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p-HARMONIC EQUATION AND QUASIREGULAR MAPPINGS

B. BOJARSKI and T. IWANIEC

Institute of Mathematics, Warsaw University, Warsaw, Poland

Introduction

In many parts of analysis, particularly in partial differential equations and in the theory of quasiconformal mappings, we are concerned with the variational integrals of the form

$$I_p[u] = \int_\Omega |\nabla u(x)|^p dx, \qquad p \geqslant 2, \tag{1}$$

where Ω is an open set in the Euclidean n-dimensional space $\boldsymbol{R}^n$ with the Lebesgue volume element $dx = dx_1 \ldots dx_n$. The domain of the functional I_p is the Sobolev space $W_p^1(\Omega)$. The stationary points of I_p are the object of this article. They are weak solutions of the Euler–Lagrange equation corresponding to (1), called in this setting the *p-harmonic equation*:

$$\operatorname{div} |\nabla u|^{p-2} \nabla u = 0. \tag{2}$$

In what follows, *p-harmonic function* will be a solution of (2) in $W_{p,\mathrm{loc}}^1(\Omega)$. By this we mean that for each test function $\varphi \in W_p^1(\Omega)$ with compact support the formula

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle = 0 \tag{3}$$

holds, where $\langle\ ,\ \rangle$ stands for the inner product in $\boldsymbol{R}^n$. For $p = 2$ we get the Laplace equation and harmonic functions. For $n = 2$ the connections with conformal mappings are classical. For arbitrary dimension $n \geqslant 2$ the case $p = n$ in (2) is also of crucial importance for conformal mappings in $\boldsymbol{R}^n$.

If $f = (f^1, \ldots, f^n)$ is conformal each component $u = f^i$, $i = 1, \ldots, n$, as well as the function $v = \log|f|$ (if $f \neq 0$), are solutions of the n-harmonic equation

$$\operatorname{div} |\nabla u|^{n-2} \nabla u = 0. \tag{4}$$

This paper arose from our interest in the properties of solutions of (4). This is motivated by a number of applications and some remarkable properties of n-harmonic functions. Note that (4) is conformally invariant and the functional I_n is basic for the concept of conformal capacity. However, I_p with $p \neq n$ is also useful for the treatment of quasiregular mappings, and if p is an integer greater than 2, every p-harmonic function $u = u(x_1, x_2)$ in a domain $\Omega \subset \mathbf{R}^2 \subset \mathbf{R}^2 \times \mathbf{R}^{p-2}$ extends trivially to the p-harmonic function $v(x_1, \ldots, x_p) = u(x_1, x_2)$ in $\Omega \times \mathbf{R}^{p-2} \subset \mathbf{R}^p$.

This observation allows one to construct n-harmonic functions in $\mathbf{R}^n$ by extension from $\mathbf{R}^2$. We shall see that the treatment of the two-dimensional case is considerably simplified by the use of quasiregular mappings and the corresponding uniformly elliptic systems of the first order.

The functional I_p, $p \neq 2$, is also the typical example in the Calculus of Variations of the so-called quasiregular multiple integrals. The theory of these, particularly the problem of C^∞ regularity, is delicate since the ellipticity of (2) at the zeros of the gradient breaks down.

In this paper we are concerned principally with examples of nonsmooth solutions of (2) for every $p > 2$. There are two approaches to this problem. The first is based on the study of the complex gradient $f = u_x - iu_y$ (for $n = 2$) which turns out to be a quasiregular mapping. Specifically, f satisfies the quasilinear elliptic system

$$(5) \qquad f_{\bar{z}} = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\bar{f}}{f} f_z + \frac{f}{\bar{f}} \bar{f}_z\right), \qquad z = x + iy,$$

and the corresponding linear system in the hodograph plane. The solutions of (5) are found explicitly. Consequently we obtain

PROPOSITION 1. *For each $p = 3, 4, \ldots$ there exists a p-harmonic function in $C^{[p/3]+1}$ which is not in $C^{[p/3]+2}$.*

In the second approach we exploit some ideas of Krol', Maz'ya [6], [7] and Tolksdorf [11], [12] who examined the particular solutions of (2) of the form

$$(6) \qquad u(x, y) = r^\lambda \Phi(\sigma)$$

in polar coordinates $x + iy = re^{i\sigma}$.

This leads to a nonlinear ordinary differential equation with parameter λ for the function $\Phi \in C^2(\mathbf{R}^1)$. The parameter λ is exactly the exponent of regularity of u. This is determined by the condition for Φ to be 2π periodic (the nonlinear eigenvalue problem). We present here a slightly different technique of finding Φ and λ. The result is the existence of a p-harmonic function ($p > 2$) in the class $C^{1,1/(p-1)}(\mathbf{R}^2)$ but not belonging to $C^2(\mathbf{R}^2)$.

Both these examples indicate that it is not justified to use the argument of the smoothness of generalized solutions of equation (4) in the study of the regularity of conformal mappings in space and in other related problems, e.g. see [10], p. 130.

Together with these examples we discuss the classical $C^{1,\varepsilon}$ regularity results in the two-dimensional case. We shall show that the singular set (the set where the solution is not C^∞) consists of isolated points. In particular, the singularities of a p-harmonic function trivially extended from $\boldsymbol{R}^2$ to $\boldsymbol{R}^n$ lie on $(n-2)$-dimensional hyperplanes. This improves the classical results on the subject.

Some facts concerning degenerate elliptic equations are more or less known. We give them for completeness.

1. Second order differentiability

The existence of the second order weak derivatives of p-harmonic functions and their interior estimates may be deduced via consideration of difference quotients.

Let $u \in W_p^1(\Omega)$ be a p-harmonic function in Ω. We consider the mapping $F\colon \Omega \to \boldsymbol{R}^n$, $F(x) = |\nabla u(x)|^{(p-2)/2} \cdot \nabla u(x)$, $F \in L_2(\Omega)$.

PROPOSITION 2. *$F \in W^1_{2,\mathrm{loc}}(\Omega)$ and for each compact subset $G \subset \Omega$ the following uniform estimate holds*:

$$\|DF\|_{2,G} \leqslant \frac{C_{n,p}}{\operatorname{dist}(G, \partial\Omega)} \|F\|_{2,\Omega}, \tag{7}$$

where DF is the differential of F (the Jacobi matrix) defined almost everywhere and $\| \ \|_{2,\Omega}$ denotes the $L_2(\Omega)$ norm.

Proof. For each $h \in \boldsymbol{R}^n$ such that $|h| < \operatorname{dist}(G, \partial\Omega)$, by (3) applied to $u(x)$ and $u(x+h)$ we obtain

$$\int_\Omega \langle |\nabla u(x+h)|^{p-2} \nabla u(x+h) - |\nabla u(x)|^{p-2} \nabla u(x), \nabla\varphi(x)\rangle \, dx = 0$$

for every admissible test function φ. In particular, we can put $\varphi(x) = \eta^2(x)[u(x+h)-u(x)]$, where $\eta \in C_0^\infty(\Omega)$, and we get

$$\begin{aligned}\int_\Omega \eta^2(x) \langle |\nabla u(x+h)|^{p-2} \nabla u(x+h) - |\nabla u(x)|^{p-2} \nabla u(x), \nabla u(x+h) - \nabla u(x)\rangle \, dx \\ = -2\int \eta(x)[u(x+h)-u(x)] \\ \times \langle |\nabla u(x+h)|^{p-2} \nabla u(x+h) - |\nabla u(x)|^{p-2} \nabla u(x), \nabla\eta(x)\rangle \, dx.\end{aligned}$$

Now we use elementary inequalities: for $\xi, \zeta \in \boldsymbol{R}^n$

$$\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\rangle \geqslant \frac{4}{p^2}\left||\xi|^{(p-2)/2}\xi - |\zeta|^{(p-2)/2}\zeta\right|^2,$$

$$\left||\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right| \leqslant \frac{2(p-1)}{p}(|\xi|^p + |\zeta|^p)^{(p-2)/(2p)}\left||\xi|^{(p-2)/2}\xi - |\zeta|^{(p-2)/2}\zeta\right|.$$

We then obtain

$$\int \eta^2(x)\,|F(x+h)-F(x)|^2\,dx$$
$$\leqslant p(p-1)\int |\eta(x)|\,|u(x+h)-u(x)|\,\big(|\nabla u(x+h)|^p+|\nabla u(x)|^p\big)^{(p-2)/(2p)}$$
$$\times|\nabla\eta(x)|\,|F(x+h)-F(x)|\,dx.$$

Hence, by the Hölder inequality

$$\int \eta^2(x)\,|F(x+h)-F(x)|^2\,dx \tag{8}$$
$$\leqslant p(p-1)\Big(\int|\nabla\eta(x)|^2\,|u(x+h)-u(x)|^p\,dx\Big)^{1/p}$$
$$\times\Big(\int|\nabla\eta(x)|^2\big(|\nabla u(x+h)|^p+|\nabla u(x)|^p\big)\,dx\Big)^{(p-2)/(2p)}$$
$$\times\Big(\int\eta^2(x)\,|F(x+h)-F(x)|^2\,dx\Big)^{1/2}.$$

Since $u\in W_p^1(\Omega)$ the first integral on the right-hand side of (8) is of order $O(|h|)$, the second integral is bounded and we get

$$\Big(\int\eta^2(x)\,|F(x+h)-F(x)|^2\,dx\Big)^{1/2}=O(|h|)$$

which shows that $F\in W^1_{2,\mathrm{loc}}(\Omega)$. The uniform estimate (7) is obtained by an appropriate choice of the test function η such that $\eta\equiv 1$ in G, $|\nabla\eta|\leqslant C(n)/\mathrm{dist}(G,\partial\Omega)$.

2. First order elliptic systems

The connections of second order elliptic equations of divergence type in the plane with elliptic systems of the first order are well known. In full analogy with the calculations concerning the continuity equation

$$\big(\varrho(|\nabla u|)\,u_x\big)_x+\big(\varrho(|\nabla u|)\,u_y\big)_y=0$$

for the potential $u=u(x,y)$ of a two-dimensional barotropic flow [2], [3] we derive the elliptic system (5) for the complex gradient $f=u_x-iu_y$. We also examine the nonlinear counterparts $F_a=|f|^a f$, $a>-1$, of the complex gradient f. We know that $F_{(p-2)/2}\in W^1_{2,\mathrm{loc}}(\Omega)$. At this moment, the formal calculations below are substantial only for $a=(p-2)/2$. However, they will be justified at the end of this section for any $a>-1$.

From the definition of f and F_a we see that

$$2u_x=|F_a|^{-a/(a+1)}(F_a+\bar F_a),\qquad 2u_y=i\,|F_a|^{-a/(a+1)}(F_a-\bar F_a). \tag{9}$$

Therefore the equality

$$\frac{\partial}{\partial y}\big[|F_a|^{-a/(a+1)}(F_a+\bar F_a)\big]=i\frac{\partial}{\partial x}\big[|F_a|^{-a/(a+1)}(F_a-\bar F_a)\big]$$

holds in the sense of distributions. Equivalently

$$\operatorname{Im}\frac{\partial}{\partial\bar{z}}(|F_a|^{-a/(a+1)}F_a) = 0, \qquad z = x+iy.$$

For $a = (p-2)/2$, F_a is differentiable almost everywhere and the last equality reduces to the complex equation

$$\frac{\partial}{\partial\bar{z}}F_a - \overline{\frac{\partial}{\partial\bar{z}}F_a} = -\frac{a}{a+2}\left[\frac{\bar{F}_a}{F_a}\cdot\frac{\partial}{\partial z}F_a - \frac{F_a}{\bar{F}_a}\cdot\overline{\frac{\partial}{\partial z}F_a}\right]. \tag{10}$$

On the other hand, our equation (2) says that the vector field

$$2|\nabla u|^{p-2}\nabla u = |F_a|^{(p-2-a)/(a+1)}[(F_a+\bar{F}_a), i\,(F_a-\bar{F}_a)]$$

is divergence free. This means that

$$\frac{\partial}{\partial x}[|F_a|^{(p-2-a)/(a+1)}(F_a+\bar{F}_a)] + i\frac{\partial}{\partial y}[|F_a|^{(p-2-a)/(a+1)}(F_a-\bar{F}_a)] = 0$$

or in the complex notation

$$\operatorname{Re}\frac{\partial}{\partial\bar{z}}(|F_a|^{(p-2-a)/(a+1)}F_a) = 0.$$

If F_a is differentiable almost everywhere this can be written in the form

$$\frac{\partial}{\partial\bar{z}}F_a + \overline{\frac{\partial}{\partial\bar{z}}F_a} = -\frac{p-2-a}{a+p}\left[\frac{\bar{F}_a}{F_a}\frac{\partial}{\partial z}F_a + \frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right]. \tag{11}$$

Adding (10) and (11) we get (cf. [3], eq. 1.4)

$$\frac{\partial}{\partial\bar{z}}F_a = q_1\frac{\partial}{\partial z}F_a + q_2\overline{\frac{\partial}{\partial z}F_a}, \tag{12}$$

where

$$q_1 = -\frac{1}{2}\left(\frac{p-2-a}{p+a}+\frac{a}{a+2}\right)\frac{\bar{F}_a}{F_a}, \qquad q_2 = -\frac{1}{2}\left(\frac{p-2-a}{p+a}-\frac{a}{a+2}\right)\frac{F_a}{\bar{F}_a}.$$

The system (12) is uniformly elliptic. In fact, for $p \geqslant 2$ and $a > -1$ we have

$$|q_1|+|q_2| = \max\left\{\left|\frac{p-2-a}{p+a}\right|, \left|\frac{a}{a+2}\right|\right\} = q_0 < 1.$$

This shows that F_a with $a = (p-2)/2$ is a quasiregular mapping. Now, observe that each F_a, $a > -1$, is the composition of $F_{(p-2)/2}$ with the radial quasiconformal map $\xi \to |\xi|^{\alpha}\xi$, $\alpha = 2(a+1)/p-1 > -1$. Thus F_a is a quasiregular map. In particular, $F_a \in W^1_{2,\mathrm{loc}}(\Omega)$ and we are justified in writing the system (12) for each $a > -1$.

The case $a = 0$ is of particular interest.

PROPOSITION 3. *The complex gradient* $f = u_x - iu_y$ *of a p-harmonic function u is a quasiregular mapping and satisfies the system*

$$f_{\bar z} = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\bar f}{f} f_z + \frac{f}{\bar f} \bar f_z\right), \tag{13}$$

$$|f_{\bar z}(z)| \leqslant \left(1 - \frac{2}{p}\right)|f_z(z)| \quad \textit{for almost every } z \in \Omega. \tag{14}$$

We note that for $a = \sqrt{p-1} - 1$ the function F_a satisfies the quasilinear system of Beltrami's type

$$\frac{\partial F_a}{\partial \bar z} = \frac{1-\sqrt{p-1}}{1+\sqrt{p-1}} \frac{\bar F_a}{F_a} \frac{\partial F_a}{\partial z}.$$

For no $a > -1$ does the coefficient q_1 in (12) vanish.

3. Hölder continuity of the gradient of p-harmonic functions in two variables

Every quasiregular mapping in the plane is Hölder continuous with Hölder exponent $\alpha = (1-q_0)/(1+q_0)$, where q_0 is the Beltrami coefficient [1], [4], [9]. Applying this result to the inequality (14) we immediately obtain

PROPOSITION 4. *Every p-harmonic function in a two-dimensional domain* Ω *is of class* $C^{1,\alpha}$ *with* $\alpha \geqslant 1/(p-1)$.

Remark. Actually, every q_0-quasiregular mapping belongs to the Sobolev space $W^1_{r,\mathrm{loc}}$ for each $2 \leqslant r < 1 + q_0^{-1}$ [5]. Consequently, the second derivatives of p-harmonic functions are locally L_s integrable with exponent $2 \leqslant s < 2 + 2/(p-2)$.

4. Smoothness except at isolated points

The zeros of a nonzero quasiregular mapping are isolated. Therefore the complex gradient f does not vanish except at isolated points. Since the coefficients q_1 and q_2 in (13) are real analytic in $\Omega \times \boldsymbol{R}^2 \setminus \{0\}$ the solution f is real analytic in Ω except in the set of zeros of f. Let S denote the singular set, i.e. the minimal set such that f is C^∞ smooth in $\Omega \setminus S$. We then have

PROPOSITION 5. *The singular set of every p-harmonic function consists of isolated points only.*

5. The equation (13) in the hodograph plane

In order to perform the hodograph transformation of (13) in the whole plane we must assume that $f = f(z)$ is one-to-one. With this assumption the inverse map $h(\xi) = f^{-1}(\xi)$ satisfies the linear equation of the form

$$(15)\qquad h_{\bar\xi} = \left(\frac{1}{2}-\frac{1}{p}\right)\left(\frac{\xi}{\bar\xi}h_\xi + \frac{\bar\xi}{\xi}\bar h_\xi\right).$$

This follows from the transformation formulas

$$f_{\bar z}(z) = -J(z,f)\,h_{\bar\xi}(\xi), \qquad f_z(z) = J(z,f)\,\overline{h_\xi(\xi)},$$

where $\xi = f(z)$ and $J(z,f) = |f_z|^2 - |f_{\bar z}|^2$ is the Jacobian of f, positive almost everywhere.

We shall study (15) with $h \in W^1_{2,\mathrm{loc}}(\Omega')$, $\Omega' \subset \boldsymbol{R}^2$ (not necessarily a homeomorphism).

One can easily verify that for each integer $k \geqslant 1$ the functions

$$(16)\qquad h_k(\xi) = |\xi|^{\gamma_k}\left[a_k(\gamma_k+k+1)\left(\frac{\xi}{|\xi|}\right)^{k-1} + \bar a_k(\gamma_k-k+1)\left(\frac{\bar\xi}{|\xi|}\right)^{k+1}\right],$$

where

$$(17)\qquad \gamma_k = -p/2+\sqrt{k^2-1+p^2/4} \qquad \left(\lim_{k\to\infty}(\gamma_k/k) = 1\right)$$

and a_k are arbitrary complex numbers, are particular solutions of the system (15). For $k = \pm 1$ we obtain the constant solutions $h_1 = 2a_1$, $h_{-1} = 2\bar a_{-1}$, and we do not get essentially new solutions by letting in (16) $k = -2, -3, \ldots$ For $k = 0$, (16) defines the singular solution

$$h_0(\xi) = (\gamma_0+1)(a_0+\bar a_0)|\xi|^{\gamma_0-1}\bar\xi, \qquad \gamma_0 < 0,$$

except the case $p = 2$ when $\gamma_0 = -1$ or a_0 is purely imaginary. Other singular solutions can be defined by formula (16) when we choose another branch of the root in (17), namely $\gamma_k^- = -p/2-\sqrt{k^2-1+p^2/4}$.

Remark. It can be shown that every regular solution in a neighbourhood of $\xi = 0$ expands into a series $h(\xi) = \sum_{k=1}^{\infty} h_k(\xi)$. The radius R of convergence is given by the same formula as for power series, $R^{-1} = \overline{\lim_{k\to\infty}}\,|a_k|^{1/k}$.

In other words, the h_k form a complete basis in the space of regular solutions of (15).

In the sequel we are only interested in homeomorphic solutions of (15). Among the particular solutions (16) only $h_2(\xi)$ is a good candidate. The argument principle (which is valid for quasiregular mappings) allows one to describe the obstructions for $h_k(\xi)$, $k > 2$, to be a homeomorphism.

For $k = 2$ we have $0 < \gamma_2 = (\sqrt{p^2+12}-p)/2 < 1$ and $h_2(\xi)$ can be written in the form

$$h_2(\xi) = |\xi|^{\gamma_2 - 1}\xi\left(1-\mu\frac{\bar{\xi}^2}{\xi^2}\right), \tag{18}$$

where

$$0 < \mu = \frac{1-\gamma_2}{3+\gamma_2} < \tfrac{1}{3}.$$

Clearly, $h_2 = h_2(\xi)$ is the composition of the radial quasiconformal map $\zeta = |\xi|^{\gamma_2 - 1}\xi$ with the quasiisometric map $H(\zeta) = \zeta - \mu\bar{\zeta}^2/\zeta$,

$$(1-3\mu)|\zeta_1 - \zeta_2| \leqslant |H(\zeta_1)-H(\zeta_2)| \leqslant (1+3\mu)|\zeta_1-\zeta_2|.$$

Hence h_2 is a quasiconformal mapping in $\boldsymbol{R}^2$.

6. The proof of Proposition 1

Restricted to the real line, h_2 is a homeomorphism of $\boldsymbol{R}^1$ onto itself and has the form

$$h_2(\xi) = (1-\mu)|\xi|^{\gamma_2-1}\xi \quad \text{for } \xi \in \boldsymbol{R}^1.$$

Now consider the inverse map $f(z) = h_2^{-1}(\xi)$, $\xi = f(z)$. This is the complex gradient of a p-harmonic function. Our goal is to show that f is of class $C^{[p/3]}$ but not of class $C^{[p/3]+1}$, for p an integer $\geqslant 3$. Obviously, f is of class $C^{[1/\gamma_2]}$. On the other hand, f restricted to the real line has the form

$$f(x) = (1-\mu)^{-1/\gamma_2}\cdot(|x|^{1/\gamma_2}x)$$

and it is not in $C^{[1/\gamma_2]+1}$ whenever $1/\gamma_2$ is not an integer. Therefore it suffices to prove that

$$\left[\frac{p}{3}\right] < \frac{1}{\gamma_2} < \left[\frac{p}{3}\right]+1.$$

We have $1/\gamma_2 = (\sqrt{p^2+12}+p)/6 > p/3 \geqslant [p/3]$. Suppose by contradiction that

$$\frac{\sqrt{p^2+12}+p}{6} \geqslant \left[\frac{p}{3}\right]+1.$$

Then we would have $3([p/3]+1)^2 - p([p/3]+1) \leqslant 1$. Since p is an integer the following two cases are possible:

Case 1. $3([p/3]+1)^2 - p([p/3]+1) = 1.$

Hence $p = 3([p/3]+1) - 1/([p/3]+1)$ which is not an integer for $p \geqslant 3$.

Case 2. $3([p/3]+1)^2 - p([p/3]+1) \leqslant 0$,

and we get a contradiction as follows:

$$p \geqslant 3([p/3]+1) > 3 \cdot p/3 = p.$$

This completes the proof of Proposition 1.

7. Quasiradial solutions

In this section we deal with solutions of (2) in two variables $(x, y) \in \boldsymbol{R}^2$ which in polar coordinates have the form

$$u(x, y) = r^{\lambda} \Phi(\sigma), \qquad x+iy = re^{i\sigma}, \tag{19}$$

where Φ is a 2π periodic function and λ is an unknown parameter. The considerations above show that some restrictions for Φ and λ must be taken into account. Thus, since u is a homogeneous function and the singular points of u cannot cover the full ray $\{re^{i\sigma_0}: 0 < r < \infty\}$ a singularity can occur only at the origin $r = 0$. In particular, Φ has to be C^{∞} smooth. In order to get u of class $C^{1,1/(p-1)}$ (see Proposition 4) we should take $\lambda \geqslant p/(p-1)$ or $\lambda = 1$.

For u to be of class $W^2_{2,\mathrm{loc}}(\boldsymbol{R}^n)$, see Proposition 3, the case $\lambda = 1$ is possible if and only if u is a linear function. These we treat as trivial solutions of (2) and we exclude them from our considerations.

We observe that a p-harmonic function of the form (19) with $\lambda > 1$ and C^{∞} in $\boldsymbol{R}^2 \setminus \{0\}$ is necessarily a weak solution of (2). Therefore the p-harmonic equation for the solution u can be written as the following second order equation:

$$|\nabla u|^2 \Delta u + (p-2)(u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}) = 0. \tag{20}$$

Using the following transformation of partial derivatives:

$$\partial_x = \cos\sigma\, \partial_r - \frac{\sin\sigma}{r} \partial_\sigma, \qquad \partial_y = \sin\sigma\, \partial_r + \frac{\cos\sigma}{r} \partial_\sigma$$

we easily get

$$|\nabla u|^2 = r^{2(\lambda-1)}(\lambda^2 \Phi^2 + \dot{\Phi}^2)$$

which shows that Φ and its derivative $\dot{\Phi}$ cannot vanish simultaneously (the zeros of ∇u are isolated):

$$\lambda^2 \Phi^2(\sigma) + \dot{\Phi}^2(\sigma) > 0 \quad \text{for each } \sigma. \tag{21}$$

Further calculations lead to the formulas:

$$\Delta u = r^{\lambda-2}(\lambda^2 \Phi + \ddot{\Phi}),$$

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = r^{2\lambda-4}[\ddot{\Phi}\dot{\Phi}^2 + (2\lambda^2 - \lambda)\dot{\Phi}^2 \Phi + \Phi^3(\lambda-1)\lambda^3].$$

Hence (20) is equivalent to the ordinary differential equation for the function Φ

$$\ddot{\Phi}+A(\Phi,\dot{\Phi})\Phi=0, \tag{22}$$

where, for $P^2+Q^2\neq 0$, we have

$$A(P,Q)=\frac{\lambda^3[\lambda(p-1)-(p-2)]P^2+\lambda[\lambda(2p-3)-(p-2)]Q^2}{\lambda^2P^2+(p-1)Q^2}.$$

Clearly

$$0<\frac{\lambda}{p-1}[\lambda(2p-3)-(p-2)]\leqslant A(P,Q)\leqslant\lambda[\lambda(p-1)-(p-2)].$$

It follows from (22) that the zeros of Φ coincide with the zeros of $\ddot{\Phi}$. From (21) we also easily deduce that

(23) *The zeros of* $\dot{\Phi}$ *are isolated.*

Another point in our study of (22) is that this equation reduces to the following first order equation:

$$\left[\dot{\Phi}^2+\lambda\left(\lambda-\frac{p-2}{p-1}\right)\Phi^2\right]^{\lambda-1}[\dot{\Phi}^2+\lambda^2\Phi^2]^{-\lambda}=\text{const}>0. \tag{24}$$

This can be verified by differentation of the left-hand side of (24). For convenience we normalize Φ, multiplying by a factor, in such a way that

$$\left[\frac{\dot{\Phi}^2}{\lambda[\lambda-(p-2)/(p-1)]}+\Phi^2\right]^{\lambda-1}=\left[\frac{\dot{\Phi}^2}{\lambda^2}+\Phi^2\right]^{\lambda}. \tag{25}$$

Although we are mostly interested in periodic solutions of (25) and $\lambda\geqslant p/(p-1)$ it is advantageous to discard these assumptions for a moment. From now on, Φ will be supposed to satisfy the conditions (21), (23) and (25) and $\lambda>(p-2)/(p-1)$. Obviously if such a solution is of class C^2 then in view of (22) it is C^∞ smooth automatically.

By (25) and (21) we infer that

$$\dot{\Phi}(\sigma_0)=0 \quad \text{if and only if} \quad \Phi(\sigma_0)=\pm 1. \tag{26}$$

The other direct consequence of (25) is the following reformulation of (21):

$$1\leqslant\frac{\dot{\Phi}^2}{\lambda(\lambda-(p-2)/(p-1))}+\Phi^2\leqslant\left[\frac{\lambda}{\lambda-(p-2)/(p-1)}\right]^{\lambda}. \tag{27}$$

In particular, Φ and $\dot{\Phi}$ are bounded functions. Now we shall show that

$$\limsup_{\sigma\to\infty}|\Phi(\sigma)|=1. \tag{28}$$

In fact, if this is not the case then by (26) for sufficiently large σ the values of $|\Phi(\sigma)|$ are uniformly separated from 1. In view of (25) and (27), for these σ's, $\dot{\Phi}(\sigma)$ is uniformly separated from zero, which cannot occur for Φ bounded. Similarly one shows that

$$\limsup_{\sigma \to -\infty} |\Phi(\sigma)| = 1. \tag{29}$$

Actually we have

$$\sup_{-\infty < \sigma < \infty} |\Phi(\sigma)| = 1. \tag{30}$$

Indeed, contradicting this, in view of (28), (29) the supremum in (30) has to be attained at a finite point σ_0 at which $\dot{\Phi}(\sigma_0) = 0$ and by (26), $\Phi(\sigma_0) = \pm 1$.

The latter equality can be specified as follows:

$$-1 = \inf \Phi < \sup \Phi = 1. \tag{31}$$

One of these equalities, say $\inf \Phi = -1$, follows from (28), (29) and (30). Contradicting the second one we would have $\sup \Phi < 1$ and by (28), (29),

$$\liminf_{\sigma \to -\infty} \Phi(\sigma) = \liminf_{\sigma \to \infty} \Phi(\sigma) = -1.$$

Consequently, there would exist σ_0 such that $\dot{\Phi}(\sigma_0) = 0$, $\Phi(\sigma_0) > -1$, i.e. by (26), $\Phi(\sigma_0) = 1$, which contradicts the supposition that $\sup \Phi < 1$.

Now, in connection with (25), (27) and (31) the following algebraic problem arises: prove the existence and uniqueness of a function $F = F(s) = F_\lambda(s)$, for $0 \leqslant s \leqslant 1$, such that

$$1 \leqslant \frac{F(s)}{\lambda[\lambda-(p-2)/(p-1)]} + s \leqslant \left[\frac{\lambda}{\lambda-(p-2)/(p-1)}\right]^\lambda \tag{32}$$

and

$$\left(\frac{F(s)}{\lambda[\lambda-(p-2)/(p-1)]} + s\right)^{\lambda-1} = \left(\frac{F(s)}{\lambda^2} + s\right)^\lambda. \tag{33}$$

The substitution

$$t(s) = \frac{F(s)}{\lambda[\lambda-(p-2)/(p-1)]} + s$$

reduces the problem to inverting the equation

$$s = \frac{\lambda(p-1)}{(p-2)} t^{(\lambda-1)/\lambda} + \left(1 - \frac{\lambda(p-1)}{(p-2)}\right) t, \quad 1 \leqslant t \leqslant \left(\frac{\lambda}{\lambda-(p-2)/(p-1)}\right)^\lambda. \tag{35}$$

Regarded as a function of t, the left-hand side of (35), $s = s(t)$, has negative derivative in the interval

$$1 \leqslant t \leqslant \left(\frac{\lambda}{\lambda-(p-2)/(p-1)}\right)^\lambda$$

and

$$s(1) = 1, \quad s\left\{\left(\frac{\lambda}{\lambda-(p-2)/(p-1)}\right)^{\lambda}\right\} = 0.$$

Therefore $s = s(t)$ is invertible and the inverse function $t = t(s)$ is real-analytic in $s \in [0, 1]$ and decreases from $\left(\frac{\lambda}{\lambda-(p-2)/(p-1)}\right)^{\lambda}$ to 1. This function also depends analytically on the parameter $\lambda > (p-2)/(p-1)$.

Thus our function

$$F(s) = \lambda\left(\lambda - \frac{p-2}{p-1}\right)(t(s) - s)$$

satisfies the conditions (32) and (33). Moreover, F decreases with respect to $s \in [0, 1]$ and by (33)

$$F(s) = \lambda^2\left[\left(\frac{F(s)}{\lambda^2} + s\right) - s\right] = \lambda^2\left[(t(s))^{(\lambda-1)/\lambda} - s\right] \geqslant \lambda^2(1-s). \tag{36}$$

We also compute that

$$\left.\frac{\partial F(s)}{\partial s}\right|_{s=1} = -\lambda[\lambda(p-1)-(p-2)]. \tag{37}$$

The equation (25) is equivalent to

$$\dot{\Phi}^2(\sigma) = F_\lambda(\Phi^2(\sigma)), \qquad -1 \leqslant \Phi(\sigma) \leqslant 1. \tag{38}$$

We now introduce a parameter $\varkappa = \varkappa(\lambda, p)$ by

$$\varkappa = \int_{-1}^{1} \frac{ds}{\sqrt{F_\lambda(s^2)}} \leqslant \frac{1}{\lambda}\int_{-1}^{1}\frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{\lambda}. \tag{39}$$

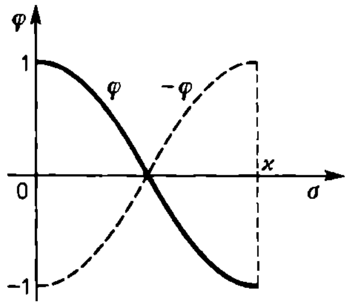
The latter inequality follows from (36). One can see that for $\lambda > 1$ and $s \in [0, 1)$, $F(s) > \lambda^2(1-s)$, thus

$$\begin{aligned} &\varkappa(\lambda, p) < \pi/\lambda \quad \text{for } \lambda > 1,\\ &\varkappa(1, p) = \pi. \end{aligned} \tag{40}$$

Obviously $\varkappa = \varkappa(\lambda, p)$ depends continuously on $\lambda > (p-2)/(p-1)$. We also introduce the auxiliary function $\varphi = \varphi(\sigma) = \varphi_\lambda(\sigma)$ defined on the interval $0 \leqslant \sigma \leqslant \varkappa$ as the inverse to the function

$$\sigma = \sigma(\varphi) = \int_{\varphi}^{1} \frac{ds}{\sqrt{F_\lambda(s^2)}}, \qquad -1 \leqslant \varphi \leqslant 1.$$

By the definition we see that $\varphi = \varphi(\sigma)$ decreases from 1 to -1. Both φ and $-\varphi$ are solutions of (38) in the interval $0 \leqslant \sigma \leqslant \varkappa$.



Notice that from (38) and (37) it follows that

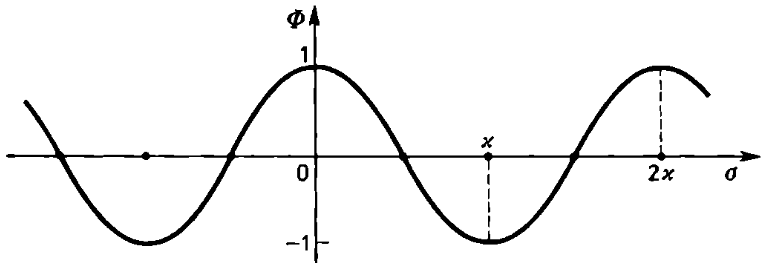
(41) $\dot{\varphi}(0) = \dot{\varphi}(\varkappa) = 0,$

(42) $-\ddot{\varphi}(0) = \ddot{\varphi}(\varkappa) = (-1)\cdot\left.\dfrac{\partial F}{\partial s}\right|_{s=1} = \lambda[\lambda(p-1)-(p-2)] > 0.$

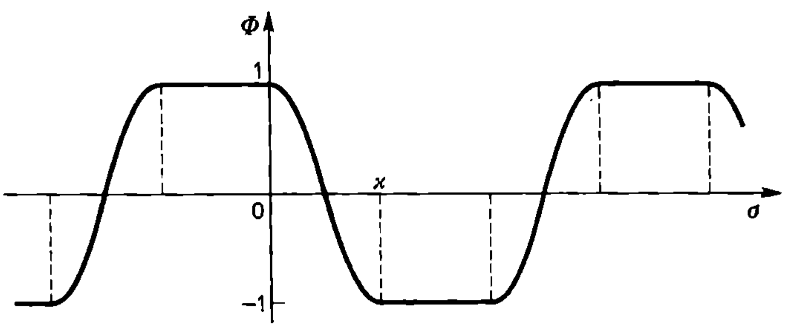
Glueing these particular solutions, φ and $-\varphi$, we are able to construct $C^2(\boldsymbol{R})$ solutions of (22),

(43) $\Phi_\lambda(\sigma) = (-1)^k \varphi_\lambda(\sigma - k\varkappa)$ for $k\varkappa \leqslant \sigma \leqslant (k+1)\varkappa$, $k = 0, \pm 1, \pm 2, \ldots$

This function is $2\varkappa$ periodic.



Remark. It can be proved that each other solution of the type considered arises from the above one by shifting the argument σ. For that, the condition (23) is essential. Without this condition equation (25) admits C^1 solutions different from (43). All of them may be obtained by glueing φ, $-\varphi$ and the constant solutions $+1$ and -1 as in the graph below.



Finally we analyse the period $2\varkappa$ of the solution (43) as a function of λ. From (40) we have

$$2\varkappa(2, p) < \pi \quad \text{and} \quad 2\varkappa(1, p) = 2\pi.$$

By the continuity of $\varkappa(\lambda, p)$ there exists $\lambda_0 \in (1, 2)$ such that

$$2\varkappa(\lambda_0, p) = \pi.$$

The solution (19) corresponding to λ_0 and Φ_{λ_0} is a p-harmonic function of class C^{1,λ_0-1} but not C^2.

Let us remark at the end that the solution (19) cannot be obtained automatically from the considerations of Section 5, because its complex gradient is not a homeomorphism. Its inverse is a double-valued function and in general the solutions (19) can be recovered from the multivalued solutions $h_k(\xi)$ of (16), e.g. for $k = \frac{3}{5}, \frac{5}{2}, \frac{7}{2}$, etc.

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