1 y_i^{e_i}$, $\varepsilon_0 y_i^{e_i} = c_i(0)$ for $i = 1, \ldots, r$.

Let $\overline{\pi}_{xx'}$, $x, x' \in X_0$, stand for the set of all the paths in X with initial point x and endpoint x'. An associative *composition rule* $\overline{\pi}_{xx'} \times \overline{\pi}_{x'x''} \to \overline{\pi}_{xx''}$ is given by

$$(4.1) \qquad (c_0, y_i^{e_i}, \dots, y_r^{e_r}, c_r)\cdot(\overline{c}_0, \overline{y}_1^{\bar{e}_1}, \dots, \overline{y}_s^{\bar{e}_s}, \overline{c}_s) = (c_0, y_1^{e_1}, \dots, y_r^{e_r}, c_r \cdot \overline{c}_0, \overline{y}_1^{\bar{e}_1}, \dots, \overline{y}_s^{\bar{e}_s}, \overline{c}_s)\,.$$

DEFINITION 4.2. For any $x, x' \in X_0$, we say that paths $C, C' \in \overline{\pi}_{xx'}$ are *homotopic* (notation $C \simeq C'$) if they are equivalent in the sense of the smallest equivalence relation generated in $\overline{\pi}_{xx'}$ by the following *elementary homotopies*:

(i) $(c_0, y_1^{e_1}, c_1, \dots, c_{r-1}, y_r^{e_r}, c_r) \simeq (\overline{c}_0, y_1^{e_1}, \overline{c}_1, \dots, \overline{c}_{r-1}, y_r^{e_r}, \overline{c}_r)$

if c_i and $\overline{c}_i$ are homotopic paths in X_0 for every i;

(ii) $(\dots, c_{i-1}, y^e, c_i, \dots) \simeq (\dots, c_{i-1} \cdot \varepsilon_1(c^e), \overline{y}^e, \varepsilon_0(c^e)^{-1} \cdot c_i, \dots)$

if $c : [0,1] \to X_1$ is any path such that $c(0) = y$, $c(1) = \overline{y}$;

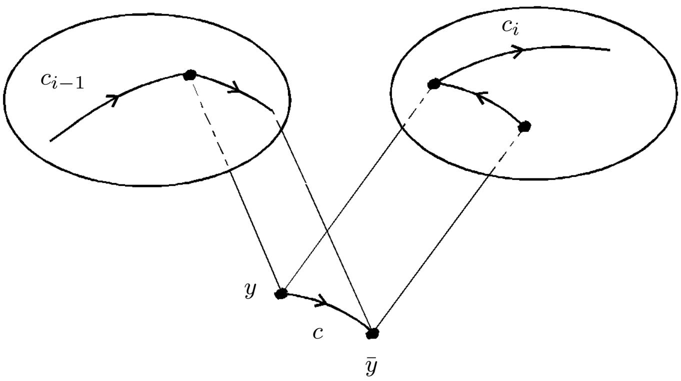


Fig. 2. Elementary homotopy of type (ii)

(iii) $(\dots, c_{i-1}, y^e, c_i, \dots) \simeq (\dots, c_{i-1} \cdot c_i, \dots)$ if $y \in \eta_0 X_0$;

(iv) $(\dots, c_{i-1}, y^e, c_i, \overline{y}^e, c_{i+1}, \dots) \simeq (\dots, c_{i-1}, (\varepsilon_1 z)^e, c_{i+1}, \dots)$

if c_i is a constant path and $z \in X_2$ is any element such that either $y = \varepsilon_2 z$, $\overline{y} = \varepsilon_0 z$ (if $e = 1$) or $y = \varepsilon_0 z$, $\overline{y} = \varepsilon_2 z$ (if $e = -1$);

(v) $(\dots, c_{i-1}, y, c_i, \overline{y}^{-1}, c_{i+1}, \dots) \simeq (\dots, c_{i-1}, (\varepsilon_2 z)^e, c_{i+1}, \dots)$

if c_i is a constant path and $z \in X_2$ is such that either $y = \varepsilon_1 z$, $\overline{y} = \varepsilon_0 z$ (then $e = 1$) or $y = \varepsilon_0 z$, $\overline{y} = \varepsilon_1 z$ (then $e = -1$);

(vi) $(\ldots, c_{i-1}, y^{-1}, c_i, \overline{y}, c_{i+1}, \ldots) \simeq (\ldots, c_{i-1}, (\varepsilon_0 z)^e, c_{i+1}, \ldots)$

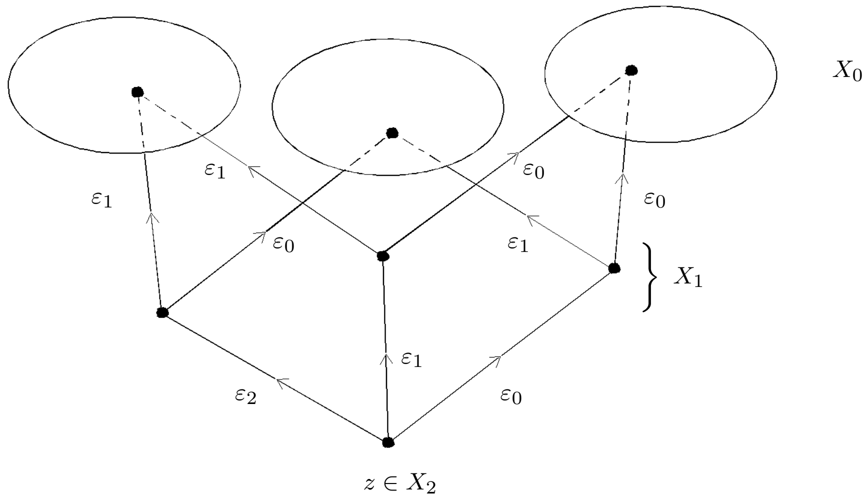


Fig. 3. Elementary homotopies of types (iv)–(vi)

if c_i is a constant path and $z \in X_2$ is such that either $y = \varepsilon_2 z$, $\overline{y} = \varepsilon_1 z$ (then $e = 1$) or $y = \varepsilon_1 z$, $\overline{y} = \varepsilon_2 z$ (then $e = -1$).

For $x \in X_0$ let ω_x denote the constant path at x.

LEMMA 4.3.

$$(c_0, y_1^{e_1}, c_1, \ldots, c_{r-1}, y_r^{e_r}, c_r) \cdot (c_r^{-1}, y_r^{-e_r}, c_{r-1}^{-1}, \ldots, c_1^{-1}, y_1^{-e_1}, c_0^{-1}) \simeq \omega_{c(0)}\,.$$

Proof. One successively applies conditions (i), (v)–(vi), and (iii) of Definition 4.2. In fact, every element $y \in X_1$ is of the form

$$y = \varepsilon_0(\eta_0 y) = \varepsilon_1(\eta_0 y) = \varepsilon_1(\eta_1 y) = \varepsilon_2(\eta_1 y)\,. \quad \blacksquare$$

For $x, x' \in X_0$, let

$$\pi_{xx'}(X) = \overline{\pi}_{xx'}(X)/\simeq$$

be the quotient set. As the composition rule (4.1) descends to the quotients, the union

$$\pi(X) = \coprod_{(x,x')\in X_0\times X_0} \pi_{xx'}(X)$$

is a small category with units

$$1_x = [\omega_x] \in \pi_{xx}(X) \quad \text{for } x \in X_0$$

and inverses (cf. Lemma 4.3)

$$[c_0, y_1^{e_1}, \ldots, y_r^{e_r}, c_r]^{-1} = [c_r^{-1}, y_r^{-e_r}, \ldots, y_1^{-e_1}, c_0^{-1}]$$

where the brackets [ ] indicate the homotopy classes of the paths. Note that the *source* and *target maps* of $\pi(X)$ are given by

$$\begin{array}{ccc} & [c_0, y_1^{e_1}, \dots, y_r^{e_r}, c_r] & \\ {}^{\beta}\swarrow & & \searrow^{\alpha} \\ c_0(0) & & c_r(1) \end{array}$$

DEFINITION 4.4. $\pi(X)$ is the *fundamental groupoid* of X. For $x \in X_0$, the group $\pi_x(X) = \pi_{xx}(X)$ is the *fundamental* (or *Poincaré*) *group* of X at x.

Remark 4.5. The groupoid $\pi(X)$ comes from the fundamental groupoid $\pi(X_0)$ of X_0 by attaching new *invertible* generators

$$[y] := [\omega_{\varepsilon_1 y}, y, \omega_{\varepsilon_0 y}]$$

such that $\alpha[y] = \varepsilon_0 y$, $\beta[y] = \varepsilon_1 y$, for $y \in X_1$, and relations

$$[c(1)] = [\varepsilon_1 c]^{-1}[c(0)][\varepsilon_0 c]$$

for any path $c : [0,1] \to X_1$, and

$$[\varepsilon_2 z][\varepsilon_0 z] = [\varepsilon_1 z]$$

for $z \in X_2$ (cf. Figs. 2, 3). The formal exponents $e = \pm 1$ are involved because of the invertibility requirement. If, in particular, $X = \mathcal{N}\mathit{\Gamma}$ is the nerve of a groupoid then the exponent $e = -1$ takes on its real meaning—the inverse in $\mathit{\Gamma}$—since then $[y][y^{-1}] = [\varepsilon_1(y, y^{-1})] = 1_{\beta y}$ for $y \in X_1 (= \mathit{\Gamma})$.

The fundamental groupoid of an ss-manifold X carries a canonical differentiable structure. In order to describe it, observe first that $\pi_{xx'}(X) \neq \emptyset$ iff $x \sim x'$ in the equivalence relation which distinguished between the connected components of X (cf. the proof of Proposition 1.15). Thus

$$\pi(X) = \coprod_{\alpha} \pi(X^{\alpha}) \tag{4.2}$$

if $X = \coprod_{\alpha} X^{\alpha}$ is the decomposition of X into its connected components.

Let now X be any *connected* ss-manifold, so that $\pi(X)$ is *transitive* on X_0, and $\pi_x(X)$, $x \in X_0$, are *isomorphic groups*. For any two connected simply connected open subsets $U, V \subset X_0$ we fix a reference point $x_0 \in X_0$ and homotopy classes of paths $u, v \in \pi(X)$ such that

$$\alpha u \in U, \qquad \alpha v \in V, \qquad \beta u = \beta v = x_0 \,. \tag{4.3}$$

Since for any $x \in U$ there is a unique homotopy class of paths in U connecting αu with x, the composition of paths extends u to a canonical section $\widetilde{u} : U \to \pi(X)$ of α (i.e. $\alpha\widetilde{u} = \mathrm{id}_U$, $\widetilde{u}(\alpha u) = u$). If, moreover, $\widetilde{v} : V \to \pi(X)$ is the extension of v, then for every $w \in \beta^{-1}U \cap \alpha^{-1}V \subset \pi(X)$ one has

$\widetilde{u}(\beta w)w\widetilde{v}(\alpha w)^{-1} \in \pi_{x_0}(X)$; consequently, there is a bijection

$$\Psi_{UV}^{uv} : \beta^{-1}U \cap \alpha^{-1}V \to U \times \pi_{x_0}(X) \times V\,, \quad w \to (\beta w, \widetilde{u}(\beta w)w\widetilde{v}(\alpha w)^{-1}, \alpha w)\,. \tag{4.4}$$

PROPOSITION 4.6. *There is a unique differentiable structure on* $\pi(X)$ *such that the fundamental groups* $\pi_{x_0}(X) \subset \pi(X)$, $x_0 \in X_0$, *are discrete and all the maps* Ψ_{UV}^{uv} (cf. (4.3)–(4.4)) *are diffeomorphisms* (*with open domains*). *When equipped with this differentiable structure*, $\pi(X)$ *is a differentiable groupoid such that* $(\beta, \alpha) : \pi(X) \to X_0 \times X_0$ *is a covering map.*

P r o o f. The connectedness of X ensures that the domains of the Ψ_{UV}^{uv}'s cover $\pi(X)$. Given any u_i, v_i, U_i, and V_i such that $\beta u_i = \beta v_i = x_i$, $i = 1, 2$, the transition map $\Psi_{U_2V_2}^{u_2v_2}(\Psi_{U_1V_1}^{u_1v_1})^{-1} : (U_1 \cap U_2) \times \pi_{x_1}(X) \times (V_1 \cap V_2) \to (U_1 \cap U_2) \times \pi_{x_2}(X) \times (V_1 \cap V_2)$,

$$(x, w, x') \to (x, \widetilde{u}_2(x)\widetilde{u}_1(x)^{-1}w\widetilde{v}_1(x')\widetilde{v}_2(x')^{-1}, x')\,,$$

is a diffeomorphism, for the compositions $\widetilde{u}_2(\cdot)\widetilde{u}_1(\cdot)^{-1}$ and $\widetilde{v}_1(\cdot)\widetilde{v}_2(\cdot)^{-1}$ are locally constant. Once we have got charts, the other properties of $\pi(X)$ are readily seen. ∎

The maps Ψ_{UV}^{uv} also make sense if X is not necessarily connected, and induce on $\pi(X)$ the manifold structure of the disjoint union (4.2), which is still a differentiable groupoid over X_0.

COROLLARY 4.7. *If an ss-manifold* X *is connected then for every* $x \in X_0$ *the submanifold* $\alpha^{-1}(x) \subset \pi(X)$ *endowed with its natural structure of a principal* (*right*) $\pi_x(X)$*-bundle over* X_0 *and the* (*left*) $\pi(X)$*-action yields an* equivalence *of groupoids*

$$\alpha^{-1}(x) : \pi(x) \overset{\approx}{\rightharpoondown} \pi_x(X) \tag{4.5}$$

inverse to the morphism generated by inclusion. ∎

One should note that the superpositions

$$\pi_x(X) \hookrightarrow \pi(X) \approx \pi_{x'}(X), \qquad x, x' \in X_0\,,$$

are well defined equivalences to groups, i.e. conjugacy classes of isomorphisms. In general, there is no canonical isomorphism of the groups.

R e m a r k 4.8. The above corollary reflects a well-known property of the so-called *Galois groupoids*. Any such groupoid Γ over N is characterized by the requirement that the map $(\alpha, \beta) : \Gamma \to N \times N$ be surjective and étale (a local diffeomorphism). A Galois groupoid is canonically equivalent—via inclusion—to each of the (discrete) *structural groups* of Γ,

$$\Gamma_x = (\beta, \alpha)^{-1}(x, x) \subset \Gamma, \qquad x \in N\,.$$

Returning to an arbitrary ss-manifold X, note the following

LEMMA 4.9. *The map* $X_1 \ni y \xrightarrow{\gamma} [y] \in \pi(X)$ *is a* $\pi(X)$*-cocycle on* X *with respect to the trivial covering* $\{X_0\}$ (cf. Example 1.5).

Proof. The identity $[\varepsilon_2 z][\varepsilon_0 z] = [\varepsilon_1 z]$ for $z \in X_2$, found in Remark 4.5, is exactly the cocycle condition (1.2). In order to check the smoothness of the cocycle it suffices to consider any chart Ψ_{UV}^{uv} (cf. (4.3)–(4.4)) and observe that the $\pi_{x_0}(X)$-coordinate of γ is $y \to \widetilde{u}(\varepsilon_1 y)[y]\widetilde{v}(\varepsilon_0 y)$. This function is constant on the connected components of $\varepsilon_1^{-1}U \cap \varepsilon_0^{-1}V$, for there is an elementary homotopy of type (ii) between the paths involved. ■

In view of the lemma, for any ss-manifold X there exists a canonical ss-map $\pi_X : X \to \mathcal{N}\pi(X)$,

$$X_0 \ni x \to 1_x \in X_0 \hookrightarrow \pi(X)\,,$$
$$X_n \ni x \to ([\varepsilon_2{}^{n-1}x], [\varepsilon_2{}^{n-2}\varepsilon_0 x], \ldots, [\varepsilon_0{}^{n-1}x]) \in \mathcal{N}_n\pi(X)$$

for $n \geq 1$ (cf. (1.3)).

DEFINITION 4.10. π_X is the *fundamental ss-map* for X. Each of the ss-morphisms

$$\Pi_X = [\pi_X] : X \rightharpoonup \mathcal{N}\pi(X)$$

and (if X is connected; cf. (4.5))

$$\Pi_{X,x} : X \overset{\Pi_X}{\rightharpoonup} \mathcal{N}\pi(X) \overset{\approx}{\rightharpoonup} \mathcal{N}\pi_x(X), \qquad x \in X_0\,,$$

is a *fundamental ss-morphism* for X.

In order to characterize principal bundles classified by the fundamental ss-morphisms, we recall Proposition 2.13. Clearly, the $\pi(X)$-bundle $E_X = \pi_X^*\overline{\mathcal{N}}\pi(X)$ can be constructed from $\pi(X) \xrightarrow{\beta} X_0$ over the 0-level, and from

$$\overline{\varepsilon}_1 : X_1 \times_{(\varepsilon_0,\beta)} \pi(X) \to \pi(X), \quad (y,u) \to [y]u\,. \tag{4.6}$$

By (4.5), the corresponding $\pi_x(X)$-bundle $\widetilde{X}_x \to X$ (if X is connected) comes from $\alpha^{-1}(x) \xrightarrow{\beta} X_0$ and from a suitable restriction of $\overline{\varepsilon}_1$. It will be shown in the sequel that the last principal bundle over X plays the role of a universal covering space.

THEOREM 4.11. *Let* X *be a connected ss-manifold, and* x *a point of* X_0. *For every* discrete *group* G *and any ss-morphism* $\mathbf{f} : X \rightharpoonup \mathcal{N}G$ *there is a* unique *morphism of groups* $\pi_x(X) \rightharpoonup G$ *such that the following triangle commutes*:

$$\begin{array}{ccc} X & & \\ {\scriptstyle \Pi_{X,x}}\downharpoonleft & \searrow^{\mathbf{f}} & \\ & & \mathcal{N}G \\ & \nearrow & \\ \mathcal{N}\pi_x(X) & & \end{array} \tag{4.7}$$

COROLLARY 4.12. *Both the fundamental ss-morphisms and the fundamental groups of a connected ss-manifold X are completely characterized* (*up to equivalence of groups*) *as solutions to the universal factorization problem* (4.7).

Proof of Theorem 4.11. We give up symmetry and shall search for a morphism $\pi(X) \rightharpoonup G$.

Existence. For any $\mathbf{f} : X \rightharpoonup \mathcal{N}G$ let $P = (P_n)$ be a principal G-bundle over X classified by $\mathbf{f}$; we intend to show that P_0 admits a canonical left $\pi(X)$-action (with respect to the bundle projection $\pi_0 : P_0 \to X_0$). So let $C = (c_0, y_1^{e_1}, \ldots, y_r^{e_r}, c_r)$ be a path in X. In order to associate with C a G-equivariant isomorphism $\pi_0^{-1}(c_r(1)) \to \pi_0^{-1}(c_0(0))$ of the fibres of P_0 we fix $u \in P_0$ over $c_r(1) = \alpha[C]$ and proceed as follows:

1° As π_0 is a covering map, c_r admits a unique (continuous) lift $\widetilde{c}_r : [0,1] \to P_0$ such that $\widetilde{c}_r(1) = u$.

2° Assume we have already constructed a lift $\widetilde{c}_k : [0,1] \to P_0$ of c_k, for some $k \geq 0$. If $k > 0$, then the pair $(y_k, \widetilde{c}_k(0))$ is an element of either $\varepsilon_0^* P_0$ (if $e_k = 1$) or $\varepsilon_1^* P_0$ (if $e_k = -1$). In view of the canonical isomorphisms $\varepsilon_0^* P_0 \cong P_1 \cong \varepsilon_1^* P_0$, the pair comes from a point $v_k \in \pi_1^{-1}(y_k) \subset P_1$ and thus gives rise to $\varepsilon_1(v_k^{e_k})$, a uniquely defined point in the fibre of P_0 over $\varepsilon_1(y_k^{e_k})$. $\widetilde{c}_{k-1} : [0,1] \to P_0$ is the only lift of c_{k-1} such that $\widetilde{c}_{k-1}(1) = \varepsilon_1(v_k^{e_k})$.

3° By induction, we eventually get a lift $\widetilde{c}_0 : [0,1] \to P_0$ of c_0.

The above *lifting procedure* 1°–3° gives rise to a path

$$\widetilde{C} = (\widetilde{c}_0, v_1^{e_1}, \ldots, v_r^{e_r}, \widetilde{c}_r)$$

in P, which ends at u and is a lift of C in an obvious sense. We set

$$C \cdot u := \widetilde{c}_0(0) = \text{initial point of } \widetilde{C}\,.$$

This action is clearly G-equvariant and commutes with composition of paths. In order to prove its dependence upon the homotopy class of the path only, it suffices to observe that each elementary homotopy between paths in X leads to an analogous elementary homotopy between the lifted paths in P. Hence the initial points of the lifted paths remain fixed.

We have shown that the action of paths descends to an action of the fundamental groupoid. When endowed with that action (its smoothness is obvious), P_0 represents a morphism of groupoids, $P_0 : \pi(X) \rightharpoonup G$.

Factorization. It remains to establish an isomorphism of P and the G-bundle $E_X(P_0)$ (cf. Corollary 3.7). At the 0-level, $\pi(X) \times_{\pi(X)} P_0 \cong P_0$ is a canonical isomorphism concealed in the multiplicative notation of elements of the first G-bundle. In view of Corollary 2.14, one has to consider the extension of that isomorphism to

$$E_X(1) \times_{\pi(X)} P_0 \cong \varepsilon_0^*(\pi(X) \times_{\pi(X)} P_0) = \varepsilon_0^* P_0 \xrightarrow{(\pi_1, \varepsilon_0)^{-1}} P_1\,.$$

The only criterion of extendability of the canonical isomorphism to the whole G-bundles over X is now the equality (cf. (4.6))

$$([y][C]) \cdot u = \varepsilon_1(\pi_1, \varepsilon_0)^{-1}(y, [C]u) \tag{4.8}$$

for any $(y, [C]) \in E_X(1) = X_1 \times_{(\varepsilon_0, \beta)} \pi(X)$ and $u \in P_0$ such that $\alpha[C] = \pi_0(u)$. Since $([y][C]) \cdot u = [y] \cdot ([C] \cdot u)$, (4.8) follows from step 2° of the lifting procedure.

Uniqueness. Let $\Sigma : \pi(X) \rightharpoonup G$ be any solution to the examined factorization problem. By Corollary 3.7, there is an isomorphism of principal G-bundles over X, $P \xrightarrow{\cong} \Sigma_*(E_X)$. Since over X_0 the isomorphism reads

$$P_0 \cong \pi(X) \times_{\pi(X)} \Sigma \cong \Sigma$$

we may, and do, represent the morphism by P_0 endowed with an appropriate left $\pi(X)$-action. It remains to show that there is at most one G-equivariant $\pi(X)$-action on P_0 which satisfies (4.8). To this end, observe that $\pi(X)$ is generated by $[y]$, $y \in X_1$, and the homotopy classes $[c]$ of small paths $c : [0,1] \to U \subset X_0$, U being any simply connected open subset of X_0. The action of $[y]$, $y \in X_1$, is completely characterized by (4.8). As for the other generators, we see that if $U \subset X_0$ is a connected simply connected open subset and $u \in P_0$ is any point such that $\pi_0(u) \in U$, then

— there is a unique continuous section $\tau : U \to \alpha^{-1}(\pi_0(u)) \subset \pi(X)$ of β which passes through the unit at $\pi_0(u)$;

— as any action requires $1_{\pi_0(u)} \cdot u = u$, every continuous action gives rise to a continuous local section of P_0, $U \ni x \to \tau(x) \cdot u \in P_0$, through u. The uniqueness of the action follows from the uniqueness of the section. ∎

COROLLARY 4.13. *For every ss-morphism* $\mathbf{f} : X \rightharpoonup Y$ *of connected ss-manifolds, and any points* $x \in X_0$, $y \in Y_0$, *there is a* unique *morphism of groups* $\pi_{yx}(\mathbf{f}) : \pi_x(X) \rightharpoonup \pi_y(Y)$ *such that* $\Pi_{Y,y} \circ \mathbf{f} = \mathcal{N}\pi_{yx}(\mathbf{f}) \circ \Pi_{X,x}$. *Furthermore, there exists a* unique *morphism of fundamental groupoids* $\pi(\mathbf{f}) : \pi(X) \rightharpoonup \pi(Y)$ *such that the following square commutes*:

$$\begin{array}{ccc} X & \xrightharpoonup{\mathbf{f}} & Y \\ \Pi_X \downharpoonleft & & \downharpoonleft \Pi_Y \\ \mathcal{N}\pi(X) & \xrightharpoonup{\mathcal{N}\pi(\mathbf{f})} & \mathcal{N}\pi(Y) \end{array} \tag{4.9}$$

The assignment $\mathbf{f} \rightsquigarrow \pi(\mathbf{f})$ *is a covariant functor.* ∎

Throughout the rest of this section we assume all ss-manifolds to be connected. In order to get an explicit description of the morphisms $\pi(\mathbf{f})$ we recall Theorem 1.12 and consider the case of an arbitrary *ss-map* $f : X \to Y$

first. Clearly, f induces a *homomorphism* of groupoids $\pi(f) : \pi(X) \to \pi(Y)$,

$$(4.10) \qquad [c_0, y_1^{e_1}, \dots, y_r^{e_r}, c_r] \to [f_0c_0, (f_1y_1)^{e_1}, \dots, (f_1y_r)^{e_r}, f_0c_r]\,,$$

which restricts to group homomorphisms $\pi_x(f) : \pi_x(X) \to \pi_{f_0x}(Y)$ for $x \in X_0$.

PROPOSITION 4.14. (i) *For any ss-map* $f : X \to Y$ *one has* (cf. (3.1)) $\pi([f]) = [\pi(f)] : \pi(X) \rightharpoonup \pi(Y)$.

(ii) *Every gluing projection* $\lambda : X_{\mathcal{U}} \to X$ *gives rise to a canonical* equivalence *of the fundamental groupoids*, $[\pi(\lambda)] : \pi(X_{\mathcal{U}}) \overset{\approx}{\rightharpoonup} \pi(X)$.

(iii) *Let* $f : X_{\mathcal{U}} \to Y$ *be any representative of an ss-morphism* $\mathbf{f} : X \rightharpoonup Y$, *and* $\lambda : X_{\mathcal{U}} \to X$ *the gluing projection. Then* $\pi(\mathbf{f}) = [\pi(f)] \circ [\pi(\lambda)]^{-1}$.

Proof. By Corollary 4.13, the first assertion follows from the commutativity of a square analogous to (4.9) on the level of ss-maps already. Parts (ii)–(iii) are immediate consequences of (i), the functoriality of $\pi(\cdot)$, and Theorem 1.12. ∎

COROLLARY 4.15. *If* $\mathcal{U}$ *is any open covering of* X_0 *then*

$$\pi_u(\lambda) : \pi_u(X_{\mathcal{U}}) \to \pi_{\lambda_0 u}(X), \qquad u \in X_{\mathcal{U}}(0)\,,$$

is an isomorphism *of the fundamental groups. Furthermore*, *if* $f : X_{\mathcal{U}} \to Y$ *is any representative of* $\mathbf{f} : X \rightharpoonup Y$ *then for each* $u \in X_{\mathcal{U}}(0)$ *the morphism of groups*

$$\pi_{f_0u,\lambda_0u}(\mathbf{f}) : \pi_{\lambda_o u}(X) \rightharpoonup \pi_{f_0u}(Y)$$

comes from the homomorphism (cf. (3.1))

$$\pi_{\lambda_o u}(X) \xrightarrow{\pi_u(\lambda)^{-1}} \pi_u(X_{\mathcal{U}}) \xrightarrow{\pi_u(f)} \pi_{f_0u}(Y)\,. \quad ∎$$

Remark 4.16. Part (ii) of Proposition 4.14 together with Corollary 4.15 make the computation of $\pi(X)$ and $\pi_x(X)$ a purely combinatorial matter. Namely, one first replaces X with its localization to any *simply connected* open covering, and then shrinks every connected component of each X_n, $n \geq 0$, to a point. The Poincaré groupoid (group) of the resulting semi-simplicial *set* is canonically equivalent to $\pi(X)$ (resp., $\pi_x(X)$).

EXAMPLE 4.17. For any manifold M, $\pi(\mathcal{N}M)$ is the classical fundamental (Poincaré) groupoid $\pi(M)$ of the manifold. Thus we get a recipe for computing $\pi(M)$ starting from any simply connected open covering of M.

EXAMPLE 4.18. For any Lie group $G, \pi(\mathcal{N}G) = G/G_0$ where $G_0 \subset G$ is the identity component. The fundamental ss-map $\pi_{\mathcal{N}G} : \mathcal{N}G \to \mathcal{N}(G/G_0)$ comes from the projection $G \to G/G_0$.

EXAMPLE 4.19. Let Γ be any Galois groupoid on a manifold M (cf. Remark 4.8). For every $x \in M$, the canonical equivalence $\Gamma_x \hookrightarrow \Gamma$ induces an equivalence of the fundamental groups $\pi_x(\mathcal{N}\Gamma_x) \xrightarrow{\approx} \pi_x(\mathcal{N}\Gamma)$ where $\pi_x(\mathcal{N}\Gamma_x) = \Gamma_x$ by Example 4.18. The equivalence is generated by an *isomorphism* $\Gamma_x \xrightarrow{\cong} \pi_x(\mathcal{N}\Gamma)$, $g \to [g]$.

We close the present section with a theorem which asserts that every connected ss-manifold admits a *simply connected covering ss-manifold*.

PROPOSITION 4.20. *Let X be a connected ss-manifold, x any point of X_0, and $\widetilde{X} = \widetilde{X}_x$ a principal $\pi_x(X)$-bundle over X classified by the fundamental ss-morphism. Then*

(i) *the ss-manifold $\widetilde{X}$ is connected, and*
(ii) $\pi(\widetilde{X}) = \widetilde{X}_0 \times \widetilde{X}_0$, *i.e.* $\pi_u(\widetilde{X}) = \{1_u\}$ *for* $u \in \widetilde{X}_0$.

Proof. Recall that $\widetilde{X}_0 = \alpha^{-1}(x) \subset \pi(X)$, $\widetilde{X}_1 = \varepsilon_0^*\widetilde{X}_0 = X_1 \times_{(\varepsilon_0,\beta)} \alpha^{-1}(x)$, and the face operators $\widetilde{X}_1 \rightrightarrows \widetilde{X}_0$ are

$$(y,[C]) \begin{array}{l} \overset{\varepsilon_0}{\nearrow} [C] \\ \underset{\varepsilon_1}{\searrow} [y][C] \end{array}$$

(cf. (4.6)). As $\widetilde{X} \to X$ is a principal $\pi_x(X)$-bundle, every path C in X admits a canonical lifting to a path $\widetilde{C}$ in $\widetilde{X}$ which ends at an arbitrary but fixed point over the endpoint of C (cf. the existence part of the proof of Theorem 4.11). In order to prove (i) it suffices to observe that for any path C such that $[C] \in \widetilde{X}_0 = \alpha^{-1}(x)$ the lift $\widetilde{C}$ which ends at the unit $1_x \in \widetilde{X}_0$ has the homotopy class $[C]$ for its initial point. The proof goes by induction on the *length* of C, i.e. the number of the elements of X_1 involved. Namely, if $C = c_0 : [0,1] \to X_0$ then the lift of C is a path $\widetilde{c}_0 : [0,1] \to \widetilde{X}_0$, $t \to [c_0^{(t)}]$, where

$$c_0^{(t)}(\tau) = \begin{cases} c_0(t) & \text{for } 0 \le \tau \le t\,, \\ c_0(\tau) & \text{for } t \le \tau \le 1\,, \end{cases} \tag{4.11}$$

so that $c_0^{(1)} = \omega_x$. In general, if $C = (c_0, y_1^{e_1}, c_1, \dots, c_r)$, and the assertion is true for $C' = (c_1, \dots, c_r)$, then we consider two cases:

For $e_1 = 1$, $(y_1,[C']) \in \widetilde{X}_1$ is the element of the fibre over y_1 which joins $[C']$ (= initial point of $\widetilde{C}'$) to that point over $c_0(1)$ to which c_0 should be lifted. As $\varepsilon_1(y_1,[C']) = [\omega_{\varepsilon_1 y_1}, y_1, c_1, \dots, c_r]$, the lift of c_0 is $\widetilde{c}_0 : [0,1] \to \widetilde{X}_0$, $t \to [c_0^{(t)}, y_1, c_1, \dots, c_r]$, where $c_0^{(t)}$ is given by (4.11). Hence $\widetilde{c}_0(0) = [C]$.

If $e_1 = -1$, then $[C'] = \varepsilon_1 w$ for $w = (y_1, [y_1]^{-1}[C']) \in \widetilde{X}_1$, so that $\varepsilon_0 w = [\omega_{\varepsilon_0 y_1}, y_1^{-1}, c_1, \dots, c_r]$. The rest goes as above.

Let now $P \to \widetilde{X}$ be a principal bundle with a discrete structure group G. By Corollary 4.12, the assertion (ii) will be demonstrated if we find a *global section* $s : \widetilde{X} \to P$ of the G-bundle, since then every ss-morphism $\mathbf{f} : \widetilde{X} \rightharpoonup \mathcal{N}G$ will be shown to factorize through (the nerve of) the trivial group. In order to get a section of P, we fix $u \in P_0$ above $1_x \in \widetilde{X}_0$ and apply the lifting procedure for paths. Namely, for any path C in X whose homotopy class $[C]$ is in $\widetilde{X}_0$ we first take the lift $\widetilde{C}$ of C to $\widetilde{X}$ (with endpoint 1_x; cf. the proof of (i)) and then the lift $\widetilde{\widetilde{C}}$ of $\widetilde{C}$ to P with endpoint u. This gives rise to a map

$$\widetilde{X}_0 \ni [C] \overset{s_0}{\to} \text{ initial point of } \widetilde{\widetilde{C}} \in P_0 \,.$$

Indeed, if two paths C and C' represent the same element of $\widetilde{X}_0$ then any sequence of elementary homotopies connecting C and C' carries over to the lifts and yields a homotopy between $\widetilde{C}$ and $\widetilde{C}'$ as well as a homotopy between $\widetilde{\widetilde{C}}$ and $\widetilde{\widetilde{C}}'$. Thus s_0 is a well defined section of P_0 (its smoothness is an immediate consequence of the fact that $P_0 \to \widetilde{X}_0$ is a covering).

Climbing up to the 1-level, we find two sections of P_1 induced by s_0 and the isomorphisms $\varepsilon_0^* P_0 \cong P_1 \cong \varepsilon_1^* P_0$. Our claim is that the two sections are actually equal, i.e.

$$(\pi_1, \varepsilon_0)^{-1}((y,[C]), s_0([C])) = (\pi_1, \varepsilon_1)^{-1}((y,[C]), s_0([y][C])) \tag{4.12}$$

for any $(y,[C]) \in \widetilde{X}_1$. Indeed, for $C = (c_0, y_1^{e_1}, \dots, c_r)$ let

$$C' = (\omega_{\varepsilon_1 y}, y, C) := (\omega_{\varepsilon_1 y}, y, c_0, y_1^{e_1}, \dots, c_r)$$

be a path representing the homotopy class $[y][C]$; its lift to X is $\widetilde{C}' = (\omega_{\varepsilon_1 \tilde{y}}, \widetilde{y}, \widetilde{C})$ for $\widetilde{y} = (y,[C]) \in \widetilde{X}_1$, and the lifting procedure yields $\widetilde{\widetilde{C}}' = (\omega_{\varepsilon_1 \tilde{\tilde{y}}}, \widetilde{\widetilde{y}}, \widetilde{\widetilde{C}})$ in P, where $\widetilde{\widetilde{y}} = (\pi_1, \varepsilon_0)^{-1}(\widetilde{y}, s_0([C]))$. Now the defining equality $s_0([C']) = \varepsilon_1 \widetilde{\widetilde{y}}$ proves (4.12).

We have shown that for any principal G-bundle P over $\widetilde{X}$ there is a section $s_0 : \widetilde{X}_0 \to P_0$ which satisfies (4.12). In order to derive from this a factorization of the classifying ss-morphism through the trivial group, it remains to compare (4.12) with (2.5). In fact, (4.12) ensures the extendability of s_0 to a global section $s : \widetilde{X} \to P$. ∎

Corollary 4.21. *Given an ss-morphism* $\mathbf{f} : X \rightharpoonup Y$ *of connected ss-manifolds, the induced morphism* $\pi_{yx}(\mathbf{f}) : \pi_x(X) \rightharpoonup \pi_y(Y)$ $x \in X_0$, $y \in Y_0$, *is generated by* epimorphisms *of the groups iff* $\mathbf{f}$ *pulls back the simply connected* $\pi_y(Y)$*-bundle* $\widetilde{Y} \to Y$ *to a* connected *one.*

Note that since every morphism of groups is a conjugacy class of homomorphisms, either all the homomorphisms are epimorphisms or none of them is.

P r o o f. By Proposition 2.10, Corollary 4.13, and Corollary 3.7, we successively have

$$\mathbf{f}^*\widetilde{Y} = \mathbf{f}^* \Pi^*_{Y,y} \overline{\mathcal{N}} \pi_y(Y) = (\Pi_{Y,y} \circ \mathbf{f})^* \overline{\mathcal{N}} \pi_y(Y)$$
$$= (\mathcal{N}\pi_{yx}(\mathbf{f}) \circ \Pi_{X,x})^* \overline{\mathcal{N}} \pi_y(Y) = \pi_{yx}(\mathbf{f})_* (\Pi^*_{X,x} \overline{\mathcal{N}} \pi_x(X)) = \pi_{yx}(\mathbf{f})_* \widetilde{X}$$

for the $\pi_x(X)$-bundle $\widetilde{X} = \widetilde{X}_x$ over X. In other words, if a homomorphism $h : \pi_x(X) \to \pi_y(Y)$ generates $\pi_{yx}(\mathbf{f})$ then the h-extension $h_*\widetilde{X}$ (cf. Example 3.4) represents the pull-back $\mathbf{f}^*\widetilde{Y}$. Consequently, one has to examine the connectedness of $h_*\widetilde{X}$ and relate it to the surjectivity of h.

If h is onto, then the canonical ss-map $\widetilde{X} \to h_*\widetilde{X}$,

$$\widetilde{X}_n \ni u \to u \cdot 1_y \in \widetilde{X}_n \times_h \pi_y(Y)\,, \tag{4.13}$$

is surjective at every level; thus the connectedness of $\widetilde{X}$ is inherited by $h_*\widetilde{X}$.

If h is not onto, then the ss-map (4.13) sends $\widetilde{X}$ to a connected component of $h_*\widetilde{X}$. As the group $\pi_y(Y)$ is discrete, the other components correspond to the left cosets of $h(\pi_x(X)) \subset \pi_y(Y)$. ∎

II. Foliations of semi-simplicial manifolds

We enlarge the category considered in Part I by admitting foliations on ss-manifolds. Fortunately, there is a natural way of distinguishing those ss-morphisms which should be understood as morphisms of the extended category, as well as those which are transverse to foliations. In the case of classifying ss-morphisms of the form $X \rightharpoonup \mathcal{N}\Gamma$ these properties are expressed in terms of the associated Γ-bundles (Thm. 1.17). Section 2 contains a classification theorem for foliations (Thm. 2.1) and a general approach to transverse projections. For an arbitrary foliation we prove the existence and uniqueness of a minimal transverse projection (Thm. 3.2). Intuitively, a minimal transverse projection for a foliated ss-manifold $(X, \mathcal{F})$ is a "surmersion" $X \rightharpoonup X/\mathcal{F}$ which induces $\mathcal{F}$ from the discrete foliation of a certain ss-manifold $X/\mathcal{F}$; every foliation of X weaker than $\mathcal{F}$ projects to a foliation of $X/\mathcal{F}$ (Thm. 3.13). It is shown that the minimal transverse projection induces an epimorphism of fundamental groups (Thm. 4.1). Furthermore, the projection contracts every leaf L of $\mathcal{F}$ to its holonomy group (Thm. 4.5). Section 5 presents implicit Γ-foliations given by geometric structures over foliated ss-manifolds. We construct the universal foliated G-bundle for foliated ss-manifolds of any fixed codimension, and the universal G-foliation (G-structure).

II.1. Foliated ss-manifolds. A *q-codimensional foliation* F of an n-dimensional manifold $M (n \geq q \geq 0)$ is a topology in M, stronger than the original one, such that every point of M admits a chart $\varphi : U \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$ which induces a homeomorphism of $(U, F|U)$ into $\mathbb{R}^{n-q} \times (\mathbb{R}^q, \delta)$, δ being the discrete topology. Alternatively, F is characterized by a maximal collection $\mathrm{Subm}(F)$ of submersions $M \supset U \to \mathbb{R}^q$ continuous in the foliation topologies, i.e. inducing a map $(U, F|U) \to (\mathbb{R}^q, \delta)$; on the overlaps, any two such submersions differ (locally) by a diffeomorphism of $\mathbb{R}^q$. The last property distinguishes the sets of submersions which come from foliations (see [18], [20], [22], [23], [25], for a more complete treatment).

Each q-codimensional foliation F induces on set M a structure of an $(n-q)$-dimensional manifold M^F such that the identity map $i : M^F \to M$ is an immersion; the connected components of M^F are *leaves* of F. The involutive subbundle $TF := i_* TM^F \subset TM$ is the *tangent bundle* of the foliation. According to the classical *Frobenius Theorem*, any involutive subbundle of TM is the tangent bundle of a (unique) foliation of M.

Any pair (M, F) composed of a manifold and its foliation is a *foliated manifold*. We shall regard manifolds as foliated by the only 0-codimensional foliation. A *map* of foliated manifolds $f : (M', F') \to (M, F)$ is any smooth map $f : M' \to M$ continuous in the foliation topologies. Infinitesimally, this reads

$$f_*(TF') \subset TF\,. \tag{1.1}$$

For every such f the induced map $f' : M'^{F'} \to M^F$ is smooth.

A pair of foliations (F, F') of the same manifold M is a *flag* if $TF \subset TF'$ or, equivalently, if $\mathrm{id}_M : (M, F) \to (M, F')$.

For any map $f : M' \to M$ *transverse* to a foliation F of M (i.e. to M^F; notation $f \pitchfork F$) the pull-back topology $f^{-1}F$ together with the manifold topology of M' generate a *pull-back foliation* f^*F of M'. We recall that the transversality means

$$f_*(T_x M') + T_{f(x)}F = T_{f(x)}M \tag{1.2}$$

for $x \in M'$. Then

$$T(f^*F) = f_*^{-1}(TF) \tag{1.3}$$

so that $f : (M', f^*F) \to (M, F)$.

The following property is classical and follows from (1.1)–(1.3).

Proposition 1.1. *For any foliated manifold* (M, F) *and every map* $f : M' \to M$ *transverse to* F,

(i) *a map* $g : M'' \to M'$ *is transverse to* f^*F *iff the composition* fg *is transverse to* F, *and* $g^*f^*F = (fg)^*F$,

(ii) *given any foliated manifold* (M'', F'') *and a map* $g : M'' \to M'$, *one has* $g : (M'', F'') \to (M', f^*F)$ *iff* $fg : (M'', F'') \to (M, F)$. ∎

The pull-back of foliations agrees with that of immersed submanifolds. Namely, we recall that for any immersed submanifold $i : N \hookrightarrow M$ each map $f : M' \to M$ transverse to N pulls it back to an immersed submanifold $f^{-1}N \hookrightarrow M'$, which is diffeomorphic (via (id, f)) to the fibre product $M' \times_{(f,i)} N \subset M' \times N$. In this sense, for $f \pitchfork F$ the pull-back foliation f^*F induces on M' the manifold structure of $f^{-1}(M^F)$.

PROPOSITION 1.2. *Let $f : (M, F) \to (M', F')$ be a map of foliated manifolds, and $g : N \to M'$ a submersion. Then both the projection $\bar{f} : M \times_{(f,g)} N \to M$ and the induced map $g' : N^{g^*F'} \to M'^{F'}$ are submersions, and*

$$(M \times_{(f,g)} N)^{\bar{f}^*F} = M^F \times_{(f',g')} N^{g^*F'} .$$

P r o o f. That $\bar{f}$ and g' are submersions is evident. In order to examine the two manifold structures on $\mathrm{set}\, M \times_{(f,g)} \mathrm{set}\, N$, we recall that the manifold on the left is $\bar{f}^{-1}(M^F)$, and its differentiable structure is characterized by its imbedding in $(M \times_{(f,g)} N) \times M^F$. Since the immersions $M^F \hookrightarrow M$, $N^{g^*F'} \hookrightarrow N$ lead to a smooth map of $M^F \times_{(f',g')} N^{g^*F'}$ to $(M \times_{(f,g)} N) \times M^F$ whose image equals that of $\bar{f}^{-1}(M^F)$, the identity map

$$M^F \times_{(f',g')} N^{g^*F'} \to (M \times_{(f,g)} N)^{\bar{f}^*F}$$

is smooth. To prove that it is actually a diffeomorphism, it remains to compare the respective tangent spaces in $M \times_{(f,g)} N$. One has

$$(v, w) \in T_{(x,y)}(M^F \times_{(f',g')} N^{g^*F'}) = T_x M^F \times_{(f'_*,g'_*)} T_y N^{g^*F'}$$

iff $v \in T_x F$, $w \in T_y(g^*F')$, and $f_*v = g_*w$. Since $f_*(T_xF) \subset T_{f(x)}F'$, an equivalent condition is

$$(v, w) \in T_{(x,y)}(M \times_{(f,g)} N) \quad \text{and} \quad v = \bar{f}_*(v, w) \in T_x F ,$$

i.e. $(v, w) \in T_{(x,y)}(\bar{f}^*F)$. ■

We shall say that an immersed submanifold $N \hookrightarrow M$ is *tangent* to a foliation F of M if

$$T_x F \subset T_x N \tag{1.4}$$

for $x \in N$. If this is the case then there is a unique foliation $F|N$ of N (the *restriction* of F to N) such that

$$T(F|N) = TF|N . \tag{1.5}$$

PROPOSITION 1.3. *Let $f : M' \to M$ be a map transverse to a foliation F of M, and $N \hookrightarrow M$ an immersed submanifold tangent to F. Then*

(i) *the submanifold $N' = f^{-1}N \hookrightarrow M'$ is tangent to f^*F,*
(ii) *the restriction $f|N' : N' \to N$ is transverse to $F|N$, and*
(iii) $(f|N')^*(F|N) = (f^*F)|N'$.

Proof. Apply (1.2)–(1.5). ∎

We are now ready to pass to semi-simplicial manifolds.

Definition 1.4. A sequence of foliations $F = (F_n)_{n\ge 0}$ is a *q-codimensional foliation* of an ss-manifold $X = (X_n)_{n\ge 0}$ if each F_n is a q-codimensional foliation of X_n, for $n \ge 0$, and

$$\varepsilon_i^* F_{n-1} = F_n$$

for $i \le n$, $n \ge 1$.

According to Proposition 1.1(i), the equalities $\varepsilon_i \eta_i = \mathrm{id}$ guarantee that $\eta_i^* F_n = F_{n-1}$, $i < n$, for every foliation F of X. Thus the leaf manifolds $X_n^{F_n}$ form an ss-manifold

$$X^F = (X_n^{F_n})_{n\ge 0}$$

and there is a canonical ss-map $i : X^F \to X$ composed of the appropriate immersions. The connected components of X^F are *leaves* of F.

Any foliation F of X is completely determined by F_0. Conversely, a foliation F_0 of X_0 generates a foliation of X iff $\varepsilon_0^* F_0 = \varepsilon_1^* F_0$ on X_1 and all the maps $\varepsilon_0^n : X_n \to X_0$, $n \ge 1$, are transverse to F_0 (cf. Proposition 1.1(i)).

Definition 1.5. A *foliated ss-manifold* is any pair (X, F) composed of an ss-manifold and a foliation of it. An ss-map $f : Y \to X$ gives rise to an *ss-map of foliated ss-manifolds* $f : (Y, F') \to (X, F)$ if $f_n : (Y_n, F'_n) \to (X_n, F_n)$ for $n \ge 0$.

In view of Proposition 1.1(ii), $f : (Y, F') \to (X, F)$ iff $f_0 : (Y_0, F'_0) \to (X_0, F_0)$; any such f yields a commuting square

$$\begin{array}{ccc} (Y, F') & \xrightarrow{f} & (X, F) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ Y^{F'} & \xrightarrow{f'} & X^F \end{array} \tag{1.6}$$

where f' is induced by f, and the bottom ss-manifolds are considered with the 0-codimensional foliations. This convention will be frequently used in the sequel.

Definition 1.6. An ss-map $f : Y \to X$ is *transverse* to a foliation F of X (notation $f \pitchfork F$) if $f_n : Y_n \to X_n$ is transverse to F_n for every $n \ge 0$. For any $f \pitchfork F$ the sequence

$$f^* F = (f_n^* F_n)_{n\ge 0}$$

is a *pull-back foliation* of F by f.

By Proposition 1.1(i), $f \pitchfork F$ iff $f_0 \pitchfork F_0$ and the pull-back foliation $f_0^* F_0$ of Y_0 generates a foliation of Y.

EXAMPLE 1.7. If all the levels X_n of X are of the same dimension m, and the structure operators are local diffeomorphisms, then X admits a unique *discrete* (m-codimensional) *foliation* given by the discrete topology on every X_n. We shall denote this discrete foliation by F_δ.

EXAMPLE 1.8. Any foliation F of the nerve $\mathcal{N}\Gamma$ of a groupoid Γ comes from a foliation F_0 of the manifold of units $N = \mathcal{N}_0\Gamma$ such that $\alpha^* F_0 = \beta^* F_0$ ($= F_1$) on $\Gamma = \mathcal{N}_1\Gamma$. The manifold Γ^{F_1} inherits from Γ a groupoid structure over N^{F_0}, and one has $(\mathcal{N}\Gamma)^F = \mathcal{N}(\Gamma^{F_1})$ (cf. Proposition 1.2).

EXAMPLE 1.9. For any foliation F of X and every open covering $\mathcal{U}$ of X_0, the gluing projection $\lambda : X_{\mathcal{U}} \to X$ is transverse to F; the pull-back foliation $F_{\mathcal{U}} := \lambda^* F$ is the *localization* of F to $\mathcal{U}$.

Our next goal is to pull foliations back by some ss-morphisms.

LEMMA 1.10. *For any foliation F' of $X_{\mathcal{U}}$, there exists exactly one foliation F of X such that $F' = F_{\mathcal{U}}$.*

Proof. Let $\mathcal{U} = (U_a)_{a\in A}$. For any a the inclusion map $i_a : U_a \hookrightarrow X_{\mathcal{U}}(0)$ pulls back F'_0 to a foliation of U_a; the resulting foliations come from a global foliation F_0 of X_0 if they agree on the overlaps. So let $a, b \in A$, and let $\eta : U_a \cap U_b \to X_{\mathcal{U}}(1)$ be the map $x \to (a, b; \eta_0 x)$. Since $i_a | U_a \cap U_b = \varepsilon_1 \eta$ and $i_b | U_a \cap U_b = \varepsilon_0 \eta$, η is transverse to F'_1, and moreover $i_a^* F'_0 | U_a \cap U_b = \eta^* F_1 = i_b^* F'_0 | U_a \cap U_b$.

It remains to check that F_0 does generate a foliation of X. This is so, for every composition

$$i_{a_h} \circ {\varepsilon_1}^{n-h} {\varepsilon_0}^h | ({\varepsilon_1}^n)^{-1} U_{a_0} \cap \ldots \cap ({\varepsilon_0}^n)^{-1} U_{a_n} = {\varepsilon_1}^{n-h} {\varepsilon_0}^h \circ i_{a_0 \ldots a_n}$$

is transverse to F'_0, $i_{a_0 \ldots a_n}$ being the inclusion in $X_{\mathcal{U}}(n)$, and

$$\varepsilon_1^* F_0 | \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b = i_{ab}^* F'_1 = \varepsilon_0^* F_0 | \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b \,. \quad \blacksquare$$

PROPOSITION 1.11 ([2]). *Let F be a foliation of an ss-manifold X. For every ss-morphism $\mathbf{f} : Y \rightharpoonup X$ the following two conditions are equivalent*:

(i) *there exists an ss-map $f : Y_{\mathcal{U}} \to X$ representing $\mathbf{f}$ and transverse to F*;

(ii) *there is a foliation F' of Y such that every representative $f : Y_{\mathcal{U}} \to X$ of $\mathbf{f}$ is transverse to F and pulls it back to a localization of F'.*

PROPOSITION 1.12. *For any two foliated ss-manifolds (X, F) and (X, F') and every ss-morphism $\mathbf{f} : Y \rightharpoonup X$ the following conditions are equivalent*:

(i) *there is a representative $f : Y_{\mathcal{U}} \to X$ of $\mathbf{f}$ such that $f : (X_{\mathcal{U}}, F'_{\mathcal{U}}) \to (X, F)$*;

(ii) *every representative $f : Y_{\mathcal{U}} \to X$ of $\mathbf{f}$ gives rise to an ss-map of foliated ss-manifolds $f : (Y_{\mathcal{U}}, F'_{\mathcal{U}}) \to (X, F)$.*

P r o o f o f P r o p o s i t i o n 1.11. Evidently, (ii)⇒(i). So, assume that an ss-map $f : Y_{\mathcal{U}} \to X$ is transverse to F. By Lemma 1.10, $f^*F = F'_{\mathcal{U}}$ for a foliation F' of Y. It remains to check that any ss-map $g : Y_{\mathcal{V}} \to X$ elementarily equivalent to f is also transverse to F, and the pull-back foliation is $F'_{\mathcal{V}}$. If $g = f\varrho_\#$, then the transversality of g to F is obvious. Thus we assume $f = g\varrho_\#$, so that $\mathcal{U} = (U_a)_{a\in A}$ is a refinement of $\mathcal{V} = (V_i)_{i\in I}$, and $\varrho : A \to I$ denotes a refinement map. Our claim is that each component

$$g_{n,i_0\ldots i_n} : ({\varepsilon_1}^n)^{-1}V_{i_0} \cap \ldots \cap ({\varepsilon_0}^n)^{-1}V_{i_n} \to X_n$$

of g_n is transverse to F_n, for $n \geq 0$. Since transversality is a local property, it suffices to demonstrate it on each $W = ({\varepsilon_1}^n)^{-1}(V_{i_0} \cap U_{a_0}) \cap \ldots \ldots \cap ({\varepsilon_0}^n)^{-1}(V_{i_n} \cap U_{a_n})$, $a_0, \ldots, a_n \in A$. Now the map

$$W \ni x \xrightarrow{\eta} (i_0, \varrho(a_0), \ldots, i_n, \varrho(a_n); \eta_0\eta_1 \ldots \eta_n x) \in Y_{\mathcal{V}}(2n+1)$$

is clearly well defined and satisfies

$$\varepsilon_{n+1}\varepsilon_n \ldots \varepsilon_1\eta(x) = (i_0, \ldots, i_n; x), \quad \varepsilon_n \ldots \varepsilon_1\varepsilon_0\eta(x) = (\varrho(a_0), \ldots, \varrho(a_n); x)\,,$$

so that

$$(1.7) \quad g_{n,i_0\ldots i_n}|W = \varepsilon_{n+1} \ldots \varepsilon_1 \circ g_{2n+1}\eta, \quad f_{n,a_0\ldots a_n}|W = \varepsilon_n \ldots \varepsilon_0 \circ g_{2n+1}\eta\,.$$

By Proposition 1.1(i), the fact that the last map is transverse to F_n implies the transversality of $g_{2n+1}\eta$ to

$$(\varepsilon_n \ldots \varepsilon_0)^*F_n = F_{2n+1} = (\varepsilon_{n+1} \ldots \varepsilon_1)^*F_n$$

and finally the transversality of $\varepsilon_{n+1} \ldots \varepsilon_1 \circ g_{2n+1}\eta$ to F_n, as was to be shown. Since $\varrho_\#^*(g^*F) = f^*F$, the two pull-back foliations are localizations of a common one. ■

P r o o f o f P r o p o s i t i o n 1.12. As in the above proof, it suffices to consider two representatives $f : Y_{\mathcal{U}} \to X$ and $g : Y_{\mathcal{V}} \to X$ of $\mathbf{f}$ such that $f = g\varrho_\#$ and $f : (Y_{\mathcal{U}}, F'_{\mathcal{U}}) \to (X, F)$. Then (1.7) (for $n = 0$) together with Proposition 1.1(ii) imply first that $g_1\eta : (V_i \cap U_a, F'_0|V_i \cap U_a) \to (X_1, F_1)$ for all i, a, and then $g_{0i} : (V_i, F'_0|V_i) \to (X_0, F_0)$ for $i \in I$. ■

DEFINITION 1.13. Any ss-morphism $\mathbf{f} : Y \rightharpoonup X$ which satisfies the equivalent conditions (i)–(ii) of Proposition 1.11 is *transverse* to F (notation $\mathbf{f} \pitchfork F$); the unique foliation F' of Y characterized by (ii) is the *pull-back* of F by $\mathbf{f}$, to be denoted by $\mathbf{f}^*F$.

An *ss-morphism of foliated ss-manifolds* $\mathbf{f} : (Y, F') \rightharpoonup (X, F)$ is any ss-morphism $\mathbf{f} : Y \rightharpoonup X$ satisfying conditions (i)–(ii) of Proposition 1.12.

Clearly,

$$(1.8) \qquad \mathbf{f} : (Y, f^*F) \rightharpoonup (X, F)$$

if $\mathbf{f} \pitchfork F$. One can easily see that foliated ss-manifolds and their ss-morphisms constitute a category. Foliated ss-manifolds isomorphic in that category will be called *equivalent*.

PROPOSITION 1.14. *Every ss-morphism of foliated ss-manifolds* $\mathbf{f} : (Y, F') \rightharpoonup (X, F)$ *descends functorially to an ss-morphism* $\mathbf{f}' : Y^{F'} \rightharpoonup X^F$ *such that the square*

$$\begin{array}{ccc} (Y,F') & \xrightarrow{\mathbf{f}} & (X,F) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ Y^{F'} & \xrightarrow{\mathbf{f}'} & X^F \end{array}$$

commutes (*after passing to ss-morphisms*).

P r o o f. By (1.6), every representative $f : Y_{\mathcal{U}} \to X$ of $\mathbf{f}$ gives rise to a commuting square

$$\begin{array}{ccc} Y_{\mathcal{U}} & \xrightarrow{f} & X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ (Y^{F'})_{\mathcal{U}} = (Y_{\mathcal{U}})^{F'_{\mathcal{U}}} & \xrightarrow{f'} & X^F \end{array}$$

where the covering $\mathcal{U}$ of Y_0 is also an open covering of $Y_0^{F'_0} = Y^{F'}(0)$. If f ranges over $\mathbf{f}$, the respective ss-maps f' are all equivalent to one another and represent an ss-morphism $\mathbf{f}' : Y^{F'} \rightharpoonup X^F$. The functoriality is evident. ■

COROLLARY 1.15. *The leaves of equivalent foliated ss-manifolds are mutually equivalent.*

P r o o f. Cf. Proposition I.1.19. ■

The next assertion generalizes Proposition 1.1.

PROPOSITION 1.16. *Let* $\mathbf{f} : Y \rightharpoonup X$ *be an ss-morphism transverse to a foliation* F *of* X, *and* $\mathbf{g} : Z \rightharpoonup Y$ *another ss-morphism. Then*

(i) $\mathbf{f} \circ \mathbf{g} \pitchfork F$ *iff* $\mathbf{g} \pitchfork \mathbf{f}^*F$, *and*
(ii) $(\mathbf{f} \circ \mathbf{g})^*F = \mathbf{g}^*\mathbf{f}^*F$.

If, *moreover*, Z *is foliated by a foliation* F', *then*

(iii) $\mathbf{f} \circ \mathbf{g} : (Z, F') \rightharpoonup (X, F)$ *iff* $\mathbf{g} : (Z, F') \rightharpoonup (Y, \mathbf{f}^*F)$.

P r o o f. For any representatives $f : Y_{\mathcal{U}} \to X$ of $\mathbf{f}$ and $g : Z_{\mathcal{V}} \to Y$ of $\mathbf{g}$, the local character of the examined properties ensures that $g \pitchfork \mathbf{f}^*F$ iff $g_{\mathcal{U}} \pitchfork f^*F$ $(= (\mathbf{f}^*F)_{\mathcal{U}})$ and that $g : (Z_{\mathcal{V}}, F'_{\mathcal{V}}) \to (Y, \mathbf{f}^*F)$ iff $g_{\mathcal{U}} : (Z_{g^{-1}\mathcal{U}}, F'_{g^{-1}\mathcal{U}}) \to (Y_{\mathcal{U}}, f^*F)$. This reduces all the assertions to Proposition 1.1. ■

In view of Theorem I.2.12, any ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ of an ss-manifold X to the nerve of a groupoid Γ can be identified with an isomorphy class of principal Γ-bundles over X. The question of whether $\mathbf{f}$ is transverse to a foliation of $\mathcal{N}\Gamma$ or is an ss-morphism of foliated ss-manifolds leads to the following useful characterization.

THEOREM 1.17. *Let Γ be a groupoid (over a manifold N), $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ an ss-morphism to $\mathcal{N}\Gamma$, and $\pi : E \to X$ a principal Γ-bundle classified by $\mathbf{f}$.*

(i) *For any foliation $\widehat{F}$ of $\mathcal{N}\Gamma$, $\mathbf{f}$ is transverse to $\widehat{F}$ iff the ss-map $\alpha : E \to \mathcal{N}N$ composed of the source maps $\alpha_n : E_n \to N$, $n \geq 0$, is transverse to $\widehat{F}_0$ trivially extended to $\mathcal{N}N$. If this is the case then the pull-back foliation $F = \mathbf{f}^*\widehat{F}$ is characterized by the identity $\pi^*F = \alpha^*\widehat{F}_0$ (on E).*

(ii) *For any foliation F of X and $\widehat{F}$ of $\mathcal{N}\Gamma$, $\mathbf{f}$ is an ss-morphism of foliated ss-manifolds iff $\alpha : (E, \pi^*F) \to (\mathcal{N}N, \widehat{F}_0)$. If this is the case then the induced ss-morphism $\mathbf{f}' : X^F \rightharpoonup (\mathcal{N}\Gamma)^{\hat{F}} = \mathcal{N}(\Gamma^{\hat{F}_1})$ is the classifying ss-morphism for the principal $\Gamma^{\hat{F}_1}$-bundle $\pi' : E^{\pi^*F} \to X^F$* (cf. Example 1.8).

P r o o f. Let $s_a : U_a \to E_0$, $a \in A$, be any collection of local sections of E_0 over an open covering $\mathcal{U} = (U_a)_{a\in A}$ of X_0. By the proof of Theorem I.2.12 (see also Example I.1.6), $\mathbf{f}$ is represented by an $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$ such that

$$f_0 = \sum_a \alpha_0 \circ s_a : \coprod_a U_a \to N$$

while the $f_n, n \geq 1$, come from the corresponding Γ-cocycle (cf. I(1.3)).

(i) $\mathbf{f} \pitchfork \widehat{F}$ iff $f \pitchfork \widehat{F}$, i.e. iff $f_n \pitchfork \widehat{F}_n$ for $n \geq 0$. As $\widehat{F}_n = ({\varepsilon_0}^n)^*\widehat{F}_0$, $f_n \pitchfork F_n$, iff $f_n \circ {\varepsilon_0}^n \pitchfork \widehat{F}_0$. Consequently, the transversality of $\mathbf{f}$ is equivalent to the condition

$$\alpha_0 \circ s \circ {\varepsilon_0}^n \pitchfork \widehat{F}_0, \quad n \geq 0\,, \tag{1.8}$$

for every local section $s : X_0 \supset U \to E_0$. Clearly, $\alpha_0 s \pitchfork \widehat{F}_0$ for every s implies $\alpha_0 \pitchfork \widehat{F}_0$; by commutativity of the diagram

$$\begin{array}{ccccc} X_0 \supset U & \xrightarrow{\ s\ } & E_0 & & \\ & & & \searrow^{\alpha_0} & \\ {\scriptstyle {\varepsilon_0}^n}\big\uparrow & & {\scriptstyle {\varepsilon_0}^n}\big\uparrow & & N \\ & & & \nearrow_{\alpha_n} & \\ X_n \supset ({\varepsilon_0}^n)^{-1}U & \xrightarrow{\ s^{(n)}\ } & E_n & & \end{array} \tag{1.9}$$

where $s^{(n)}$ is the induced section of $E_n \cong ({\varepsilon_0}^n)^*E_0$, (1.8) ensures that $\alpha_n \pitchfork \widehat{F}_0$ for every $n \geq 0$. Hence $\mathbf{f} \pitchfork \widehat{F}$ implies $\alpha \pitchfork \widehat{F}_0$.

To prove the converse, consider a principal Γ-bundle E' over a *manifold* M. Any local section $s : M \supset U \to E'$ yields a trivialization $U \times_{(\alpha s, \beta)} \Gamma \xrightarrow{\cong}$

$\pi^{-1}(U) \subset E'$ compatible with both the bundle projection π and the source map α. If $\alpha \pitchfork \widehat{F}_0$, then

$$\alpha_* T_{(x,y)}(U \times_{(\alpha s,\beta)} \Gamma) + T_{\alpha g}\widehat{F}_0 = T_{\alpha g} N$$

for every $(x,g) \in U \times_{(\alpha s,\beta)} \Gamma$. Assuming this, let us show that then $\alpha \circ s \pitchfork \widehat{F}_0$. Indeed, for any $w \in T_{\alpha s(x)} N$, $x \in U$, there is a $w' \in T_{\alpha s(x)}\widehat{F}_0$ and a vector

$$(u,v) \in T_{(x,\alpha s(x))}(U \times_{(\alpha s,\beta)} \Gamma) = T_x U \times_{((\alpha s)_*,\beta_*)} T_{\alpha s(x)}\Gamma \tag{1.10}$$

such that

$$w = \alpha_*(u,v) + w' = \alpha_* v + w' = (\alpha s)_* u + (\alpha_* v - \beta_* v) + w'$$

by (1.10). Since for the inclusion map $i : N \hookrightarrow \Gamma$, $v - i_*\beta_* v \in \ker \beta_*$ is an element of $(\beta_{*\alpha s(x)})^{-1} T_{\alpha s(x)}\widehat{F}_0 = T_{\alpha s(x)}\widehat{F}_1$, one has

$$\alpha_* v - \beta_* v = \alpha_*(v - i_*\beta_* v) \in T_{\alpha s(x)}\widehat{F}_0$$

and the decomposition of w ensures the transversality of $\alpha \circ s$ to $\widehat{F}_0$.

By the above reasoning (for $E' = E_n$) and diagram (1.9), $\alpha_n \pitchfork \widehat{F}_0$ for $n \geq 0$ implies (1.8), i.e. $\mathbf{f} \pitchfork \widehat{F}$.

In order to establish the equality which characterizes the pull-back foliation, we lift $\mathbf{f}$ to an ss-morphism $\overline{\mathbf{f}}$ of Γ-bundles and consider the following diagram which commutes after passing to ss-morphisms:

$$\begin{array}{ccc} & \mathcal{N}N & \\ {}^{\alpha}\nearrow & & \nwarrow^{\alpha} \\ E & \xrightarrow{\bar{\mathbf{f}}} & \overline{\mathcal{N}}\Gamma \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow{\mathbf{f}} & \mathcal{N}\Gamma \end{array}$$

One has

$$\pi^*\mathbf{f}^*\widehat{F} = \overline{\mathbf{f}}^*\pi^*\widehat{F} = \overline{\mathbf{f}}^*\alpha^*\widehat{F}_0 = \alpha^*\widehat{F}_0$$

where $\pi^*\widehat{F} = \alpha^*\widehat{F}_0$ on $\overline{\mathcal{N}}\Gamma$ since both the foliations are equal (to $\widehat{F}_1$) at the 0-level.

(ii) We shall show that $\alpha_0 : (E_0, \pi_0^* F_0) \to (N, \widehat{F}_0)$ iff f_0 is foliation-preserving, i.e. iff $\alpha_0 s : (U, F_0|U) \to (N, \widehat{F}_0)$ for every local section $s : X_0 \supset U \to E_0$. Since $s \pitchfork \pi_0^* F_0$ and $F_0|U = s^*\pi_0^* F_0$, the "only if" part is trivial. To prove the other part assume that $s : U \to E_0$ is any section such that

$$(\alpha_0 s)_*(T_x F_0) \subset T_{\alpha_0 s(x)}\widehat{F}_0 \quad \text{for } x \in U\,; \tag{1.11}$$

we may hope that $\alpha_0|\pi_0^{-1}U$ is then foliation-preserving. This is clearly equivalent to

$$\alpha : (U \times_{(\alpha_0 s,\beta)} \Gamma, \pi^*(F_0|U)) \to (N, \widehat{F}_0)\,. \tag{1.12}$$

By (1.3), a vector $(u,v) \in T_xU \times_{((\alpha_0 s)_*,\beta_*)} T_g\Gamma$ is in $T_{(x,g)}\pi^*(F_0|U)$, with $(x,g) \in U \times_{(\alpha_0 s,\beta)} \Gamma$, iff $u = \pi_*(u,v) \in T_xF_0$. Then, successively, $\beta_* v = (\alpha_0 s)_* u \in T_{\beta g}\widehat{F}_0$, by (1.11), $v \in (\beta_{*g})^{-1} T_{\beta g}\widehat{F}_0 = T_g\widehat{F}_1$, and $\alpha_* v \in T_{\alpha g}\widehat{F}_0$. Since $\alpha_*(u,v) = \alpha_* v$, this yields (1.12) and implies the first assertion of (ii).

Observe now that for any principal Γ-bundle $\pi : E' \to M$ over a manifold M endowed with a foliation F', if $\alpha : (E', \pi^*F') \to (N, \widehat{F}_0)$ then there are well defined induced maps

$$\begin{array}{ccc} E'^{\pi^*F'} & \xrightarrow{\alpha'} & N^{\hat{F}_0} \\ {\scriptstyle \pi'}\big\downarrow & & \\ M^{F'} & & \end{array}$$

Furthermore, every local section $M \supset U \xrightarrow{s} E'$ gives rise to $s' : M^{F'} \supset U^{F'|U} \to E'^{\pi^*F'}$, a section of π', and—in the same vein—the isomorphism of foliated manifolds

$$(U \times_{(\alpha s,\beta)} \Gamma, \pi^*(F'|U)) \xrightarrow{\cong} (\pi^{-1}U, \pi^*F'|\pi^{-1}U)$$

induces a diffeomorphism

$$(U \times_{(\alpha s,\beta)} \Gamma)^{\pi^*(F'|U)} \xrightarrow{\cong} (\pi^{-1}U)^{\pi^*F'|\pi^{-1}U},$$

i.e. (cf. Propositions 1.2 and 1.3(iii))

$$U^{F'|U} \times_{(\alpha' s',\beta')} \Gamma^{\widehat{F}_1} \xrightarrow{\cong} \pi'^{-1}(U^{F'|U}) \subset E'^{\pi^*F'}.$$

This local triviality condition ensures that the Γ-bundle structure on E' descends to a principal $\Gamma^{\widehat{F}_1}$-bundle structure on $E'^{\pi^*F'}$.

Returning to the considered principal Γ-bundle $E \to X$ and applying the above reasoning to every level, we conclude that $E^{\pi^*F} \to X^F$ is a principal $\Gamma^{\widehat{F}_1}$-bundle. Moreover, as every section $s_a : U_a \to E_0$, $a \in A$, induces a section of $E^{\pi^*F}(0)$, there is a representative $\widetilde{f} : (X^F)_{\mathcal{U}'} \to \mathcal{N}\Gamma^{\widehat{F}_1}$, $\mathcal{U}' = (U_a^{F_0|U_a})_{a\in A}$, of the classifying ss-morphism $\widetilde{\mathbf{f}}$ for E^{π^*F} which—on the set level—consists of the same maps as f does. Consequently, f is an ss-map of foliated ss-manifolds, $\widetilde{f} = f'$ is the induced ss-map, and $\widetilde{\mathbf{f}} = \mathbf{f}'$ is the induced ss-morphism (cf. Proposition 1.14). ∎

COROLLARY 1.18. *Let Γ and Γ' be equivalent groupoids, and $\mathbf{f} : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma'$ an equivalence. Then*

(i) $\mathbf{f}$ *is transverse to every foliation F' of $\mathcal{N}\Gamma'$, and the assignment $F' \rightsquigarrow \mathbf{f}^*F'$ is a bijection of the set of foliations of $\mathcal{N}\Gamma'$ onto the set of foliations of $\mathcal{N}\Gamma$;*

(ii) *if F and F' are foliations of $\mathcal{N}\Gamma$ and $\mathcal{N}\Gamma'$, respectively, such that $\mathbf{f} : (\mathcal{N}\Gamma, F) \rightharpoonup (\mathcal{N}\Gamma', F')$ is an equivalence of foliated ss-manifolds, then $F = \mathbf{f}^*F'$.*

P r o o f. Let $\Sigma : \Gamma \rightharpoonup \Gamma'$ be the unique morphism of groupoids such that $\mathbf{f} = \mathcal{N}\Sigma$; it is represented by a principal Γ'-bundle Σ over N, the units of Γ, equipped with a Γ'-equivariant left Γ-action, while the principal Γ'-bundle E classified by $\mathbf{f}$ is an extension of $E_0 = \Sigma$ to the whole of $\mathcal{N}\Gamma$. According to Theorem I.3.2, the invertibility of $\mathbf{f}$ implies that $\alpha : \Sigma \to N'$ (N' the units of Γ') is a submersion, and is therefore transverse to every foliation of N'. The same reasoning applied to $\mathbf{f}^{-1}$ yields (i). Since $\mathbf{f}^{-1} = \mathcal{N}\,{}^t\Sigma$ comes from the transposition of Σ, the assumption of (ii) implies

$$\alpha : (\Sigma, \pi^* F_0) \to (N', F_0'), \qquad \pi : (\Sigma, \alpha^* F_0') \to (N, F_0)\,,$$

so that $T(\pi^* F_0) = T(\alpha^* F_0')$. Equality of the two foliations of $\Sigma = E_0$ ensures that $\pi^* F = \alpha^* F'$ on E, as was to be shown. ∎

II.2. Foliations modelled on a pseudogroup. Let Γ_q denote the groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$, $q \geq 0$. Γ_q carries a canonical structure of a (highly non-Hausdorff) q-dimensional manifold, and so do all the levels of its semi-simplicial nerve. Since the structure operators of $\mathcal{N}\Gamma_q$ are local diffeomorphisms, this ss-manifold admits a discrete foliation F_δ.

Theorem 2.1. *For every ss-manifold X and any q-codimensional foliation F of X there exists a* unique *ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma_q$ transverse to F_δ such that $F = \mathbf{f}^* F_\delta$.*

Definition 2.2. The unique ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma_q$ such that $\mathbf{f}^* F_\delta = F$ is the *classifying ss-morphism* for F.

Before demonstrating the theorem in general consider a q-codimensional foliation F of a manifold M. We shall denote by E_F the manifold of germs

$$E_F = \{[\varphi, x];\ \varphi \in \mathrm{Subm}(F),\ x \in \mathrm{Domain}(\varphi)\}\,,$$

which carries the sheaf manifold structure with respect to the projection $E_F \ni [\varphi, x] \to x \in M$. When endowed with the left Γ_q-action

$$[\gamma, y] \cdot [\varphi, x] = [\gamma\varphi, x] \quad \text{iff} \quad y = \varphi(x)$$

E_F becomes the *canonical* (left) *principal Γ_q-bundle* over M associated with the foliation (cf. [2]). Every transverse map $f : M' \to M$ admits a Γ_q-equivariant lift $\widetilde{f} : E_{f^*F} \to E_F$,

$$[\varphi \circ f, x] \xrightarrow{\tilde{f}} [\varphi, f(x)]\,. \tag{2.1}$$

Consequently, for each q-codimensional foliation $F = (F_n)$ of an ss-manifold X, $E_F := (E_{F_n})$ inherits from X the structure operators and is the *canonical principal Γ_q-bundle* over X *associated with* F.

Proof of Theorem 2.1. Let F be a q-codimensional foliation of X. We claim that the classifying ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma_q$ for E_F is transverse to F_δ and pulls it back to F. Indeed, for any submersion $\varphi : X_0 \supset U \to \mathbb{R}^q$, if $\varphi \in \mathrm{Subm}(F_0)$ then $\varphi \circ {\varepsilon_0}^n \in \mathrm{Subm}(F_n)$, $n \geq 1$. In view of Theorem 1.17(i), this implies $\mathbf{f} \pitchfork F_\delta$, $\mathbf{f}^* F_\delta = F$.

To prove the uniqueness of the ss-morphism, consider any Γ_q-cocycle (γ_{ab}) on X with respect to a covering $\mathcal{U} = (U_a)_{a \in A}$, such that the associated ss-map $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma_q$ is transverse to the discrete foliation and pulls it back to $F_{\mathcal{U}}$. Since $f_0(a, \cdot) = \gamma_{aa}\eta_0$ and $f_1(a, b; \cdot) = \gamma_{ab}$ (cf. I(1.3)), one has $\varphi_a := \gamma_{aa}\eta_0 \in \mathrm{Subm}(F_0)$, while the equalities

$$\alpha \circ \gamma_{ab} = \varepsilon_0 \circ f_1(a, b; \cdot) = \varphi_b \circ \varepsilon_0 ,$$
$$\beta \circ \gamma_{ab} = \varepsilon_1 \circ f_1(a, b; \cdot) = \varphi_a \circ \varepsilon_1$$

imply

$$[\varphi_a \circ \varepsilon_1, \cdot] = \gamma_{ab}[\varphi_b \circ \varepsilon_0, \cdot] .$$

Thus the Γ_q-cocycle is a cocycle description of the canonical Γ_q-bundle E_F (with respect to the sections $s_a = [\varphi_a, \cdot]$, $a \in A$; cf. I(2.5)), and f represents the classifying ss-morphism $\mathbf{f}$. ■

Example 2.3. For any q-dimensional manifold N let Γ_N be the groupoid of germs of local diffeomorphisms of N (equipped with the sheaf topology and the corresponding differentiable structure). The classifying ss-morphism $\mathcal{N}\Gamma_N \rightharpoonup \mathcal{N}\Gamma_q$ for the discrete foliation is an *equivalence*. This is the only equivalence of those ss-manifolds, for every invertible ss-morphism $\mathbf{f} : \mathcal{N}\Gamma_N \rightharpoonup \mathcal{N}\Gamma_q$ must be transverse to F_δ (cf. Corollary 1.18), and $\mathbf{f}^* F_\delta$ is the only q-codimensional foliation of $\mathcal{N}\Gamma_N$—the discrete foliation.

We already know that every q-codimensional foliation F of an ss-manifold X is a pull-back of a universal (discrete) foliation by a transverse ss-morphism $X \rightharpoonup \mathcal{N}\Gamma_q$ (equivalently, $X \rightharpoonup \mathcal{N}\Gamma_N$ for any manifold N of dimension q). The transverse ss-morphism can be regarded as a submersion shrinking the leaves; its "range" should characterize the complexity of the foliation. Since submersions are open mappings, it is reasonable to look for the range among open subgroupoids of some Γ_N.

Any open subgroupoid $\Gamma \subset \Gamma_N$, N a manifold, consists of the germs of elements of a uniquely defined pseudogroup of diffeomorphisms of N. More generally, by a *groupoid of germs* we shall mean any groupoid Γ whose source and target maps $\alpha, \beta : \Gamma \to N$ are local diffeomorphisms, and which can be identified with an open subgroupoid of Γ_N via a map

$$\Gamma \ni g \xrightarrow{i} [\beta \circ (\alpha|U^g)^{-1}, \alpha g] \in \Gamma_N \tag{2.2}$$

where $U^g \subset \Gamma$ denotes any neighbourhood of g such that $\alpha|U^g$ is a diffeomorphism. The pseudogroup generated by the compositions $\beta \circ (\alpha|U)^{-1}$ will be referred to as the *underlying pseudogroup* for Γ.

DEFINITION 2.4. An ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ is a *transverse projection* for a foliation F of X if Γ is a groupoid of germs, $\mathbf{f}$ is transverse to the discrete foliation F_δ of $\mathcal{N}\Gamma$, and $F = \mathbf{f}^* F_\delta$.

A *Γ-foliation* on X is any pair $(F, \mathbf{f})$ such that F is a foliation of X and $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ a transverse projection for F.

By abuse of language, a foliation F of X is called a Γ-foliation if it admits a transverse projection to $\mathcal{N}\Gamma$, i.e. if its classifying ss-morphism admits a factorization $X \rightharpoonup \mathcal{N}\Gamma \overset{[\mathcal{N}i]}{\rightharpoonup} \mathcal{N}\Gamma_N$.

For any principal Γ-bundle $\pi : E \to X$, Γ a groupoid of germs, the bundle projection consists of local diffeomorphisms, for so are the source and target maps of Γ. This motivates

DEFINITION 2.5. Given a groupoid of germs Γ, a principal Γ-bundle $\pi : E \to X$ is an *unrolling Γ-bundle* for a foliation F of X if the pull-back foliation $\pi^* F$ of E comes—via the source map α—from the discrete foliation $F_\delta^{(N)}$ of the units N of Γ, i.e. if $\alpha \pitchfork F_\delta^{(N)}$ and $\pi^* F = \alpha^* F_\delta^{(N)}$.

In view of Theorem 1.17(i), the notions of transverse projection and unrolling Γ-bundle are dual to each other:

PROPOSITION 2.6. *For any groupoid of germs Γ, a principal Γ-bundle E over X is unrolling for a foliation F of X iff the classifying ss-morphism $\mathbf{f}_E : X \rightharpoonup \mathcal{N}\Gamma$ is a transverse projection for Γ.* ■

A Γ-bundle E over a manifold M is unrolling for a foliation of M if so is the trivial extension $\mathcal{N}E \to \mathcal{N}M$. Clearly, the structure of E is then completely determined by the so-called *distinguished submersions*,

$$\varphi : M \supset U \xrightarrow{s} E \xrightarrow{\alpha} N$$

where s ranges over local sections of E. More precisely, the sheaf $\widetilde{E}$ of germs of all distinguished submersions for E is a left principal Γ-bundle (an $i(\Gamma)$-bundle; cf. (2.2)) over M, and the map

$$E \ni s(x)g \to ig^{-1}[\alpha s, x] \in \widetilde{E} \tag{2.3}$$

is a well defined canonical isomorphism. $\widetilde{E}$ is thus a *canonical form* of E. The next lemma is straightforward.

LEMMA 2.7. *If $f : M' \to M$ is transverse to F, and $E \to M$ an unrolling Γ-bundle for F, then*

(i) *the pull-back $f^* E$ is unrolling for the pull-back foliation $f^* F$ of M';*

(ii) *the distinguished submersions for $f^* E$ are all maps locally of the form $\varphi \circ f$, where φ ranges over the distinguished submersions for E;*

(iii) *in terms of the canonical forms of the* Γ*-bundles*, *the lift* $f^*E \to E$ *of* f *reads* $[\varphi \circ f, x] \to [\varphi, f(x)]$ (cf. (2.1)). ∎

COROLLARY 2.8. *If a principal* Γ*-bundle* E *over an ss-manifold* X *is unrolling for a foliation* $F = (F_n)$, *then each* E_n *is an unrolling* Γ*-bundle for* F_n, $n \geq 0$. *Furthermore*, *the sequence* $\widetilde{E} = (\widetilde{E}_n)$ *of the respective canonical* Γ*-bundles equipped with the structure operators*

$$[\varphi \circ \varepsilon_i, x] \xrightarrow{\varepsilon_i} [\varphi, \varepsilon_i x], \quad [\psi \circ \eta_j, x] \xrightarrow{\eta_j} [\psi, \eta_j x]$$

is a principal Γ*-bundle isomorphic to* E. ∎

Remark 2.9. When dealing with a groupoid of germs Γ it is convenient—for notational reasons—to consider *left* rather than *right* principal Γ-bundles. We recall that a left bundle is endowed with a *target map* $\beta : E \to N$ (N the units of Γ), and the action $(g, z) \to gz$ is defined on the fibre product $\Gamma \times_{(\alpha,\beta)} E$.

Let $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ be a transverse projection for a foliation F of an ss-manifold X. Among all isomorphic pull-backs of $\overline{\mathcal{N}\Gamma}$ by $\mathbf{f}$ there is a canonical Γ-bundle $\widetilde{E}$ given by Corollary 2.8. The intrinsic character of the structure operators of $\widetilde{E}$ ensures that the whole Γ-bundle is completely determined by $\widetilde{E}_0$.

DEFINITION 2.10. Without referring directly to the classified Γ-bundle, the distinguished submersions for $\widetilde{E}_0$ (i.e. local sections followed by the target map) will be called *distinguished submersions for* $\mathbf{f}$.

The notion of distinguished submersion is useful in constructing transverse projections.

THEOREM 2.11. *Let* $\mathcal{G}$ *be a pseudogroup of diffeomorphisms of a manifold* N, Γ *the associated groupoid of germs*, *and* (X, F) *a foliated ss-manifold of codimension* $q = \dim N$. *If* $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ *is a transverse projection for* F (*if there exists any*), *then the collection* $\Phi = \Phi_{\mathbf{f}}$ *of all the distinguished submersions* $\varphi : X_0 \supset U_\varphi \to N$ *for* $\mathbf{f}$ *satisfies the following properties*:

(i) $\bigcup_{\varphi \in \Phi} U_\varphi = X_0$;

(ii) *if* $\varphi, \psi \in \Phi$ *then for every* $x \in \varepsilon_1^{-1} U_\varphi \cap \varepsilon_0^{-1} U_\psi \subset X_1$ *there is a* $\gamma \in \mathcal{G}$ *such that* $\varphi \circ \varepsilon_1 = \gamma \circ \psi \circ \varepsilon_0$ *on a neighbourhood of* x;

(iii) $\gamma \circ \varphi \in \Phi$ *if* $\gamma \in \mathcal{G}$, $\varphi \in \Phi$;

(iv) Φ *is a maximal set of submersions* (*of subsets of* X_0 *to* N) *which satisfies* (i)–(iii);

(v) $\varphi^* F_\delta^{(N)} = F_0 | U_\varphi$ *for* $\varphi \in \Phi$.

If, *conversely*, Φ_0 *is any set of submersions satisfying* (i), (ii), (v), *then there exists exactly one transverse projection* $X \rightharpoonup \mathcal{N}\Gamma$ *for* F *whose collection* Φ *of distinguished submersions contains* Φ_0; *the completion* Φ *of*

Φ_0 consists of all maps which are locally compositions of some $\gamma \in \mathcal{G}$ and $\varphi \in \Phi_0$.

P r o o f. For any transverse projection $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ and the corresponding canonical pull-back Γ-bundle $\widetilde{E}$ one has: (i) is evident, (ii) follows from a comparison between two sections $[\varphi \circ \varepsilon_1, \cdot]$ and $[\psi \circ \varepsilon_0, \cdot]$ of $\widetilde{E}_1$ (cf. Lemma 2.7(ii)), (iii) reflects the fact that Γ acts on $\widetilde{E}_0$ by the composition of germs, (iv) will follow from the second part of the theorem, and (v) is the pull-back of the identity $\pi^* F_0 = \alpha^* F_\delta^{(N)}$ (on $\widetilde{E}_0$) by the section $[\varphi, \cdot]$, $\varphi \in \Phi$.

In order to demonstrate the second, converse assertion of the theorem observe first that for any Φ_0 there is exactly one Γ-cocycle $(\gamma_{\varphi\psi})$ on X with respect to the covering $\mathcal{U} := (U_\varphi)_{\varphi \in \Phi_0}$ such that $\gamma_{\varphi\varphi} = \varphi : U_\varphi \to N$ for every $\varphi \in \Phi_0$. Indeed, the cocycle condition requires that $\alpha \circ \gamma_{\varphi\psi} = \psi \circ \varepsilon_0$, $\beta \circ \gamma_{\varphi\psi} = \varphi \circ \varepsilon_1$ (suitably restricted), for $\varphi, \psi \in \Phi_0$. Since Γ is a groupoid of germs, this implies

$$\gamma_{\varphi\psi}(x) = [\gamma^{(x)}, \psi(\varepsilon_0 x)] \quad \text{for} \quad x \in \varepsilon_1^{-1} U_\varphi \cap \varepsilon_0^{-1} U_\psi \tag{2.4}$$

where $\gamma^{(x)}$ is any element of $\mathcal{G}$ such that $\varphi \circ \varepsilon_1 = \gamma^{(x)} \circ \psi \circ \varepsilon_0$ on a neighbourhood of x. Actually, formula (2.4) does define a collection of smooth maps $(\gamma_{\varphi\psi})_{\varphi,\psi \in \Phi_0}$, and the cocycle condition I(1.2) for that collection is a straightforward consequence of the commutation relations $\varepsilon_0\varepsilon_0 = \varepsilon_0\varepsilon_1$, $\varepsilon_0\varepsilon_2 = \varepsilon_1\varepsilon_0$, and $\varepsilon_1\varepsilon_1 = \varepsilon_1\varepsilon_2$.

Let now $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma$ denote the extension of the Γ-cocycle $(\gamma_{\varphi\psi})$ to an ss-map (cf. I(1.3)), and let $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ be the ss-morphism represented by f. As f pulls F_δ back to $F_{\mathcal{U}}$, $\mathbf{f}$ is a transverse projection for F. To conclude the proof it suffices to observe that the set Φ of all maps which are locally of the form $\gamma \circ \varphi$ is the largest set satisfying (i)–(iii) and containing Φ_0, and that $E = (E_n)_{n \geq 0}$ with

$$E_0 := \{[\varphi, x]; \varphi \in \Phi, x \in U_\varphi\},$$
$$E_1 := \{[\varphi \circ \varepsilon_1, x]; \varphi \in \Phi, x \in \varepsilon_1^{-1} U_\varphi\} = \{[\psi \circ \varepsilon_0, x]; \psi \in \Phi, x \in \varepsilon_0^{-1} U_\psi\} \text{ etc.}$$

is the canonical unrolling Γ-bundle classified by $\mathbf{f}$. ∎

Example 2.12. *Morphisms of pseudogroups* (cf. [15]). Let $\mathcal{G}$ and $\mathcal{G}'$ be pseudogroups of diffeomorphisms of manifolds N and N', respectively, such that $\dim N = \dim N'$. A *morphism* Φ of $\mathcal{G}$ to $\mathcal{G}'$ (to be denoted $\Phi : \mathcal{G} \rightharpoonup \mathcal{G}'$) is any maximal collection of diffeomorphisms $\varphi : N \supset U_\varphi \to N'$ of open subsets of N onto open subsets of N', such that

(i) $N = \bigcup U_\varphi$,
(ii) $\varphi\gamma\psi^{-1} \in \mathcal{G}'$ if $\varphi, \psi \in \Phi$ and $\gamma \in \mathcal{G}$, and
(iii) $\gamma'\varphi\gamma \in \Phi$ if $\varphi \in \Phi$, $\gamma \in \mathcal{G}$, and $\gamma' \in \mathcal{G}'$.

The *composition* of $\Phi : \mathcal{G} \rightharpoonup \mathcal{G}'$ and $\Psi : \mathcal{G}' \rightharpoonup \mathcal{G}''$ is the unique morphism of pseudogroups $\Psi \circ \Phi : \mathcal{G} \rightharpoonup \mathcal{G}''$ generated by the compositions $\psi \circ \varphi$ as φ and ψ range over Φ and Ψ, respectively.

Every morphism of pseudogroups $\Phi : \mathcal{G} \rightharpoonup \mathcal{G}'$ yields functorially a morphism of the associated groupoids of germs, $|\Phi| : \Gamma \rightharpoonup \Gamma'$, where

$$|\Phi| = \{[\varphi, x]; \varphi \in \Phi, x \in U_\varphi\}$$

and the grupoid actions come from the composition of germs. Furthermore, the associated ss-morphism $\mathcal{N}|\Phi| : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma'$ is transverse to the discrete foliation of $\mathcal{N}\Gamma'$ and pulls it back to the discrete foliation of $\mathcal{N}\Gamma$.

Conversely, every ss-morphism $\mathbf{f} : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma'$ such that $\mathbf{f}^* F_\delta = F_\delta$ is of the form $\mathbf{f} = \mathcal{N}|\Phi|$ for a uniquely determined morphism $\Phi : \mathcal{G} \rightharpoonup \mathcal{G}'$. The set Φ consists of all the single-valued distinguished submersions for $\mathbf{f}$.

PROPOSITION 2.13. *Let Γ and Γ' be groupoids of germs, of pseudogroups $\mathcal{G}$ and $\mathcal{G}'$, respectively. Any morphism $\Phi : \mathcal{G} \rightharpoonup \mathcal{G}'$ transfers every transverse projection $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ to a transverse projection $\mathbf{f}' = \mathcal{N}|\Phi| \circ \mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma'$ for the same foliation of X. If Ψ is the set of distinguished submersions for $\mathbf{f}$ then the set $\Psi_0' = \{\varphi \circ \psi; \varphi \in \Phi, \psi \in \Psi\}$ generates the distinguished submersions for $\mathbf{f}'$.*

Proof. By Corollary I.3.7, if E is a Γ-bundle over X classified by $\mathbf{f}$, then $E' = |\Phi|_* E$ is classified by $\mathbf{f}'$. Consequently, for any $\varphi \in \Phi$ and $\psi \in \Psi$, and a section s of E_0 such that $\psi = \alpha s$, the map $x \to s(x) \cdot [\varphi, \alpha s(x)] \in E_0 \times_\Gamma |\Phi|$ is a local section s' of E_0' such that $\alpha s' = \varphi \circ \psi$. ∎

II.3. Holonomy and the transverse structure. In connection with the previous section it is important to know the real range of any transverse projection $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$, i.e. the smallest (in appriopriate sense) groupoid of germs Γ_0 such that $\mathbf{f}$ can be reduced to a projection $X \rightharpoonup \mathcal{N}\Gamma_0$. A solution to this problem requires good understanding of the transverse geometry of foliations. As standard courses assume the paracompactness of the foliated manifolds, we recall some classical notions in their generality (cf. [11], [25]).

Let F be any q-codimensional foliation of a manifold M. A *local transversal* for F at $x \in M$ is any q-dimensional imbedded submanifold $T \hookrightarrow M$ passing through x and transverse to M^F. A *transversal* for F is any immersion $i : T \to M$ transverse to F such that T and F are of complementary dimensions. Whenever this causes no confusion, we shall call T itself a transversal for the foliation.

An open submanifold $U \subset M$ is *simply foliated* by F if there is a manifold $\widehat{U}$ and a submersion $p : U \to \widehat{U}$ such that the leaves of $F|U$ are exactly the fibres of p. If this is the case then for any two local transversals $T, T' \hookrightarrow U$

at points x and x' of the same leaf $p^{-1}(\widehat{x})$ the foliation induces a local diffeomorphism h of a neighbourhood $V' \subset T'$ of x' onto a neighbourhood $V \subset T$ of x such that $p \circ h = p|V'$; h is a *holonomy map* from T' into T with respect to the simply foliated submanifold. Clearly, the germ of h at x' depends on $F|U$, the local transversals, and the source and target points x' and x only.

Given any (continuous) path $c : [0,1] \to M$ in a leaf of F, a *holonomy chain* along c is any sequence $(U_0, T_1, U_1, \ldots, T_n, U_n)$, $n \geq 0$, such that

(i) the U_i's are simply foliated open subsets of M,

(ii) there is a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_{n+1} = 1$ of the unit interval such that $c[\tau_i, \tau_{i+1}] \subset U_i$ and $T_i \hookrightarrow U_{i-1} \cap U_i$ is a local transversal at $c(\tau_i)$, for $i \leq n$.

For arbitrary local transversals $T_0 = T$ at $c(0)$ and $T_{n+1} = T'$ at $c(1)$, the holonomy chain gives rise to a *holonomy map* along c (from T' into T) which is obtained by successive composition of the holonomy maps from T_{i+1} into T_i with respect to U_i, $i = 0, 1, \ldots, n$. It can be shown that the germ $h_{T,c,T'}$ of the holonomy map is independent of the defining holonomy chain. Furthermore, when regarded as a function of the path c, $h_{T,c,T'}$ is *continuous* (in, respectively, the compact-open and the sheaf topology), and thus locally constant if c remains in the same leaf. The last property means that $h_{T,c,T'}$ depends in fact on the *homotopy class* of c in the leaf. The notation is chosen so as to have

$$h_{T,c,T'} h_{T',c',T''} = h_{T,c\cdot c',T''} \tag{3.1}$$

whenever $c(1) = c'(0)$.

More generally, for any transversal $i : T \to M$ and every path c in a leaf of F such that $c(a) = i(t_a)$ for some $t_a \in T$, $a = 0, 1$, there are neighbourhoods $V_a \subset T$ of the t_a's which are local transversals at the ends of c. In this sense

$$h^i_{t_0,c,t_1} := h_{V_0,c,V_1}$$

is a well defined *holonomy germ* along c, with source t_1 and target t_0.

Any map $f : M' \to M$ transverse to F carries transversals for the pull-back foliation f^*F to transversals for F. If $c : [0,1] \to M'$ is any path in a leaf of f^*F then

$$h^i_{t_0,c,t_1} = h^{f\circ i}_{t_0,f\circ c,t_1} \tag{3.2}$$

for every transversal $i : T \to M'$ which passes through the ends of c. The equality of the respective holonomy germs follows easily from the fact that f carries leaves to leaves and that every foliation pulled-back from a simply foliated manifold inherits the simplicity (cf. [25]).

In order to grasp the transverse geometry of a fixed foliated manifold (M,F) one could consider the totality of local transversals for F and the whole pseudogroup of diffeomorphisms generated on the transversals by the holonomy maps. Fortunately, this is not necessary, for an equivalent pseudogroup is generated on every *complete transversal*, i.e. a transversal $i : T \to M$ such that $i(T)$ meets every leaf. For further reference, we shall denote that pseudogroup by $\mathcal{G}_{F,i}$, or simply $\mathcal{G}_{F,T}$ if i is fixed; this is the *holonomy pseudogroup* of F with respect to the transversal T. The associated open subgroupoid of germs $\Gamma_{F,i} \subset \Gamma_T$ (denoted also by $\Gamma_{F,T}$) consists of all the holonomy germs $h^i_{t,c,t'}$, $t,t' \in T$, where c ranges over paths in M^F connecting points of the complete transversal; $\Gamma_{F,i}$ is the *holonomy groupoid* of F with respect to i.

Given a complete transversal $i : T \to M$ for F, one can project a neighbourhood of every $x \in M$ along any path c in M^F which joins $x = c(1)$ to a point of T (i.e. $c(0) = i(t)$, $t \in T$). More precisely, if $\varphi \in \operatorname{Subm}(F)$ is any submersion of a neighbourhood of x to $\mathbb{R}^q$ locally constant on the leaves of F, then every section j of φ through x is a local transversal at x. When followed by a holonomy map along c (from j to i) φ yields a submersion of a neighbourhood of x to T, a *holonomy projection* along c, and its germ

$$H^i_{t,c} := h^{i\amalg j}_{t,c,\varphi(x)}[\varphi, x] \tag{3.3}$$

is well defined. It is readily seen that the sheaf $E_{F,T}$ of all the germs of the above form is a principal $\Gamma_{F,T}$-bundle over M unrolling the foliation; the distinguished submersions for $E_{F,T}$ will be called *holonomy projections* along F (on the transversal). According to Proposition 2.6, the classifying ss-morphism $\mathcal{N}M \rightharpoonup \mathcal{N}\Gamma_{F,T}$ is a transverse projection for F.

The significance of the notion of holonomy is reflected in the universality of its connection with arbitrary transverse projections. Namely, for any groupoid of germs Γ let $\pi : E \to M$ be a principal Γ-bundle unrolling a foliation F of M. From the local triviality of E, it follows that $\pi' : E^{\pi^*F} \to M^F$ is a covering map, so that leaves of the pull-back foliation π^*F of E are covering spaces of the leaves of F. Consequently, every path c in a leaf of F can be uniquely lifted to a path in E which starts from an arbitrary but fixed point above $c(0)$. This yields a Γ-equivariant bijective map (the *holonomy translation* in E),

$$h^*_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(1))\,,$$

which is an invariant description of the holonomy for foliated manifolds (cf. [13]).

Lemma 3.1. *Let $\widetilde{E}$ be the canonical form of a principal Γ-bundle E unrolling F* (cf. (2.3)). *For any path c in a leaf of F and every germ*

$[\varphi, c(0)] \in \pi^{-1}(c(0)) \subset \widetilde{E}$, *one has*

$$h_c^*[\varphi, c(0)] = [\psi, c(1)]$$

where

$$[\psi|T', c(1)] = [\varphi|T, c(0)]h_{T,c,T'}\,,$$

T *and* T' *being any local transversals at* $c(0)$ *and* $c(1)$, *respectively.*

Proof. Let $c : [0,1] \to M$ be a path in a leaf of F. There exists a sequence $\varphi_i : U_i \to N$, $i = 0, 1, \ldots, r$, of distinguished submersions for E (N the units of Γ) defined on simply foliated domains, and a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_{r+1} = 1$ of the unit segment, such that $c[\tau_i, \tau_{i+1}] \subset U_i$ for $i \le r$. In order to get a holonomy chain along c we choose a local transversal T_i ($T_0 = T$ and $T_{r+1} = T'$) at every point $c(\tau_i)$. Then, on the germ level,

$$[\varphi_i|T_i, c(\tau_i)]^{-1}[\varphi_i|T_{i+1}, c(\tau_{i+1})] = h_{T_i, c|[\tau_i, \tau_{i+1}], T_{i+1}}$$

for $i \le r$, and therefore

$$\begin{aligned} h_{T,c,T'} &= h_{T,c|[0,\tau_1],T_1} h_{T_1,c|[\tau_1,\tau_2],T_2} \ldots h_{T_r,c|[\tau_1,1],T'} \\ &= [\varphi_0|T, c(0)]^{-1}[\varphi_0|T_1, c(\tau_1)][\varphi_1|T_1, c(\tau_1)]^{-1} \ldots [\varphi_r|T', c(1)] \\ &= [\varphi_0|T, c(0)]^{-1}\gamma_{01}(c(\tau_1)) \ldots \gamma_{r-1,r}(c(\tau_r))[\varphi_r|T', c(1)] \end{aligned}$$

where $\gamma_{ij} : U_i \cap U_j \to \Gamma$ are given by

$$[\varphi_i, x] = \gamma_{ij}(x)[\varphi_j, x] \quad \text{in } \widetilde{E}\,.$$

On the other hand, we are able to write down explicitly the lift $\widetilde{c}$ of c to $\widetilde{E}$ starting from $\widetilde{c}(0) = [\varphi_0, c(0)]$. Namely,

$$\widetilde{c}(\tau) = \gamma_{01}(c(\tau_1)) \ldots \gamma_{i-1,i}(c(\tau_i))[\varphi_i, c(\tau)]$$

for $\tau \in [\tau_i, \tau_{i+1}]$, $i \le r$; hence

$$h_c^*[\varphi_0, c(0)] = \widetilde{c}(1) = \gamma_{01}(c(\tau_1)) \ldots \gamma_{r-1,r}(c(\tau_r))[\varphi_r, c(1)]\,.$$

Since the germ $[\varphi_0, c(0)]$ is in fact an arbitrary element of the fibre $\pi^{-1}(c(0))$, the resulting two formulas conclude the proof. ∎

After the above introduction to holonomy we pass to foliated ss-manifolds. The following theorem has already been announced at the beginning of the present section.

THEOREM 3.2 ([2]–[3]). *Let* (X, F) *be a foliated ss-manifold.*

(i) *There is a groupoid of germs* Γ_F *and a transverse projection* $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ (*for* F) *such that every transverse projection* $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ *for* F *descends to a* unique *ss-morphism* $\mathcal{N}\Gamma_F \rightharpoonup \mathcal{N}\Gamma$ *which makes the following* *triangle commutative*:

(3.4)
$$\begin{array}{ccc} & X & \\ \Pi_F \swarrow & & \searrow \mathbf{f} \\ \mathcal{N}\Gamma_F & \dashrightarrow & \mathcal{N}\Gamma \end{array}$$

(ii) *If* $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ *and* $\widetilde{\Pi}_F : X \rightharpoonup \mathcal{N}\widetilde{\Gamma}_F$ *are two transverse projections satisfying* (i)*, then there exists exactly one* equivalence *of the groupoids which transfers* Π_F *to* $\widetilde{\Pi}_F$.

Furthermore, the ss-morphisms which exist according to (i)–(ii) *are transverse to the discrete foliations, and come from the unique morphisms of the underlying pseudogroups of germs* (cf. Example 2.12).

Proof. *Construction.* Fix a complete transversal $i : T \to X_1$ for F_1. Since there are two structure maps from X_1 to X_0 transverse to F_0, we take any two disjoint copies T^0, T^1 of T (e.g. $T^a = \{a\} \times T$; elements of T^a will be denoted by t^a where $t \in T$, $a = 0, 1$) and consider the immersion

$$i' := \varepsilon_0 i \cup \varepsilon_1 i : T^0 \cup T^1 \to X_0 ,$$

which is a complete transversal for F_0. Indeed, for any leaf L of F_0 there is a leaf $\overline{L}$ of F_1 containing $\eta_0 L$; since both $\varepsilon_0\overline{L}$ and $\varepsilon_1\overline{L}$ contain L and are contained in a leaf of $F_0, \varepsilon_0\overline{L} = L = \varepsilon_1\overline{L}$. Thus $\overline{L} \cap i(T) \neq \emptyset$ implies $L \cap \varepsilon_a i(T) \neq \emptyset$ for $a = 0, 1$.

In order to combine the transverse structure of F_0 and the combinatorial structure of the foliated ss-manifold we enlarge the holonomy pseudogroup $\mathcal{G}_{F_0, T^0 \cup T^1}$ (for F_0) by adjoining the *identification maps*

$$T^0 \ni t^0 \underset{\mathrm{id}_1^0}{\overset{\mathrm{id}_0^1}{\rightleftarrows}} t^1 \in T^1 .$$

The smallest pseudogroup of local diffeomorphisms of $T^0 \cup T^1$ generated by $\mathcal{G}_{F_0, T^0 \cup T^1}$ and the maps id_0^1, id_1^0 will be denoted by $\mathcal{G}'_{F,T}$. The associated groupoid of germs, $\Gamma'_{F,T}$, is our candidate for Γ_F.

Searching for a transverse projection $\Pi' : X \rightharpoonup \mathcal{N}\Gamma'_{F,T}$, we recall Theorem 2.11, and consider the set of all the holonomy projections of X_0 on $T^0 \cup T^1$ along F_0. This set generates a transverse projection Π' for F if it satisfies condition (ii) of the theorem. On the germ level, (ii) is equivalent to the existence of a $g \in \Gamma'_{F,T}$ such that

(3.5)
$$H^{i'}_{t_1, c_1}[\varepsilon_1, x] = g H^{i'}_{t_0, c_0}[\varepsilon_0, x]$$

for an arbitrary but fixed $x \in X_1$ and paths c_a, $a = 0, 1$, joining $\varepsilon_a x$ to some $i'(t_a) \in i'(T^0 \cup T^1)$.

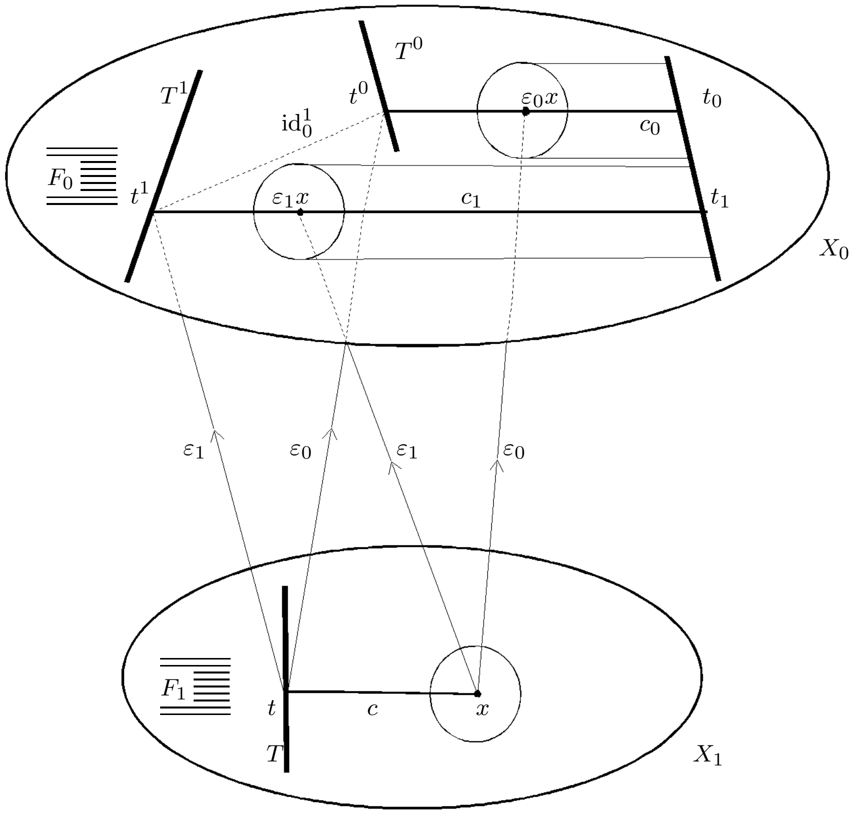


Fig. 4

To find the g we fix a path c in a leaf of F_1 such that $c(1) = x$ and $c(0) = i(t)$ for a $t \in T$ (cf. Fig. 4). Then (3.2)–(3.3) imply

$$H^{i'}_{t^1,\varepsilon_1 c}[\varepsilon_1, x] = [\mathrm{id}^1, t] H^{i}_{t,c} = [\mathrm{id}^1_0, t^0] H^{i'}_{t^0,\varepsilon_0 c}[\varepsilon_0, x]$$

where $\mathrm{id}^1 : T \to T^1$ is the identification map. Consequently, properties (3.1)–(3.3) ensure that (3.5) holds true iff

$$g = h^{i'}_{t_1, c_1\cdot(\varepsilon_1 c)^{-1}, t^1} [\mathrm{id}^1_0, t^0] h^{i'}_{t^0, (\varepsilon_0 c)\cdot c_0^{-1}, t_0} ,$$

which is clearly an element of $\Gamma'_{F,T}$.

Factorization. In order to factorize an arbitrary transverse projection $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ through Π' we consider the distinguished submersions for $\mathbf{f}$ and restrict them to the transversal $i' : T^0 \cup T^1 \to X_0$. According to Example 2.12, the resulting set Φ_0 of diffeomorphisms of open subsets of $T^0 \cup T^1$ into the units of Γ generates a morphism of the underlying pseudogroups,

$\Phi : \mathcal{G}'_{F,T} \rightharpoonup \mathcal{G}$, iff $\varphi\gamma\psi^{-1} \in \mathcal{G}$ for any $\gamma \in \mathcal{G}'_{F,T}$ and $\varphi, \psi \in \Phi_0$. By definition of $\mathcal{G}'_{F,T}$, there are essentially two cases to consider:

1° $\gamma \in \mathcal{G}_{F_0,T^0\cup T^1}$, so that γ is locally a holonomy map along a path c in a leaf of F_0, such that $c(a) = i'(t_a) \in i'(T^0 \cup T^1)$ for $a = 0, 1$. We assume that $\varphi = \overline{\varphi} \circ i'$ and $\psi = \overline{\psi} \circ i'$ are any elements of Φ_0 defined on neighbourhoods of t_0 and t_1, respectively, $\overline{\varphi}$ and $\overline{\psi}$ being distinguished submersions for $\mathbf{f}$. Passing to the canonical form $E = (E_n)$ of the Γ-bundle classified by $\mathbf{f}$, we get $[\overline{\varphi}, c(0)], [\overline{\psi}, c(1)] \in E_0$. By Lemma 3.1, the germ $[\omega, c(1)] := h_c^*[\overline{\varphi}, c(0)] \in E_0$ is characterized by

$$[\omega \circ i', t_1] = [\overline{\varphi} \circ i', t_0] h^{i'}_{t_0,c,t_1} = [\varphi \circ \gamma, t_1]\,.$$

On the other hand, being in the fibre of E_0 over $c(1)$, this germ is of the form $g[\overline{\psi}, c(1)]$ for a $g \in \Gamma$ so that

$$[\varphi \circ \gamma, t_1] = [\omega, c(1)][i', t_1] = g[\overline{\psi}, c(1)][i', t_1] = g[\psi, t_1]\,.$$

This implies $[\varphi \circ \gamma \circ \psi^{-1}, \psi(t_1)] = g \in \Gamma$.

2° $\gamma = \mathrm{id}_0^1$, and $\varphi = \overline{\varphi} \circ i'$, $\psi = \overline{\psi} \circ i'$, where $\overline{\varphi}$ and $\overline{\psi}$ are distinguished submersions defined on neighbourhoods of $i'(t^1)$ and $i'(t^0)$, respectively, for a $t \in T$. By Lemma 2.7(ii), $\overline{\varphi} \circ \varepsilon_1$ and $\overline{\psi} \circ \varepsilon_0$ are distinguished submersions for E_1, and $[\overline{\varphi} \circ \varepsilon_1, i(t)] = g[\overline{\psi} \circ \varepsilon_0, i(t)]$ for a $g \in \Gamma$. Since g is a germ of $\varphi \circ \mathrm{id}_0^1 \circ \psi^{-1}$, we are done.

We have shown that any transverse projection $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ for F gives rise to a morphism of pseudogroups $\Phi : \mathcal{G}'_{F,T} \rightharpoonup \mathcal{G}$ and so to an ss-morphism $\mathcal{N}|\Phi| : \mathcal{N}\Gamma'_{F,T} \rightharpoonup \mathcal{N}\Gamma$. By Proposition 2.13, any distinguished submersion for the composition $\mathcal{N}|\Phi| \circ \Pi' : X \rightharpoonup \mathcal{N}\Gamma$ is locally of the form $(\overline{\varphi} \circ i') \circ H$, where $\overline{\varphi}$ is a distinguished submersion for $\mathbf{f}$, and H a holonomy projection along F_0. Passing to germs at an arbitrary $x \in X_0$ we find a path c (from an $i'(t)$ to x) such that

$$[H, x] = H^{i'}_{t,c} = h^{i'\amalg j}_{t,c,\psi(x)}[\psi, x]$$

for any $\psi \in \mathrm{Subm}(F_0)$ about x, and a local section j of ψ through x. In view of Lemma 3.1,

$$\begin{aligned}[\overline{\varphi} \circ i' \circ H, x] &= [\overline{\varphi} \circ i', t] h^{i'\amalg j}_{t,c,\psi(x)}[\psi, x] \\ &= h_c^*[\overline{\varphi}, i'(t)][j, \psi(x)][\psi, x] = h_c^*[\overline{\varphi}, i'(t)] \in E_0\end{aligned}$$

so that $(\overline{\varphi} \circ i') \circ H$ is a distinguished submersion for $\mathbf{f}$. Now Theorem 2.11 ensures that $\mathcal{N}|\Phi| \circ \Pi' = \mathbf{f}$.

Uniqueness. We wish to deduce the uniqueness of the decomposition of $\mathbf{f}$ in our category. By Corollary I.3.7, any decomposition $\mathbf{f} = \mathcal{N}\Sigma \circ \Pi'$,

$\Sigma : \Gamma'_{F,T} \rightharpoonup \Gamma$, gives rise to an isomorphism of principal Γ-bundles

$$E_0 \cong \Gamma'_{F,T} \backslash (E'_0 \times_{(\beta,\pi)} \Sigma)$$

where $E = (E_n)$ is any Γ-bundle classified by $\mathbf{f}$, and $E' = (E'_n)$ is the canonical left $\Gamma'_{F,T}$-bundle classified by Π'. If s is a local section of Σ, and H a holonomy projection (along F_0), then the map

$$x \to \Gamma'_{F,T}([H, x], s(Hx))$$

is a section of the Γ-bundle isomorphic to E_0; the distinguished submersion for $\mathbf{f}$ it defines is the composition $\alpha \circ sH$. As H is arbitrary, we conclude that $\alpha \circ s$ is always a submersion. Thus the Γ-bundle $\Sigma \to T^0 \cup T^1$ unrolls a foliation. Choosing H to be a (local) left inverse to the immersion i', we get $\alpha \circ s = (\alpha \circ sH) \circ i'$ so that every distinguished submersion for Σ is locally the restriction to i' of a distinguished submersion for $\mathbf{f}$. In order to prove the uniqueness of the Γ-equivariant left $\Gamma'_{F,T}$-action, we pass to the canonical form $\widetilde{\Sigma}$ of the Γ-bundle (cf. (2.3)) and observe that for any $\varphi \in \mathcal{G}'_{F,T}$ and any distinguished submersion φ for Σ the action yields a $\widetilde{\Sigma}$-valued map $\psi = [\gamma, \cdot] \cdot [\varphi, \cdot]$ such that $\pi \circ \psi = \gamma$ and $\alpha \circ \psi = \varphi$. By continuity of ψ, we conclude that $\psi = [\varphi\gamma^{-1}, \gamma(\cdot)]$; hence the action must be $[\gamma, x] \cdot [\varphi, x] = [\varphi\gamma^{-1}, \gamma(x)]$.

Part (ii) of the theorem is an immediate consequence of (i). We have already shown that for $\Gamma_F = \Gamma'_{F,T}$ the ss-morphism $\mathcal{N}\Gamma_F \rightharpoonup \mathcal{N}\Gamma$ which appears in the factorization (3.4) is transverse to the discrete foliation and pulls it back to another discrete foliation. As equivalences of nerves of groupoids preserve transversality (cf. Corollary 1.18), the last assertion of the theorem remains true for every solution to the universal factorization problem. ∎

Definition 3.3. Every solution $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ to the universal factorization problem (3.4) will be called a *minimal transverse projection* for (X, F). The groupoid of germs Γ_F is then the *holonomy groupoid* for F.

Example 3.4. If Γ is any groupoid of germs then the identity ss-morphism $\mathbf{1}_{\mathcal{N}\Gamma} : \mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma$ is evidently a minimal transverse projection for the discretely foliated $\mathcal{N}\Gamma$. In particular, Γ is a holonomy groupoid for the discrete foliation.

More generally, let Γ be any groupoid *equivalent* to a groupoid of germs Γ'. Then $\mathcal{N}\Gamma$ carries a canonical foliation F that comes from the discrete foliation of $\mathcal{N}\Gamma'$ (cf. Corollary 1.18(i); F does not depend on the equivalence!). Every minimal transverse projection $\mathcal{N}\Gamma \rightharpoonup \mathcal{N}\Gamma_F$ for F is an equivalence of ss-manifolds.

In course of the proof of Theorem 3.2 we have shown that every complete transversal $i : T \to X_1$ gives rise to a particular holonomy groupoid $\Gamma'_{F,T}$ and to a particular minimal transverse projection Π'. If $\mathcal{G}'_{F,T}$ is the underlying pseudogroup of diffeomorphisms of $T^0 \cup T^1$ then we may, and do, reduce it to a pseudogroup $\mathcal{G}_{F,T}$ of those local diffeomorphisms of T which are transferred into $\mathcal{G}'_{F,T}$ by the identification maps $\mathrm{id}^a : T \to T^a$, $a = 0, 1$; the associated groupoid of germs $\Gamma_{F,T}$ is canonically equivalent to $\Gamma'_{F,T}$ and thus leads to a minimal transverse projection $\Pi : X \rightharpoonup \mathcal{N}\Gamma_{F,T}$.

DEFINITION 3.5. $\Gamma_{F,T}$ and $\Gamma'_{F,T}$ are, respectively, the *reduced* and the *non-reduced holonomy groupoids* for F with respect to the transversal. The corresponding minimal projections Π and Π' are the *natural transverse projections* with respect to T.

Remark 3.6. According to Proposition I.2.13, a principal $\Gamma'_{F,T}$-bundle $E'_{F,T}$ over X classified by Π' can be described by a $\Gamma'_{F,T}$-extension of the canonical $\Gamma_{F_0,T^0\cup T^1}$-bundle over X_0,

$$E'_{F,T}(0) = \Gamma'_{F,T} \times_{\Gamma_{F_0,T^0\cup T^1}} E_{F_0,T^0\cup T^1}$$

(in multiplicative notation), and an $\varepsilon_1 : E'_{F,T}(1) = \varepsilon_0^* E'_{F,T}(0) \to E'_{F,T}(0)$ such that

$$(x, gH^{i'}_{t^0,\varepsilon_0 c}) \xrightarrow{\varepsilon_1} g[\mathrm{id}^0_1, t^1] H^{i'}_{t^1,\varepsilon_1 c} \tag{3.6}$$

where $g \in \Gamma'_{F,T}$, and for every $x \in X_1$, c is any path in $X_1^{F_1}$ joining $c(0) = i(t) \in i(T)$ to $c(1) = x$ (cf. (3.5), for $c_a = \varepsilon_a c$, $a = 0, 1$).

EXAMPLE 3.7. Any foliation F_0 of a manifold M extends trivially to a foliation F of the nerve $\mathcal{N}M$. Every complete transversal $i : T \to \mathcal{N}_1 M = M$ is a complete transversal for F_0, and one has $\Gamma_{F,T} = \Gamma_{F_0,T}$. Furthermore, the corresponding natural transverse projection for F is the classifying ss-morphism for the principal $\Gamma_{F_0,T}$-bundle $E_{F_0,T}$ over M. Thus the abstract characterization of holonomy via Theorem 3.2 agrees with the classical one.

EXAMPLE 3.8. Let (X, F) be a foliated ss-manifold. If $i : T \to X_1$ and $j : S \to X_1$ are two complete transversals for F_1, then

(i) the holonomy maps from T to S along paths in the leaves of F_1 generate an equivalence of pseudogroups $\mathcal{G}_{F,T} \overset{\approx}{\rightharpoonup} \mathcal{G}_{F,S}$, and

(ii) the holonomy maps from $T^0 \cup T^1$ to $S^0 \cup S^1$ along paths in the leaves of F_0 generate an equivalence $\mathcal{G}'_{F,T} \overset{\approx}{\rightharpoonup} \mathcal{G}'_{F,S}$.

The associated canonical equivalences of the holonomy groupoids are the only equivalences which transfer the natural transverse projections to each other.

EXAMPLE 3.9. Let $\mathbf{f} : Y \rightharpoonup X$ be an ss-morphism transverse to a foliation F of X, and let $F' := \mathbf{f}^* F$. For any minimal transverse projections $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ and $\Pi_{F'} : Y \rightharpoonup \mathcal{N}\Gamma_{F'}$ there is a *unique* morphism of holonomy groupoids, $\Sigma : \Gamma_{F'} \rightharpoonup \Gamma_F$, such that the square

$$\begin{array}{ccc} Y & \xrightarrow{\mathbf{f}} & X \\ \Pi_{F'} \downarrow & & \downarrow \Pi_F \\ \mathcal{N}\Gamma_{F'} & \xrightarrow{\mathcal{N}\Sigma} & \mathcal{N}\Gamma_F \end{array}$$

commutes. If, in particular, $\mathbf{f} = [f]$ comes from an ss-map $f : Y \to X$, and $i : T' \to Y_1$ is a complete transversal for F'_1, then $f_1 \circ i : T' \to X_1$ can be enlarged to a complete transversal T for F_1. $\Sigma : \Gamma_{F',T'} \rightharpoonup \Gamma_{F,T}$ is then generated by the inclusion $T' \hookrightarrow T$.

One should be aware of the fact that an ss-morphism $\mathbf{f} : (X', F') \rightharpoonup (X, F)$ does not necessarily lead to a natural morphism $\Gamma_{F'} \rightharpoonup \Gamma_F$ of the respective holonomy groupoids unless $\mathbf{f}$ is transverse to F. If $\mathbf{f} \pitchfork F$, then there is a decomposition of $\mathbf{f}$,

$$(X', F') \to (X', \mathbf{f}^* F) \xrightarrow{f} (X, F) ,$$

and the required morphism is the composition of $\Gamma_{\mathbf{f}^* F} \rightharpoonup \Gamma_F$ and a morphism characterized in Corollary 3.14(ii) below.

R e m a r k 3.10. For any foliated manifold (M, F) the set of all the holonomy translations h^*_c in E_F carries a canonical structure of a (differentiable) groupoid $\Gamma(F)$ over M called the *graph* of F (cf. [20]). Although $\Gamma(F)$ is not a groupoid of germs (unless F is discrete), the ss-morphism $\mathcal{N}M \rightharpoonup \mathcal{N}\Gamma(F)$ induced by the identification of M with the units is also a solution to the universal factorization problem (3.4). In fact, $\Gamma(F)$ is canonically equivalent to any $\Gamma_{F,T}$, and the ss-manifold $\mathcal{N}\Gamma(F)$ can be regarded as a canonical form of the mutually equivalent $\mathcal{N}\Gamma_F$'s.

Any minimal transverse projection $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ for a foliated ss-manifold (X, F) can be interpreted as a "projection" $X \to X/F$ where the *quotient* of X by F is an ss-manifold admitting a discrete foliation and defined up to equivalence. This point of view will be justified in the sequel.

Definition 3.11. A pair (F, F') of foliations of an ss-manifold X is a *flag* on X if $\mathrm{id}_X : (X, F) \to (X, F')$.

Clearly, (F, F') is a flag iff (F_0, F'_0) is a flag of foliations on X_0. Furthermore, if (F, F') is a flag on X, and $\mathbf{f} : Y \rightharpoonup X$ is an ss-morphism transverse to F, then $\mathbf{f} \pitchfork F'$ and the pull-backs by $\mathbf{f}$ form a flag $(\mathbf{f}^* F, \mathbf{f}^* F')$ on Y (cf. Proposition 1.16(iii)).

In order to establish a global relationship between the transverse structures of foliations which form a flag, we need an appropriate local result first.

Lemma 3.12. *Let (F, F') be a flag of foliations on a manifold M, c :*

$[0,1] \to M$ any path in a leaf of F, and $T_0, T_1 \subset M$ local transversals at the ends of c such that there is a holonomy map for F, $h : T_1 \to T_0$, along c. If $F'|T_a$, $a = 0, 1$, stand for the pull-back foliations of the transversals, then $h^*(F'|T_0) = F'|T_1$. If, moreover, $T_1' \subset T_1$ is any local transversal for $F'|T_1$ at the endpoint of c, and $T_0' = h(T_1')$ is a local transversal for $F'|T_0$ at the initial point, then

$$[i_0, c(0)] h_{T_0', c, T_1'} = h_{T_0, c, T_1} [i_1, c(1)] , \tag{3.7}$$

$i_a : T_a' \hookrightarrow T_a$, $a = 0, 1$, being the inclusion maps. In other words, $h|T_1' : T_1' \to T_0'$ is a holonomy map along c for F'.

Proof. By definition, h comes from a holonomy chain along c and is therefore a composition of holonomy maps along portions of the path contained in simply foliated open subsets of M. Since it suffices to prove the assertion of the lemma for each of the portions separately we may, and do, assume that the whole manifold M is simply foliated by F. If now $U_1, \ldots, U_k \subset M$ are open subsets simply foliated by F' and such that $c[\tau_{i-1}, \tau_i] \subset U_i$, $i \le k$, for a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_k = 1$ of the path, then the U_i's are still simply foliated by F, and any sequence of local transversals $T_{\tau_i} \subset U_i \cap U_{i+1}$ at $c(\tau_i)$, $i = 1, \ldots, k-1$, yields a decomposition of h (about $c(1)$) into holonomy maps from T_{τ_i} to $T_{\tau_{i-1}}$ with respect to the U_i, $i \le k$. As the lemma is local, it suffices to consider the simplest case of $U \subset M$ open and simply foliated with respect to both F and F'. For such U, let $p : U \to \widehat{U}$ and $p' : U \to \widehat{U}'$ be surmersions whose fibres are leaves of, respectively, $F|U$ and $F'|U$. Since every surmersion uniquely characterizes both the topology and the differentiable structure of its image, the only map $q : \widehat{U} \to \widehat{U}'$ such that $p' = q \circ p$ is again a surmersion; hence $F'|U$ descends to a foliation of $\widehat{U}$. Furthermore, if $T_0', T_1' \subset U$ are local transversals for F' such that $p(T_0') = p(T_1')$ (and $p'|T_0'$ is invertible) then the identity

$$(p'|T_0')^{-1}(p'|T_1') = (p|T_0')^{-1}(p|T_1')$$

yields (3.7). ∎

THEOREM 3.13 ([2]). *Let (F, F') be any flag on an ss-manifold X, and $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ a minimal transverse projection for F. Given a transverse projection $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma$ for F', there exists a unique morphism of groupoids $\Sigma : \Gamma_F \rightharpoonup \Gamma$ such that the triangle*

$$\begin{array}{ccc} X & & \\ {\scriptstyle \Pi_F}\downarrow & \searrow^{\mathbf{f}} & \\ & & \mathcal{N}\Gamma \\ & \nearrow_{\mathcal{N}\Sigma} & \\ \mathcal{N}\Gamma_F & & \end{array}$$

commutes. The ss-morphism $\mathcal{N}\Sigma : \mathcal{N}\Gamma_F \rightharpoonup \mathcal{N}\Gamma$ is a transverse projection for a foliation $\widetilde{F}$ of $\mathcal{N}\Gamma_F$ such that $F' = \Pi_F^* \widetilde{F}$.

P r o o f. According to Corollary 1.18(i), we may assume without loosing generality that Π_F is the natural transverse projection $\Pi' : X \rightharpoonup \mathcal{N}\Gamma'_{F,T}$ with respect to a complete transversal $i : T \to X_1$; then we proceed as in the proof of Theorem 3.2. Namely, in the same vein we show that for any factorization $\mathbf{f} = \mathcal{N}\Sigma \circ \Pi'$ (if it exists) the source map α for Σ is a submersion. This implies (cf. Example I.3.5) that the source ss-map for the Γ-bundle classified by $\mathcal{N}\Sigma$ is transverse to the discrete foliation, and so $\mathcal{N}\Sigma : \mathcal{N}\Gamma'_{F,T} \rightharpoonup \mathcal{N}\Gamma$ is a transverse projection for a foliation, say $\widetilde{F}$, of $\mathcal{N}\Gamma'_{F,T}$ (cf. Theorem 1.17(i)). Clearly,

$$\Pi'^*\widetilde{F} = \Pi'^*(\mathcal{N}\Sigma)^*F_\delta = \mathbf{f}^*F_\delta = F' .$$

The factorization is unique, for the distinguished submersions for $\mathcal{N}\Sigma$ (i.e. for Σ) are generated by the distinguished submersions for $\mathbf{f}$ restricted to the transversal i'.

In order to prove the existence, observe that the set $\Phi_0 = \{\overline{\varphi} \circ i'; \overline{\varphi} \in \Phi_{\mathbf{f}}\}$, where $\Phi_{\mathbf{f}}$ stands for the distinguished submersions for $\mathbf{f}$, consists of submersions $T^0 \cup T^1 \supset U_\varphi \xrightarrow{\varphi} N$ $(= \mathcal{N}_0\Gamma)$ such that $\varphi^*F_\delta^{(N)} = (i'^*F'_0)|U_\varphi$. We wish to show that Φ_0 satisfies (ii) of Theorem 2.11, since this condition implies $\varepsilon_1^*(i'^*F'_0) = \varepsilon_0^*(i'^*F'_0)$ so that $\widetilde{F}_0 := i'^*F'_0$ generates a foliation $\widetilde{F}$ of $\mathcal{N}\Gamma'_{F,T}$, and Φ_0 yields a transverse projection for $\widetilde{F}$. In terms of the underlying pseudogroups: $\mathcal{G}'_{F,T}$ for $\Gamma'_{F,T}$, and $\mathcal{G}$ for Γ, the condition reads: for any $\varphi, \psi \in \Phi_0$, $t \in U_\psi$, and $h \in \mathcal{G}'_{F,T}$ such that $h(t) \in U_\varphi$, there is a $\gamma \in \mathcal{G}$ such that

$$\varphi \circ h = \gamma \circ \psi \tag{3.8}$$

on a neighbourhood of t.

There are two cases to consider:

1° h is a holonomy map along a path c in a leaf of F_0 such that $c(1) = i'(t)$ and $c(0) = i'h(t)$, and $\varphi = \overline{\varphi} \circ i'$, $\psi = \overline{\psi} \circ i'$. At the 0-level of the canonical Γ-bundle classified by $\mathbf{f}$, one has

$$h_c^*[\overline{\varphi}, c(0)] = [\gamma \circ \overline{\psi}, c(1)]$$

for a $\gamma \in \mathcal{G}$. By Lemma 3.12, if $T'_0 \subset T^0 \cup T^1$ is any local transversal for $i'^*F'_0$, and $T'_1 = h^{-1}T_0$ (cf. Lemma 3.12), then

$$[\gamma \circ \psi | T'_1, t] = [\varphi | T'_0, h(t)] h_{T'_0, c, T'_1} = [\varphi \circ h | T'_1, t]$$

where the holonomy germ is taken for F'_0, and the last equality holds true by (3.7).

2° $h = \mathrm{id}_0^1$. In this case, the proof of (3.8) is completely analogous to the corresponding part of the proof of Theorem 3.2.

It remains to check that the only transverse projection $\mathbf{g} : \mathcal{N}\Gamma'_{F,T} \rightharpoonup \mathcal{N}\Gamma$ for $\widetilde{F}$ whose set of distinguished submersions contains Φ_0 factorizes $\mathbf{f}$ into

$\mathbf{g} \circ \Pi'$. To this end, recall that in the uniqueness part of the proof we have actually shown that the distinguished submersions for $\mathbf{g} \circ \Pi'$ are (locally) compositions of distinguished submersions for $\mathbf{g}$ and for Π'. If H is any holonomy projection on $T^0 \cup T^1$ then its every germ is of the form

$$[H, x] = H^{i'}_{t,c} = h^{i' \amalg j}_{t,c,\psi(x)}[\psi, x]$$

for a path c in a leaf of F_0, and some $\psi \in \mathrm{Subm}(F_0)$ (cf. (3.3)). In view of Lemma 3.12, there are local transversals for F'_0, $T'_0 \subset T^0 \cup T^1$ and $T'_1 \subset \mathrm{image}(j)$ such that

$$h^{i' \amalg j}_{t,c,\psi(x)}[\psi | T'_1, x] = [i_0, t] h_{T'_0, c, T'_1},$$

$i_0 : T'_0 \hookrightarrow T^0 \cup T^1$ being the inclusion. Consequently, for every distinguished submersion $\overline{\varphi} \circ i' \in \Phi_0$ whose domain contains $t = H(x)$,

$$[(\overline{\varphi} \circ i') \circ H | T'_1, x] = [\overline{\varphi} | T'_0, t] h_{T'_0, c, T'_1}$$

is the restriction to T'_1 of $h^*_c[\overline{\varphi}, i'(t)]$ (cf. Lemma 3.1). This proves that all the compositions $(\overline{\varphi} \circ i') \circ H$ are distinguished submersions for $\mathbf{f}$, and therefore $\mathbf{g} \circ \Pi' = \mathbf{f}$ as was to be shown. ∎

COROLLARY 3.14. *For any flag (F, F') on an ss-manifold X and every minimal transverse projection $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ for F, there exists a unique foliation $\widetilde{F}$ of $\mathcal{N}\Gamma_F$ such that $F' = \Pi^*_F \widetilde{F}$. The transverse structure of (X, F') equals that of $(\mathcal{N}\Gamma_F, \widetilde{F})$ in the following sense*:

(i) *if $\Pi_{\widetilde{F}} : \mathcal{N}\Gamma_F \rightharpoonup \mathcal{N}\Gamma_{\widetilde{F}}$ is a minimal transverse projection for $\widetilde{F}$ then $\Pi_{\widetilde{F}} \circ \Pi_F : X \rightharpoonup \mathcal{N}\Gamma_{\widetilde{F}}$ is a minimal transverse projection for F'*;

(ii) *if $\Pi_{F'} : X \rightharpoonup \mathcal{N}\Gamma_{F'}$ is a minimal transverse projection for F' then the only ss-morphism $\widetilde{\Pi} : \mathcal{N}\Gamma_F \rightharpoonup \mathcal{N}\Gamma_{F'}$ such that $\Pi_{F'} = \widetilde{\Pi} \circ \Pi_F$ is a minimal transverse projection for $\widetilde{F}$.*

Proof. The uniqueness of $\widetilde{F}$ such that $F' = \Pi^*_F \widetilde{F}$ follows from the unique factorization (through Π_F) of the classifying ss-morphism $X \rightharpoonup \mathcal{N}\Gamma_{q'}$ for F' (cf. Theorem 2.1). Assertions (i)–(ii) can be easily verified by standard diagram-chasing and require some patience in applying Theorems 3.2 and 3.13. ∎

Remark 3.15. The above corollary ensures, roughly speaking, that the ss-morphism $X \rightharpoonup X/F$ project every foliation F' weaker than F to a foliation—say F'/F—of the quotient. Furthermore, the successive quotients lead to the formula

$$(X/F)/(F'/F) = X/F'.$$

II.4. A relationship with fundamental groups. In this section we complete Theorem 3.2, which characterizes minimal transverse projections and holonomy grupoids for foliations of arbitrary ss-manifolds, with an assertion on 1-connectedness of every projection $X \rightharpoonup X/F$ (cf. [12]). Moreover, after passing to the leaves we define holonomy groups and relate them with the fundamental groups of the leaves.

THEOREM 4.1. *Let F be a foliation of a* connected *ss-manifold X, and $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ a minimal transverse projection for F. Then*

(i) *the ss-manifold $\mathcal{N}\Gamma_F$ is* connected, *and*

(ii) *the induced morphism of the fundamental groupoids $\pi(\Pi_F) : \pi(X) \rightharpoonup \pi(\mathcal{N}\Gamma_F)$ comes from an* epimorphism *of fundamental groups.*

LEMMA 4.2. *For any connected manifold M endowed with a foliation F, and every complete transversal $i : T \to M$ for F, the nerve of the holonomy grupoid $\Gamma_{F,T}$ is connected.*

P r o o f. Let $\mathcal{N}\Gamma^\circ$ be a connected component of $\mathcal{N}\Gamma_{F,T}$. By connectedness of M, it suffices to show that the union $M^\circ \subset M$ of those leaves of F which meet the units $T^\circ = \mathcal{N}_0\Gamma^\circ \subset T$ is both open and closed in M, since this will immediately imply $T^\circ = T$, $\Gamma^\circ = \Gamma_{F,T}$ (cf. the proof of Proposition I.1.15). If $x \in M$ then for any path c in the leaf through x such that $c(0) \in i(T)$, $c(1) = x$, there is a holonomy projection $H : M \supset U \to T$ along c defined on a connected neighbourhood of x. For x in the closure of M°, there is a $y \in U \cap M^\circ$ so that $H(y) \in T^\circ$. By connectedness of $H(U)$, this implies $H(U) \subset T^\circ$, $U \subset M^\circ$. ■

P r o o f o f T h e o r e m 4.1. We restrict ourselves to the case of the natural transverse projection $\Pi' : X \rightharpoonup \mathcal{N}\Gamma'_{F,T}$ with respect to a fixed complete transversal $i : T \to X_1$.

(i) Let $t, t' \in T^0 \cup T^1$ be arbitrary points of the induced transversal for F_0. The connectedness of X ensures that there is a path $C = (c_0, y_1^{e_1}, c_1, \ldots, y_r^{e_r}, c_r)$ in X such that $c_0(0) = i'(t)$ and $c_r(1) = i'(t')$. Since i is a complete transversal for F_1, its image meets every connected component of X_1. Consequently, an elementary homotopy of type (ii) (cf. Definition I.4.2) deforms C to a $C' = (c'_0, i(t_1)^{e_1}, c'_1, \ldots, i(t_r)^{e_r}, c'_r)$, $t_1, \ldots, t_r \in T$. By Lemma 4.2, each c'_k, $k \le r$, gives rise to a path in $\mathcal{N}\Gamma_{F_0, T^0 \cup T^1}$ (hence in $\mathcal{N}\Gamma'_{F,T}$), while the germs $[\mathrm{id}_0^1, t_k^0]^{e_k} \in \Gamma'_{F,T}$, $k \le r$, link the paths together. The resulting path in $\Gamma'_{F,T}$ connects t and t'.

(ii) For a fixed reference point $t_0 \in T^0 \cup T^1$, let E be the canonical simply connected $\pi_{t_0}(\mathcal{N}\Gamma'_{F,T})$-bundle over $\mathcal{N}\Gamma'_{F,T}$. By Corollary I.4.21, $\pi(\Pi')$ is generated by an epimorphism of fundamental groups iff the pull-back Π'^*E is connected. In order to construct a pull-back $\overline{X}$ of E by Π' we apply Proposition I.3.8(i). Namely, as the morphism of groupoids associated with

E is the composition

$$\Gamma'_{F,T} \xrightarrow{[\ ]} \pi(\mathcal{N}\Gamma'_{F,T}) \xrightarrow{\alpha^{-1}(t_0)} \pi_{t_0}(\mathcal{N}\Gamma'_{F,T})$$

(cf. I.3 and I(4.5)–I(4.6)), and the left $\Gamma'_{F,T}$-bundle $E'_{F,T}$ classified by Π' is generated by $E'_{F,T}(0) = \Gamma'_{F,T} \times_{\Gamma_{F_0,T^0\cup T^1}} E_{F_0,T^0\cup T^1}$ (cf. Remark 3.6), $\overline{X}$ is generated by

$$\overline{X}_0 = E'_{F,T}(0) \times_{\Gamma'_{F,T}} \alpha^{-1}(t_0) \cong \Gamma_{F_0,T^0\cup T^1}\backslash(E_{F_0,T^0\cup T^1} \times_{(\beta,\beta)} \alpha^{-1}(t_0))$$

where the groupoid acts on $\alpha^{-1}(t_0) \subset \pi(\mathcal{N}\Gamma'_{F,T})$ via the homomorphism [ ]. Furthermore, for $\overline{X}_1 = \varepsilon_0^*\overline{X}_0$, $\varepsilon_1 : \overline{X}_1 \to \overline{X}_0$ is given by

$$(x, ([H] : [C])) \xrightarrow{\varepsilon_1} (\varepsilon_1(x, [H]) : [C])$$

(cf. (3.6)), where we adopt the homogeneous notation (:) for the cosets—elements of $\overline{X}_0$.

Each element of $\overline{X}_0$ is of the form $(H^{i'}_{t,c} : [C])$, where c is a path in $X_0^{F_0}$ such that $c(0) = i'(t)$, $t \in T^0\cup T^1$, and C is a path in $\mathcal{N}\Gamma'_{F,T}$ with initial point t and endpoint t_0. We wish to join this element to $(H_{t_0}, \omega_{i'(t_0)} : 1_{t_0}) \in \overline{X}_0$ by a suitable path in $\overline{X}$. Let

$$H^{i'}_t := H^{i'}_{t,\omega_{i'(t)}} \in E_{F_0,T^0\cup T^1}$$

denote the holonomy projection along the constant path at $i'(t)$, for $t \in T^0 \cup T^1$; clearly, $H^{i'}_t$ is a continuous function of t. To start with, observe that the path

$$[0,1] \ni \tau \to (H^{i'}_{t,c|[0,\tau]} : [C]) \in \overline{X}_0$$

connects $(H^{i'}_{t,c} : [C])$ and $(H^{i'}_t : [C])$.

Lemma 4.3. *The following pairs of points of $\overline{X}_0$ are the initial point and the endpoint of a certain path in $\overline{X}$*:

(i) $(H^{i'}_{c(0)} : [c][C])$ *and* $(H^{i'}_{c(1)} : [C])$, *for any path c in $T^0 \cup T^1$*;

(ii) $(H^{i'}_t : [h^{i'}_{t,c,t'}][C])$ *and* $(H^{i'}_{t'} : [C])$, *for any holonomy germ* $h^{i'}_{t,c,t'} \in \Gamma_{F_0,T^0\cup T^1} \subset \Gamma'_{F,T}$;

(iii) $(H^{i'}_{t^1} : [[\mathrm{id}^1_0, t^0]][C])$ *and* $(H^{i'}_{t^0} : [C])$, *for $t \in T$*;

(iv) $(H^{i'}_{t^0} : [[\mathrm{id}^0_1, t^1]][C])$ *and* $(H^{i'}_{t^1} : [C])$, *for $t \in T$*.

Proof. The respective paths are given by:

(i) $\tau \to (H^{i'}_{c(\tau)} : [c|[\tau, 1]][C])$,

(ii) $\tau \to (H^{i'}_{t,c|[0,\tau]} : [h^{i'}_{t,c,t'}][C])$,

(iii) $(i(t), (H^{i'}_{t^0} : [C])) \in \overline{X}_1$ (cf. (3.6)), and

(iv) the formal inverse $(i(t), (H^{i'}_{t^0} : [[\mathrm{id}^0_1, t^1]][C]))^{-1}$. ∎

Returning to the proof of Theorem 4.1 we recall that every homotopy class $[C] \in \pi(\mathcal{N}\Gamma'_{F,T})$ is a composition of homotopy classes $[c]$ of paths in the units $T^0 \cup T^1$ and classes $[g]$, $g \in \Gamma'_{F,T}$ (cf. Remark I.4.5). As every element of the non-reduced holonomy groupoid $\Gamma'_{F,T}$ is a composition of germs of the form either $h^{i'}_{t,c,t'}$ or $[\mathrm{id}^1_0, t^0]^{\pm 1}$, Lemma 4.3 provides a recipe for successive reduction of an arbitrary $[C] \in \alpha^{-1}(t_0)$ to the unit 1_{t_0}. This ensures the connectedness of $\overline{X} = \Pi'^* E$ and implies, as we observed earlier, the surjectivity of the group homomorphisms generating $\pi(\Pi')$.

Since connectedness of ss-manifolds is preserved by equivalences, and π is a functor, properties (i)–(ii) are inherited by every minimal transverse projection Π_F for F. ∎

Each minimal transverse projection $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ for a foliated ss-manifold (X, F) induces an ss-morphism

$$\Pi'_F : X^F \rightharpoonup (\mathcal{N}\Gamma_F)^{F_\delta} = \mathcal{N}(\Gamma_F)^{F_\delta},$$

which is (up to equivalence) uniquely associated with (X, F) (cf. Proposition 1.14 and Corollary 1.15). In order to examine this ss-morphism in detail, we need

LEMMA 4.4. *For any groupoid of germs Γ, the leaves of the discrete foliation F_δ of $\mathcal{N}\Gamma$ are in canonical one-to-one correspondence with the orbits of Γ and are equivalent to the nerves of the (discrete) structural groups $\Gamma_x = \alpha^{-1}(x) \cap \beta^{-1}(x) \subset \Gamma$.*

Proof. The leaves of Γ are the nerves of some discrete transitive subgroupoids, i.e. Galois groupoids over the orbits. According to Remark I.4.8, each of those groupoids is equivalent to any of its structural groups. ∎

Let L be a leaf of a foliated ss-manifold (X, F), and $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ a minimal transverse projection for F. In view of Proposition I.1.19, there is a unique leaf of $(\mathcal{N}\Gamma_F, F_\delta)$, to be denoted by $\mathcal{N}\Gamma_F|L$, such that Π_F descends to an ss-morphism $\Pi_F|L : L \rightharpoonup \mathcal{N}\Gamma_F|L$.

THEOREM 4.5. *Let (X, F) be a foliated ss-manifold and $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ a minimal transverse projection for F. Then*

(i) *the assignment $L \rightsquigarrow \mathcal{N}\Gamma_F|L$ is a bijection between the leaves of F and the leaves of the discrete foliation of $\mathcal{N}\Gamma_F$. The equivalence class of $\mathcal{N}\Gamma_F|L$ depends of F and L only.*

Furthermore, for each leaf L of F,

(ii) *there is a unique morphism of groupoids $h_L : \pi(L) \rightharpoonup \Gamma_F|L$ which* *closes the triangle*

$$\begin{array}{ccc} & L & \\ {\scriptstyle \varPi_L}\swarrow & & \searrow{\scriptstyle \varPi_F|L} \\ \mathcal{N}\pi(L) & \underset{\mathcal{N}h_L}{\longrightarrow} & \mathcal{N}\varGamma_F|L \end{array}$$

(iii) *for any point $x \in L_0$, and every unit z of $\varGamma_F|L$, the canonical equivalences $\pi(L) \approx \pi_x(L)$ and $\varGamma_F|L \approx \varGamma_{F,z}$ (the structural group at z) transfer h_L to*

$$h'_L : \pi_x(L) \rightharpoonup \varGamma_{F,z}$$

which is generated by a group epimorphism;

(iv) *there is a natural commuting square*

$$\begin{array}{ccc} \pi(L) & \xrightarrow{\pi(i)} & \pi(X) \\ {\scriptstyle h_L}\downarrow & & \downarrow{\scriptstyle \pi(\varPi_F)} \\ \varGamma_F|L & \xrightarrow{j_L} & \pi(\mathcal{N}\varGamma_F) \end{array}$$

in which j_L is a restriction of the fundamental morphism $\varGamma_F \rightharpoonup \pi(\mathcal{N}\varGamma_F)$; in the diagram, each one of the groupoids is equivalent to some discrete group, and the two vertical morphisms are represented by group epimorphisms.

The naturality of the diagram should be understood as follows: if an equivalence of holonomy groupoids $\varSigma : \varGamma_F \rightharpoonup \widetilde{\varGamma}_F$ transfers $\varPi_F$ to, say, $\widetilde{\varPi}_F$ then the associated equivalences $\varGamma_F|L \approx \widetilde{\varGamma}_F|L$ and $\pi(\mathcal{N}\varGamma_F) \approx \pi(\mathcal{N}\widetilde{\varGamma}_F)$ transfer the square to the appropriate square for $\widetilde{\varGamma}_F$.

Proof. The uniqueness part of Proposition I.1.19 together with Proposition 1.14 reduces the proof of (i) to the case of any particular minimal transverse projection, e.g. to the natural projection $\varPi' : X \rightharpoonup \mathcal{N}\varGamma'_{F,T}$ associated with a complete transversal $i : T \to X_1$. Since the leaves of $(\mathcal{N}\varGamma'_{F,T}, F_\delta)$ come from the partition of $T^0 \cup T^1$ into orbits of the groupoid, $\varPi'$ sends each leaf L of F to the leaf over the set T_L (with the discrete topology) of all points $H(x) \in T^0 \cup T^1$, where H ranges over the distinguished submersions for $\varPi'$, and x over L_0. In order to check that $T_L \neq T_{L'}$ for leaves $L \neq L'$ it suffices to observe that the evidently pairwise disjoint subsets

$$T'_L = \{t \in T^0 \cup T^1; i'(t) \in L_0\} \subset T_L$$

are precisely the orbits. Indeed, passing to generators of the non-reduced holonomy groupoid we see that:

1° if $h^{i'}_{t,c,t'} \in \varGamma'_{F,T}$ and $i'(t) \in L_0$ then the whole path c is in L_0 and thus $i'(t') = c(1) \in L_0$;

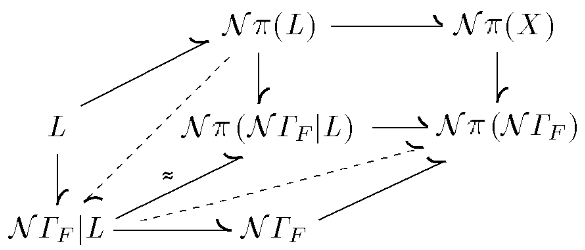
2° if $[\mathrm{id}_0^1, t^0] \in \varGamma'_{F,T}$, $t \in T$, then $\varepsilon_0 i(t) = i'(t^0) \in L_0$ iff $i(t) \in L_1$, and so iff $\varepsilon_1 i(t) = i'(t^1) \in L_0$.

(ii) is a consequence of Theorem I.4.11 and Lemma 4.4. In terms of the natural projection $\varPi'$, property (iii) is equivalent to the conectedness of a covering ss-manifold $\overline{L} = (\overline{L}_n)$ (over L) defined by $\overline{L}_n = \beta_n^{-1}(t_0) \subset E'_{F,T}(n)$ for $n \geq 0$, $t_0 \in T_L$ being any fixed unit of $\varGamma'_{F,T}|L$. The last property can be verified as in the proof of Theorem 4.1(i). We omit the details here, as the property is an immediate consequence of Theorem 4.8(ii) below.

(iv) The fundamental groupoid functor π, when applied to the square

$$\begin{array}{ccc} L & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{N}\varGamma_F|L & \longrightarrow & \mathcal{N}\varGamma_F \end{array}$$

gives rise to the commuting diagram



in which the morphism $\varGamma_F|L \rightharpoonup \pi(\mathcal{N}\varGamma_F|L)$ turns out to be an equivalence (cf. Example I.4.19). Clearly, any equivalence of holonomy groupoids transfers the bottom parallelepiped to the appropriate one—for the new groupoid. This explains the naturality of the whole diagram and concludes the proof. ∎

For any leaf L of F, and every $x \in L_0$, Theorem 4.5(iii) gives us a family of epimorphisms of $\pi_x(L)$ onto the structural group of $\varGamma_F|L$. Since the epimorphisms differ from each other by an isomorphism of their image, it is the common kernel, say $\mathcal{H}_x(F) \subset \pi_x(L)$, which characterizes the family. The structural groups are all isomorphic to the quotient $\pi_x(L)/\mathcal{H}_x(F)$.

Definition 4.6. The discrete group

$$\mathrm{Hol}_x(F) := \pi_x(L)/\mathcal{H}_x(F)$$

will be called the *holonomy group* of F at x (the holonomy group of the leaf through x). The projection $\pi_x(L) \to \mathrm{Hol}_x(F)$ is the *holonomy homomorphism* for L (at x), while $h_L : \pi(L) \rightharpoonup \varGamma_F|L$ the *holonomy morphism* for L (with respect to $\varPi_F$).

We conclude this section with a construction of a natural collection of holonomy groupoids for foliated manifolds which come from arbitrary complete transversals at the 0-level. The construction reflects the fact that holonomy groupoids are—in a sense—unions of holonomy groups of the leaves.

Let (X, F) be a foliated ss-manifold, and $i : T \to X_0$ a complete transversal for F_0. For any path

$$C = (c_0, y_1^{e_1}, \dots, y_r^{e_r}, c_r)$$

in a leaf L of F such that $c_0(0) = i(t_0)$ and $c_r(1) = i(t_1)$, $t_0, t_1 \in T$, we define a *holonomy germ* $h^i_{t_0,C,t_1} \in \Gamma_T$ (along C, with source t_1 and target t_0) as follows: let $V_j \subset X_1$ be a local transversal for F_1 at y_j, $j = 1, \dots, r$, and $U_0, U_1 \subset T$ local transversals at the ends of C; we set (see Fig. 5)

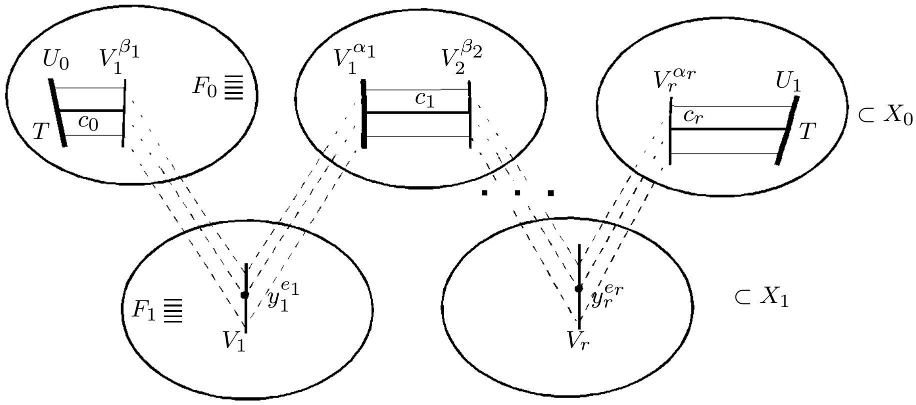


Fig. 5

$$(4.1)\qquad h^i_{t_0,C,t_1} = h_{U_0,c_0,V_1^{\beta_1}}[(\mathrm{id}_{V_1})^1_0, (y_1)^0]^{e_1} h_{V_1^{\alpha_1},c_1,V_2^{\beta_2}} \cdots$$
$$\cdots [(\mathrm{id}_{V_r})^1_0, (y_r)^0]^{e_r} h_{V_r^{\alpha_r},c_r,U_1}$$

where the indices $\alpha_j = (1 - e_j)/2$, $\beta_j = (1 + e_j)/2$ (i.e. either 0 or 1 for $e_j = \pm 1$) indicate the way of passing from $V_j \subset X_1$ to X_0.

PROPOSITION 4.7. *The holonomy germs are independent of the choices involved. Furthermore*,

(i) $h^i_{t_0,C,t_1} = h^i_{t_0,C',t_1}$ *if* C' *is any path homotopic to* C *in the leaf*, *and*

(ii) $h^i_{t_0,C,t_1} h^i_{t_1,C',t_2} = h^i_{t_0,C\cdot C',t_2}$ *for any two composable paths* C *and* C' *in the leaf.*

Proof. If $V, W \subset X_1$ are any two local transversals at a $y \in X_1$ then

$$[(\mathrm{id}_W)^1_0, y^0] h_{W^0,\omega_{\varepsilon_0 y},V^0} = h_{W^1,\omega_{\varepsilon_1 y},V^1}[(\mathrm{id}_V)^1_0, y^0]$$

since both the germs come from $h_{W,\omega_y,V}$ (cf. (3.2)). Here ω_z stands for the appropriate constant path. This equality applied to y_j, $j = 1, \ldots, r$, ensures the correctness of the definition.

In order to prove (i) it suffices to assume that C' is elementarily homotopic to C. We consider separately the different types of elementary homotopies according to Definition I.4.2(i)–(vi).

(i) The holonomy germs for F_0 depend already on the homotopy classes of the paths.

(ii) One has

$$[(\mathrm{id}_{\bar{V}})^1_0, \overline{y}^0] h_{\bar{V}^0,\omega_{\varepsilon_0 c},V^0} = h_{\bar{V}^1,\omega_{\varepsilon_1 c},V^1}[(\mathrm{id}_V)^1_0, y^0]$$

for local transversals $V \ni y$ and $\overline{V} \ni \overline{y}$.

(iii) If $y = \eta_0 x$, and $V = \eta_0 U$ comes from a local transversal $U \ni x$ for F_0, then $[(\mathrm{id}_V)^1_0, y^0] = h_{V^1,\omega_x,V^0}$ and the corresponding factor in (4.1) reduces to the holonomy germ along $c_{i-1} \cdot \omega_x \cdot c_i \simeq c_{i-1} \cdot c_i$.

(iv)–(vi) If W is any sufficiently small local transversal for F_2 at a $z \in X_2$, then $W_i = \varepsilon_i W \ni \varepsilon_i z$, $i = 0, 1, 2$, are local transversals for F_1. The commutation axioms for face operators imply

$$\begin{aligned}[(\mathrm{id}_{W_2})^1_0, (\varepsilon_2 z)^0] h_{(W_2)^0,\omega_{\varepsilon_1\varepsilon_0 z},(W_0)^1}[(\mathrm{id}_{W_0})^1_0, (\varepsilon_0 z)^0] \\ = h_{(W_2)^1,\omega_{\varepsilon_1\varepsilon_1 z},(W_1)^1}[(\mathrm{id}_{W_1})^1_0, (\varepsilon_1 z)^0] h_{(W_1)^0,\omega_{\varepsilon_0\varepsilon_0 z},(W_0)^0}\end{aligned}$$

where the holonomy germs are just the canonical identifications of the transversals.

The above particular cases yield assertion (i) of the proposition. The last assertion (multiplicativity) follows from I(4.1). ∎

Theorem 4.8. *Let (X, F) be a foliated ss-manifold, and $i : T \to X_0$ a complete transversal for F_0. The collection $\Gamma_{F,T} \subset \Gamma_T$ (more precisely, $\Gamma_{F,i}$) of all holonomy germs along paths in X^F ending in T is an open groupoid. Furthermore,*

(i) *$\Gamma_{F,T}$ is a holonomy groupoid for F; there is a natural minimal transverse projection $\Pi_{F,T} : X \rightharpoonup \mathcal{N}\Gamma_{F,T}$ such that for every collection of holonomy projections $H_a : U_a \to T$ (for F_0) over a covering $\mathcal{U} = (U_a)_{a\in A}$ of X_0, $\Pi_{F,T}$ is represented by an ss-map $X_{\mathcal{U}} \to \mathcal{N}\Gamma_{F,T}$ given—at the 1-level—by a cocycle $(\gamma_{ab})_{a,b\in A}$,*

$$\gamma^{(y)}_{ab} = h^i_{H_a(\varepsilon_1 y),C(y),H_b(\varepsilon_0 y)} \tag{4.2}$$

for $y \in \varepsilon_1^{-1} U_a \cap \varepsilon_0^{-1} U_b$. Here (cf. Fig. 6)

$$C(y) = (c_a^{(\varepsilon_1 y)}, y, c_b^{(\varepsilon_0 y)_{-1}})$$

is any path in the leaf of F through y whose portions $c_a^{(\cdot)}$ and $c_b^{(\cdot)}$ are char-

acterized by the requirement that

$$[H_d, x] = H^i_{H_d(x), c_d^{(\cdot)}}$$

for $d \in A$, $x \in U_d$ (cf. (3.3)).

(ii) *For any leaf* L *of* F, *any reference point* $x_0 \in L_0 \cap i(T)$, *and* $t_0 \in i^{-1}(x_0)$ *the epimorphism*

$$\pi_{x_0}(L) \ni [C] \to h^i_{t_0, C, t_0} \in (\Gamma_{F,T})_{t_0}$$

represents the holonomy morphism $h_L : \pi(L) \rightharpoonup \Gamma_{F,T}|L$.

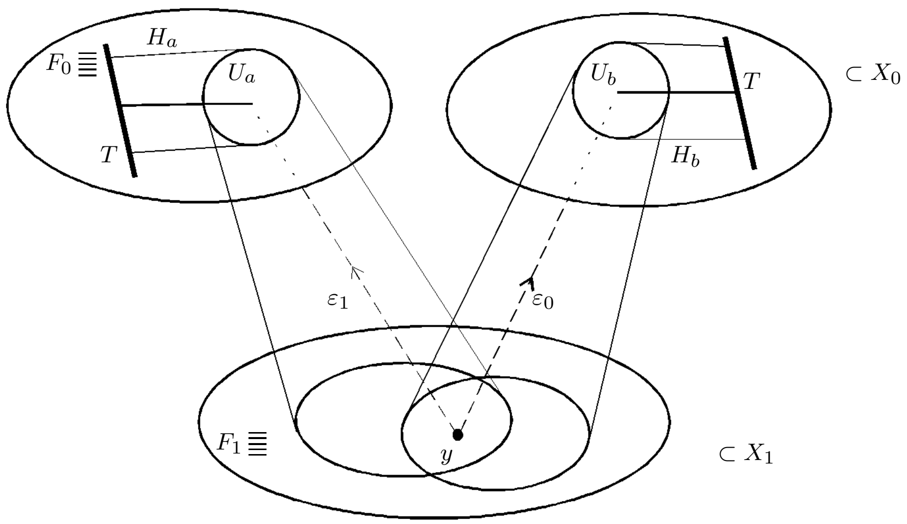


Fig. 6

P r o o f. Fix a collection of local transversals $y \in S_y \subset X_1$, for every $y \in X_1$, and set $S := \coprod S_y \amalg T$; there exists a canonical extension of $\eta_0 \circ i : T \to X_1$ to a complete transversal $j : S \to X_1$ for F_1. We claim that

$$\Gamma_{F,T} = \{g \in \Gamma_{F,S};\ \alpha g, \beta g \in T\}\,. \tag{4.3}$$

Indeed, if $C = (c_0, y_1^{e_1}, \ldots, c_r)$ is any path in a leaf of F with ends $i(t_0)$, $i(t_1)$, for $t_0, t_1 \in T$, then for $V_i = S_{y_i}$, $i \le r$, formula (4.1) defines $h^i_{t_0,C,t_1}$ as a composition of elements of the non-reduced holonomy groupoid $\Gamma'_{F,S}$, except at the ends, where the identification $T^0 \approx T \approx T^1$ places the whole germ in $\Gamma_{F,S}$. Conversely, every element of $\Gamma'_{F,S}$ is—by definition—expressible as a composition

$$g = h^{j'}_{s_0, c_0, s_1'} [\mathrm{id}_0^1, {s_1}^0]^{e_1} h^{j'}_{s_1'', c_1, s_2'} [\mathrm{id}_0^1, {s_2}^0]^{e_2} \ldots h^{j'}_{s_r'', c_r, s_{r+1}}$$

where c_i, $i \le r$, are paths in $X_0^{F_0}$, $e_i = \pm 1$, and s_i' and s_i'' are equal to the appropriate copy ${s_i}^0$ or ${s_i}^1$ of $s_i \in S$, for $i = 1, \ldots, r$. The germ becomes an element of $\Gamma_{F,S}$ if one identifies its source and target, $s_{r+1}, s_0 \in S^0 \amalg S^1$, with their originals in S. If, in particular, the originals are in $T \subset S$ then, according to (4.1),

$$g = h^i_{s_0,C,s_{r+1}} \in \Gamma_{F,T}$$

where

$$C = (c_0, j(s_1)^{e_1}, c_1, j(s_2)^{e_2}, \dots, c_r)\,.$$

In view of (4.3), $\Gamma_{F,T}$ is open in $\Gamma_{F,S}$ and is therefore a groupoid of germs; the inclusion homomorphism is an equivalence, for T is complete. Furthermore, if $(H_a)_{a\in A}$ is any collection of holonomy projections of elements of a covering $\mathcal{U} = (U_a)$ onto T, then the unique extension of the collection to a representative for the natural projections $\Pi_{F,S} : X \rightharpoonup \mathcal{N}\Gamma_{F,S}$ is in fact $\Gamma_{F,T}$-valued; the ss-morphism it represents, $X \rightharpoonup \mathcal{N}\Gamma_{F,T}$, differs from $\Pi_{F,S}$ by an equivalence of groupoids and so is a minimal transverse projection for F. It remains to show that for $y \in \varepsilon_1^{-1}U_a \cap \varepsilon_0^{-1}U_b$, $\gamma_{ab}(y)$ given by (4.2) satisfies

$$[H_a \circ \varepsilon_1, y] = \gamma_{ab}(y)[H_b \circ \varepsilon_0, y]$$

(cf. (2.4) in the proof of Theorem 2.11), i.e.

$$H^i_{H_a(\varepsilon_1 y), c_a^{(\cdot)}}[\varepsilon_1, y] = \gamma_{ab}(y) H^i_{H_b(\varepsilon_0 y), c_b^{(\cdot)}}[\varepsilon_0, y]$$

where the dots mean $\varepsilon_1 y$ and $\varepsilon_0 y$, respectively. When restricted to any local transversal through y, the above equality immediately implies (4.2), and so assertion (i) is proved.

In order to prove (ii) we denote the epimorphism considered by h and abbreviate $\Gamma_{F,T}$ with Γ. Since the canonical equivalence of grupoids $\pi(L) \rightharpoonup \pi_{x_0}(L)$ is represented by $\alpha^{-1}(x_0) \subset \pi(L)$ (cf. Corollary I.4.7), the composition

$$\pi(L) \rightharpoonup \pi_{x_0}(L) \xrightarrow{h} \Gamma_{t_0} \hookrightarrow \Gamma|L$$

is represented by a $(\Gamma|L)$-extension, $\Sigma := \alpha^{-1}(x_0) \times_h (\Gamma|L)$, of the principal $\pi_{x_0}(L)$-bundle $\alpha^{-1}(x_0)$. Elements of Σ are equivalence classes of pairs $([C'], h^i_{t_0,C'',t})$ such that both C' and C'' are paths in L, and $\alpha[C'] = i(t_0) = \beta[C'']$; the equivalence relation is

$$([C'C], h^i_{t_0,C'',t}) \sim ([C'], h^i_{t_0,CC'',t})$$

for $[C] \in \pi_{x_0}(L)$, so that multiplicative notation for equivalence classes is plausible. It is also readily seen that the turning point $t_0 \in T$ where homotopy changes into holonomy is inessential—every pair $([C'], h^i_{\tau,C'',t})$ gives rise to a well defined product $[C']h^i_{\tau,C'',t} \in \Sigma$ provided C' and C'' are composable paths in L and $i(\tau)$ is their common end.

According to Theorem 4.5(ii), we have to verify the equality $\Pi_{F,T}|L = \mathcal{N}\Sigma \circ \Pi_L$, which is—by Proposition I.1.19—equivalent to the commutativity of the square

$$\begin{array}{ccc} L & \longrightarrow & X^F \\ \mathbf{f} \downarrow & & \downarrow (\Pi_{F,T})' \\ \mathcal{N}\Gamma|L & \longrightarrow & \mathcal{N}\Gamma^{F_\delta} \end{array}$$

Here $\mathbf{f}$ denotes the composition $\mathcal{N}\Sigma \circ \Pi_L$. We intend to show that the diagram does commute by comparing the principal Γ^{F_δ}-bundles over L classified by the two compositions.

In view of I(4.6) and Corollary I.3.7, $\mathbf{f}$ classifies a $(\Gamma|L)$-bundle equal to $\Sigma \to L_0$ at the 0-level and characterized by $\varepsilon_1 : \varepsilon_0^* \Sigma \to \Sigma$,

$$L_1 \times_{(\varepsilon_0, \pi)} \Sigma \ni (y, \sigma) \to [y]\sigma \in \Sigma\,,$$

at the 1-level. On the other hand, the restriction of $\Pi'_{F,T}$ to L is known from its cocycle description (4.2). More precisely, if we denote by U_a^L the counter-image of U_a via the immersion $L_0 \to X_0$, for $a \in A$, then both H_a and γ_{ab} descend to continuous maps

$$H'_a : U_a^L \to (T, \delta), \qquad \gamma'_{ab} : L_1 \supset \varepsilon_1^{-1} U_a^L \cap \varepsilon_0^{-1} U_b^L \to (\Gamma, \delta) = \Gamma^{F_\delta}$$

for $a, b \in A$, δ being the discrete topology. Starting from the new cocycle, we can construct a Γ^{F_δ}-bundle $E = (E_n)$ it classifies—as in Remark I.2.16. In particular,

$$E_0 = \coprod_a U_a^L \times_{(H'_a, \beta)} \Gamma^{F_\delta} / \sim$$

where

$$(b, x, g) \sim (a, x, \gamma_{ab}(\eta_0 x) g)$$

for $x \in U_a^L \cap U_b^L$. Note that outside $\Gamma|L$, a transitive component of Γ^{F_δ}, the action of the groupoid is only hypothetical.

We claim that Σ and E_0 are isomorphic principal $(\Gamma|L)$-bundles over L_0. Namely, we shall show that the assignment

$$E_0 \ni [a, x, g] \xrightarrow{I_0} [c_a^{(x)}]^{-1} g \in \Sigma$$

is a well defined isomorphism:

— if c and c' are two paths in L_0 which yield the same holonomy projection H_a about x, then $[c']^{-1} g = [c]^{-1} g$, for the loop $c'c$ has then trivial holonomy germ;

— for $x \in U_a^L \cap U_b^L$, one has

$$\gamma_{ab}(\eta_0 x) = h^i_{H_a(x), C(\eta_0 x), H_b(x)} \quad \text{where} \quad [C(\eta_0 x)] = [c_a^{(x)}][c_b^{(x)}]^{-1}\,.$$

Consequently, we get

$$[c_a^{(x)}]^{-1} \gamma_{ab}(\eta_0 x) g = [c_a^{(x)}]^{-1} [C(\eta_0 x)] g = [c_b^{(x)}]^{-1} g$$

so that I_0 is correctly defined. Its smoothness follows from the fact that E_0 (as well as Σ) is a covering manifold, and from the way of introducing the differential structure on $\pi(L)$. Being evidently $(\Gamma|T)$-equivariant, I_0 must be an isomorphism.

In order to conclude the proof it now remains to use Corollary I.2.14 and verify that

$$\begin{aligned}\varepsilon_1 I_1(y,[b,\varepsilon_0 y,g]) &= \varepsilon_1(y,[c_b^{(\varepsilon_0 y)}]^{-1}g) = [y][c_b^{(\varepsilon_0 y)}]^{-1}g\\ &= [c_a^{(\varepsilon_1 y)}]^{-1}\gamma_{ab}(y)g = I_0[a,\varepsilon_1 y,\gamma_{ab}(y)g]\\ &= I_0\varepsilon_1(y,[b,\varepsilon_0 y,g])\end{aligned}$$

at the 1-levels. ∎

II.5. Foliated bundles and G-structures. A large class of Γ-foliations is supplied by specific geometric structures over the foliated ss-manifold. In fact, explicitly given transverse projections are very rare in the mathematical Nature. In this section we concentrate on G-structures, which provide an elegant way of making transverse projections implicit.

Given a q-codimensional foliation F of a manifold M, and a positive integer k, we recall that the *kth order normal bundle* $\pi^k : P_F^k \to M$ is the manifold of k-jets

$$P_F^k = \{j_x^k\varphi; \varphi \in \mathrm{Subm}(F),\ \varphi(x)=0\}\,,$$

together with the projection $j_x^k\varphi \to x$; the structure group of P_F^k is the *k-th order linear group*

$$\mathrm{GL}_k(q) = \{j_0^k\gamma; \gamma : \mathbb{R}^q \supset u \xrightarrow{\cong} v \subset \mathbb{R}^q,\ \gamma(0)=0\}$$

which acts by composition of jets.

For $k \geq l$, there are canonical *order projections*,

$$\pi_l^k : P_F^k \to P_F^l, \qquad \mathrm{GL}_k(q) \to \mathrm{GL}_l(q)\,,$$

which lower the order of jets. Furthermore, any map $f : M' \to M$ transverse to the foliation gives rise to a homomorphism of $\mathrm{GL}_k(q)$-bundles called the *k-th prolongation* $f^{(k)}$ of f,

$$P_{f^*F}^k \ni j_x^k(\varphi\circ f) \to j_{f(x)}^k\varphi \in P_F^k$$

for $k \geq 1$. In particular, for every $\varphi \in \mathrm{Subm}(F)$, $\varphi : U \to \mathbb{R}^q$, its prolongation $\varphi^{(k)} : P_F^k|U \to P_{\mathbb{R}^q}^k$ is a submersion onto the kth order frame bundle over $\mathbb{R}^q$.

LEMMA 5.1. *For every $k \geq 1$ the prolongations $\varphi^{(k)}$ of submersions $\varphi \in \mathrm{Subm}(F)$ pull the discrete foliation of $P_{\mathbb{R}^q}^k$ back to portions of a global foliation F^k of P_F^k. The latter foliation is characterized by the property that* *the canonical homomorphism*

$$E_F \ni [\varphi, x] \xrightarrow{J^k} j^k_x(\mathtt{t}_{-\varphi(x)} \circ \varphi) \in P^k_F$$

induces a map of foliated manifolds,

$$J^k : (E_F, \pi^* F) \to (P^k_F, F^k)\,,$$

which is a local diffeomorphism of the leaves. Here $\mathtt{t}_a$, $a \in \mathbb{R}^q$, *stands for the translation* $z \to z + a$.

Furthermore, *for any map* $f : M' \to M$ *transverse to* F,

(5.1) $$f^{(k)} \pitchfork F^k, \quad f^{(k)*} F^k = (f^* F)^k \quad \textit{on } P^k_{f^*F}\,.$$

The foliation F^k of P^k_F will be referred to as the *k-th prolongation* of F.

Proof. The submersions $\varphi^{(k)}$, $\varphi \in \mathrm{Subm}(F)$, define a foliation of the kth order normal bundle, for every relationship of the form $\psi = \gamma \circ \varphi$ between elements φ and ψ of $\mathrm{Subm}(F)$ implies $\psi^{(k)} = \gamma^{(k)} \circ \varphi^{(k)}$, where $\gamma^{(k)}$ is a local automorphism of the frame bundle.

If $f \pitchfork F$ then $\varphi \circ f \in \mathrm{Subm}(f^* F)$ for every $\varphi \in \mathrm{Subm}(F)$. As $(\varphi \circ f)^{(k)} = \varphi^{(k)} \circ f^{(k)}$, the last assertion of the lemma follows from Proposition 1.1(i).

In order to prove that $J^k : E_F \to P^k_F$ preserves the foliations it suffices to restrict the map to any section, $U \ni x \xrightarrow{\widetilde{\varphi}} [\varphi, x] \in E_F$, where $\varphi : M \supset U \to \mathbb{R}^q$ is any element of $\mathrm{Subm}(F)$ (we recall that $\pi : E_F \to M$ is a local diffeomorphism). Clearly,

$$\varphi^{(k)} \circ J^k([\varphi, x]) = j^k_{\varphi(x)} \mathtt{t}_{-\varphi(x)} \quad \text{for } x \in U\,,$$

which is locally constant on the leaves of F; by Proposition 1.1(ii), we get

$$J^k : (\widetilde{\varphi}(U), \pi^* F | \widetilde{\varphi}(U)) \to (P^k_F, F^k)\,.$$

As φ was arbitrary, and the leaves of both $\pi^* F$ and F^k are coverings of leaves of F, this proves the lemma. ■

DEFINITION 5.2. For $k \geq 1$ and a q-codimensional foliation $F = (F_n)$ of an ss-manifold X the *k-th order normal* $\mathrm{GL}_k(q)$-*bundle* for F is the sequence $P^k_F = (P^k_{F_n})$ of the respective normal bundles endowed with the structure operators $\varepsilon_i = \varepsilon^{(k)}_i$, $\eta_i = \eta^{(k)}_i$, and with the canonical bundle projection, to be denoted by $\pi^k : P^k_F \to X$. The prolongations $F^k_n = (F_n)^k$, $n \geq 0$, form the *k-th prolongation* F^k of the foliation.

EXAMPLE 5.3. If the structure operators of an ss-manifold X are local diffeomorphisms, then for each k, X admits a *k-th order frame bundle* $P^k_X = (P^k_{X_n})$, which is the normal bundle for the discrete foliation. The prolongations of F are all discrete.

EXAMPLE 5.4. For every ss-morphism $\mathbf{f} : Y \rightharpoonup X$ transverse to a foliation F of X, and $k \geq 1$, there is a canonical form $(P^k_{\mathbf{f}^* F}, \overline{\mathbf{f}})$ of the pull-back of P^k_F

by $\mathbf{f}$ such that the distinguished lifts of representatives $f \in \mathbf{f}$, $f : Y_{\mathcal{U}} \to X$, are the prolongations

$$(P^k_{\mathbf{f}^*F})_{\pi^{-1}\mathcal{U}}(n) \ni (a_0, \dots, a_n; u) \to f^{(k)}_{n,a_0\dots a_n} u \in P^k_F \tag{5.2}$$

where $f_{n,a_0\dots a_n}$ stands for the appropriate restriction of f_n, $n \geq 0$.

For any closed subgroup $G \subset \mathrm{GL}_k(q)$, $k \geq 1$, a *G-structure* on a foliated manifold (M, F) (in the transverse direction) is an arbitrary G-subbundle $P \subset P^k_F$ tangent to F^k. The restriction $F^k|P$ is then the *horizontal foliation* of P. According to Proposition 1.3, for any map $f : M' \to M$ transverse to F the G-reduction $f^{-1}P := f^{(k)-1}P \subset P^k_{f^*F}$,

$$f^{-1}P = \{j^k_x(\varphi \circ f) \in P^k_{f^*F}; j^k_{f(x)}\varphi \in P\}, \tag{5.3}$$

is tangent to the prolongation of f^*F (cf. (5.1)). This is the *counter-image G-structure* on (M', f^*F).

Definition 5.5. A *G-structure on a foliated ss-manifold* (X, F) ($G \subset \mathrm{GL}_k(q), q = \operatorname{codim} F$) is any principal G-bundle $P = (P_n)$ over X such that

(i) for every n, $P_n \subset P^k_{F_n}$ is a G-structure on (X_n, F_n), and
(ii) the structure operators of P are restrictions of those of P^k_F.

The integer k is the *order* of the G-structure. The horizontal foliations of the levels constitute the *horizontal foliation* of P.

It is readily seen that every G-structure P on (X, F) is completely characterized by $P_0 \subset P^k_{F_0}$ such that $\varepsilon_0^{-1}P_0 = \varepsilon_1^{-1}P_0$ on (X_1, F_1).

Example 5.6. For any groupoid of germs Γ, every G-structure P on $(\mathcal{N}\Gamma, F_\delta)$ comes from a G-structure P_0 on the manifold of units. We have $\alpha^{-1}P_0 = \beta^{-1}P_0$ iff P_0 is invariant under (the prolongation of) the pseudogroup that underlies Γ.

Definition 5.7. For $G \subset \mathrm{GL}_k(q)$, a *G-foliation* on an ss-manifold X is any pair (F, P) composed of a q-codimensional foliation F of X and a G-structure P on (X, F).

Usually F itself is called a G-foliation if (X, F) admits any G-structure.

The class of G-structures is closed—as we shall see later in this section—with respect to pull-backs (counter-images) by ss-morphisms transverse to the foliations. In fact, the notion turns out to be equivalent to that of Γ_G-structure, for an appropriate groupoid of germs Γ_G. On the other hand, considering arbitrary ss-morphisms of foliated ss-manifolds as well as arbitrary modifications of the structure group requires more general objects called foliated G-bundles. This notion is originally due to P. Molino [19].

Recall that for G a Lie group, a foliated manifold $(P, \overline{F})$ is a *foliated G-bundle* over another foliated manifold (M, F) if P is a principal G-bundle over M, G acts on P via automorphisms of $(P, \overline{F})$, and the bundle projection $\pi : P \to M$ induces a covering map $\pi' : P^{\bar{F}} \to M^F$. These properties of $\overline{F}$ justify the name *flat partial connection* for its tangent bundle $T\overline{F} \subset TP$. A connection ω in P is *adapted* to $\overline{F}$ if it extends $T\overline{F}$ over the whole of TM, i.e. if the leaves of $\overline{F}$ are horizontal. For M paracompact adapted connections always exist.

Definition 5.8. A *foliated G-bundle over a foliated ss-manifold* (X, F) is any foliated ss-manifold $(P, \overline{F})$ such that

(i) P is a principal G-bundle over X, and
(ii) for every n, $(P_n, \overline{F}_n)$ is a foliated G-bundle over (X_n, F_n).

If $\operatorname{codim} F = 0$, $(P, \overline{F})$ is also called a *flat G-bundle* over X.

Example 5.9. Every G-structure P on (X, F) together with its horizontal foliation is a foliated G-bundle. The restriction of P to any leaf L of F is a flat G-bundle.

The next theorem illustrates the naturality of the examined notion.

Theorem 5.10. *Let $(P, \overline{F})$ be a foliated G-bundle over a foliated ss-manifold (X, F).*

(i) *Given any ss-morphism of foliated ss-manifolds $\mathbf{f} : (Y, F') \rightharpoonup (X, F)$, for every pull-back $(P', \overline{\mathbf{f}})$ of P by $\mathbf{f}$ there exists exactly one foliation $\overline{F}'$ of P' which makes P' a foliated G-bundle over (Y, F') such that $\overline{\mathbf{f}} : (P', \overline{F}') \rightharpoonup (P, \overline{F})$. If, in particular, $\mathbf{f} \pitchfork F$ and $F' = \mathbf{f}^* F$, then $\overline{\mathbf{f}} \pitchfork \overline{F}$ and $\overline{F}' = \overline{\mathbf{f}}^* \overline{F}$.*

(ii) *For any homomorphism of Lie groups $h : G \to G'$ and every h-homomorphism $\overline{h} : P \to P'$ of P to a principal G'-bundle P' over X, there exists exactly one foliation $\overline{F}'$ of P' such that $(P', \overline{F}')$ is a foliated G'-bundle over (X, F), and $\overline{h} : (P, \overline{F}) \to (P', \overline{F}')$.*

Definition 5.11. Under the hypothesis of Theorem 5.10(i), $(P', \overline{F}')$ will be called a *relative pull-back foliated G-bundle* of $(P, \overline{F})$ by $\mathbf{f}$ with respect to F'.

As P' is defined up to isomorphism, so is the pull-back $(P', \overline{F}')$. If $F' = \mathbf{f}^* F$, the adjective "relative" is superfluous.

We shall first prove the following particular case of the theorem.

Lemma 5.12 [1]. (i) *Let $f : (M', F') \to (M, F)$ be a map of foliated manifolds covered by a homomorphism $\bar{f} : P' \to P$ of principal G-bundles. For any foliated bundle structure $\overline{F}$ on P the subbundle*

$$T\overline{F}' := \pi_*^{-1} TF' \cap \bar{f}_*{}^{-1} T\overline{F} \subset TP' \tag{5.4}$$

defines a unique foliation $\overline{F}'$ of P' such that $(P', \overline{F}')$ is a foliated G-bundle and $\bar{f} : (P', \overline{F}') \to (P, \overline{F})$. If ω is any adapted connection for $(P, \overline{F})$ then its *pull-back by* $\bar f$ *is an adapted connection for* $(P', \overline{F}')$. *Furthermore,* $f \pitchfork F$ *and* $F' = f^*F$ *imply* $\bar f \pitchfork \overline{F}$ *and* $\overline{F}' = \bar f^* \overline{F}$.

(ii) *Let* $(P, \overline{F})$ *be a foliated* G*-bundle over a foliated manifold* (M, F). *For any Lie group homomorphism* $h : G \to G'$ *and every* h*-homomorphism* $\overline{h}$ *of* P *to a principal* G'*-bundle* P' *over* M, $\overline{h}$ *transfers* $\overline{F}$ *to a unique foliation* $\overline{F}'$ *of* P' *such that* $(P', \overline{F}')$ *is a foliated* G'*-bundle.* $\overline{F}'$ *is characterized by its tangent bundle,*

$$T_{\overline{h}(u)}\overline{F}' = \overline{h}_*(T_u\overline{F}) \tag{5.5.}$$

for $u \in P$. *If* ω *is any adapted connection for* $(P, \overline{F})$, *then there is an adapted connection* ω' *for* $(P', \overline{F}')$ *such that* $\overline{h}^*\omega' = h_* \circ \omega$.

P r o o f. The assertions are local, so we may, and do, assume the base manifolds to be paracompact.

(i) If $(P, \overline{F})$ is a foliated bundle over (M, F), and $\omega \in A^1(P) \otimes \mathfrak{g}$ ($\mathfrak{g}$ the Lie algebra of G) is any adapted connection, so that $T\overline{F} \subset \ker\omega$, then [17], Proposition II.6.2, ensures that the pull-back $\omega' = \bar f^*\omega$ is a connection in P', and $\bar f_*{}^{-1}T\overline{F} \subset \ker\omega'$. In particular, for every $u \in P'$ the subspace $(\bar f_{*u})^{-1}T_{\bar f(u)}\overline{F} \subset T_uP'$ contains no vertical vectors; we claim that the projection of this subspace to $T_{\pi u}M'$ covers $T_{\pi u}F'$, i.e.

$$T_{\pi u}F' \subset \pi_*(\bar f_{*u})^{-1}T_{\bar f(u)}\overline{F}\,. \tag{5.6}$$

Indeed, for every $v \in T_{\pi u}F'$ the horizontal lift $w \in T_{\bar f(u)}\overline{F}$ of $f_*v \in T_{f(\pi u)}F$ is (f, π)-related to v (i.e. $\pi_*w = f_*v$). As P' is isomorphic to the fibre product $M' \times_{(f,\pi)} P$, the pair (v, w) gives rise to a unique vector $w' \in T_uP'$ such that $v = \pi_*w'$ and $\bar f_*w' = w \in T_{\bar f(u)}\overline{F}$.

In view of (5.6), $T\overline{F}'$, as defined in (5.4), equals $\pi_*^{-1}TF' \cap \ker\omega'$ and is therefore a smooth vector bundle; since

$$\bar f_*(\pi_*^{-1}TF') \subset \pi_*^{-1}f_*TF' \subset \pi_*^{-1}TF$$

the flatness of ω over TF—which is equivalent to the involutivity of $T\overline{F}$—implies the flatness of ω' over TF'. Hence (5.4) does define the required foliation. Its uniqueness is an immediate consequence of the fact that the requirements for $\overline{F}'$ imply $\pi_*T\overline{F}' \subset TF'$, $\bar f_*T|\overline{F}' \subset T\overline{F}$.

If $f \pitchfork F$ and $w \in T_{\bar f(u)}P$, then there is a decomposition $\pi_*w = f_*v' + v$, where $v' \in T_{\pi u}M'$, $v \in T_{f(\pi u)}F$. Taking the horizontal lift $\overline{v} \in T_{\bar f(u)}\overline{F}$ of v we get $\pi_*(w - v) = f_*v'$, from which it follows that there is a $w' \in T_uP'$ such that $\bar f_*w' = w - v$. Hence we conclude

$$w = \bar f_*w' + \overline{v} \in \bar f_*T_uP' + T_{\bar f(u)}\overline{F}\,.$$

The tangent bundle $T\bar f^*\overline{F} = \bar f_*^{-1}T\overline{F}$ equals $T\overline{F}'$, for it is contained in $\pi_*^{-1}TF' = \pi_*^{-1}f_*^{-1}TF$.

(ii) Let $\omega \in A^1(P)\otimes\mathfrak{g}$, $T\overline{F} \subset \ker\omega$, be any adapted connection in P, and ω' its extension to P' such that $\overline{h}^*\omega' = h_* \circ \omega$ (cf. [17], Proposition II.6.1). Then

$$T\overline{F}' := \ker\omega' \cap \pi_*^{-1}TF$$

is a smooth vector bundle such that (5.5) is satisfied. Since the curvature forms Ω and Ω' of the two connections are related via $\overline{h}^*\Omega' = h_* \circ \Omega$, Ω' annihilates the bundle $T\overline{F}'$ over the image of $\overline{h}$; as $T\overline{F}'$ is G'-invariant, we get $\Omega'|T\overline{F}' = 0$, which yields the involutivity of TF'. The uniqueness of $\overline{F}'$ follows from the fact that the requirements for $\overline{F}'$ imply $T\overline{F}' \subset \ker\omega'$, $\pi_*T\overline{F} \subset TF$. ∎

Proof of Theorem 5.10. (i) Assume first that $\mathbf{f}$ is represented by an ss-map $f : Y \to X$. Any distinguished lift $\bar{f}$ of f is a sequence of homomorphisms $\bar{f}_n : P'_n \to P_n$ of principal G-bundles. Since, for each $n \geq 0$, $\bar{f}_n$ projects to $f_n : (Y_n, F'_n) \to (X_n, F_n)$, Lemma 5.12(i) yields a sequence of foliated G-bundles $(P'_n, \overline{F}'_n)$ over (Y_n, F'_n). By (5.4), the face operators of P' are maps of foliated manifolds,

$$\varepsilon_i : (P'_n, \overline{F}'_n) \to (P'_{n-1}, \overline{F}'_{n-1}),$$

and, therefore, $\overline{F}'_n = \varepsilon_i^*\overline{F}'_{n-1}$ for $i \leq n$, $n = 1, 2, \ldots$

Passing to the general case we see that for any representative $f : Y_{\mathcal{U}} \to X$ of $\mathbf{f}$ and any distinguished lift $\bar{f}$ of f there is a unique foliated G-bundle $(P', \overline{F}')$ over (Y, F') such that

$$\bar{f} : (P'_{\pi^{-1}\mathcal{U}}, \overline{F}'_{\pi^{-1}\mathcal{U}}) \to (P, \overline{F})$$

(cf. Lemma 1.10). Equivalent representatives of $\mathbf{f}$ and their distinguished lifts are easily seen to determine the same foliation of P'. Furthermore, $\mathbf{f} \pitchfork F$ iff $f \pitchfork F$, and $\overline{\mathbf{f}} \pitchfork \overline{F}$ iff $\bar{f} \pitchfork \overline{F}$; if this is the case then

$$\overline{F}'_{\pi^{-1}\mathcal{U}} = \bar{f}^*\overline{F} = (\overline{\mathbf{f}}^*\overline{F})_{\pi^{-1}\mathcal{U}}.$$

(ii) According to Lemma 5.12(ii), there is a unique sequence of foliations $\overline{F}'_n$ on P'_n, $n \geq 0$, such that, for each n, $(P'_n, \overline{F}'_n)$ is a foliated G-bundle over (X_n, F_n), and $\overline{h}_{n*}T\overline{F}_n \subset T\overline{F}'_n$. For $n \geq 1$ and $i \leq n$ the pull-back foliation $\varepsilon_i^*\overline{F}'_{n-1}$ makes P'_n a foliated bundle such that

$$\overline{h}_n : (P_n, \overline{F}_n) \to (P'_n, \varepsilon_i^*\overline{F}'_{n-1}).$$

By uniqueness, $\varepsilon_i^*\overline{F}'_{n-1}$ equals $\overline{F}'_n$; hence $\overline{F}' = (\overline{F}'_n)$ is the required foliation of P'. ∎

For a Lie group G and an integer $q \geq 0$ let $\pi : \underline{C}^\infty(\mathbb{R}^q, G) \to \mathbb{R}^q$ be the sheaf of germs of smooth mappings of $\mathbb{R}^q$ to G. The fibre product

$$\Gamma_{q,G} = \Gamma_q \times_{(\alpha,\pi)} \underline{C}^\infty(\mathbb{R}^q, G)$$

equipped with the product operation

$$([\gamma',x'],[\psi',x'])\cdot([\gamma,x],[\psi,x]) = ([\gamma'\circ\gamma,x],[(\psi'\circ\gamma)\psi,x])$$

for $x'=\gamma(x)$ is a differentiable groupoid whose source and target maps $\alpha,\beta : \Gamma_{q,G}\to\mathbb{R}^q$ are local diffeomorphisms. In order to construct a *universal foliated G-bundle* over $(\mathcal{N}\Gamma_{q,G},F_\delta)$, we consider a groupoid of germs over $\mathbb{R}^q\times G$, $\overline{\Gamma}_{q,G}\subset\Gamma_{\mathbb{R}^q\times G}$, associated with the pseudogroup of local automorphisms of the trivial G-bundle $\mathbb{R}^q\times G\to\mathbb{R}^q$, i.e. diffeomorphisms of the form

$$\mathbb{R}^q\times G\ni(x,g)\xrightarrow{(\gamma,\psi)}(\gamma(x),\psi(x)g)\in\mathbb{R}^q\times G\,.$$

G acts on the nerve $\mathcal{N}\overline{\Gamma}_{q,G}$ by translating the initial points of the germs:

$$[(\gamma,\psi),(x,g)]\cdot g' = [(\gamma,\psi),(x,gg')]\,,$$

and there is a projection

$$\overline{\Gamma}_{q,G}\ni[(\gamma,\psi),(x,g)]\to([\gamma,x],[\psi,x])\in\Gamma_{q,G}$$

which makes $\mathcal{N}\overline{\Gamma}_{q,G}$ a principal G-bundle over $\mathcal{N}\Gamma_{q,G}$ (foliated by the discrete foliation).

THEOREM 5.13. *For any foliated G-bundle $(P,\overline{F})$ over a q-codimensional foliated manifold (X,F) there is exactly one ss-morphism $\mathbf{f}_{(P,\bar F)} : X \rightharpoonup \mathcal{N}\Gamma_{q,G}$ which is transverse to the discrete foliation and pulls the universal foliated G-bundle $(\mathcal{N}\overline{\Gamma}_{q,G},F_\delta)$ back to $(P,\overline{F})$.*

We start with the following important

LEMMA 5.14. *For every foliated bundle $(P,\overline{F})$ over a foliated manifold (M,F) there exists a local section through any point of P, continuous in the foliation topologies.*

Proof. Let $U\subset M$ be any open subset such that $(U,F|U)\cong\mathbb{R}^{n-q}\times(\mathbb{R}^q,\delta)$. We fix a transversal $T\cong\mathbb{R}^q\hookrightarrow U$ which meets every leaf in exactly one point, and lift it to a $\widetilde{T}\hookrightarrow P$. Since the leaves of $\overline{F}$ which pass through $\widetilde{T}$ are coverings of those leaves of F which meet T, their restrictions to $\pi^{-1}(U)\subset P$ extend $\widetilde{T}$ to the desired section $\sigma : U\to P$. The smoothness of σ is a consequence of the standard fact that solutions to a smooth first order differential equation depend smoothly on the initial conditions. ■

Proof of Theorem 5.13. Choose sections $s_a : U_a\to P_0$ as in Lemma 5.14, and submersions $\varphi_a : U_a\to\mathbb{R}^q$ in $\mathrm{Subm}(F_0)$, both over a covering $\mathcal{U}=(U_a)_{a\in A}$ of X_0. As the sections send leaves to leaves so do the induced local sections of P_1, and the associated G-cocycle consists of functions which are locally constant on the leaves of F_1. Consequently, the basic data, (s_a) and (φ_a), give rise to a $\Gamma_{q,G}$-cocycle,

$$\varepsilon_1^{-1}U_a\cap\varepsilon_0^{-1}U_b\ni x\to([\gamma_{ab}^{(x)},\varphi_b\varepsilon_0 x],[g_{ab}^{(x)},\varphi_b\varepsilon_0 x])\in\Gamma_{q,G}$$

such that

$$\gamma_{ab}^{(x)} \circ (\varphi_b \varepsilon_0) = \varphi_a \varepsilon_1 \,,$$
$$(\pi, \varepsilon_0)^{-1}(\mathrm{id}, s_b \varepsilon_0) = (\pi, \varepsilon_1)^{-1}(\mathrm{id}, s_a \varepsilon_1) \cdot (g_{ab}^{(x)} \cdot \varphi_b \varepsilon_0)$$

in a neighbourhood of each x. Clearly, the morphism $\mathbf{f}_{(P,\bar{F})}$ represented by the corresponding ss-map $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma_{q,G}$ is independent of the choices involved.

An ss-morphism $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma_{q,G}$ classifies $(P, \overline{F})$ iff there is a pull-back $(P, \overline{\mathbf{f}})$ of $\mathcal{N}\overline{\Gamma}_{q,G}$ by $\mathbf{f}$ such that $\overline{\mathbf{f}}^* F_\delta = \overline{F}$. If this is the case then every representative $f : X_{\mathcal{U}} \to \mathcal{N}\Gamma_{q,G}$ yields a commuting square

$$\begin{array}{ccc} P_{\pi^{-1}\mathcal{U}} & \xrightarrow{\bar{f}} & \mathcal{N}\overline{\Gamma}_{q,G} \\ \downarrow & & \downarrow \\ X_{\mathcal{U}} & \xrightarrow{f} & \mathcal{N}\Gamma_{q,G} \end{array}$$

where $\bar{f} \in \overline{\mathbf{f}}$ is a distinguished lift of f. Denoting by φ_a, $a \in A$, the components of f_0 we get a collection of submersions over $\mathcal{U}$ such that $\overline{f}_0$ establishes an isomorphism of foliated G-bundles,

$$(P_0|U_a, \overline{F}_0|U_a) \cong \varphi_a^*(\mathbb{R}^q \times G, F_\delta)\,,$$

for every a; the local sections $s_a : U_a \to P_0$ that come from

$$U_a \ni x \to (x, (\varphi_a(x), e)) \in \varphi_a^*(\mathbb{R}^q \times G)$$

carry leaves to leaves. It is now easy to see that f is characterized by (s_a) and (φ_a) as in the former part of the proof; hence $\mathbf{f} = \mathbf{f}_{(P,\bar{F})}$.

Reversing the above argument we conclude that $\mathbf{f}_{(P,\bar{F})}$ does classify the foliated G-bundle $(P, \overline{F})$. ∎

Remark 5.15. It follows from the above proof that a principal G-bundle P over an ss-manifold X foliated by F carries a foliated bundle structure over (X, F) iff the classifying ss-morphism $\mathbf{f}_P : X \rightharpoonup \mathcal{N}G$ has a representative f such that every f_n is locally constant on the leaves of F_n. Those representatives which correspond to the same foliated bundle structure are easily seen to form a connected collection with respect to elementary equivalences.

Remark 5.16. For $0 \le q' < q$, an identification $\mathbb{R}^q = \mathbb{R}^{q'} \times \mathbb{R}^{q-q'}$ gives rise to a homomorphism of groupoids,

$$\Gamma_{q',G} \ni ([\gamma, x], [\psi, x]) \xrightarrow{j} ([\gamma \times \mathrm{id}_{\mathbb{R}^{q-q'}}, (x, 0)], [\psi \circ \mathrm{pr}, (x, 0)]) \in \Gamma_{q,G}\,,$$

where $\mathrm{pr} : \mathbb{R}^q \to \mathbb{R}^{q'}$ stands for the projection. The ss-map $\mathcal{N}j$ pulls $\mathcal{N}\overline{\Gamma}_{q,G}$

back to $\mathcal{N}\overline{\Gamma}_{q',G}$, for j admits a G-equivariant lift,

$$\mathcal{N}\overline{\Gamma}_{q',G} \ni [(\gamma,\psi),(x,g)] \xrightarrow{\bar{j}} [(\gamma \times \mathrm{id}_{\mathbb{R}^{q-q'}}, \psi \circ \mathrm{pr}),((x,0),g)] \in \overline{\Gamma}_{q,G}\,;$$

hence $\mathcal{N}j : (\mathcal{N}\Gamma_{q',G}, F_\delta) \to (\mathcal{N}\Gamma_{q,G}, F_\delta)$ pulls back one of the universal foliated G-bundles to the other. In conclusion: every foliated G-bundle over a foliated ss-manifold of codimension $\leq q$ is a relative pull-back of $(\mathcal{N}\overline{\Gamma}_{q,G}, F_\delta)$.

Returning to G-structures, we now specify those particular cases of Theorem 5.10 in which the new foliated bundles still remain G-structures.

PROPOSITION 5.17. (i) *There is a natural counter-image operation such that for any G-structure P on a foliated ss-manifold (X,F), and every ss-morphism $\mathbf{f} : Y \rightharpoonup X$ transverse to F, the counter-image of P by $\mathbf{f}$ is a G-structure $\mathbf{f}^{-1}P$ on $(Y,\mathbf{f}^*F)$; the naturality $\mathbf{g}^{-1}\mathbf{f}^{-1}P = (\mathbf{f}\circ\mathbf{g})^{-1}P$ together with the requirement that*

$$[f]^{-1}P = (f_n^{-1}P_n) \tag{5.7}$$

for any ss-map $f : Y \to X$ transverse to the base foliation determine the pull-back operation in a unique manner. For any P and $\mathbf{f}$ the G-structure $\mathbf{f}^{-1}P$ together with the appropriate prolongation $\overline{\mathbf{f}}$ of $\mathbf{f}$ and with the horizontal foliation is a pull-back of the foliated G-bundle P.

(ii) *For any closed subgroups $G \subset \mathrm{GL}_k(q)$ and $\overline{G} \subset \mathrm{GL}_{\bar{k}}(q)$ such that $\pi^k_{\bar{k}}G \subset \overline{G}$, $k \geq \bar{k}$, and every G-structure P on a foliated ss-manifold (X,F), there is a natural $\overline{G}$-extension of P, i.e. a $\overline{G}$-structure $P \cdot \overline{G}$ on (X,F) such that every level $P_n \cdot \overline{G}$ of $P \cdot \overline{G}$ is the unique $\overline{G}$-structure containing $\pi^k_{\bar{k}}P_n \subset P^{\bar{k}}_{F_n}$.*

The two operations on G-structures described in (i) *and* (ii) *commute with each other.*

Proof. (i) If $f : Y_{\mathcal{U}} \to X$ is a representative of an ss-morphism $\mathbf{f} : Y \rightharpoonup X$ transverse to a foliation F of X, then for any G-structure P on (X,F), (5.7) defines a G-structure on $(Y_{\mathcal{U}}, (\mathbf{f}^*F)_{\mathcal{U}})$. Now a reasoning completely analogous to our proof of Lemma 1.10 ensures that the G-structure on the localization of $(Y,\mathbf{f}^*F)$ is a counter-image—by $[\lambda] : Y_{\mathcal{U}} \rightharpoonup Y$—of a global G-structure P' on $(Y,\mathbf{f}^*F)$. Since P' is uniquely characterized by $[\lambda]^{-1}P' = [f]^{-1}P$, it is independent of the representative of $\mathbf{f}$. On the other hand, the representative admits a natural lift,

$$\bar{f} : P'_{\pi^{-1}\mathcal{U}} \cong [\lambda]^{-1}P' = [f]^{-1}P \xrightarrow{f^{(k)}} P\,;$$

since the collection $\overline{\mathbf{f}}$ of all the lifts is connected with respect to elementary equivalences, $(P',\overline{\mathbf{f}})$ is a pull-back of P by $\mathbf{f}$. Being restrictions of the prolongations (5.2), the lifts preserve the respective horizontal foliations.

(ii) Observe first that for every $n \geq 0$ the subspace

$$P_n \cdot \overline{G} := \{j_x^{\bar{k}}\psi \cdot \overline{g} \in P_{F_n}^{\bar{k}};\ j_x^k\psi \in P_n,\ \overline{g} \in \overline{G}\}$$

is a $\overline{G}$-structure on (X_n, F_n). Indeed, any submersion $\varphi : U \to \mathbb{R}^q$ in $\mathrm{Subm}(F_n)$ gives rise to a simultaneous trivialization of all the normal bundles,

$$U \times \mathrm{GL}_h(q) \ni (x, j_0^h\gamma) \to j_x^h(\gamma^{-1}\mathtt{t}_{-\varphi(x)}\varphi) \in P_{F_n}^h|U\,,$$

$h = 1, 2, \dots$; in terms of the trivializations, the order projection $\pi_{\bar{k}}^k$ for the bundles reduces to the projection for the higher order linear groups. Consequently, if the G-subbundle $P_n|U \subset P_{F_n}^k|U$ is characterized by a section $U \to P_{F_n}^k$,

$$U \ni x \xrightarrow{\sigma} (x, g(x)) \in U \times \mathrm{GL}_k(q)\,,$$

then its image in $P_{F_n}^{\bar{k}}|U$ is described by

$$U \ni x \xrightarrow{\sigma'} (x, \pi_{\bar{k}}^k g(x)) \in U \times \mathrm{GL}_{\bar{k}}(q)\,,$$

which is a section over U of a $\overline{G}$-subbundle. If, moreover, the component g of σ is chosen locally constant on the leaves of F_n (cf. Lemma 5.14) then the same is true for $\pi_{\bar{k}}^k g$ and σ'. This means that the projection $P_n \to P_n \cdot \overline{G}$ sends $T_{\sigma(x)}F_n^k$ onto $T_{\sigma'(x)}F_n^{\bar{k}}$, for $x \in U$. By group-invariance of the prolongations of F_n, we conclude that $P_n \cdot \overline{G}$ is tangent to $F_n^{\bar{k}}$.

By construction of the levels, $P \cdot \overline{G} := (P_n \cdot \overline{G})$ is a $\overline{G}$-structure on (X, F). The last assertion of the proposition is obvious. ■

Before we formulate a classification theorem for G-structures, let us recall that a kth order G-structure P on a foliated manifold (M, F) is called *integrable* if for every $x \in M$ there is an *integrating submersion* $\varphi : U \to \mathbb{R}^q$ about x, i.e. a $\varphi \in \mathrm{Subm}(F)$ such that the map

$$j^k\varphi : U \ni y \to j_y^k(\mathtt{t}_{-\varphi(y)}\varphi) \in P_F^k$$

is a section of P, $\mathtt{t}_a$ being the translation $z \to z + a$ in $\mathbb{R}^q$. As integrability is preserved by counter-images by maps transverse to the foliations, for any G-structure $P = (P_n)$ on a foliated ss-manifold the integrability of P_0 is inherited by the other levels.

DEFINITION 5.18. A G-structure P on a foliated ss-manifold (X, F) is *integrable* if each P_n is integrable; P is *$\overline{G}$-integrable* if its $\overline{G}$-extension $P \cdot \overline{G}$ is integrable. If this is the case, the G-foliation (F, P) will be called *integrable* (resp., *$\overline{G}$-integrable*).

PROPOSITION 5.19. *If P is a $\overline{G}$-integrable G-structure on a foliated ss-manifold (X, F), and $\mathbf{f} : Y \rightharpoonup X$ an ss-morphism transverse to F, then the counter-image $\mathbf{f}^{-1}P$ is a $\overline{G}$-integrable G-structure on $(Y, \mathbf{f}^*F)$.*

P r o o f. According to Proposition 5.17, the $\overline{G}$-extension $(\mathbf{f}^{-1}P)\cdot\overline{G} = \mathbf{f}^{-1}(P\cdot\overline{G})$ inherits integrability from $P\cdot\overline{G}$. ∎

The following classification theorem asserts that $\overline{G}$-integrable G-foliations are in fact Γ-foliations. For arbitrary G-foliations (the case $\overline{G} = \mathrm{GL}(q)$) the construction of a universal G-structure is due to T. E. Duchamp [9].

THEOREM 5.20. *Let $G \subset \mathrm{GL}_k(q)$ and $\overline{G} \subset \mathrm{GL}_{\bar{k}}(q)$, $k \geq \overline{k}$, be closed subgroups such that G projects into $\overline{G}$. There exist a groupoid of germs $\Gamma_{\bar{G},G}$ and a universal G-structure $P_{\bar{G},G}$ on $(\mathcal{N}\Gamma_{\bar{G},G}, F_\delta)$ such that*

(i) *the $\overline{G}$-extension of $P_{\bar{G},G}$ is integrable*, *and*

(ii) *for every $\overline{G}$-integrable G-structure P on a foliated ss-manifold (X,F) there is a unique ss-morphism $\mathbf{f}_P : X \rightharpoonup \mathcal{N}\Gamma_{\bar{G},G}$ transverse to the discrete foliation such that $\mathbf{f}^*_P F_\delta = F$ and $\mathbf{f}^{-1}_P P_{\bar{G},G} = P$ (the* classifying ss-morphism *for P).*

COROLLARY 5.21. *Given a minimal transverse projection $\Pi_F : X \rightharpoonup \mathcal{N}\Gamma_F$ for a foliated ss-manifold (X,F), every G-structure P on (X,F) projects to a unique G-structure $\widetilde{P}$ on $(\mathcal{N}\Gamma_F, F_\delta)$ such that $P = \Pi_F^{-1}\widetilde{P}$.*

$\widetilde{P}$ is $\overline{G}$-integrable iff so is P.

P r o o f. If $\mathbf{f}_P$ classifies P as a G-structure, then the unique factorization $\mathbf{f}_P = \mathbf{f}\circ\Pi_F$ (cf. Theorem 3.2(i)) is equivalent to $P = \Pi_F^{-1}(\mathbf{f}^{-1}P_{\bar{G},G})$ where $\overline{G} = \mathrm{GL}(q)$. For $\overline{G}$ arbitrary, the above equality yields the second assertion. ∎

R e m a r k 5.22. By a careful examination of the prolonged ss-morphism $\overline{\Pi}_F : P \rightharpoonup \widetilde{P} \cong \mathcal{N}(\Gamma_F \times_{(\alpha,\pi)} \widetilde{P}_0)$ (cf. I(3.2)), it can be shown that the groupoid $\Gamma_F \times_{(\alpha,\pi)} \widetilde{P}_0$ is a groupoid of germs of local automorphisms of $\widetilde{P}_0$, and that $\overline{\Pi}_F$ is in fact a *minimal transverse projection* for the horizontal foliation of P. We shall not need this result in the sequel, so the details are left to the reader.

In order to prove Theorem 5.20, we shall construct canonical candidates for both $\Gamma_{\bar{G},G}$ and $P_{\bar{G},G}$. To this end we let $N(\overline{G},G)$ be the sheaf of germs of smooth maps $\mathbb{R}^q \to (\pi^k_{\bar{k}})^{-1}\overline{G}/G$; the projection $\pi : N(\overline{G},G) \to \mathbb{R}^q$ induces on it a natural structure of a q-dimensional (non-Hausdorff, in general) differentiable manifold. The manifold carries a canonical G-structure $P_{\bar{G},G}(0)$ such that above each $\sigma = [s,x] \in N(\overline{G},G)$, we have

$$(5.8)\qquad j^k_\sigma(\mathrm{t}_{-x}\pi)\overline{g} \in P_{\bar{G},G}(0) \quad \text{iff} \quad \pi^k_{\bar{k}}\overline{g} \in \overline{G} \text{ and } \overline{g}G = s(x)\,.$$

Clearly, the $\overline{G}$-extension of $P_{\bar{G},G}(0)$ is nothing but the counter-image by π

of the *flat* $\overline{G}$*–structure*

$$P_{\bar G}(0) := \{ j^{\bar k}_x \varphi \in P^{\bar k}_{\mathbb{R}^q};\ j^{\bar k}_0(\varphi \mathrm{t}_x) \in \overline{G} \} . \tag{5.8$'$}$$

In order to obtain $P_{\bar G,G} = (P_{\bar G,G}(n))_{n\geq 0}$ it now suffices (cf. Example 5.6) to define $\Gamma_{\bar G,G}$ to be the groupoid of germs of all automorphisms of the G-structure $P_{\bar G,G}(0)$. The groupoid can be identified (via π) with the fibre product

$$\Gamma_{\bar G,G} = \Gamma_{\bar G} \times_{(\alpha,\pi)} N(\overline{G}, G)$$

where $\Gamma_{\bar G} \subset \Gamma_q$ is the groupoid of germs of local automorphisms of $P_{\bar G}(0)$. The product operation in $\Gamma_{\bar G,G}$,

$$(g', \sigma')(g, \sigma) = (g'g, \sigma) \quad \text{iff} \quad \sigma' = g\sigma$$

for $(g', \sigma'), (g, \sigma) \in \Gamma_{\bar G,G}$, is an extension of a natural $\Gamma_{\bar G}$-action on $N(\overline{G}, G)$,

$$[\gamma, x][s, x] = [(D^k\gamma \cdot s) \circ \gamma^{-1}, \gamma(x)] \qquad \text{for } [\gamma, x] \in \Gamma_{\bar G}\,, \tag{5.9}$$

where $D^k\gamma$ stands for the map

$$\mathrm{Domain}\,\gamma \ni y \to j^k_0(\mathsf{t}_{-\gamma(y)} \circ \gamma \circ \mathsf{t}_y) \in (\pi^k_{\bar k})^{-1}\overline{G}\,.$$

Definition 5.23. $\Gamma_{\bar G,G}$ is the *classifying groupoid for* $\overline{G}$*-integrable* G*-structures*. In particular, $\Gamma_G := \Gamma_{G,G}$ is the *classifying groupoid for integrable* G*-structures*, and $\widetilde{\Gamma}_G := \Gamma_{\mathrm{GL}(q),G}$—*for* (*arbitrary*) G*-structures*.

Proof of Theorem 5.20. Let P be a $\overline{G}$-integrable G-structure on a foliated manifold (M, F). Every integrating submersion $\varphi : M \supset U \to \mathbb{R}^q$ for $P \cdot \overline{G}$ can be naturally lifted to a submersion $\widehat{\varphi} : U \to N(\overline{G}, G)$ such that

$$P|U = \widehat{\varphi}^{-1} P_{\bar G,G}(0)\,. \tag{5.10}$$

Indeed, there is a well defined map $s_\varphi : U \to (\pi^k_{\bar k})^{-1}\overline{G}/G$ such that

$$s_\varphi(x) = \overline{g}G \quad \text{iff} \quad j^k_x(\mathsf{t}_{-\varphi(x)}\varphi)\overline{g} \in P\,. \tag{5.11}$$

As P is a union of leaves of F^k, s_φ is locally constant on the leaves of F. In order to define $\widehat{\varphi}$ we pick a local section of φ through an arbitrary $x \in U$, $i^{(x)} : \mathbb{R}^q \supset V \to U$, and set

$$\widehat{\varphi}(x) = [s_\varphi \circ i^{(x)}, \varphi(x)]$$

for $x \in U$. For every jet $u = j^k_x(\mathsf{t}_{-\varphi(x)}\varphi)\overline{g}$ in $P|U$,

$$\widehat{\varphi}^{(k)}(u) = \widehat{\varphi}^{(k)} j^k_x(\mathsf{t}_{-\varphi(x)}\pi \circ \widehat{\varphi})\overline{g} = j^k_{\widehat{\varphi}(x)}(\mathsf{t}_{-\varphi(x)}\pi)\overline{g} \in P_{\bar G,G}(0)$$

by (5.8). The resulting inclusion $\widehat{\varphi}^{(k)}(P|U) \subset P_{\bar G,G}(0)$ proves (5.10).

One easily verifies that if φ and $\psi = \chi \circ \varphi$ are any two integrating submersions for $P \cdot \overline{G}$ then χ is locally in the pseudogroup underlying $\Gamma_{\bar G}$, and

$$\widehat{\psi}(x) = [\chi, \varphi(x)]\widehat{\varphi}(x) \tag{5.12}$$

for every x (cf. (5.9)).

In addition to (5.10), we claim that for any submersion $f : M \supset U \to N(\overline{G}, G)$,

(5.13) $$f = (\pi \circ f)^\wedge \quad \text{if} \quad f^{-1}P_{\bar{G},G}(0) = P|U\,.$$

To prove this, observe that the equality

$$P \cdot \overline{G}|U = f^{-1}(P_{\bar{G},G}(0) \cdot \overline{G}) = f^{-1}\pi^{-1}P_{\bar{G}}(0)$$

ensures that $\varphi := \pi \circ f$ is an integrating submersion for $P \cdot \overline{G}$. As (5.8) and (5.11) imply that s_φ is the composition of f and the evaluation map $N(\overline{G}, G) \to (\pi^k_{\bar{k}})^{-1}\overline{G}/G$, we conclude that $\widehat{\varphi} = f$.

Let now $P = (P_n)$ be a $\overline{G}$-integrable G-structure on a foliated ss-manifold (X, F). As shown above, the integrating submersions for $P_0 \cdot \overline{G}$ give rise to a collection of submersions of the form $\widehat{\varphi} : X_0 \supset U \to N(\overline{G}, G)$. According to Theorem 2.11, the collection will generate a transverse projection for F, $\mathbf{f}_P : X \rightharpoonup \mathcal{N}\Gamma_{\bar{G},G}$ (such that $\mathbf{f}_P^{-1}P_{\bar{G},G} = P$, by (5.10)), if we show that for any integrating submersions $\varphi : U \to \mathbb{R}^q$ and $\psi : V \to \mathbb{R}^q$ there exists an automorphism $\widetilde{\chi}$ of $P_{\bar{G},G}(0)$ such that

$$\widehat{\psi} \circ \varepsilon_0 = \widetilde{\chi} \circ \widehat{\varphi} \circ \varepsilon_1$$

locally on $\varepsilon_1^{-1}U \cap \varepsilon_0^{-1}V \subset X_1$. Since the counter-image G-structures are

$$(\widehat{\varphi} \circ \varepsilon_1)^{-1}P_{\bar{G},G}(0) = \varepsilon_1^{-1}(P_0|U) = P_1|\varepsilon_1^{-1}U$$

and

$$(\widehat{\psi} \circ \varepsilon_0)^{-1}P_{\bar{G},G}(0) = P_1|\varepsilon_0^{-1}V$$

(5.13) implies

$$\widehat{\varphi} \circ \varepsilon_1 = (\varphi \circ \varepsilon_1)^\wedge, \qquad \widehat{\psi} \circ \varepsilon_0 = (\psi \circ \varepsilon_0)^\wedge\,.$$

By (5.12), $\widetilde{\chi}$ is the lift (to the pseudogroup underlying $\Gamma_{\bar{G},G}$) of a χ such that $\psi \circ \varepsilon_0 = \chi \circ \varphi \circ \varepsilon_1$.

If $\mathbf{f} : X \rightharpoonup \mathcal{N}\Gamma_{\bar{G},G}$ is any transverse projection for F such that $\mathbf{f}^{-1}P_{\bar{G},G} = P$ then, by (5.13), its distinguished submersions must all be of the form $\widehat{\varphi}$. Hence $\mathbf{f} = \mathbf{f}_P$ is unique. ∎

Remark 5.24. We have concentrated our attention on $\overline{G}$-integrable G-structures, for the class contains both the integrable and non-integrable G-structures, as well as e.g. *complex G-structures* which are—by definition—$\mathrm{GL}(q, \mathbb{C})$-integrable. One can expect that more complicated structures can be classified in a similar manner. For example, *Kähler structures*, i.e. $U(q)$-structures which are both complex and symplectic ($\mathrm{Sp}(q)$-integrable) are classified by a restriction of $\Gamma_{\mathrm{GL}(q,\mathbb{C}),U(q)}$ to an open submanifold of units.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ŁÓDŹ BRANCH
NARUTOWICZA 56
90-136 ŁÓDŹ, POLAND

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