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DISTRIBUTION OF VALUES OF PERMANENT AND STANDARD FORMS OF (0, 1)-MATRICES

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1. Introduction

Let $A = (a_{ij})$ be an $m \times n$ matrix. The *permanent* of A, denoted by $\operatorname{per} A$, is defined as follows:

$$\operatorname{per} A = \begin{cases} \sum\limits_{i_1, \ldots, i_m \in P_m^n} a_{1i_1} a_{2i_2} \ldots a_{mi_m} & \text{if } m \leqslant n, \\ \operatorname{per} A^T & \text{if } m > n, \end{cases}$$

where P_m^n is the set of m-permutations on $\{1, \ldots, n\}$. Being viewed as a function of matrices, the permanent has the following properties:

(1) For any $m \times n$ matrix A, $\operatorname{per} A = \operatorname{per} A^T$.

(2) Permuting the rows and columns of A does not alter its permanent, i.e., for any permutation matrices P and Q of order m and n respectively, we have $\operatorname{per}(PAQ) = \operatorname{per} A$.

In this paper we only investigate $m \times n$ (0, 1)-matrices with $m \leqslant n$. The term "$m \times n$ matrix" will mean $m \times n$ (0, 1)-matrix, $1 \leqslant m \leqslant n$, in the rest of the paper.

Let A and B be two $m \times n$ matrices. They are called *equivalent* (written $A \sim B$) if B can be obtained from A by permuting rows and columns. Evidently, $\sim$ is an equivalence relation. Thus all the $m \times n$ matrices are partitioned into equivalence classes and the permanent of the matrices in an equivalence class has the same value. A representative in each class that has the simplest form is called the *standard form* of the matrices in this class. The problem of finding the standard forms of square matrices with a given value of permanent was first investigated by Gordon *et al.* [1], and later by Li [2].

Before proceeding to study the standard forms of matrices with a given value of permanent, we should first have some knowledge of the values the permanent of matrices can take. This is called the *distribution problem* of the values of permanent. In this paper, we treat first the distribution problem, and then extend the results on the standard forms of square matrices to that of rectangular matrices. We also give the standard forms of matrices for some other cases.

2. Distribution of values of permanent

Let $[x, y]$ denote the set of integers between x and y both inclusive, and $N_{m,n}$ the set of the values of the permanent of $m \times n$ matrices:

$$N_{m,n} = \{\operatorname{per} A \mid A \text{ is an } m \times n \text{ matrix}\}.$$

We also write $N_{n,n}$ as N_n.

The distribution problem of the values of permanent is to determine the set $N_{m,n}$. We observe that the $N_{m,n}$ have the following properties:

(i) $N_m \subseteq N_{m,m+s}$ for $m \geqslant 1$, $s \geqslant 0$;
(ii) $N_{n_1} \subseteq N_{n_2}$ if $n_1 \leqslant n_2$;
(iii) $s! \cdot N_{m,m+s} \subseteq N_{m+s}$ for $m \geqslant 1$, $s \geqslant 0$,

where $s! \cdot N_{m,m+s} = \{s! \cdot x \mid x \in N_{m,m+s}\}$;

(iv) $N_{n_1} \cdot N_{n_2} \subseteq N_{n_1+n_2}$,

where $N_{n_1} \cdot N_{n_2} = \{xy \mid x \in N_{n_1},\ Y \in N_{n_2}\}$.

EXAMPLE 1. It is easy to check that

$N_2 = [0, 2]$, $N_3 = [0, 4] \cup \{6\}$.

$N_{1,2} = [0, 2]$, $N_{2,3} = [0, 4] \cup \{6\}$,

$N_{2,4} = [0, 7] \cup \{9, 12\}$, $N_{2,5} = [0, 10] \cup [12, 13] \cup \{16, 20\}$,

$N_{2,6} = [0, 17] \cup [20, 21] \cup \{25, 30\}$,

$N_{2,7} = [0, 18] \cup [20, 22] \cup [24, 26] \cup [30, 31] \cup \{36, 42\}$.

EXAMPLE 2. Since $\operatorname{per} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = 9$,

$$\operatorname{per} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 10, \quad \operatorname{per} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 11,$$

$$\operatorname{per}\begin{bmatrix}0&0&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}=12,\quad \operatorname{per}\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}=14,$$

$$\operatorname{per}\begin{bmatrix}0&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}=18,\quad \operatorname{per}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}=24,$$

and $[0, 8] \subseteq N_4$ by Theorem 2.1 below, we have

$$N_4 = [0, 12] \cup \{14, 18, 24\}.$$

Generally, for fixed m and n, $N_{m,n}$ contains some maximal segments of consecutive integers. The problem of determining $N_{m,n}$ seems to be hard. We consider here the first maximal segment of $N_{m,n}$.

LEMMA 2.1. *Let* $H_1 = [1]$, $H_2 = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$, *and*

$$H_m = (h_{ij}) = \begin{bmatrix}1&1&1&\dots&1&1\\1&1&1&\dots&1&1\\&1&1&\dots&1&1\\&&\ddots&&\vdots&\vdots\\&&&\ddots&\vdots&\vdots\\&&&&\ddots&\vdots\\&&&&1&1\\&&&&1&1\end{bmatrix},\qquad m \geqslant 3,$$

be an $m \times m$ *matrix whose element* h_{ij} *is* 0 *if and only if* $i > j+1$. *Then*

$$\operatorname{per} H_m = 2^{m-1}, \quad m \geqslant 1. \tag{2.1}$$

Proof. The cases of $m = 1$ and 2 are trivial. Expanding the permanent of H_m by the first column, we have $\operatorname{per} H_m = 2 \operatorname{per} H_{m-1}$. Hence (2.1) follows by induction on m.

THEOREM 2.1. *For any* m, n *with* $1 \leqslant m \leqslant n$,

$$[0, 2^{m-1}] \subseteq N_{m,n}.$$

Proof. Since $N_m \subseteq N_{m,n}$, it is sufficient to prove that

$$[0, 2^{m-1}] \subseteq N_m. \tag{2.2}$$

Let

$$K_m(\delta_1, \delta_2, \ldots, \delta_m) = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ & 1 & \ldots & 1 & 1 \\ & & & \vdots & \vdots \\ & & & 1 & 1 \\ \delta_m & \delta_{m-1} & & \delta_2 & \delta_1 \end{bmatrix}$$

where $\delta_i = 0$ or 1. In particular, $K_1(\delta_1) = (\delta_1)$, $K_2(\delta_1, \delta_2) = \left[\begin{smallmatrix} 1 & 1 \\ \delta_2 & \delta_1 \end{smallmatrix}\right]$. Obviously, $\operatorname{per} K_1(\delta_1) = \delta_1$, $\operatorname{per} K_2(\delta_1, \delta_2) = \delta_2 + \delta_1$.

By expanding $\operatorname{per} K_m(\delta_1, \ldots, \delta_m)$ by the first column it follows from Lemma 2.1 that

$$\begin{aligned} \operatorname{per} K_m(\delta_1, \ldots, \delta_m) &= \delta_m \operatorname{per} H_{m-1} + \operatorname{per} K_{m-1}(\delta_1, \ldots, \delta_{m-1}) \\ &= \delta_m 2^{m-2} + \operatorname{per} K_{m-1}(\delta_1, \ldots, \delta_{m-1}). \end{aligned}$$

By induction on m we have

$$\operatorname{per} K_m(\delta_1, \ldots, \delta_m) = \delta_m \cdot 2^{m-2} + \delta_{m-1} \cdot 2^{m-3} + \ldots + \delta_3 \cdot 2 + \delta_2 + \delta_1. \tag{2.3}$$

When δ_i $(1 \leqslant i \leqslant m)$ take 0 and 1 independently, the value of the right-hand side of (2.3) runs through $[0, 2^{m-1}]$. Hence (2.2). This completes the proof.

The permanent of the matrices with relatively small number of 0's often has relatively large values. For example, when J is a matrix whose all entries are ones, then

$$\operatorname{per} J_{m \times n} = \binom{n}{m} m!, \tag{2.4}$$

$$\begin{aligned} \operatorname{per} \begin{array}{c} \begin{array}{cc} 1 & n-1 \end{array} \\ \begin{bmatrix} 0 & J \\ J & J \end{bmatrix} \end{array} \begin{array}{l} \\ 1 \\ m-1 \end{array} &= \binom{n}{m} m! - \binom{n-1}{m-1}(m-1)! \\ &= (n-1)\binom{n-1}{m-1}(m-1)!, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \operatorname{per} \begin{array}{c} \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & J \\ J & J \end{bmatrix} \end{array} \begin{array}{l} \\ 2 \\ m-2 \end{array} &= \binom{n}{m} m! - 2\binom{n-1}{m-1}(m-1)! + \binom{n-2}{m-2}(m-2)! \\ &= (n^2 - 3n + 3)\binom{n-2}{m-2}(m-2)!, \end{aligned} \tag{2.6}$$

$$(2.7)\qquad \operatorname{per}\begin{array}{cc} \begin{array}{cc} 2 & n-2 \end{array} & \\ \begin{bmatrix} [0\ 0] & J \\ J & J \end{bmatrix} & \begin{array}{l} 1 \\ m-1 \end{array} \end{array} = \binom{n}{m}m! - 2\binom{n-1}{m-1}(m-1)!$$

$$= (n^2-3n+2)\binom{n-2}{m-2}(m-2)!,$$

$$(2.8)\qquad \operatorname{per}\begin{array}{cc} \begin{array}{cc} 3 & n-3 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 3 \\ m-3 \end{array} \end{array} = \binom{n}{m}m! - 3\binom{n-1}{m-1}(m-1)!$$

$$+3\binom{n-2}{m-2}(m-2)! - \binom{n-3}{m-3}(m-3)!$$

$$= (n^3-6n^2+14n-13)\binom{n-3}{m-3}(m-3)!,$$

$$(2.9)\qquad \operatorname{per}\begin{array}{cc} \begin{array}{cc} 3 & n-3 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 2 \\ m-2 \end{array} \end{array} = \binom{n}{m}m! - 3\binom{n-1}{m-1}(m-1)! + 2\binom{n-2}{m-2}(m-2)!$$

$$= (n^3-6n^2+13n-10)\binom{n-3}{m-3}(m-3)!,$$

$$(2.10)\qquad \operatorname{per}\begin{array}{cc} \begin{array}{cc} 2 & n-2 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 2 \\ m-2 \end{array} \end{array} = \binom{n}{m}m! - 3\binom{n-1}{m-1}(m-1)! + \binom{n-2}{m-2}(m-2)!$$

$$= (n^3-6n^2+12n-8)\binom{n-3}{m-3}(m-3)!,$$

$$(2.11)\qquad \operatorname{per}\begin{array}{cc} \begin{array}{cc} 3 & n-3 \end{array} & \\ \begin{bmatrix} [0\ 0\ 0] & J \\ J & J \end{bmatrix} & \begin{array}{l} 1 \\ m-1 \end{array} \end{array} = \binom{n}{m}m! - 3\binom{n-1}{m-1}(m-1)!$$

$$= (n^3-6n^2+11n-6)\binom{n-3}{m-3}(m-3)!,$$

etc. When $m \geqslant 3$, $n \geqslant 4$, the above values of permanent are strictly decreasing from (2.4) to (2.11). And when $m \geqslant 3$, $n \geqslant 5$, no two of them can be consecutive.

We now turn to a related problem. Let $k \leqslant n$ be a given integer and

$$\mathfrak{M}_k = \{A \mid A \text{ is an } m\times n \text{ matrix with } k \text{ 0's}\}.$$

We shall determine the extremal values of permanent of matrices in $\mathfrak{M}_k$.

THEOREM 2.2. $\min\limits_{A\in\mathfrak{M}_k} \operatorname{per} A = \binom{n}{m}m! - k\binom{n-1}{m-1}(m-1)!.$

The minimum is reached by any matrix A whose k 0's are in the same row. When $k \leqslant m$, it is also reached by the matrices in $\mathfrak{M}_k$ whose k 0's are in the same column.

Proof. Let $A \in \mathfrak{M}_k$. There are $\binom{n}{m}m!$ terms in the expansion of $\operatorname{per} A$. Every term is a product of m elements of A which are in different rows and columns. Denoting by P_A the set of terms whose values are 1 and by Q_A the set of terms whose values are 0, we have

$$P_A \cap Q_A = \emptyset, \qquad |P_A| + |Q_A| = \binom{n}{m}m!, \qquad \operatorname{per} A = |P_A|.$$

Note that every 0 of A appears in exactly $\binom{n-1}{m-1}(m-1)!$ terms in the expansion of $\operatorname{per} A$, so the number of terms that contain 0's is at most $k\binom{n-1}{m-1}(m-1)!$. That is, there are at most $k\binom{n-1}{m-1}(m-1)!$ terms with value 0 in the expansion of $\operatorname{per} A$. Hence

$$\operatorname{per} A = \binom{n}{m}m! - |Q_A| \geqslant \binom{n}{m}m! - k\binom{n-1}{m-1}(m-1)!.$$

Let A_0 be any matrix in $\mathfrak{M}_k$ whose k 0's are in the same row when $m < k \leqslant n$ and in the same row or column when $0 \leqslant k \leqslant m$. Each term in the expansion of $\operatorname{per} A_0$ has at most one 0 as its factor. There are exactly $k\binom{n-1}{m-1}(m-1)!$ terms in the expansion of $\operatorname{per} A_0$ whose values equal 0. So

$$\operatorname{per} A_0 = \binom{n}{m}m! - k\binom{n-1}{m-1}(m-1)!.$$

This completes the proof.

To find the maximal value of permanent, we first prove

LEMMA 2.2. *Let $u_1 \geqslant 2$, $u_1 > u_2 \geqslant 0$, $u_1 + u_2 \leqslant n$, and*

$$C_1 = \begin{bmatrix} B_1 \\ B \end{bmatrix}, \qquad C_2 = \begin{bmatrix} B_2 \\ B \end{bmatrix}$$

where B is an $(m-2)\times n$ matrix whose entries in the first $u_1 + u_2$ columns are all 1's when $u_2 \geqslant 1$ and arbitrary when $u_2 = 0$ and

$$B_1 = \begin{bmatrix} \overbrace{0 \ldots 0}^{u_1} & \overbrace{1 \ldots 1}^{u_2} & \overbrace{1 \ldots 1}^{n-u_1-u_2} \\ 1 \ldots 1 & 0 \ldots 0 & 1 \ldots 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \overbrace{0 \ldots 0}^{u_1-1} & \overbrace{1 \ldots 1}^{u_2+1} & \overbrace{1 \ldots 1}^{n-u_1-u_2} \\ 1 \ldots 1 & 0 \ldots 0 & 1 \ldots 1 \end{bmatrix}.$$

Then $\operatorname{per} C_1 \leqslant \operatorname{per} C_2$.

Proof. Expanding $\operatorname{per} C_1$ by its first two rows, we have

$$\begin{aligned}\operatorname{per} C_1 &= \sum_{\substack{1 \leqslant i \leqslant u_1 \\ u_1+1 \leqslant j \leqslant n}} \operatorname{per} B(-|i, j) + \sum_{\substack{u_1+u_2+1 \leqslant i \leqslant n \\ u_1+1 \leqslant j \leqslant n, j \neq i}} \operatorname{per} B(-|i, j) \\ &= \sum_{\substack{1 \leqslant i \leqslant u_1-1 \\ u_1+1 \leqslant j \leqslant n}} \operatorname{per} B(-|i, j) + \sum_{\substack{u_1+u_2+1 \leqslant i \leqslant n \\ u_1+1 \leqslant j \leqslant n, j \neq i}} \operatorname{per} B(-|i, j) \\ &\quad + \sum_{u_1+1 \leqslant j \leqslant n} \operatorname{per} B(-|u_1, j),\end{aligned}$$

where $B(-|i, j)$ denotes the submatrix obtained from B by deleting the ith and jth columns. Similarly,

$$\begin{aligned}\operatorname{per} C_2 &= \sum_{\substack{1 \leqslant i \leqslant u_1-1 \\ u_1 \leqslant j \leqslant n}} \operatorname{per} B(-|i, j) + \sum_{\substack{u_1+u_2+1 \leqslant i \leqslant n \\ u_1 \leqslant j \leqslant n, j \neq i}} \operatorname{per} B(-|i, j) \\ &= \sum_{\substack{1 \leqslant i \leqslant u_1-1 \\ u_1+1 \leqslant j \leqslant n}} \operatorname{per} B(-|i, j) + \sum_{\substack{u_1+u_2+1 \leqslant i \leqslant n \\ u_1+1 \leqslant j \leqslant n, j \neq i}} \operatorname{per} B(-|i, j) \\ &\quad + \sum_{1 \leqslant i \leqslant u_1-1} \operatorname{per} B(-|i, u_1) + \sum_{u_1+u_2+1 \leqslant i \leqslant n} \operatorname{per} B(-|i, u_1).\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{per} C_2 - \operatorname{per} C_1 &= \sum_{1 \leqslant i \leqslant u_1-1} \operatorname{per} B(-|i, u_1) + \sum_{u_1+u_2+1 \leqslant i \leqslant n} \operatorname{per} B(-|i, u_1) \\ &\quad - \sum_{u_1+1 \leqslant j \leqslant n} \operatorname{per} B(-|u_1, j) \\ &= \sum_{1 \leqslant i \leqslant u_1-1} \operatorname{per} B(-|i, u_1) - \sum_{u_1+1 \leqslant i \leqslant u_1+u_2} \operatorname{per} B(-|i, u_1).\end{aligned}$$

When $u_2 = 0$,

$$\operatorname{per} C_2 - \operatorname{per} C_1 = \sum_{1 \leqslant i \leqslant u_1-1} \operatorname{per} B(-|i, u_1) \geqslant 0.$$

When $u_2 \geqslant 1$, since the entries in the first u_1+u_2 columns of B are all 1's, $\operatorname{per} B(-|i, u_1) = \operatorname{per} B(-|j, u_1)$ for any i, j in $[1, u_1+u_2]$. So

$$\begin{aligned}\operatorname{per} C_2 - \operatorname{per} C_1 &= (u_1-1) \operatorname{per} B(-|1, u_1) - u_2 \operatorname{per} B(-|1, u_1) \\ &= (u_1-u_2-1) \operatorname{per} B(-|1, u_1) \geqslant 0.\end{aligned}$$

This completes the proof.

LEMMA 2.3. *Let $u \geqslant 2$ and*

$$D_1 = [F_1 F_2 E], \quad D_2 = [F_3 F_4 E]$$

where E is an $m \times (n-2)$ matrix and

$$[F_1 F_2] = \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix}}\right\} u \\ \\ \\ \\ \end{matrix}, \quad [F_3 F_4] = \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix}}\right\} u \\ \\ \\ \\ \end{matrix}$$

Then $\operatorname{per} D_1 \leqslant \operatorname{per} D_2$.

Proof. From the definition of permanent, we have

$$\begin{aligned} \operatorname{per} D_1 = & \sum_{1 \leqslant i_1 < \ldots < i_m \leqslant n-2} \operatorname{per} E[-|i_1, \ldots, i_m] \\ & + \sum_{1 \leqslant i_1 < \ldots < i_{m-2} \leqslant n-2} \operatorname{per}[F_1 F_2 E[-|i_1, \ldots, i_{m-2}]] \\ & + \sum_{1 \leqslant i_1 < \ldots < i_{m-1} \leqslant n-2} \operatorname{per}[F_1 E[-|i_1, \ldots, i_{m-1}]] \\ & + \sum_{1 \leqslant i_1 < \ldots < i_{m-1} \leqslant n-2} \operatorname{per}[F_2 E[-|i_1, \ldots, i_{m-1}]], \\ \operatorname{per} D_2 = & \sum_{1 \leqslant i_1 < \ldots < i_m \leqslant n-2} \operatorname{per} E[-|i_1, \ldots, i_m] \\ & + \sum_{1 \leqslant i_1 < \ldots < i_{m-2} \leqslant n-2} \operatorname{per}[F_3 F_4 E[-|i_1, \ldots, i_{m-2}]] \\ & + \sum_{1 \leqslant i_1 < \ldots < i_{m-1} \leqslant n-2} \operatorname{per}[F_3 E[-|i_1, \ldots, i_{m-1}]] \\ & + \sum_{1 \leqslant i_1 < \ldots < i_{m-1} \leqslant n-2} \operatorname{per}[F_4 E[-|i_1, \ldots, i_{m-1}]] \end{aligned}$$

where $E[-|i_1, \ldots, i_m]$ denotes the submatrix consisting of the i_1-th, $\ldots$, i_m-th columns of E. By Lemma 2.2

$$\begin{aligned} \operatorname{per}[F_3 F_4 E[-|i_1, \ldots, i_{m-2}]] &= \operatorname{per}[F_3 F_4 E[-|i_1, \ldots, i_{m-2}]]^T \\ &\geqslant \operatorname{per}[F_1 F_2 E[-|i_1, \ldots, i_{m-2}]]^T \\ &= \operatorname{per}[F_1 F_2 E[-|i_1, \ldots, i_{m-2}]]. \end{aligned}$$

On the other hand,

$$\begin{aligned}
&\operatorname{per}[F_1 E[-|i_1, \ldots, i_{m-1}]] + \operatorname{per}[F_2 E[-|i_1, \ldots, i_{m-1}]] \\
&\qquad = 2 \sum_{u+1 \leqslant d \leqslant m} \operatorname{per} E(d|i_1, \ldots, i_{m-1}] + \sum_{1 \leqslant d \leqslant u} E(d|i_1, \ldots, i_{m-1}] \\
&\qquad = \operatorname{per}[F_3 E[-|i_1, \ldots, i_{m-1}]] + \operatorname{per}[F_4 E[-|i_1, \ldots, i_{m-1}]]
\end{aligned}$$

where $E(d|i_1, \ldots, i_m]$ denotes the submatrix obtained from $E[-|i_1, \ldots, i_m]$ by deleting the dth row. Therefore

$$\operatorname{per} D_1 \leqslant \operatorname{per} D_2.$$

This completes the proof.

For the next theorem we write $k = mq + r$, $q \geqslant 0$, $m > r \geqslant 0$.

THEOREM 2.3. *Let* $0 \leqslant k \leqslant n$. *Then*

$$\max_{A \in \mathfrak{M}_k} \operatorname{per} A = \sum_{i=0}^{m} \sum_{t=0}^{i} (-1)^i \binom{r}{t} \binom{m-r}{i-t} (q+1)^t q^{i-t} \binom{n-i}{m-i} (m-i)!.$$

The maximum is reached by the matrices in $\mathfrak{M}_k$ *that can by permuting columns be changed to the following form:*

$$\begin{array}{cc}
 & \overbrace{\overset{q+1}{\overbrace{\hphantom{0\ldots0}}}\quad\cdots\quad\overset{q+1}{\overbrace{\hphantom{0\ldots0}}}}^{r}\ \ \overbrace{\overset{q}{\overbrace{\hphantom{0\ldots0}}}\quad\cdots\quad\overset{q}{\overbrace{\hphantom{0\ldots0}}}}^{m-r}\ \ \overset{n-k}{\overbrace{\hphantom{0\ldots0}}} \\
A_0 = & \left[\begin{array}{c|c|c|c|c|c|c}
0\ldots0 & & & & & & \\
 & \ddots & & & & & \\
 & & 0\ldots0 & & & & \\
 & & & 0\ldots0 & & & \\
 & & & & \ddots & & \\
 & & & & & 0\ldots0 &
\end{array}\right]
\end{array}$$

The entries in blank positions are 1*'s.*

Proof. Let $A \in \mathfrak{M}_k$. If there are more than one 0 in some column of A (say the ith), then A has at least one column (say the jth) with all entries equal to 1. Let A' be the matrix obtained by interchanging a 0 in the ith column with the 1 in the jth column and in the same row as the 0. By Lemma 2.3, $\operatorname{per} A' \geqslant \operatorname{per} A$. Note that the number of columns of A' that contains 0's exceeds by one the same number for A, and the number of 0's in each row of A remains unchanged. After a finite number of such interchanges, we can get a matrix A_1 whose k 0's are in different columns and $\operatorname{per} A_1 \geqslant \operatorname{per} A$.

Obviously, by permuting columns, A_1 can be changed to the following form:

$$\begin{array}{cc}
 & \begin{array}{cccccc} u_1 & u_2 & & u_m & n-k \end{array} \\
A_2 = & \left[\begin{array}{c|c|c|c|c}
0\ldots0 & & & & \\
 & 0\ldots0 & & & \\
 & & \ddots & & \\
 & & & 0\ldots0 &
\end{array}\right]
\end{array}$$

where u_i is the number of 0's in the ith row of A_1. Obviously $\operatorname{per} A_2 = \operatorname{per} A_1 \geqslant \operatorname{per} A$.

If there are u_i and u_j such that $u_i > u_j$, then we interchange a 0 in the ith row of A_2 with the 1 in the jth row and in the column which the 0 is in. By Lemma 2.2, $\operatorname{per} A_2$ does not decrease. Thus we can make the sizes of u_i in A_2 as uniform as possible while the value of permanent keeps nondecreasing. Since $k = qm+r$, $q \geqslant 0$, $0 \leqslant r < m$, by a finite number of such interchanges, A_2 can be changed to A_0. Thus $\operatorname{per} A_0 \geqslant \operatorname{per} A_2 \geqslant \operatorname{per} A$.

In the expansion of $\operatorname{per} A_0$, the number of terms that contain i given 0's in different rows and columns of A_0 is $\binom{n-i}{m-i}(m-i)!$, and there are

$$\sum_{t=0}^{i} \binom{r}{t}\binom{m-r}{i-t}(q+1)^t q^{i-t}$$

ways of choosing such i 0's. Applying the inclusion-exclusion principle we find that

$$\operatorname{per} A_0 = \sum_{i=0}^{m} \sum_{t=0}^{i} (-1)^i \binom{r}{t}\binom{m-r}{i-t}(q+1)^t q^{i-t}\binom{n-i}{m-i}(m-i)!.$$

This completes the proof.

3. Standard forms

Gordon *et al.* [1] find the standard forms of square matrices the values of whose permanent are 1, 2 and 3 by the methods of graph theory. Li [2] re-establishes the same results by the methods of matrix theory, and finds standard forms of matrices with permanent 4. We list their results as follows.

THEOREM 3.1 ([1], [2]). *Let A be an $m \times m$ matrix. Then* $\operatorname{per} A = 1, 2$ *and* 3 *if and only if A is equivalent to the following standard forms respectively*:

(3.1) $$A_1 = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

(3.2) $$A_2 = \begin{bmatrix} 1 & & & & & & & & & * \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{bmatrix} & & & & & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ 0 & & & & & & 1 & & & \end{bmatrix},$$

$$
(3.3) \qquad A_3 = \left[\begin{array}{c|c|c|c}
\begin{matrix} 1 & & * \\ & \ddots & \\ & & 1 \end{matrix} & * & * & * \\
\hline
0 & \begin{matrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{matrix} & \begin{matrix} 1 & & \\ & & \\ & & \\ & & \end{matrix} & * \\
\hline
0 & \begin{matrix} & & \\ & & \\ & & \\ 1 & & \end{matrix} & \begin{matrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{matrix} & * \\
\hline
0 & 0 & 0 & \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix}
\end{array}\right].
$$

The entries the asterisks $*$ *represent in these matrices may be either* 0 *or* 1.

THEOREM 3.2 ([2]). *Let A be an $m \times m$ matrix. Then* $\operatorname{per} A = 4$ *if and only if A is equivalent to*

$$
(3.4) \qquad \begin{bmatrix} E_1 & & * \\ & E_2 & \\ 0 & & E_3 \end{bmatrix}
$$

where E_1 and E_3 are of the form (3.1) *and E_2 is one of the following six forms:*

$$
\begin{bmatrix}
\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} & 1 & \\
& \left[\begin{array}{c|c}
\begin{matrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{matrix} & \begin{matrix} 1 & \\ & \\ & \\ & \end{matrix} \\
\hline
\begin{matrix} & & & \\ & & & \\ & & & \\ 1 & & & \end{matrix} & \begin{matrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{matrix}
\end{array}\right] & 1 \\
\begin{bmatrix} & & \\ & & \\ 1 & & \end{bmatrix} & & \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}
\end{bmatrix},
$$

$$
\begin{bmatrix}
\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix} & & * \\
& \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\
& & \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix}
\end{bmatrix},
\quad
\begin{bmatrix}
\left[\begin{array}{cccc|cccc}
1 & 1 & & & 1 & & & \\
& \ddots & \ddots & & & & & \\
& & 1 & 1 & & & & \\
1 & & & 1 & & & & \\ \hline
& & & & 1 & 1 & & \\
& & & & & \ddots & \ddots & \\
& & & & & & 1 & 1 \\
1 & & & & & & & 1
\end{array}\right] & \begin{matrix} \\ \\ \\ 1 \\ \\ \\ \\ \\ \end{matrix} \\
\begin{matrix} & & & \\ & & & \\ & & & \\ 1 & & & \end{matrix} & \begin{matrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{matrix}
\end{bmatrix}
$$

$$
\left[\begin{array}{cccc|cccc}
1 & 1 & & & & & & \\
& \ddots & \ddots & & 1 & & & \\
& & 1 & 1 & 1 & & & \\
1 & & & 1 & & & & \\ \hline
& & & & 1 & 1 & & \\
& & & & & \ddots & \ddots & \\
& & & & & & 1 & 1 \\
& & 1 & & & & & 1
\end{array}\right],
\quad
\begin{bmatrix}
\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} & \begin{matrix} \\ \\ \\ 1 \end{matrix} & \\
& \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix} & \begin{matrix} 1 \\ \\ 1 \\ \\ \end{matrix} \\
\begin{bmatrix} & & & \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} & & \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\left[\begin{array}{cccc|cccc}
1 & 1 & & & & & & \\
& \ddots & \ddots & & 1 & & & \\
& & 1 & 1 & & & & \\
1 & & & 1 & & & & \\ \hline
& & & & 1 & 1 & & \\
& & & & & \ddots & \ddots & \\
& & & & & & 1 & 1 \\
1 & & & & & & & 1
\end{array}\right] & \begin{matrix} \\ \\ \\ 1 \\ \\ \\ \\ \\ \end{matrix} \\
\begin{matrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & 1 & & & \end{matrix} & \begin{matrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{matrix}
\end{bmatrix}.
$$

We assume that A is an $m\times n$ matrix with $m<n$ from now on. Every A_i $(1\leqslant i\leqslant 3)$ appearing hereafter will have the corresponding form in (3.1)–(3.3).

THEOREM 3.3. *$\operatorname{per} A=1$ if and only if A is equivalent to*

$$[0_{m\times(n-m)}\,A_1].$$

Proof. The permanent of $[0_{m\times(n-m)}\,A_1]$ is obviously 1. Now let $\operatorname{per} A=1$. Then A must have a submatrix A_0 of order m with $\operatorname{per} A_0=1$. Since $A_0\sim A_1$, A can be changed to the form $[B\,A_1]$ by permuting rows and columns of A, where $B=(b_{ij})$ is an $m\times(n-m)$ matrix. We claim that $B=0$. Otherwise there would be some $b_{ij}=1$. Expanding $\operatorname{per}[B\,A_1]$ by the ith row, we have $\operatorname{per} A=\operatorname{per}[B\,A_1]\geqslant 2$, a contradiction. This completes the proof.

THEOREM 3.4. *$\operatorname{per} A=2$ if and only if A is equivalent to one of the following two forms:*

$$(3.5)\qquad [0_{m\times(n-m)}\,A_2],$$

$$(3.6)\qquad \begin{array}{c} \begin{array}{ccc} n-m-1 & 1 & \quad m \end{array} \\ \left[\begin{array}{c|c|ccc} & 1 & 1 & & * \\ 0 & 0 & & \ddots & \\ & \vdots & & & \\ & 0 & 0 & & 1 \end{array}\right] m \end{array}.$$

The matrices (3.5) *and* (3.6) *are inequivalent.*

Proof. Let $\operatorname{per} A=2$. Then $\max_{1\leqslant i_1<\dots<i_m\leqslant n}\operatorname{per} A[-|i_1,\dots,i_m]=2$ or 1. When $\max\operatorname{per} A[-|i_1,\dots,i_m]=2$, A is equivalent to (3.5) by a similar argument to that used in the proof of Theorem 3.3. When $\max\operatorname{per} A[-|i_1,\dots,i_m]=1$, A is equivalent to $[B\,A_1]$ where $B=(b_{ij})\neq 0$. We claim that B cannot have two 1's or more. Otherwise there would be $b_{ij}=b_{kl}=1$, $(i,j)\neq(k,l)$. When $i=k$, we expand $\operatorname{per}[B\,A_1]$ by the ith row; when $i\neq k$, we expand it by the ith row and the kth row. In both cases, we have $\operatorname{per} A=\operatorname{per}[B\,A_1]\geqslant 3$, a contradiction. Hence B contains exactly one 1 (say $b_{ij}=1$). Let $A_1=(a_{pq})$. We assert further that $a_{pi}=0$ for $p=1,\dots,i-1$. Otherwise there would be $a_{pi}=1$ for some $p\in[1,i-1]$. By interchanging the jth column with the $(n-m+i)$th column, the matrix $[B\,A_1]$ would become

$$\begin{array}{c} \begin{array}{cc} n-m & \quad m \end{array} \\ \left[\begin{array}{c|ccc} 1 & 1 & & * \\ \vdots & & \ddots & \\ 1 & 0 & & 1 \end{array}\right] m \end{array}.$$

This is imposible. Hence

$$(3.7)\qquad [BA_1] = \left[\begin{array}{c|cccccccc} & 1 & & * & 0 & & & & * \\ & & \ddots & & \vdots & & & & \\ & & & 1 & 0 & & & & \\ \hline 1 & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & 1 \end{array}\right]$$

which is equivalent to (3.6).

On the other hand, the values of the permanent of (3.5) and (3.6) are both 2. And (3.5) and (3.6) are evidently inequivalent. This completes the proof.

THEOREM 3.5. $\operatorname{per} A = 3$ *if and only if A is equivalent to one of the following three forms*:

$$(3.8)\qquad [0_{m\times(n-m)}A_3],$$

$$(3.9)\qquad \begin{array}{c} \begin{array}{ccc} n-m-1 & \quad 3 \quad & m-2 \end{array} \\ \left[\begin{array}{c|c|c} 0 & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} & * \\ \hline 0 & 0 & \begin{array}{ccc} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{array} \end{array}\right] \end{array} \begin{array}{l} 2 \\ \\ m-2 \end{array},$$

$$(3.10)\qquad \begin{array}{c} \begin{array}{ccc} n-m-2 & \quad 2 \quad & m \end{array} \\ \left[\begin{array}{c|cc|cccc} & 1 & 1 & 1 & & & * \\ 0 & 0 & 0 & & 1 & & \\ & \vdots & \vdots & & & \ddots & \\ & 0 & 0 & 0 & & & 1 \end{array}\right] \end{array} m.$$

These matrices are inequivalent to one another.

Proof. All the values of the permanent of (3.8)–(3.10) are 3. Let $\operatorname{per} A = 3$. Then $\max_{1\leq i_1<\dots<i_m\leq n} \operatorname{per} A[-|i_1,\dots,i_m] = 3$, 2 or 1.

Case 1: $\max \operatorname{per} A[-|i_1,\dots,i_m] = 3$. In this case, A is equivalent to (3.8) by an argument analogous to the one used in the proof of Theorem 3.3.

Case 2: $\max \operatorname{per} A[-|i_1,\dots,i_m] = 2$. We shall prove that this is impossible. Otherwise A would be equivalent to $[BA_2]$ where $B = (b_{ij}) \neq 0$. Let $b_{ij} = 1$ and

$$A_2 = \begin{array}{c} \begin{array}{ccc} k & u & m-k-u \end{array} \\ \left[\begin{array}{c|c|c} \begin{matrix} 1 & & * \\ & \ddots & \\ & & 1 \end{matrix} & * & * \\ \hline 0 & \begin{matrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{matrix} & * \\ \hline 0 & 0 & \begin{matrix} 1 & & * \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right] \end{array} \begin{array}{l} k \\ u \\ m-k-u \end{array}$$

Expanding $\operatorname{per}[B\,A_2]$ by the ith row, when $i \leqslant k$ or $i \geqslant k+u+1$, we have $\operatorname{per} A = \operatorname{per}[B\,A_2] \geqslant 2+2 = 4$; when $k < i \leqslant k+u$ we have $\operatorname{per} A = \operatorname{per}[B\,A_2] \geqslant 1+1+2 = 4$. Both are impossible.

Case 3: $\max \operatorname{per} A[-|i_1, \ldots, i_m] = 1$. In this case, A is equivalent to $[B\,A_1]$, where $B = (b_{ij}) \neq 0$, and B contains no more than two 1's. Let $A_1 = (a_{ku})$.

(a) *B contains one* 1. Let $b_{ij} = 1$. Obviously at least one of a_{ki}, $k = 1, \ldots, i-1$, is 1. We assert that there is only one 1 among them. Otherwise assume that $a_{ti} = a_{si} = 1$ where $1 \leqslant t < s \leqslant i-1$. By interchanging the jth column with the $(n-m+i)$th column $[B\,A_1]$ would become

$$\left[\begin{array}{c|ccc} 1 & 1 & & * \\ 1 & & \ddots & \\ 1 & 0 & & 1 \end{array}\right]$$

This is impossible.

Let $a_{ui} = 1$. We claim further that $a_{1u} = a_{2u} = \ldots = a_{u-1,u} = 0$. If $a_{ru} = 1$ for some $r \in [1, u-1]$, then expanding $\operatorname{per} A$ by its rth row, uth row and ith row gives that $\operatorname{per}[B\,A_1] \geqslant 1+1+1+1 = 4$. Hence when B contains one 1, $[B\,A_1]$ must be

$$(3.11) \qquad \left[\begin{array}{c|cccccccc} & 1 & * & 0 & * & 0 & & & * \\ & & \ddots & \vdots & & \vdots & & & \\ & & & 0 & & 0 & & & \\ & & & 1 & \text{---} & 1 & & & \\ & & & & \ddots & 0 & & & \\ & & & & & \vdots & & & \\ & & & & & 0 & & & \\ 1 & \text{---} & \text{---} & \text{---} & \text{---} & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & 1 \end{array}\right]$$

which is equivalent to (3.9).

(b) *B contains two* 1's. We first prove that the two 1's are in the same row or in the same column. Otherwise assume $b_{st} = b_{ku} = 1$ where $s \neq k$ and $t \neq u$. Expanding $\operatorname{per}[B\,A_1]$ by the sth row and the kth row, we have $\operatorname{per} A = \operatorname{per}[B\,A_1] \geqslant 4$, a contradiction.

When the two 1's are in one column, $[B\,A_1]$ must be

$$(3.12)\qquad \left[\begin{array}{c|cccccc}
 & 1 & * & 0 & * & 0 & * \\
 & & \ddots & \vdots & & & \vdots \\
 & & & 0 & & & \vdots \\
1 & & & 1 & & & \vdots \\
\vdots & & & & \ddots & & \\
1 & & & & & 1 & \\
 & & & & & & \ddots \\
 & & & & & & 1
\end{array}\right]$$

which is equivalent to (3.11), hence to (3.9).

When the two 1's are in one row, $[B\,A_1]$ must be

$$(3.13)\qquad \left[\begin{array}{cc|ccccc}
 & & 1 & * & 0 & & * \\
 & & & \ddots & \vdots & & \\
 & & & & 0 & & \\
1 & 1 & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{array}\right]$$

which is equivalent to (3.10). Evidently, the matrices (3.8)–(3.10) are inequivalent to one another. This completes the proof.

THEOREM 3.6. $\operatorname{per} A = 4$ *if and only if A is equivalent to one of the following seven forms*:

$$(3.14)\qquad [0_{m\times(n-m)}\,A_4],$$

where A_4 *is of the form* (3.4);

$$(3.15)\qquad \begin{array}{cl}
\begin{array}{ccc} \;\; n-m-1 & \quad 3 \quad & \qquad\qquad \end{array} & \\
\left[\begin{array}{c|ccc|ccc}
0 & 1 & 1 & 1 & & * & \\
 & 0 & 1 & 1 & & & \\ \hline
 & & & & 1 & & * \\
0 & & 0 & & & \ddots & \\
 & & & & 0 & & 1
\end{array}\right] & \begin{array}{l} 2 \\ \\ m-2 \end{array}
\end{array};$$

$$(3.16)\qquad \begin{array}{cl}
\begin{array}{ccc} n-m-1 & \quad 2 \quad & m-1 \end{array} & \\
\left[\begin{array}{c|cc|ccc}
0 & 1 & 1 & & * & \\ \hline
 & & & 1 & & * \\
0 & & 0 & & \ddots & \\
 & & & 0 & & 1
\end{array}\right] & \begin{array}{l} 1 \\ \\ m-1 \end{array}
\end{array};$$

$$\begin{array}{cc} \begin{array}{ccc} \scriptstyle n-m-1 & \scriptstyle 4 & \scriptstyle m-3 \end{array} & \\ \left[\begin{array}{c|c|c} 0 & \begin{matrix}0&1&0&1\\0&0&1&1\\1&0&0&1\end{matrix} & * \\ \hline 0 & 0 & \begin{matrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{matrix}\end{array}\right] & \begin{array}{c} \scriptstyle 3 \\[2em] \scriptstyle m-3\end{array} \end{array}; \tag{3.17}$$

$$\begin{array}{cc} \begin{array}{ccc} \scriptstyle n-m-1 & \scriptstyle 4 & \scriptstyle m-3 \end{array} & \\ \left[\begin{array}{c|c|c} 0 & \begin{matrix}0&1&1&0\\0&0&1&1\\1&0&0&1\end{matrix} & * \\ \hline 0 & 0 & \begin{matrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{matrix}\end{array}\right] & \begin{array}{c} \scriptstyle 3 \\[2em] \scriptstyle m-3\end{array} \end{array}; \tag{3.18}$$

$$\begin{array}{cc} \begin{array}{ccc} \scriptstyle n-m-2 & \scriptstyle 4 & \scriptstyle m-2 \end{array} & \\ \left[\begin{array}{c|c|c} 0 & \begin{matrix}1&0&1&0\\0&1&0&1\end{matrix} & * \\ \hline 0 & 0 & \begin{matrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{matrix}\end{array}\right] & \begin{array}{c} \scriptstyle 2 \\[2em] \scriptstyle m-2\end{array} \end{array}; \tag{3.19}$$

$$\begin{array}{cc} \begin{array}{ccc} \scriptstyle n-m-3 & \scriptstyle 3 & \scriptstyle m \end{array} & \\ \left[\begin{array}{c|c|c} 0 & \begin{matrix}1&1&1\\0&0&0\\ \vdots&\vdots&\vdots\\0&0&0\end{matrix} & \begin{matrix}1 & & *\\ & & \\ 0 & & 1\end{matrix}\end{array}\right] & \scriptstyle m \end{array}. \tag{3.20}$$

All the seven matrices are inequivalent to one another.

Proof. The values of the permanent of the matrices (3.14)–(3.20) are 4. Let $\operatorname{per} A = 4$. Then $\max_{1 \le i_1 < \ldots < i_m \le n} \operatorname{per} A[-|i_1, \ldots, i_m] = 4, 3, 2,$ or 1.

Case 1: $\max \operatorname{per} A[-|i_1, \ldots, i_m] = 4$. In this case, A is equivalent to (3.14) for some A_4.

Case 2: $\max \operatorname{per} A[-|i_1, \ldots, i_m] = 3$. By an argument analogous to Case 2 in the proof of Theorem 3.5 we can prove that this case is impossible.

Case 3: $\max \operatorname{per} A[-|i_1, \ldots, i_m] = 2$. In this case A is equivalent to

$[B\,A_2]$ where $B = (b_{ij}) \neq 0$, and B contains exactly one 1. Let $b_{ij} = 1$. If the ith row of A_2 covers a row of the block

$$
C = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{bmatrix}
$$

of A_2, then the order of C must be 2 (otherwise $\operatorname{per} A = \operatorname{per}[B\,A_2] \geqslant 5$). So $[B\,A_2]$ must be

$$
(3.21)\qquad \left[\begin{array}{c|cccccccc}
 & 1 & & * & 0 & 0 & & & * \\
 & & \ddots & & \vdots & \vdots & & & \\
 & & & 1 & 0 & 0 & & & \\
1 & & & & 1 & 1 & & & \\
 & & & & 1 & 1 & & & \\
 & & & & & & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{array}\right]
$$

which is equivålent to (3.15). If the ith row of A does not cover any row of C, then $[B\,A_2]$ must be one of the following two:

$$
(3.22)\qquad \left[\begin{array}{c|cccccccc}
 & 1 & * & 0 & & & & & * \\
 & & \ddots & \vdots & & & & & \\
 & & & 0 & & & & & \\
1 & & & 1 & \ddots & & & & \\
 & & & & & 1 & & & \\
 & & & & & & \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{bmatrix} & & \\
 & & & & & & & 1 & \\
 & & & & & & & & \ddots \\
 & & & & & & & & & 1
\end{array}\right],
$$

$$
\left[\begin{array}{c|cccccccc}
 & 1 & & & * & & & 0 & * \\
 & & \ddots & & & & & & \\
 & & & 1 & & & & & \\
 & & & & \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 1 \end{bmatrix} & & & \vdots & \\
 & & & & & 1 & & & \\
 \vdots & & & & & & \ddots & 0 & \\
 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & 1 & \\
 & & & & & & & & \ddots \\
 & & & & & & & & 1
\end{array}\right]
\tag{3.23}
$$

which are both equivalent to (3.16).

Case 4: $\max \operatorname{per} A[-|i_1, \ldots, i_m] = 1$. In this case, A is equivalent to $[B A_1]$ where B contains no more than three 1's.

(a) *B contains one* 1. Then $[B A_1]$ must be one of the following two:

$$
\left[\begin{array}{c|cccccccccc}
 & 1 & * & 0 & * & 0 & * & 0 & & * \\
 & & \ddots & \vdots & & \cdot & & \vdots & & \\
 & & & 0 & & \cdot & & 0 & & \\
 & & & 1 & \text{---} & \text{---} & \text{---} & 1 & & \\
 & & & & \ddots & \cdot & & 0 & & \\
 & & & & & \ddots & & \vdots & & \\
 & & & & & 0 & & 0 & & \\
 & & & & & 1 & \text{---} & 1 & & \\
 & & & & & & \ddots & 0 & & \\
 & & & & & & & \vdots & & \\
 & & & & & & & 0 & & \\
 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & 1 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 1
\end{array}\right],
\tag{3.24}
$$

(3.25)
$$\left[\begin{array}{c|ccccccccc}
 & 1 & * & 0 & * & 0 & * & 0 & & * \\
 & & \ddots & \vdots & & \vdots & & & & \vdots \\
 & & & 0 & & 0 & & & & \\
 & & & 1 & \text{---} & 1 & & & & \\
 & & & & \ddots & 0 & & & & \\
 & & & & & \vdots & & & & \\
 & & & & & 0 & & 0 & & \\
 & & & & & 1 & \text{---} & 1 & & \\
 & & & & & & \ddots & 0 & & \\
 & & & & & & & \vdots & & \\
 & & & & & & & 0 & & \\
1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & 1 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 1
\end{array}\right]$$

which are equivalent to (3.17) and (3.18) respectively.

(b) *B contains two* 1's. Let $A_2 = (a_{ij})$. If the two 1's are in the same row (say the ith) of B, then some of $a_{1i}, a_{2i}, \dots, a_{i-1,i}$ must be 1. And then $\operatorname{per}[B\,A_1] \geqslant 5$. This is impossible.

If the two 1's are in different columns, then $[B\,A_1]$ must be

(3.26)
$$\left[\begin{array}{cc|ccccccc}
 & & 1 & * & 0 & * & 0 & & * \\
 & & & \ddots & \vdots & & & & \vdots \\
 & & & & 0 & & & & \\
 & 1 & \text{---} & \text{---} & 1 & & & & \\
 & & & & & \ddots & 0 & & \\
1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{array}\right]$$

which is equivalent to (3.19). If they are in the same column, then $[B\,A_1]$ must be one of the following four:

(3.27)
$$\left[\begin{array}{ccccccccc}
 & 1 & * & 0 & * & 0 & * & 0 & * \\
 & & \ddots & \vdots & & \cdot & & \vdots & \\
 & & & 0 & & \cdot & & 0 & \\
 & & & 1 & \text{---} & \text{---} & \text{---} & 1 & \\
 & & & & \ddots & 0 & & 0 & \\
1 & \text{---} & \text{---} & \text{---} & \text{---} & 1 & & \vdots & \\
| & & & & & & \ddots & 0 & \\
1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & 1 & \\
 & & & & & & & & \ddots \\
 & & & & & & & & & 1
\end{array}\right],$$

$$
(3.28)\qquad
\begin{bmatrix}
 & 1 & * & 0 & * & 0 & & * \\
 & & \ddots & \vdots & & \vdots & & \\
 & & & 0 & & 0 & & \\
1 & \text{—} & \text{—} & 1 & \text{—} & 1 & & \\
 & & & & \ddots & 0 & & \\
 & & & & & \vdots & & \\
 & & & & & 0 & & \\
1 & \text{—} & \text{—} & \text{—} & \text{—} & 1 & \ddots & \\
 & & & & & & & 1
\end{bmatrix},
$$

$$
(3.29)\qquad
\begin{bmatrix}
 & 1 & * & 0 & * & 0 & * & 0 & & * \\
 & & \ddots & \vdots & & \cdot & & \cdot & & \\
1 & & & 0 & & \cdot & & \cdot & & \\
1 & \text{—} & \text{—} & 1 & \ddots & \cdot & & \cdot & & \\
 & & & & & 0 & & 0 & & \\
 & & & & & 1 & \text{—} & 1 & & \\
 & & & & & & \ddots & 0 & & \\
 & & & & & & & \vdots & & \\
 & & & & & & & 0 & & \\
1 & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & 1 & \ddots & \\
 & & & & & & & & & 1
\end{bmatrix},
$$

$$
(3.30)\qquad
\begin{bmatrix}
 & 1 & * & 0 & * & 0 & * & 0 & & * \\
 & & \ddots & \vdots & & \vdots & & \vdots & & \\
 & & & 0 & & 0 & & \vdots & & \\
 & & & 1 & \text{—} & 1 & & \vdots & & \\
 & & & & \ddots & 0 & & \vdots & & \\
 & & & & & \vdots & & \vdots & & \\
 & & & & & 0 & & \vdots & & \\
1 & \text{—} & \text{—} & \text{—} & \text{—} & 1 & \ddots & \vdots & & \\
 & & & & & & & 0 & & \\
1 & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & 1 & \ddots & \\
 & & & & & & & & & 1
\end{bmatrix}.
$$

Evidently, (3.28) is equivalent to (3.21), whence to (3.15); and (3.27), (3.29) and (3.30) are all equivalent to (3.25), whence to (3.18).

(c) *B contains exactly three* 1's. If these 1's are in the same row, then $[B\,A_1]$ must be

$$
(3.31)\qquad
\left[\begin{array}{ccc|cccc}
 & & & 1 & * & 0 & * \\
 & & & & \ddots & \vdots & \\
 & & & & & 0 & \\
1 & 1 & 1 & & & 1 & \\
 & & & & & & \ddots \\
 & & & & & & 1
\end{array}\right]
$$

which is equivalent to (3.20). If the three 1's are in the same colımn, then $[B A_1]$ must be

$$
(3.32)\qquad
\left[\begin{array}{c|cccccccc}
 & 1 & * & 0 & * & 0 & * & 0 & * \\
 & & \ddots & \vdots & & \vdots & & \vdots & \\
 & & & 0 & & \vdots & & \vdots & \\
1 & & & 1 & & \vdots & & \vdots & \\
 & & & & \ddots & 0 & & \vdots & \\
1 & & & & & 1 & & \vdots & \\
 & & & & & & \ddots & 0 & \\
1 & & & & & & & 1 & \\
 & & & & & & & & \ddots \\
 & & & & & & & & 1
\end{array}\right]
$$

which is equivalent to (3.24), whence to (3.17).

We now explain that the three 1's must be in the same row or column. Otherwise $[B A_1]$ would be one of the following:

$$
(3.33)\qquad
\begin{array}{c}
\left[\begin{array}{cc|ccccc}
 & & 1 & & & & * \\
 & & & \ddots & & & \\
 & 1 & & 1 & & & \\
 & & & & \ddots & & \\
1 & 1 & & & & 1 & \\
 & & & & & & \ddots \\
 & & & & & & 1
\end{array}\right]
\begin{array}{c}
\\ \\ k \\ \\ t \\ \\ \\
\end{array}
\end{array},
$$

$$
(3.34)\qquad
\left[\begin{array}{cc|cccccc}
 & & 1 & & & & & * \\
 & & & \ddots & & & & \\
 & 1 & & 1 & & & & \\
 & & & & \ddots & & & \\
 & 1 & & & & 1 & & \\
 & & & & & & \ddots & \\
1 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & 1
\end{array}\right],
$$

$$(3.35)\qquad \left[\begin{array}{cc|ccccc} & & 1 & & & & * \\ & & & \ddots & & & \\ & 1 & & & 1 & & \\ & & & & & \ddots & \\ 1 & 1 & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & 1 \end{array}\right],$$

$$(3.36)\qquad \left[\begin{array}{ccc|ccccc} & & & 1 & & & & * \\ & & & & \ddots & & & \\ & & 1 & & 1 & & & \\ & & & & & \ddots & & \\ 1 & & & & & 1 & & \\ & & & & & & \ddots & \\ & 1 & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & 1 \end{array}\right].$$

Expanding the permanent of (3.33) by the kth row and the ith row, we have $\operatorname{per}[B\,A_1] \geqslant 5$, which is impossible. The remaining cases can be proved similarly. Evidently the matrices (3.14) to (3.20) are inequivalent to one another. This completes the proof.

From Theorem 2.2 and the formulae (2.4)–(2.11) it follows that

THEOREM 3.7. *Let A be an $m \times n$ matrix where $m \geqslant 3$ and $n \geqslant 4$. Then*

(i) $\operatorname{per} A = \binom{n}{m} m!$ *if and only if* $A = J_{m\times n}$.

(ii) $\operatorname{per} A = (n-1)\binom{n-1}{m-1}(m-1)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 1 & n-1 \end{array} & \\ \begin{bmatrix} 0 & J \\ J & J \end{bmatrix} & \begin{array}{c} 1 \\ m-1 \end{array} \end{array}.$$

(iii) $\operatorname{per} A = (n^2-3n+3)\binom{n-2}{m-2}(m-2)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 2 & n-2 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{c} 2 \\ m-2 \end{array} \end{array}.$$

(iv) $\operatorname{per} A = (n-2)\binom{n-1}{m-1}(m-1)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 2 & n-2 \end{array} & \\ \begin{bmatrix} [0\ 0] & J \\ J & J \end{bmatrix} & \begin{array}{l} 1 \\ m-1 \end{array} \end{array} \quad or \quad A \sim \begin{array}{cc} \begin{array}{cc} 1 & n-1 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 2 \\ m-2 \end{array} \end{array}.$$

(v) $\operatorname{per} A = (n^3-6n^2+14n-13)\binom{n-3}{m-3}(m-3)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 3 & n-3 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 3 \\ m-3 \end{array} \end{array}$$

(vi) $\operatorname{per} A = (n^3-6n^2+13n-10)\binom{n-3}{m-3}(m-3)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 3 & n-3 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 2 \\ m-2 \end{array} \end{array}$$

(vii) $\operatorname{per} A = (n^3-6n^2+12n-8)\binom{n-3}{m-3}(m-3)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 2 & n-2 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 2 \\ m-2 \end{array} \end{array}.$$

(viii) $\operatorname{per} A = (n^3-6n^2+11n-6)\binom{n-3}{m-3}(m-3)! = (n-3)\binom{n-1}{m-1}(m-1)!$ *if and only if*

$$A \sim \begin{array}{cc} \begin{array}{cc} 3 & n-3 \end{array} & \\ \begin{bmatrix} [0\ 0\ 0] & J \\ J & J \end{bmatrix} & \begin{array}{l} 1 \\ m-1 \end{array} \end{array} \quad or \quad A \sim \begin{array}{cc} \begin{array}{cc} 1 & n-1 \end{array} & \\ \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & J \\ J & J \end{bmatrix} & \begin{array}{l} 3 \\ m-3 \end{array} \end{array}$$

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