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THE NUMBER OF PRIME FACTORS OF BINOMIAL COEFFICIENTS

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Given an arbitrarily large integer A, is there an n such that all the numbers in the nth row of the Pascal triangle (except the 1's) have at least A distinct prime factors? That is, can we find an n such that $\omega\left(\binom{n}{r}\right) \geqslant A$ for $1 \leqslant r \leqslant n-1$? This question arose in connexion with a problem in group theory being studied by two colleagues in Cardiff, and they wondered whether n could be chosen to be the product of the first A primes. This turns out to be the case: trivially it is necessary to chose n with $\omega(n) \geqslant A$, and this is also sufficient, as follows.

THEOREM 1. *$\omega\left(\binom{n}{r}\right) \geqslant \omega(n)$ for $1 \leqslant r \leqslant n-1$.*

Proof. The basis of the proof is Legendre's result that if p^α is a prime power dividing $\binom{n}{r}$ then $p^\alpha \leqslant n$. Because of the symmetry of the Pascal triangle we can assume that $r \leqslant \frac{1}{2}n$ from now on.

Case 1: r is large. By Legendre's result

$$\omega\left(\binom{n}{r}\right) \geqslant \log\binom{n}{r}/\log n$$
$$\geqslant r\left(1-\frac{\log r}{\log n}\right) \quad \text{(using the fact that } \binom{n}{r} \geqslant \left(\frac{n}{r}\right)^r\text{)}$$
$$> \omega(n)-1 \quad \text{for large } r.$$

In fact it is not hard to show that the last inequality holds whenever $r \geqslant \log n$.

Case 2: r is small. This case is dealt with similarly, but giving special treatment to the prime factors of n that are bigger than r. There are

$$\geqslant \omega(n)-\pi(r)$$

of these prime factors (where $\pi(r)$ is the number of primes $\leqslant r$) and, since

$$\binom{n}{r}=\frac{n(n-1)\dots(n-r+1)}{1\cdot 2\cdot\dots\cdot r},$$

each of them divides $\binom{n}{r}$ to the same power that it divides n. (It divides the first term of the numerator and no other, and divides no term of the denominator.) The contribution of these prime powers to $\log\binom{n}{r}$ is thus at most $\log n$. Applying Legendre's result to the remaining prime factors of $\binom{n}{r}$ now shows that there are

$$\geqslant \left(\log\binom{n}{r}-\log n\right)/\log n$$

of them, and hence

$$\omega\left(\binom{n}{r}\right)\geqslant r\left(1-\frac{\log r}{\log n}\right)-1+\omega(n)-\pi(r).$$

Thus

$$\omega\left(\binom{n}{r}\right)-\omega(n)\geqslant r\left(1-\frac{\log r}{\log n}\right)-\pi(r)-1,$$

and it is not hard to show that the right hand side is greater than -1 for $4\leqslant r\leqslant \log n$.

The cases $r=2$ and 3 are easily dealt with separately, completing the proof of the theorem.

This result can be expressed as

$$\omega\left(\binom{n}{r}\right)\geqslant \omega\left(\binom{n}{1}\right)\quad \text{for}\quad 1\leqslant r\leqslant n/2,$$

which suggests the question whether $\omega\left(\binom{n}{r}\right)$ is an increasing function of r for r in this range. Erdös pointed out that the answer is 'no', since $\omega\left(\binom{n}{5}\right)>\omega\left(\binom{n}{6}\right)$ for $n=p+5$, where p is any prime congruent to 1 or

65 (mod 72). More generally, for every r there are infinitely many n with

$$\omega\left(\binom{n}{r+1}\right)=\omega\left(\binom{n}{r}\right)-\omega(r+1)+1.$$

This follows from the identity

$$\binom{n}{r+1}=\frac{n-r}{r+1}\binom{n}{r},$$

which shows that if n is chosen to satisfy certain congruences modulo powers of prime factors of $r+1$ then every such prime factor divides $\binom{n}{r}$ but not $\binom{n}{r+1}$. These congruences are consistent with making $n-r$ prime, which ensures that $\binom{n}{r+1}$ contains only one prime factor that does not divide $\binom{n}{r}$. Erdös then asked whether $\omega\left(\binom{n}{r}\right)$ is an 'almost increasing' function of r in the range $1 \leqslant r \leqslant \frac{1}{2}n$, in the sense that for every r there is an $f(r)$ (depending only on r) such that

$$\omega\left(\binom{n}{r+t}\right)\geqslant\omega\left(\binom{n}{r}\right)\quad \forall t>f(r),\quad \forall n\geqslant 2(r+t). \tag{1}$$

Similarly, one could ask for a $g(r)$ such that

$$\omega\left(\binom{n}{r+t}\right)\geqslant\omega\left(\binom{n}{r}\right)-g(r)\quad \forall t\geqslant 0,\quad \forall n\geqslant 2(r+t). \tag{2}$$

We show that such functions $f(r)$ and $g(r)$ do exist and are $o(r)$, although the bounds we obtain for their sizes are probably a long way short of the truth. We also find the smallest possible values for these functions for $r \leqslant 10$. These results depend on three lemmas that are similar in form but differ markedly in their proofs. They overlap considerably, but none of them is a consequence of the others.

LEMMA 1. $\omega\left(\binom{n}{r+t}\right)-\omega\left(\binom{n}{r}\right)\geqslant t-1-\omega\left(\binom{r+t}{r}\right)$ *for* $t\geqslant 0$ *and* n *sufficiently large as a function of* r *and* t.

Proof. We use the identity

$$\binom{n}{r+t}\Big/\binom{n}{r}=\binom{n-r}{t}\Big/\binom{r+t}{t}.$$

A result of Thue ([5], Satz 12) (a consequence of his well known theorem on Diophantine approximation) says that there is only a finite number of pairs of integers whose difference is bounded and all of whose prime factors are bounded in size. Hence, for large enough n, with at most one exception the numbers $n-r$, $n-r-1, \ldots, n-r-t+1$ (that occur in the numerator of $\binom{n-r}{t}$) each have a prime factor greater than $r+t$. This gives at least $t-1$ primes that do not divide $\binom{n}{r}$ but do divide $\binom{n}{r+t}$ (since they divide the numerator on the right of the above identity but not the denominator). On the other hand, the identity shows that there are at most $\omega\left(\binom{r+t}{t}\right)$ primes that divide $\binom{n}{r}$ but not $\binom{n}{r+t}$.

Although the remaining two lemmas are similar in form to Lemma 1, their proofs are entirely elementary and do not use Thue's theorem or anything similar.

In what follows, $s = r+t$.

LEMMA 2. $\omega\left(\binom{n}{r+t}\right)-\omega\left(\binom{n}{r}\right) \geqslant \log\left\{(s!)^*\binom{n}{r+t}\Big/\binom{n}{r}\right\}\Big/\log n-\pi(r+t)$,

where

$$(s!)^* = \prod_{p \leqslant s} p$$

is the square-free kernel of $s!$.

The first term on the right hand side tends to t as $n \to \infty$, so we have

COROLLARY. $\omega\left(\binom{n}{r+t}\right)-\omega\left(\binom{n}{r}\right) \geqslant t-\pi(r+t)$ *for* $t \geqslant 0$ *and* n *sufficiently large as a function of* r *and* t.

For most r and t this corollary is weaker than Lemma 1, but for some r and t the right hand side is larger by 1 than the right hand side of Lemma 1, and this is occasionally useful.

Proof of Lemma 2. The proof is on the same lines as Case 2 of Theorem 1, in that the prime factors of $\binom{n}{r}$ and $\binom{n}{s}$ are divided into classes which are treated differently and Legendre's result is used.

For each prime p we denote the largest powers of p that divide $\binom{n}{r}$ and $\binom{n}{s}$, respectively, by p^{ϱ} and p^{σ}, and we put $\tau = \sigma-\varrho$. We need an upper bound for p^{τ} for each p. We divide the primes into classes as follows:

$S = \left\{p \mid p \text{ divides } \binom{n}{s} \text{ but not } \binom{n}{r}\right\}$, for these we have $p^\tau = p^\sigma \leqslant n$, by Legendre;

$R = \left\{p \mid p \text{ divides } \binom{n}{r} \text{ but not } \binom{n}{s}\right\}$, for these $p^\tau \leqslant 1/p$;

$U = \left\{p \mid p \text{ divides } \binom{n}{r} \text{ and } \binom{n}{s}, \text{ and } p > s\right\}$, for these $p^\tau = 1$;

$V = \left\{p \mid p \text{ divides } \binom{n}{r} \text{ and } \binom{n}{s}, \text{ and } p \leqslant s\right\}$, for these $p^\tau \leqslant n/p$, by Legendre.

We now have

$$\binom{n}{s}\bigg/\binom{n}{r} = \prod_{S,R,U,V} p^\tau \leqslant n^{|S|+|V|} \prod_{R,V} \frac{1}{p}$$

and so

$$n^{|S|-|R|} \geqslant \binom{n}{s}\binom{n}{r}^{-1} \prod_{R,V} \frac{p}{n} \geqslant \binom{n}{s}\binom{n}{r}^{-1} \prod_{p \leqslant s} \frac{p}{n}.$$

The lemma follows on taking logarithms.

THEOREM 2. *The numbers $f(r)$ and $g(r)$ exist for all r and satisfy*

$$f(r) = O(r/\log^{1/2} r) \quad \textit{and} \quad g(r) = O(r/\log^2 r).$$

Proof. The right hand side of Lemma 2 increases with n, and the smallest relevant value of n is $2s$. So replacing n by $2s$ in the right hand side gives

$$\left(\omega\left(\binom{n}{s}\right) - \omega\left(\binom{n}{r}\right)\right) \log 2s \geqslant \log\binom{2s}{s} - \log\binom{2s}{r} + \sum_{p \leqslant s} (\log p - \log 2s)$$

$$\geqslant \frac{t^2}{3s} + O\left(\frac{s}{\log s}\right),$$

using Stirling's formula and the prime number theorem. This is positive for $t > C_1 r/\log^{1/2} r$ and is $> -C_2 r/\log r$ for all t, where C_1 and C_2 are certain constants.

The third lemma is proved in the same way as Lemma 2 except that the class V is subdivided and a different estimate for p^τ is used for the primes in one of the parts. (We omit the details.)

LEMMA 3. $\omega\left(\binom{n}{r+t}\right) - \omega\left(\binom{n}{r}\right) \geqslant \log\left\{\left(\frac{s!}{r!}\right)^* \binom{n}{r+t}\bigg/ r^{\pi(r)} \binom{n}{r}\right\}\bigg/ \log n -$

$-\omega\left(\frac{(r+t)!}{r!}\right)+\eta$, *where* η *is* 1 *if* t *is a prime that divides* s *exactly once and is* 0 *otherwise.*

On letting $n \to \infty$ we have the following corollary, which is always at least as strong as the corollary to Lemma 2.

COROLLARY. $\omega\left(\binom{n}{r+t}\right)-\omega\left(\binom{n}{r}\right) \geqslant t-\omega\left(\frac{(r+t)!}{r!}\right)+\eta$ *for* $t \geqslant 0$ *and* n *sufficiently large as a function of* r *and* t.

Using these three lemmas (the third in a slightly more general form than that given here) and a Commodore Pet microcomputer the smallest possible values of $f(r)$ and $g(r)$ for $r \leqslant 10$ were calculated. That it was possible to do this for so many r was a matter of luck in the way the figures turned out: there is no algorithm for calculating the minimal values of $f(r)$ and $g(r)$ for an arbitrarily given r, even in principle. The results were as follows.

r	1–3	4	5	6	7	8	9	10
$f(r)$	0	1	4	2	1	3	5	4
$g(r)$	0	1	1	1	1	1	1	2

Since $f(r) = 0$ for $r \leqslant 3$, we have

$$\omega\left(\binom{n}{s}\right) \geqslant \omega\left(\binom{n}{r}\right) \quad \text{for } r \leqslant 3 \text{ and } r \leqslant s \leqslant n-r,$$

improving on Theorem 1. The same is true for $r \leqslant 4$ with the single exception that

$$\omega\left(\binom{10}{5}\right) = 3 < \omega\left(\binom{10}{4}\right) = 4.$$

As an illustration, the value $f(10) = 4$ is caused by the fact that

$$\binom{28}{10} = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23$$

has eight distinct prime factors but

$$\binom{28}{14} = 2^3 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 19 \cdot 23$$

has only six.

To remove the effect of isolated exceptional binomial coefficients we can ask for functions $f_\infty(r)$ and $g_\infty(r)$ that satisfy (1) and (2) not necessarily for all $n \geqslant 2(r+t)$ but for all n that are sufficiently large in terms of r (t being constrained not to exceed $\frac{1}{2}n-r$). In other words, for each r finitely many exceptional pairs (t, n) are allowed. Clearly $f_\infty(r)$ and $g_\infty(r)$ are bounded above by $f(r)$ and $g(r)$, and, in view of Lemma 1, $f_\infty(r)$ is bounded above by the largest t for which $\omega\left(\binom{r+t}{t}\right) \geqslant t$. Erdös and Selfridge [1] have given a simple proof that this largest t is $O\left((r/\log r)^{1/2}\right)$ and pointed out that this estimate could be improved by using ideas introduced by Ramachandra in [3]. When this improvement is put into effect it gives the estimate $O(r^{c+\varepsilon})$ for t (and hence for $f_\infty(r)$ and $g_\infty(r)$), where

$$c = (4\sqrt{e}-3)/(10\sqrt{e}-9) = 0.4801\ldots$$

We have used our lemmas to compute the minimal values of $f_\infty(r)$ and $g_\infty(r)$ for $r \leqslant 10$ with the following results.

r	1–4	5	6–8	9	10
$f_\infty(r)$	0	1	0	1 or 2	0, 2 or 3
$g_\infty(r)$	0	1	0	1	0 or 1

The probable value of $f_\infty(9)$ is 2, but this depends on a hypothesis similar to the existence of infinitely many Mersenne primes. (Explicitly, that there are infinitely many primes $p \equiv 7 \pmod{1980}$ for which $(2^p+1)/3$ is also prime.) On Schinzel's Hypothesis H (see [4]) $g_\infty(10) = 1$; and with a further plausible hypothesis $f_\infty(10) = 2$. The fact that these individual values of f_∞ and g_∞ depend on deep hypotheses shows that there is no algorithm for computing these functions in general.

As regards the true order of magnitude of these functions, it is a consequence of Schinzel's Hypothesis that $f_\infty(r)$ is $\Omega(r^{1/e})$ and $g_\infty(r)$ is $\Omega(r^{1/e}/\log r)$. (Without hypothesis I have only succeeded in showing that these functions are $\Omega(\log r/\log\log r)$.) It seems probable that $r^{1/e}$ is about the right maximum order of magnitude for each of the functions f, g, f_∞ and g_∞.

Finally, it is possible to obtain results corresponding to all those we have mentioned for the function $\Omega\left(\binom{n}{r}\right)$ (the total number of prime factors of $\binom{n}{r}$, counting multiplicities – not the Ω of the previous paragraph!). In particular,

$$\Omega\left(\binom{n}{r}\right) \geqslant \Omega(r) \quad \text{for } 1 \leqslant r \leqslant n-r.$$

A full account of this work and other related results is given in [2].

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