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LIMIT BEHAVIOUR OF SINGULAR SOLUTIONS OF SOME SEMILINEAR ELLIPTIC EQUATIONS

LAURENT VÉRON

Département de Mathématiques, Université de Tours
Tours, France

The aim of this paper is to give a survey of the singularity problem concerning the equation $\Delta u = g(u)$ when g is a nondecreasing function.

0. Introduction

Let Ω be an open subset of $\boldsymbol{R}^N$, $N \geqslant 2$, containing 0 and

$$A\colon \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}^N, \quad B\colon \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$$

two Carathéodory functions. Suppose u is continuous in $\Omega' = \Omega - \{0\}$, with first derivatives belonging to some local Lebesgue space $L^{\alpha}_{\mathrm{loc}}(\Omega')$, and satisfies in $\mathscr{D}'(\Omega')$

$$\operatorname{div} A(x, u, Du) + B(x, u, Du) = 0; \tag{0.1}$$

then the singularity question at 0 can be stated with these two problems.

(P1) *Under what conditions can u be extended to Ω as a continuous function $\tilde{u}$ satisfying* (0.1) *in* $\mathscr{D}'(\Omega)$? *In that case the singularity of u at* 0 *is said to be removable.*

(P2) *If u cannot be extended to Ω, what is the behaviour of u near* 0?

A very general answer to those problems has been given by Serrin in his celebrated paper [13] when A and B have the same growth, that is, when they satisfy

$$|A(x, r, p)| \leqslant a_1 |p|^{\alpha-1} + a_2 |r|^{\alpha-1} + a_3, \tag{0.2}$$

$$|B(x, r, p)| \leqslant b_1 |p|^{\alpha-1} + b_2 |r|^{\alpha-1} + b_3, \tag{0.3}$$

$$A(x, r, p)\cdot p \geqslant |p|^{\alpha} - c_2 |r|^{\alpha} - c_3, \tag{0.4}$$

where $\alpha \geqslant 1$ and $a_1, a_2, a_3, b_1, b_2, b_3, c_2, c_3$ are nonnegative constants. He proved the following

THEOREM 0.1. *Suppose u is a weak solution of* (0.1) *in Ω' such that for some $\varepsilon > 0$ the following holds*:

$$u(x) = \begin{cases} O(|x|^{(\alpha - N)/(\alpha - 1) + \varepsilon}), & N > \alpha, \\ O((\operatorname{Log}(1/|x|))^{1-\varepsilon}), & N = \alpha; \end{cases} \tag{0.5}$$

then u can be extended to Ω as a Hölder continuous function $\tilde{u}$ with first derivatives in $L^{\alpha}_{\text{loc}}(\Omega)$ and satisfying (0.1) *in $\mathscr{D}'(\Omega)$.*

THEOREM 0.2. *Suppose u is a weak solution of* (0.1) *in Ω' which is bounded below but not above. Then there exists a positive constant c such that the relations*

$$c^{-1}|x|^{(\alpha - N)/(\alpha - 1)} \leqslant u(x) \leqslant c|x|^{(\alpha - N)/(\alpha - 1)}, \quad N > \alpha, \tag{0.6}$$

$$c^{-1}\operatorname{Log}(1/|x|) \leqslant u(x) \leqslant c \operatorname{Log}(1/|x|), \quad N = \alpha, \tag{0.7}$$

hold for $|x|$ small enough.

Moreover, in the linear case $\alpha = 2$ the removability condition (0.5) can be replaced by the following weaker one:

$$u(x) = \begin{cases} o(|x|^{(\alpha - N)/(\alpha - 1)}), & N > \alpha, \\ o(\operatorname{Log}(1/|x|)), & N = \alpha. \end{cases} \tag{0.8}$$

The situation is completely different when the growth of B is bigger than that of A and in its full generality the problem is now beyond reach. However, in the last five years many results have been obtained concerning the following semilinear case:

$$\Delta u = g(u), \tag{0.9}$$

in particular when $g(u) = -u^q$, $q > 1$ and $u \geqslant 0$, or when $g(u) = u|u|^{q-1}$, $q > 1$, where many explicit computations can be done. As a simple illustration between Serrin's framework and the superlinear case let us consider the equation

$$\Delta u = u|u|^{q-1}, \tag{0.10}$$

with $q > 1$, and let us look for solutions of it in $\boldsymbol{R}^N$ of the form $x \mapsto \alpha|x|^{\beta}$. We immediately find $\beta = -2/(q-1)$ and $\alpha = \big((2/(q-1))(2q/(q-1) - N)\big)^{1/(q-1)}$ and that last quantity is defined only if $q < N/(N-2)$. So under the condition that $1 < q < N/(N-2)$ there exist singular solutions of (0.10) in $\boldsymbol{R}^N - \{0\}$ and their blow up at 0 is much bigger that the potential one, $|x|^{2-N}$, obtained in Theorem 0.2. Moreover, it must be noticed that α no longer exists when $q \geqslant N/(N-2)$ and to this corresponds a very general result due to Brézis and Véron [5].

THEOREM 0.3. *Suppose g is a continuous function on $\mathbf{R}$ such that*

$$\underset{r\to+\infty}{\underline{\lim}}\; g(r)/r^{N/(N-2)} > 0, \qquad \overline{\lim_{r\to-\infty}}\; g(r)/|r|^{N/(N-2)} < 0, \tag{0.11}$$

and $u \in C^1(\Omega')$ is a solution of (0.9) *in $\mathscr{D}'(\Omega')$. Then u can be extended to Ω as a C^1 function $\tilde{u}$ satisfying* (0.9) *in $\mathscr{D}'(\Omega)$.*

We shall not give the original proof of that result but a more general one due to Vázquez and Véron [20], [21] and using the concepts of isotropy and weak and strong singularities. When $1 < q < N/(N-2)$ there exists a complete classification of isotropic singularites of solutions of (0.10) and they are of two types:

(i) *either $|x|^{2/(q-1)}u(x)$ converges to $\pm[(2/(q-1))(2q/(q-1)-N)]^{1/(q-1)}$ as x tends to* 0 *(strong singularity),*

(ii) *or $u(x)/\mu(x)$ converges to some c, which can take any nonzero value, as x tends to zero where $\mu(x) = |x|^{2-N}$ $(N > 2)$ or* $\mathrm{Log}(1/|x|)$ $(N = 2)$ *(weak singularity),*

(iii) *or u can be extended to Ω as a C^2 solution of* (0.9) *in Ω (regular solution).*

In fact, all the singularities of solutions of (0.10) are not isotropic when $1 < q < (N+1)/(N-1)$. To see that we consider the following nonlinear eigenvalue problem on S^{N-1}:

$$-\Delta_{S^{N-1}}\,\omega + \omega|\omega|^{q-1} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\omega, \tag{0.12}$$

where $\Delta_{S^{N-1}}$ is the Laplace–Beltrami operator on S^{N-1}. As $1 < q < (N+1)/(N-1)$,

$$\frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) > N-1$$

which is the second eigenvalue of $-\Delta_{S^{N-1}}$; so (0.12) admits nonconstant solutions which also change sign on S^{N-1}. For such a solution ω the function u_ω defined in $\mathbf{R}^N - \{0\}$ by

$$u_\omega(x) = |x|^{-2/(q-1)}\,\omega(x/|x|) \tag{0.13}$$

is a solution of (0.10) with a *strong nonisotropic singularity* at 0.

In Section 1 we prove a *general isotropy result* due to Vázquez and Véron [20], [21]. A particular case of it reads as follows:

Suppose g is a nondecreasing continuous function and $u \in C^1(\Omega')$ is a solution of (0.9) *such that*

$$\lim_{x\to 0} |x|^{N-1}u(x) = 0. \tag{0.14}$$

Then $u(x)/\mu(x)$ admits a limit in $\mathbf{R} \cup \{-\infty, +\infty\}$ as x tends to 0.

For a solution of (0.9) such that $u(x)/\mu(x)$ admits a limit in $\boldsymbol{R}\cup\{-\infty, +\infty\}$ (that is, an *isotropic singularity*) we introduce two concepts of singularity generalizing the ones obtained in the power case: if the limit of $u(x)/\mu(x)$ is $+\infty$ or $-\infty$ (resp. finite but nonzero) we say that u admits a *strong* (resp. *weak*) *singularity* at 0. In Sections 2 ($N \geqslant 2$) and 3 ($N = 2$) we study those two notions, we give applications to removability problems and list some open questions.

In Section 4 we give the proof of the complete classification theorem concerning isotropic singularities of solutions of (0.10).

In Section 5 we study nonisotropic singularities of solutions of (0.10) in the plane, and in Section 6 the symmetries of singular solutions.

The contents of this survey is the following:

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1. The isotropy theorem

Let (r, σ) be the spherical coordinates in $\boldsymbol{R}^N$ ($N \geqslant 2$), $r > 0$, $\sigma \in S^{N-1}$, $|S^{N-1}|$ the $(N-1)$-measure of the unit sphere in $\boldsymbol{R}^N$, and for any function v defined in $\boldsymbol{R}^+ \times S^{N-1}$ let $\bar{v}$ be its average on S^{N-1}:

$$\bar{v}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} v(r, \sigma)\, d\sigma. \tag{1.1}$$

We also recall that $\mu(x) = |x|^{2-N}$ ($N \geqslant 3$) or $\operatorname{Log}(1/|x|)$ ($N = 2$). The isotropy result of Vázquez and Véron [20], [21] is the following:

THEOREM 1.1. *Suppose that g is a continuous nondecreasing function and $u \in C^1(\Omega')$ is a solution of*

$$\Delta u = g(u) \tag{1.2}$$

in the sense of distributions in Ω' such that

$$\lim_{r\to 0} r^{N-1} \|u(r, \cdot) - \bar{u}(r)\|_{L^2(S^{N-1})} = 0. \tag{1.3}$$

Then $u(x)/\mu(x)$ admits a limit in $\boldsymbol{R}\cup\{-\infty, +\infty\}$ as x tends to 0.

The proof in the case $N = 2$ can be found in [20] and we give here a sketch of the proof in the (more difficult) case $N \geqslant 3$ (see also [21]). Without any loss of generality we can assume that $\Omega \supset \bar{B}_1 = \{x \in \boldsymbol{R}^N \colon |x| \leqslant 1\}$. From regularity results for elliptic equations, (1.2) is satisfied a.e. and we have in spherical coordinates

$$\frac{\partial^2 u}{\partial r^2}+\frac{N-1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Delta_{S^{N-1}}u = g(u) \tag{1.4}$$

a.e. in $(0, 1] \times S^{N-1}$ where $\Delta_{S^{N-1}}$ is the Laplace–Beltrami operator. We make the following change of variables and the unknown function:

$$s = \frac{r^{N-2}}{N-2}, \qquad u(r, \sigma) = r^{2-N} v(s, \sigma); \tag{1.5}$$

so the function v satisfies

$$s^2\frac{\partial^2 v}{\partial s^2}+\frac{1}{(N-2)^2}\Delta_{S^{N-1}}v = (N-2)^{(4-N)/(N-2)}s^{N/(N-2)}g\left(\frac{v}{s(N-2)}\right) \tag{1.6}$$

a.e. in $\big(0, 1/(N-2)\big] \times S^{N-1}$. If $\bar{v}(s)$ is the average of $v(s, \cdot)$ on S^{N-1}, the following estimate is the keystone of the proof of Theorem 1.1.

LEMMA 1.1. *Under the hypotheses of Theorem* 1.1, *there exists a constant* C *independent of* s *such that*

$$\|v(s, \cdot)-\bar{v}(s)\|_{L^\infty(S^{N-1})} \leqslant Cs^{(N-1)/(N-2)} \tag{1.7}$$

for any $s \in \big(0, 1/(N-2)\big]$.

Before proving (1.7) we need the following L^2 version of it.

LEMMA 1.2. *Under the hypotheses ot Theorem* 1.1, *there exists a constant* C *independent of* s *such that*

$$\|v(s, \cdot)-\bar{v}(s)\|_{L^2(S^{N-1})} \leqslant Cs^{(N-1)/(N-2)} \tag{1.8}$$

for any $s \in \big(0, 1/(N-2)\big]$.

Proof. We let $\overline{g\big(v/[s(N-2)]\big)}$ be the average of $g\big(v/[s(N-2)]\big)$ on S^{N-1}, hence

$$s^2\frac{d^2 \bar{v}}{ds^2}+\frac{1}{(N-2)^2}\Delta_{S^{N-1}}\bar{v} = (N-2)^{(4-N)/(N-2)}s^{N/(N-2)}\overline{g\left(\frac{v}{s(N-2)}\right)}. \tag{1.9}$$

As $N-1$ is the second eigenvalue of $-\Delta_{S^{N-1}}$ and $\bar{v}$is the projection of v onto the first eigenspace [3] we have

$$-\int_{S^{N-1}}(v-\bar{v})\Delta_{S^{N-1}}(v-\bar{v})\,d\sigma \geqslant (N-1)\int_{S^{N-1}}(v-\bar{v})^2\,d\sigma. \tag{1.10}$$

Moreover, from monotonicity

$$(1.11)\qquad \int_{S^{N-1}} \left(g\left(\frac{v}{s(N-2)}\right)-\overline{g\left(\frac{v}{s(N-2)}\right)}\right)(v-\bar{v})\,d\sigma$$
$$= \int_{S^{N-1}} \left(g\left(\frac{v}{s(N-2)}\right)-g\left(\frac{\bar{v}}{s(N-2)}\right)\right)(v-\bar{v})\,d\sigma \geqslant 0,$$

so we deduce from (1.6) and (1.9)

$$(1.12)\qquad s^2 \int_{S^{N-1}} (v-\bar{v})\frac{\partial^2}{\partial s^2}(v-\bar{v})\,d\sigma-\frac{N-1}{(N-2)^2}\int_{S^{N-1}} (v-\bar{v})^2\,d\sigma \geqslant 0.$$

If we set $Y(S) = \|v(s,\cdot)-\bar{v}(s)\|_{L^2(S^{N-1})}$ and $I = \{s\in(0, 1/(N-2)]:\ Y(s)>0\}$, then I is open and we have a.e. on I

$$(1.13)\qquad Y(s)\frac{d^2 Y}{ds^2}(s) \geqslant \int_{S^{N-1}} (v-\bar{v})\frac{\partial^2}{\partial s^2}(v-\bar{v})\,d\sigma,$$

which yields

$$(1.14)\qquad s^2\frac{d^2 Y}{ds^2}-\frac{N-1}{(N-2)^2}Y \geqslant 0,$$

a.e. on I. Moreover, the condition (1.3) means $Y(s) = o(s^{-1/(N-2)})$ near 0. Set $\alpha = (N-2)^{(N-1)/(N-2)}\,Y(1/(N-2))$. For any $\varepsilon>0$

$$Y_\varepsilon(s) = \varepsilon s^{-1/(N-2)}+\alpha s^{(N-1)/(N-2)}$$

is a solution of

$$(1.15)\qquad s^2\frac{d^2 Y_\varepsilon}{ds^2}-\frac{N-1}{(N-2)^2}Y_\varepsilon = 0$$

on $(0, 1/(N-2)]$. As $Y = o(Y_\varepsilon)$ near 0 and $Y(1/(N-2)) \leqslant Y_\varepsilon(1/(N-2))$ we deduce from the maximum principle that $Y \leqslant Y_\varepsilon$ on $(0, 1/(N-2)]$. Letting ε go to 0 we get (1.8).

Proof of Lemma 1.1. We first need some a priori estimates concerning the solution ω of the following Dirichlet problem:

$$(1.16)\qquad \begin{aligned} & s^2\frac{\partial^2\omega}{\partial s^2}+\frac{1}{(N-2)^2}\Delta_{S^{N-1}}\,\omega = 0 && \text{in } (\varrho,\tau)\times S^{N-1},\\ & \omega(\varrho,\cdot)=\alpha(\cdot),\quad \omega(\tau,\cdot)=\beta(\cdot) && \text{in } S^{N-1}, \end{aligned}$$

where α and β belong to $L^2(S^{N-1})$. Introducing the semigroup T of contractions on $L^2(S^{N-1})$ generated by

$$-\left(-\frac{1}{(N-2)^2}\Delta_{S^{N-1}}+\tfrac{1}{4}\,\mathrm{Id}\right)^{1/2}$$

a straightforward computation shows that the function w defined by

$$w(s,\cdot)=\sqrt{\frac{s}{\tau}}\,T\left(\operatorname{Log}\frac{\tau}{s}\right)\beta^+(\cdot)+\sqrt{\frac{s}{\varrho}}\,T\left(\operatorname{Log}\frac{s}{\varrho}\right)\alpha^+(\cdot)$$

satisfies

$$(1.17)\qquad \begin{aligned} &s^2\frac{\partial^2 w}{\partial s^2}+\frac{1}{(N-2)^2}\Delta_{S^{N-1}}w=0 \quad \text{in } (\varrho,\tau)\times S^{N-1},\\ &w(\varrho,\cdot)\geqslant\alpha(\cdot),\qquad w(\tau,\cdot)\geqslant\beta \quad \text{in } S^{N-1}. \end{aligned}$$

Hence $w\geqslant\omega$ in $(\varrho,\tau)\times S^{N-1}$. Using regularizing effects from $L^2(S^{N-1})$ into $L^\infty(S^{N-1})$ (see [21] or [23] and [27] for details) we have for any $t>0$ and $\gamma\in L^2(S^{N-1})$

$$(1.18)\qquad \|T(t)\gamma(\cdot)\|_{L^\infty(S^{N-1})}\leqslant C\left(1+\frac{1}{t}\right)^{(N-1)/2}\exp(-t/2)\,\|\gamma(\cdot)\|_{L^2(S^{N-1})}.$$

From (1.17) and (1.18) we get the following internal estimate:

$$(1.19)\qquad \begin{aligned} \|\omega(s,\cdot)\|_{L^\infty(S^{N-1})}\leqslant C\Bigg\{&\left[1+\frac{1}{\operatorname{Log}(s/\varrho)}\right]^{(N-1)/2}\|\alpha(\cdot)\|_{L^2(S^{N-1})}\\ &+\frac{s}{\tau}\left[1+\frac{1}{\operatorname{Log}(\tau/s)}\right]^{(N-1)/2}\|\beta(\cdot)\|_{L^2(S^{N-1})}\Bigg\}. \end{aligned}$$

End of the proof. Let $a, b\in \mathbf{R}$, $0<\varrho<\tau<1/(N-2)$ and y be the solution of the following two-point problem:

$$(1.20)\qquad \begin{aligned} &s^2\frac{d^2y}{ds^2}=(N-2)^{(4-N)/(N-2)}\,s^{N/(N-2)}g\left(\frac{y}{s(N-2)}\right)\quad \text{in } (\varrho,\tau),\\ &y(\varrho)=a,\qquad y(\tau)=b. \end{aligned}$$

Set $w=v-y$, so w satisfies

$$(1.21)\qquad s^2\frac{\partial^2 w}{\partial s^2}+\frac{1}{(N-2)^2}\Delta_{S^{N-1}}w=(N-2)^{(4-N)/(N-2)}s^{N/(N-2)}hw$$

where

$$h=\begin{cases}\big(g(v/[s(N-2)])-g(y/[s(N-2)])\big)/(v-y) & \text{if } v\neq y,\\ 0 & \text{if } v=y.\end{cases}$$

From the monotonicity of g, h is nonnegative. Let ω be the solution of (1.16) with $\alpha(\cdot)=\big(v(\varrho,\cdot)-a\big)^+$ and $\beta(\cdot)=\big(v(\tau,\cdot)-b\big)^+$. As ω is nonnegative we deduce from comparison principles that $w\leqslant\omega$. If we minorize w in the same way we get from (1.19)

$$\text{(1.22)}\quad \|v(s,\cdot)-y(s)\|_{L^\infty(S^{N-1})}\leqslant C\left\{\left[1+\frac{1}{\operatorname{Log}(s/\varrho)}\right]^{(N-1)/2}\|v(\varrho,\cdot)-a\|_{L^2(S^{N-1})}\right.$$
$$\left.+\frac{s}{\tau}\left[1+\frac{1}{\operatorname{Log}(\tau/s)}\right]^{(N-1)/2}\|v(\tau,\cdot)-b\|_{L^2(S^{N-1})}\right\}.$$

As y is radially symmetric, (1.22) remains true after replacing $v(s,\cdot)$ by $\bar{v}(s)$. If we take $a=\bar{v}(\varrho)$ and $b=\bar{v}(\tau)$ we get

$$\text{(1.23)}\quad \|v(s,\cdot)-\bar{v}(s)\|_{L^\infty(S^{N-1})}$$
$$\leqslant 2C\left\{\left[1+\frac{1}{\operatorname{Log}(s/\varrho)}\right]^{(N-1)/2}\|v(\varrho,\cdot)-\bar{v}(\varrho)\|_{L^2(S^{N-1})}\right.$$
$$\left.+\frac{s}{\tau}\left[1+\frac{1}{\operatorname{Log}(\tau/s)}\right]^{(N-1)/2}\|v(\tau,\cdot)-\bar{v}(\tau)\|_{L^2(S^{N-1})}\right\}.$$

If we take $\tau=2s=4\varrho$ and use (1.8) we get (1.7).

Proof of Theorem 1.1. We shall distinguish the cases of $\{\bar{v}(s)\}$ bounded and unbounded in $(0, 1/(N-2)]$.

Step 1: Assume that $\{\bar{v}(s)\}$ remains bounded when $0<s\leqslant 1/(N-2)$, so there exists a real c and a sequence $\{s_n\}$ such that

$$\lim_{n\to\infty} s_n=0 \quad\text{and}\quad \lim_{n\to\infty}\bar{v}(s_n)=c.$$

We then claim that

$$\text{(1.24)}\qquad \lim_{s\to 0} v(s,\cdot)=c.$$

We first assume that $c>0$ (or $c<0$ in the same way). From (1.7) there exists $n_0\in N$ such that $v(s_n,\sigma)\geqslant c/2>0$ for $n\geqslant n_0$ and $\sigma\in S^{N-1}$. Let $\tilde{g}(r)=g(r)-g(0)$ and $\tilde{v}$ be the solution of

$$\text{(1.25)}\quad \begin{gathered} s^2\frac{\partial^2\tilde{v}}{\partial s^2}+\frac{1}{(N-2)^2}\Delta_{S^{N-1}}\tilde{v} \\ =(N-2)^{(4-N)/(N-2)}s^{N/(N-2)}\tilde{g}\left(\frac{\tilde{v}}{s(N-2)}\right)\quad\text{in } (s_n, s_{n_0})\times S^{N-1}, \\ \tilde{v}(s_{n_0},\cdot)=c/2,\qquad \tilde{v}(s_n,\cdot)=c/2\quad\text{in } S^{N-1}. \end{gathered}$$

From comparison principles, $\tilde{v}\geqslant 0$ in $(s_n, s_{n_0})\times S^{N-1}$. Let Λ be the solution of the following differential equation:

$$\text{(1.26)}\quad \begin{gathered} s^2\frac{d^2\Lambda}{ds^2}+(N-2)^{(4-N)/(N-2)}s^{N/(N-2)}|g(0)|=0\quad\text{in } (s_n, s_{n_0}), \\ \Lambda(s_n)=\Lambda(s_{n_0})=0. \end{gathered}$$

It is clear that $\Lambda(s) \geqslant 0$; moreover, there exists K independent of $n \geqslant n_0$ such that

$$(1.27) \qquad \Lambda(s) \leqslant Ks \quad \text{in } [s_n, s_{n_0}].$$

If we set $w = \tilde{v} - \Lambda$ we get

$$(1.28) \qquad s^2 \frac{\partial^2 w}{\partial s^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} w \geqslant (N-2)^{(4-N)/(N-2)} s^{N/(N-2)} g\left(\frac{w}{s(N-2)}\right)$$

in $(s_n, s_{n_0}) \times S^{N-1}$. From comparison principles we deduce

$$(1.29) \qquad v(s, \cdot) \geqslant w(s, \cdot) = \tilde{v}(s, \cdot) - \Lambda(s) \geqslant -Ks.$$

Hence $v_k(s, \cdot) = v + Ks$ is nonnegative in $(0, s_{n_0}] \times S^{N-1}$. Let $\bar{v}_k(s)$ be the average of $v_k(s, \cdot)$ on S^{N-1} and

$$g_k(r) = g\left(r - \frac{K}{N-2}\right) - g\left(-\frac{K}{N-2}\right).$$

We then have

$$(1.30) \qquad s^2 \frac{d^2 \bar{v}_k}{ds^2} = (N-2)^{(4-N)/(N-2)} s^{N/(N-2)} \left(g_k\left(\frac{v_k}{s(N-2)}\right) + g\left(-\frac{K}{N-2}\right)\right).$$

As $v_k \geqslant 0$ in $(0, s_{n_0}] \times S^{N-1}$ and $g_k(0) = 0$ we get

$$(1.31) \qquad s^2 \frac{d^2 \bar{v}_k}{ds^2} \geqslant (N-2)^{(4-N)/(N-2)} s^{N/(N-2)} g\left(-\frac{K}{N-2}\right),$$

which implies that the function

$$(1.32) \qquad s \mapsto \bar{v}_k(s) - \frac{1}{2N}(N-2)^{(4-N)/(N-2)} s^{N/(N-2)} g\left(-\frac{K}{N-2}\right)$$

is convex. As

$$\lim_{n\to\infty} \left(\bar{v}_k(s_n) - \frac{1}{2N}(N-2)^{(N-4)/(N-2)} s_n^{N/(N-2)} g\left(-\frac{K}{N-2}\right)\right) = c,$$

we deduce

$$\lim_{s\to 0} \left(\bar{v}_k(s) - \frac{1}{2N}(N-2)^{(4-N)/(N-2)} s^{N/(N-2)} g\left(-\frac{K}{N-2}\right)\right) = c$$

which implies

$$\lim_{s\to 0} \bar{v}(s) = c \quad \text{and} \quad \lim_{s\to 0} v(s, \sigma) = c$$

uniformly with respect to $\sigma \in S^{N-1}$, that is, $\lim_{x\to 0} |x|^{N-2} u(x) = c$.

If $c = 0$ then $\lim_{s\to 0} \bar{v}(s) = 0$. Otherwise there would exist $c' \neq 0$ and a sequence $\{s_{n'}\}$ converging to 0 such that $\lim_{n'\to\infty} \bar{v}(s_{n'}) = c'$, which would imply as before $\lim_{s\to 0} v(s, \sigma) = c'$ uniformly on S^{N-1} which would contradict $\lim_{n\to\infty} \bar{v}(s_n) = 0$. Using (1.7) this yields $\lim_{s\to 0} v(s, \sigma) = 0$ uniformly on S^{N-1}.

Step 2: Assume $\{\bar{v}(s)\}$ is unbounded near 0. There exists a sequence $\{s_n\}$ converging to 0 such that $\lim_{n\to\infty}\bar{v}(s_n) = +\infty$ (or $-\infty$ in the same way). As in Step 1 the function defined by (1.32) is convex on some interval $(0, s_{n_0}]$ which implies $\lim_{s\to 0}\bar{v}(s) = +\infty$ and $\lim_{x\to 0}|x|^{N-2}u(x) = +\infty$.

Remark 1.1. A stronger form for condition (1.3) is to assume

$$\lim_{x\to 0}|x|^{N-1}u(x) = 0. \tag{1.33}$$

This condition is satisfied as soon as $(sg(s))^{-1/2}$ is integrable at $+\infty$ and $-\infty$ and

$$\lim_{r\to+\infty} r^{1/(N-1)}\left(\int_{-\infty}^{-r}\frac{ds}{\sqrt{sg(s)}}+\int_{r}^{+\infty}\frac{ds}{\sqrt{sg(s)}}\right) = 0. \tag{1.34}$$

This is in particular the case if

$$\lim_{|r|\to+\infty}|g(r)|/|r|^{(N+1)/(N-1)} = 0$$

(see [25, Lemma 2.1]).

Remark 1.2. If $\lim_{x\to 0}u(x)/\mu(x) = 0$ then u remains bounded near 0 and it can be extended as a C^1 solution of (1.2) in Ω. To see that we consider a ball $\bar{B}_\varrho \subset \Omega$, $\varrho > 0$, and set $v_\varepsilon(x) = \varepsilon\mu(x)+K$ where K is chosen such that v_ε remains positive in $\bar{B}_\varrho - \{0\}$ and $K \geqslant \max_{|x|=\varrho}u(x)$. If ψ satisfies

$$\begin{aligned} \Delta\psi &= -|g(0)| && \text{in } B_\varrho,\\ \psi &= 0 && \text{on } \partial B_\varrho, \end{aligned} \tag{1.35}$$

then $v_\varepsilon+\psi$ is clearly a supersolution for (1.2) and $u \leqslant v_\varepsilon$ in $B_\varrho - \{0\}$. Letting $\varepsilon \to 0$ yields $u^+ \in L^\infty(B_\varrho)$. We do the same with u^- and we conclude using Theorem 0.1 and regularity results for elliptic equations.

Remark 1.3. A slightly weaker form for condition (1.3) is to assume that there exists a sequence $\{r_n\}$ converging to 0 such that

$$\lim_{n\to+\infty} r_n^{N-1}\,\|u(r_n,\cdot)-\bar{u}(r_n)\|_{L^2(S^{N-1})} = 0. \tag{1.36}$$

The same comparison arguments yield (1.8) and (1.7).

2. Weak and strong singularities in R^N, $N \geqslant 3$

In this section we suppose $\Omega \subset R^N$, $N \geqslant 3$, and we say that a solution $u \in C^1(\Omega')$ of (1.2) admits a *strong* (resp. a *weak*) *isotropic singularity at* 0 if $|x|^{N-2}u(x)$ converges to $+\infty$ or $-\infty$ (resp. to some nonzero real number) as x tends to 0. The main problems are to find conditions on g ensuring the existence of such singularities. Many of the results presented here are due to

Vázquez and Véron [21], and without loss of generality we assume that all the singularities are nonnegative.

THEOREM 2.1. *Assume g is a continuous nondecreasing real function. Then the equation* (1.2) *admits solutions with weak singularities if and only if g satisfies for some $\alpha > 0$*

$$\int_{\alpha}^{+\infty} g(s)\, s^{-2(N-1)/(N-2)}\, ds < +\infty. \tag{2.1}$$

Moreover, if g satisfies for some $\alpha' > 0$

$$\int_{\alpha'}^{+\infty} \frac{ds}{\sqrt{s g(s)}} < +\infty, \tag{2.2}$$

then there exist solutions of (1.2) *with strong singularities at* 0.

Proof. Step 1: We suppose that (2.1) holds and for $\varepsilon > 0$ and $c > 0$ we let y_ε be the solution of

$$\begin{gathered} \frac{d^2}{ds^2} y_\varepsilon = (N-2)^{(4-N)/(N-2)} (s+\varepsilon)^{N/(N-2)-2}\, g\left(\frac{y_\varepsilon}{(s+\varepsilon)(N-2)}\right) \quad \text{on } (0, 1], \\ y_\varepsilon(0) = c, \quad y_\varepsilon(1) = 0. \end{gathered} \tag{2.3}$$

For the sake of simplicity we assume that $g(0) = 0$, so y_ε is a nonnegative nonincreasing convex function and $y_\varepsilon \leqslant c$ on $[0, 1]$. Integrating (2.3) yields

$$\frac{d}{ds} y_\varepsilon(s) = \frac{d}{ds} y_\varepsilon(1) - c_1 \int_0^1 (\sigma+\varepsilon)^{N/(N-2)-2}\, g\left(\frac{y_\varepsilon}{(\sigma+\varepsilon)(N-2)}\right) d\sigma,$$

with $c_1 = (N-2)^{(4-N)/(N-2)}$. As y_ε and $d^2 y_\varepsilon/ds^2$ remain bounded on $[\frac{1}{2}, 1]$, so does $(d/ds)\, y_\varepsilon(1)$. So we get for $0 \leqslant \sigma \leqslant \tau \leqslant 1$

$$|y_\varepsilon(\tau) - y_\varepsilon(\sigma)| \leqslant c'(\tau-\sigma) + c_1 \int_\sigma^\tau \int_s^1 (\sigma+\varepsilon)^{N/(N-2)-2}\, g\left(\frac{c}{(N-2)(\sigma+\varepsilon)}\right) d\sigma\, ds.$$

If we take $\varepsilon < 1$ we have

$$\int_\sigma^\tau \int_s^1 (\sigma+\varepsilon)^{N/(N-2)-2}\, g\left(\frac{c}{(N-2)(\sigma+\varepsilon)}\right) d\sigma\, ds \leqslant \int_{\sigma+\varepsilon}^{\tau+\varepsilon} \int_s^2 \sigma^{N/(N-2)-2}\, g\left(\frac{c}{\sigma(N-2)}\right) d\sigma\, ds.$$

We define Φ on $(0, 2]$ by

$$\Phi(x) = \int_x^2 \int_s^2 \sigma^{N/(N-2)-2}\, g\left(\frac{c}{(N-2)\sigma}\right) d\sigma\, ds. \tag{2.4}$$

From (2.1), $\lim_{x\to 0} \Phi(x)$ exists so Φ can be extended to $[0, 2]$ as a continuous function $\tilde{\Phi}$, and $\tilde{\Phi}$ is uniformly continuous on $[0, 2]$ and we get for $0 \leqslant \sigma \leqslant \tau \leqslant 1$

$$|y_\varepsilon(\tau) - y_\varepsilon(\sigma)| \leqslant c'(\tau - \sigma) + c_1 |\tilde{\Phi}(\tau + \varepsilon) - \tilde{\Phi}(\sigma + \varepsilon)|. \tag{2.5}$$

Hence the family of functions $\{y_\varepsilon\}_{0<\varepsilon<1}$ is equicontinuous on $[0, 1]$. From Arzelà–Ascoli theorem there exists a sequence $\{\varepsilon_n\}$ and $y \in C^0([0, 1])$ such that $\{y_{\varepsilon_n}\}$ converges to y uniformly on $[0, 1]$ and y is then the (unique) solution of

$$\begin{gathered} \frac{d^2 y}{ds^2} = (N-2)^{(4-N)/(N-2)} s^{N/(N-2)-2} g\left(\frac{y}{s(N-2)}\right) \quad \text{on } (0, 1], \\ y(0) = c, \quad y(1) = 0. \end{gathered} \tag{2.6}$$

Clearly the function $x \mapsto |x|^{2-N} y(|x|^{N-2}/(N-2))$ is a solution of (1.2) with a weak singularity at 0.

Step 2: We suppose that (2.1) does not hold and that there exists a solution u of (1.2) such that $\lim_{x \to 0} |x|^{N-2} u(x) = c > 0$. Let $\bar{u}(r)$ be the average of $u(r, \cdot)$ on S^{N-1}. Then $\lim_{r \to 0} r^{N-2} \bar{u}(r) = c$ and

$$\frac{d}{dr}\left(r^{N-1} \frac{d\bar{u}}{dr}\right) \geqslant r^{N-1} g\left(\frac{c}{2r^{N-2}}\right) \tag{2.7}$$

holds for $0 < r < R$. Integrating (2.7) yields

$$-r^{N-1} \frac{d\bar{u}}{dr}(r) \geqslant -R^{N-1} \frac{d\bar{u}}{dr}(R) + \int_r^R s^{N-1} g\left(\frac{c}{2s^{N-2}}\right) ds,$$

which implies

$$\lim_{r \to 0} r^{N-1} \frac{d\bar{u}}{dr}(r) = -\infty \quad \text{and} \quad \lim_{r \to 0} r^{N-2} \bar{u}(r) = +\infty,$$

contradiction.

Step 3: We assume that (2.1) and (2.2) hold. From a result of Vázquez [18], for any compact subset $K \subset \Omega'$ there exists $c_K > 0$ such that for any solution u of (1.2) in Ω' we have $u(x) \leqslant c_K$ for any $x \in K$. From Step 1 we let u_c $(c > 0)$ be the solution of (1.2) in $B_1 - \{0\}$ (we may suppose that $\Omega \supset \bar{B}_1$) vanishing on ∂B_1 and such that $\lim_{x \to 0} |x|^{N-2} u_c(x) = c$. As c goes to $+\infty$, $\{u_c\}$ increases and converges to a solution of (1.2) with a strong singularity at 0, which ends the proof.

Remark 2.1. As we have seen, the strong singularities of Theorem 2.1 are obtained as the upper envelope of solutions of (1.2) with weak singularities at 0. When g is a power there is no other way to obtain strong singularities. In the case of a general g this is not always true as the following result shows:

THEOREM 2.2. *Assume g is a continuous real-valued function satisfying*

$$\int_{\alpha}^{+\infty} g(s)s^{-2(N-1)/(N-2)}\,ds = \int_{\alpha}^{+\infty} \frac{ds}{\sqrt{sg(s)}} = +\infty, \tag{2.8}$$

for any $\alpha > 0$. Then the equation (1.2) *admits no solution with a weak singularity at* 0 *but infinitely many with a strong one.*

The proof is a consequence of the following continuation lemma concerning solutions of some O.D.E.

LEMMA 2.1. *Let a be a real number and f a continuous positive function defined on $[a, +\infty)$. Every nonnegative solution θ of*

$$\frac{d^2\theta}{dt^2} = f(t)g(\theta) \tag{2.9}$$

defined in an interval $[a, a^*)$ to the right of a can be continued as a solution of (2.9) *on $[a, +\infty)$ if and only if g satisfies*

$$\int_{\alpha}^{+\infty} \frac{ds}{\sqrt{sg(s)}} = +\infty, \tag{2.10}$$

for any $\alpha > 0$.

Proof. Step 1: We first assume that θ is defined on a maximal interval $[a, a^*)$, $a^* < +\infty$. Without loss of generality we can assume that θ is convex and nondecreasing, that $g(\theta)$ is nondecreasing and that $\lim_{t\to a^*}\theta(t) = +\infty$. We set

$$j(r) = \int_0^r g(s)\,ds \quad \text{and} \quad F = \|f\|_{L^\infty(a,a^*)}$$

so we get

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right)^2 \leqslant 2F\frac{d}{dt}(j(\theta)),$$

which implies

$$\frac{\theta'(t)}{\sqrt{j(\theta(t))}} \leqslant \frac{\theta'(a)}{\sqrt{j(\theta(t))}} + 2\sqrt{F},$$

$$\int_{\theta(a)}^{\theta(t)} \frac{ds}{\sqrt{j(s)}} \leqslant \theta'(a)\int_a^t \frac{ds}{\sqrt{j(\theta(s))}} + 2\sqrt{F}\,(t-a).$$

Hence

$$\int_{\theta(a)}^{+\infty} \frac{ds}{\sqrt{j(s)}} < +\infty$$

and (2.10) is not satisfied since $j(s) \leqslant sg(s)$ (we assume $g(0) = 0$ for the sake of simplicity).

Step 2: We assume

$$\int_a^{+\infty} \frac{ds}{\sqrt{sg(s)}} < +\infty \qquad \text{for some } \alpha > 0$$

and we fix $A > a$. As f is continuous and positive we have $0 < \alpha \leqslant f(t) \leqslant \beta$ for any t in $[a, A]$, α and β being two constants. From Vázquez' a priori estimate [18] there exists a function γ defined on (a, A) such that

$$\theta(t) \leqslant \gamma(t) \qquad \forall t \in (a, A) \tag{2.11}$$

holds for any solution of (2.9) on (a, A). Moreover, γ can be supposed to be convex and $\lim_{t\to a} \gamma(t) = \lim_{t\to A} \gamma(t) = +\infty$. If θ is a solution of (2.9) on $(a, a+\varepsilon)$ such that

$$\theta(a) > \min_{a<t<A} \gamma(t) \qquad \text{and} \qquad \theta'(a) > 0,$$

it is clear that $\theta(t^*) = \gamma(t^*)$ for some $t^* < A$ and $\theta(t) > \gamma(t)$ for $t > t^*$. So θ does not satisfy (2.11) which means that θ cannot be defined on the whole (a, A) and there exists $a^* < A$ such that $\lim_{t\uparrow a^*} \theta(t) = +\infty$.

Proof of Theorem 2.2. From Theorem 2.1 there exists no solution of (1.2) with a weak singularity at 0. Let ζ be the solution of

$$\begin{aligned} -\Delta\zeta + g(\zeta) &= 0 \quad \text{in } B_1, \\ \zeta &= 0 \quad \text{on } \partial B_1. \end{aligned} \tag{2.12}$$

ζ is radial, and if we set $t = r^{2-N}$ and $\xi(t) = \zeta(r)$ then

$$t^2 \frac{d^2\xi}{dt^2} = \frac{t^{-2/(N-2)}}{(N-2)^2} g(\xi) \qquad \text{for } t > 1, \tag{2.13}$$

and ξ is the only solution of (2.13) which is bounded at infinity. For any $\alpha > (d\xi/dt)(1)$, let w_α be the solution of the Cauchy problem

$$\begin{aligned} t^2 \frac{d^2 w_\alpha}{dt^2} &= \frac{t^{-2/(N-2)}}{(N-2)^2} g(w_\alpha) \qquad \text{for } t > 1, \\ w_\alpha(1) &= 1, \qquad \frac{dw_\alpha}{dt}(1) = \alpha. \end{aligned} \tag{2.14}$$

From Lemma 1, w_α is defined on $[1, +\infty)$. Moreover, $w_\alpha \geqslant \xi$. As ξ is the unique bounded function satisfying (2.13) we must have $\lim_{t\to\infty} w_\alpha(t) = +\infty$. So w_α is asymptotically convex and $\lim_{t\to+\infty} w_\alpha(t)/t$ exists in $(0, +\infty]$. If we set $\xi_\alpha(t) = u_\alpha(|x|^{2-N})$, we have

$$\begin{aligned} -\Delta u_\alpha + g(u_\alpha) &= 0 \quad \text{in } B_1 - \{0\}, \\ u_\alpha &= 1 \quad \text{on } \partial B_1. \end{aligned} \tag{2.15}$$

Moreover, $\lim_{x\to 0}|x|^{N-2}u_\alpha(x)$ exists in $(0, +\infty]$. As the limit cannot be finite we have $\lim_{x\to 0}|x|^{N-2}u_\alpha(x) = +\infty$.

Remark 2.2. The set of nondecreasing functions g satisfying (2.8) is not empty but no function of the type

$$r \mapsto r^\alpha (\operatorname{Log} r)^\beta (\operatorname{Log}(\operatorname{Log} r))^\gamma \ldots$$

α, β, γ, ... real numbers, can satisfy (2.8).

The removability result of Brézis and Véron [5] is an immediate consequence of Theorems 2.1 and 2.2. We give here an improvement of it.

THEOREM 2.3. *Suppose g is a continuous real-valued function such that*

$$\begin{aligned}&\underline{\lim}_{r\to+\infty}\, g(r)\operatorname{Log} r/r^{N/(N-2)} > 0,\\ &\overline{\lim}_{r\to-\infty}\, g(r)\operatorname{Log}(-r)/|r|^{N/(N-2)} < 0.\end{aligned} \tag{2.16}$$

If $u\in C^1(\Omega')$ is a solution of (1.2) *in the sense of distributions in Ω' then it can be extended as a C^1 function in Ω satisfying* (1.2) *in $\mathcal{D}'(\Omega)$.*

Proof. Step 1: We claim that $|x|^{N-2}u(x)$ remains bounded near 0. From [25, Lemma 2.1] it is clear that

$$\lim_{x\to 0}|x|^{2/(q-1)}u(x) = 0, \tag{2.17}$$

for any $q\in(1, N/(N-2))$. Moreover,

$$g(r) \geqslant \alpha\frac{r^{N/(N-2)}}{\operatorname{Log} r} - \beta \quad \text{for } r \geqslant e^{(N-2)/2},$$

where $\alpha > 0$ and $\beta \geqslant 0$. For $1 < q < N/(N-2)$ we set

$$\omega_q(x) = \left(\frac{2N-4}{\alpha e(q-1)^2}\right)^{1/(q-1)}|x|^{-2/(q-1)} \tag{2.18}$$

for $x \neq 0$; ω_q satisfies the equation

$$-\Delta\omega_q + \alpha e\left(\frac{N}{N-2} - q\right)\omega_q^q = 0 \tag{2.19}$$

in $R^N - \{0\}$. For $r > e^{(N-2)/2}$ the maximum of the function $q \mapsto (N/(N-2) - q)r^q$ over $(1, N/(N-2))$ is achieved at $q^* = N/(N-2) - 1/\operatorname{Log} r$ and

$$\left(\frac{N}{N-2} - q^*\right)r^{q^*} = \frac{r^{N/(N-2)}}{e \operatorname{Log} r}.$$

If we suppose that $\bar{B}_1 \subset \Omega$ and take $\gamma > \max(e^{(N-2)/2}, \sup_{|x|=1}u(x))$ large enough, we have

$$-\Delta(\omega_q+\gamma)+\alpha\frac{(\omega_q+\gamma)^{N/(N-2)}}{\operatorname{Log}(\omega_q+\gamma)}\geqslant\beta. \tag{2.20}$$

As

$$-\Delta u+\alpha\frac{u^{N/(N-2)}}{\operatorname{Log} u}\leqslant\beta$$

a.e. on $\{x\colon u(x)>e^{(N-2)/2}\}$, from Kato's inequality $\Delta(u-\omega_q-\gamma)^+\geqslant 0$ in $\mathscr{D}'(\Omega)$ and $u\leqslant\omega_q+\gamma$ in $B_R-\{0\}$. Letting q converge to $N/(N-2)$ yields

$$u(x)\leqslant\left(\frac{(N-2)^3}{2\alpha e}\right)^{(N-2)/2}|x|^{2-N}+\gamma. \tag{2.21}$$

In the same way $u(x)$ is minorized by $-c'|x|^{2-N}$.

Step 2: We claim that u remains bounded near 0. From Step 1, $|u(x)|\leqslant c|x|^{2-N}$. Moreover,

$$g(r)\geqslant\alpha\frac{r^{N/(N-2)}}{\operatorname{Log} r}-\beta\quad\text{for } r\geqslant e^{(N-2)/2}.$$

Let φ_ε be the solution of

$$\begin{aligned}
\Delta\varphi_\varepsilon&=\alpha\frac{\varphi_\varepsilon^{N/(N-2)}}{\operatorname{Log}\varphi_\varepsilon}-\beta &&\text{in } \{x\colon \varepsilon<|x|<1\},\\
\varphi_\varepsilon(x)&=c\varepsilon^{2-N} &&\text{for } |x|=\varepsilon,\\
\varphi_\varepsilon(x)&=K &&\text{for } |x|=1,
\end{aligned} \tag{2.22}$$

where K is large enough so that $\varphi_\varepsilon\geqslant e^{(N-2)/2}$ in $\varepsilon<|x|<1$. Clearly we have in $\{x\colon \varepsilon<|x|<1\}$

$$u(x)\leqslant\varphi_\varepsilon(x)\leqslant c|x|^{2-N}+K. \tag{2.23}$$

From the compact imbedding theorem there exist $\varphi\in C^1(\bar{B}_1-\{0\})$ and a sequence $\{\varepsilon_n\}$ such that $\varphi_{\varepsilon_n}\to\varphi$ uniformly on each compact subset of $\bar{B}_1-\{0\}$ and φ satisfies

$$\Delta\varphi=\alpha\frac{\varphi^{N/(N-2)}}{\operatorname{Log}\varphi}-\beta \tag{2.24}$$

in $\mathscr{D}'(\Omega')$. From (2.23) and Theorem 1.1, $|x|^{N-2}\varphi(x)$ admits a limit in $\boldsymbol{R}$ as x tends to 0 and from Theorem 2.1 this limit must be 0 since φ cannot have a weak singularity. Using an easy comparison principle, φ must remain bounded in Ω and so u^+ is bounded near 0. We do the same with u^-, and so $u\in C^1(\Omega)$ (see Remark 1.2).

Remark 2.3. The result of Theorem 2.3 still holds if g satisfies

$$(2.25)\qquad \begin{aligned} &\underline{\lim}_{r\to+\infty}\; g(r)(\operatorname{Log} r)(\operatorname{Log}\operatorname{Log} r)/r^{N/(N-2)} > 0,\\ &\overline{\lim}_{r\to-\infty}\; g(r)(\operatorname{Log}(-r))(\operatorname{Log}\operatorname{Log}(-r))/|r|^{N/(N-2)} < 0,\end{aligned}$$

and this can be continued. However, it is an open question under what necessary and sufficient conditions on g all the isolated singularities of solutions of (1.2) are removable. Thanks to Theorem 2.1 this question reduces to the nonexistence of strong singularities. Another interesting problem is to find conditions on g which imply the uniqueness of strong singularities in the sense that if u and v are solutions of (1.2) with a strong singularity at 0, then $\lim_{x\to 0} u(x)/v(x) = 1$.

3. Weak and strong singularities in the plane

In this section we suppose that $\Omega \subset \boldsymbol{R}^2$ and g is a continuous nondecreasing function. Following [19] we define the exponential orders of growth of g:

$$(3.1)\qquad \begin{aligned} a_g^+ &= \inf\Big\{a \geqslant 0\colon \int_0^{+\infty} g(s)e^{-as}\,ds < +\infty\Big\},\\ a_g^- &= \inf\Big\{a \geqslant 0\colon \int_{-\infty}^{0} g(s)e^{as}\,ds > -\infty\Big\}.\end{aligned}$$

The following result characterizes the existence of weak singularities and the range of the c's $= \lim_{x\to 0} u(x)/\operatorname{Log}(1/|x|)$.

THEOREM 3.1. *The equation* (1.2) *admits at least a solution u_c such that $\lim_{x\to 0} u_c(x)/\operatorname{Log}(1/|x|) = c$, $c\in \boldsymbol{R}$, if and only if*

$$(3.2)\qquad -2/a_g^- \leqslant c \leqslant 2/a_g^+,$$

where $-2/a_g^- = -\infty$ if $a_g^- = 0$ and $2/a_g^+ = +\infty$ if $a_g^+ = 0$.

Proof. If $[-2/a_g^-, 2/a_g^+] = \{0\}$ the result is obvious so we assume $(0, 2/a_g^+] \neq \emptyset$ (or $[-2/a_g^-, 0) \neq \emptyset$ in the same way).

Step 1: Assume $c\in[0, 2/a_g^+)$. We claim that there exists a solution u_c of (1.2) such that

$$(3.3)\qquad \lim_{x\to 0} u_c(x)/\operatorname{Log}(1/|x|) = c.$$

For $\varepsilon > 0$ let y_ε be the solution of the following two-point problem:

$$(3.4)\qquad \begin{aligned} &\frac{d^2 y_\varepsilon}{dt^2} = (t+\varepsilon)^{-3} e^{-2/(t+\varepsilon)} g\big(y_\varepsilon/(t+\varepsilon)\big) \quad \text{in } (0, 1],\\ &y_\varepsilon(0) = c, \qquad y_\varepsilon(1) = 0.\end{aligned}$$

Assuming $g(0)=0$ yields that y_ε is increasing and convex. Integrating (3.4) implies

$$\frac{d}{dt}y_\varepsilon(t)=\frac{d}{dt}y_\varepsilon(1)-\int_{t+\varepsilon}^{1+\varepsilon}\sigma^{-3}e^{-2/\sigma}g\big(y_\varepsilon(\sigma-\varepsilon)/\sigma\big)\,d\sigma.$$

As $(d/dt)\,y_\varepsilon(1)$ remains bounded we get

$$|y_\varepsilon(\tau)-y_\varepsilon(\sigma)|\leqslant\int_\sigma^\tau\int_{t+\varepsilon}^{1+\varepsilon}\sigma^{-3}e^{-2/\sigma}g(c/\sigma)\,d\sigma\,dt+c_1(\tau-\sigma),\tag{3.5}$$

for $0\leqslant\sigma<\tau\leqslant 1$. If we set

$$\psi(x)=\int_x^2\int_t^2\sigma^{-3}e^{-2/\sigma}g(c/\sigma)\,d\sigma\,dt,\tag{3.6}$$

we then have

$$\int_\sigma^\tau\int_{t+\varepsilon}^{1+\varepsilon}\sigma^{-3}e^{-2/\sigma}g(c/\sigma)\,d\sigma\,dt\leqslant|\psi(\tau+\varepsilon)-\psi(\sigma+\varepsilon)|$$

for $0<\varepsilon<1$. Moreover,

$$\lim_{x\to 0}\psi(x)=\int_0^2\int_t^2\sigma^{-3}e^{-2/\sigma}g(c/\sigma)\,d\sigma\,dt=\int_0^2\sigma^{-2}e^{-2/\sigma}g(c/\sigma)\,d\sigma$$

$$=c^{-1}\int_{c/2}^{+\infty}e^{-2s/c}g(s)\,ds<+\infty$$

since $2/c>a_g^+$. Using again Arzelà–Ascoli theorem we conclude that there exists a continuous function y on $[0,1]$ (unique) satisfying

$$\begin{aligned}&\frac{d^2y}{dt^2}=t^{-3}e^{-2/t}g(y/t)\quad\text{in }(0,1],\\&y(0)=c,\qquad y(1)=0.\end{aligned}\tag{3.7}$$

If we set $u_c(x)=-\mathrm{Log}(|x|)\,y(-1/\mathrm{Log}\,|x|)$, then u_c satisfies

$$\begin{aligned}&\Delta u_c=g(u_c)\quad\text{in }\{x\in\mathbf{R}^2\colon\ 0<|x|<e^{-1}\},\\&u_c(x)=0\qquad\text{for }|x|=e^{-1},\end{aligned}\tag{3.8}$$

$$\lim_{x\to 0}u_c(x)/\mathrm{Log}(1/|x|)=c.$$

Step 2: Assume $c=2/a_g^+$, set $c_n=c-1/n$ and let y_n be the solution of the equation

$$\begin{aligned}&\frac{d^2y_n}{dt^2}=t^{-3}e^{-2/t}g(y_n/t)\quad\text{in }(0,1],\\&y_n(0)=c_n,\qquad y_n(1)=0.\end{aligned}\tag{3.9}$$

The sequence of functions $\{y_n\}$ is nondecreasing, bounded above by c. Moreover, y_n is convex. From Dini's theorem, $\{y_n\}$ is uniformly convergent on $[0,1]$ to some y which satisfies (3.7). We return to u as before.

Step 3: Suppose u_c is a solution of (1.2) satisfying (3.3) and suppose also $c \notin [-2/a_g^-, 2/a_g^+]$, $c > 2/a_g^+$ for example. Set $y(t, \theta) = tu_c(e^{-1/t}, \theta)$; then y satisfies

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial \theta^2} = t^{-3} e^{-2/t} g(y/t), \\ &\lim_{t \to 0} y(t, \theta) = c \quad \text{uniformly with respect to } \theta \in S^1. \end{aligned} \tag{3.10}$$

For t small enough, $y(t, \theta) \geqslant c' > 2/a_g^+$. Let $\bar{y}(t)$ be the average of $y(t, \cdot)$ on S^1; we have

$$\frac{d^2 \bar{y}}{dt^2} \geqslant t^{-3} e^{-2/t} g(c'/t), \tag{3.11}$$

for $0 < t < t_0$. Integrating (3.11) twice yields $\lim_{t \to \infty} \bar{y}(t) = +\infty$, contradiction.

Remark 3.1. Theorem 3.1 implies the existence of solutions of (1.2) with weak singularities. If we suppose moreover that $a_g^+ = 0$ and (2.2) then there exist solutions of (1.2) with a strong singularity which are obtained as the upper limit ($c \to +\infty$) of the solutions u_c with a weak singularity at 0. This is not always the case and there exist continuous nondecreasing g such that

$$\int_\alpha^{+\infty} \frac{ds}{\sqrt{sg(s)}} = \int_\alpha^{+\infty} g(s) e^{-as} ds = +\infty, \tag{3.12}$$

for any $\alpha \geqslant 0$ and $a \geqslant 0$. For such g's there exists no solution of (1.2) with a weak singularity at 0 but infinitely many with a strong one; they are obtained as in Theorem 2.2. In order to avoid that unpleasant phenomenon we introduce the following hypothesis on g:

(3.13) For any $a > 0$, $\lim_{r \to +\infty} e^{-ar} g(r)$ and $\lim_{r \to -\infty} e^{ar} g(r)$ exist in $\boldsymbol{R} \cup \{-\infty, +\infty\}$.

This implies that a_g^+ and a_g^- can be defined as follows:

$$\begin{aligned} a_g^+ &= \inf\{a \geqslant 0 : \lim_{r \to +\infty} e^{-ar} g(r) = 0\}, \\ a_g^- &= \inf\{a \geqslant 0 : \lim_{r \to -\infty} e^{ar} g(r) = 0\}. \end{aligned} \tag{3.14}$$

We then have the following:

THEOREM 3.2. *Suppose g satisfies* (3.13) *and $u \in C^1(\Omega')$ is a solution of* (1.2) *in the sense of distributions in Ω' such that $u(x)/\mathrm{Log}(1/|x|)$ admits a limit*

$c \in \boldsymbol{R} \cup \{-\infty, +\infty\}$ as x tends to 0. Then we have

$$-2/a_g^- \leqslant c \leqslant 2/a_g^+. \tag{3.15}$$

Without loss of generality we suppose $\Omega \supset \bar{B}_1$ and we have the following a priori estimate:

LEMMA 3.1. *Suppose $v \in C^1(\Omega')$ satisfies*

$$-\Delta v + ae^{\alpha v} \leqslant c \tag{3.16}$$

in the sense of distributions on $\{x\colon v(x) > 0\}$, for some constants a, α and $c > 0$. Then

$$v(x) \leqslant \frac{2}{\alpha} \operatorname{Log}(1/|x|) + B, \tag{3.17}$$

for $0 < |x| \leqslant 1$, where B depends on a, α and c.

Proof. We choose x_0 such that $0 < |x_0| \leqslant 1$ and we set

$$\psi(x) = \lambda \operatorname{Log}(1/(R^2 - |x - x_0|^2)) + \mu, \tag{3.19}$$

for $0 \leqslant |x - x_0| < R < |x_0|$; λ and μ are to be chosen such that $-\Delta\psi + ae^{\alpha\psi} \geqslant c$. A straightforward computation gives

$$-\Delta\psi + ae^{\alpha\psi} = -\frac{4\lambda R^2}{(R^2 - |x - x_0|^2)^2} + a\frac{e^{\alpha\mu}}{(R^2 - |x - x_0|^2)^{\alpha\lambda}}. \tag{3.20}$$

Taking $\lambda = 2/\alpha$ and $\mu = \alpha^{-1} \operatorname{Log}\{(\alpha c + 8) R^2/(a\alpha)\}$ gives

$$-\Delta\psi + ae^{\alpha\psi} \geqslant c \tag{3.21}$$

in $\{x\colon |x - x_0| < R\}$. As $\lim_{|x - x_0| \to R} \psi(x) = +\infty$ and v is bounded in $\{x\colon |x - x_0| \leqslant R\}$ we deduce from basic comparison results that $v(x) \leqslant \psi(x)$ if $|x - x_0| < R$. In particular,

$$v(x_0) \leqslant \psi(x_0) \leqslant \frac{2}{\alpha} \operatorname{Log}(1/R) + \frac{1}{\alpha} \operatorname{Log}((\alpha c + 8)/(a\alpha)). \tag{3.22}$$

If we let $R \to |x_0|$ we get (3.17) with $B = \alpha^{-1} \operatorname{Log}((\alpha c + 8)/(a\alpha))$.

Proof of Theorem 3.2. If $a_g^+ = 0$ (resp. $a_g^- = 0$) then $2/a_g^+ = +\infty$ (resp. $-2/a_g^- = -\infty$) and there exists no finite upper bound for c (resp. finite lower bound). So we assume $a_g^+ > 0$; for any α, $0 < \alpha < a_g^+$, there exist $a > 0$ and $c > 0$ such that $g(s) > ae^{\alpha s} - c$ for any $s > 0$. Hence

$$-\Delta u + ae^{\alpha u} \leqslant c \tag{3.23}$$

in the sense of distributions in $\{x\colon u(x) > 0\}$, which implies

$$\lim_{x \to 0} u(x)/\operatorname{Log}(1/|x|) \leqslant \frac{2}{\alpha}.$$

Letting $\alpha \to a_g^+$ we obtain the right-hand side of (3.15). The left-hand side is obtained in the same way.

As an application of Theorem 3.2 to removability problems we can prove the following result.

THEOREM 3.3. *Suppose g is a continuous real-valued function such that*

$$\lim_{r \to +\infty} e^{-ar} g(r) = +\infty, \qquad \lim_{r \to -\infty} e^{ar} g(r) = -\infty, \tag{3.24}$$

for any $a > 0$. Then any solution $u \in C^1(\Omega')$ of (1.2) in the sense of distributions in Ω' can be extended as a $C^1(\Omega)$ solution of (1.2) in $\mathscr{D}'(\Omega)$.

In [20] this result is extended to singularities lying on a C^1 compact manifold of codimension 2 in the N-dimensional space.

4. Isotropic singularities in the power case

In this section we study the classification of the singularities at 0 of solutions of

$$-\Delta u + u|u|^{q-1} = 0 \tag{4.1}$$

in $\Omega' = \Omega - \{0\}$, where $q > 1$. The two main results are the following: first, we have a complete classification of isotropic singularities and more generally of singularities of solutions with constant sign near 0; we also prove that all the singularities are isotropic (and thus classified) if $(N+1)/(N-1) \leqslant q < N/(N-2)$. The two main ingredients for such a program are the following two results due to Fowler [6], and Brézis and Lieb [4] in a particular case and Véron [22] in the general case.

LEMMA 4.1. *Suppose φ is a nonnegative solution of*

$$\frac{d^2\varphi}{dr^2} + \frac{N-1}{r}\frac{d\varphi}{dr} - \varphi^q = 0 \tag{4.2}$$

on some interval $(0, a]$ and $1 < q < N/(N-2)$. Then φ can have at most two types of behaviour as r tends to 0:

(i) *either $\varphi(r)/\mu(r)$ converges to some nonnegative real number,*

(ii) *or $r^{2/(q-1)}\varphi(r)$ converges to*

$$l_{q,N} = \left(\frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)^{1/(q-1)}.$$

LEMMA 4.2. *Suppose u is any solution of (4.1) in $B_2 - \{0\}$. Then*

$$|u(x)| \leqslant l_{q,N}|x|^{-2/(q-1)}(1 + C|x|^\tau), \quad 0 < |x| \leqslant 1, \tag{4.3}$$

where $\tau = \frac{1}{2}\left\{\sqrt{\left(N - 2\frac{q+1}{q-1}\right)^2 + 8\left(\frac{2q}{q-1} - N\right)} + 2\frac{q+1}{q-1} - N\right\}$ *and* $C = C(q, N)$.

THEOREM 4.1. *Suppose* $1 < q < N/(N-2)$, $N \geqslant 2$, *and* $u \in C^2(\Omega')$ *is a nonnegative solution of* (4.1) *in* Ω'. *Then we have the following as* x *tends to* 0:

(i) *either* $|x|^{2/(q-1)} u(x)$ *converges to* $l_{q,N}$,

(ii) *or* $u(x)/\mu(x)$ *converges to some real number which can take any positive value,*

(iii) *or* $u(x)$ *admits a finite limit and* u *can be extended as a* C^2 *solution of* (4.1) *in* Ω.

Proof. Step 1: Setting $d = u^{q-1}$ we write (4.1) as

$$-\Delta u + du = 0. \tag{4.4}$$

For any $y \in B_1 - \{0\}$ (we assume $\Omega \supset \bar{B}_2$) and $\varrho = \frac{1}{2}|y|$ we have

$$\sup_{x \in B_{\varrho/4}(y)} u(x) \leqslant K \inf_{x \in B_{\varrho/4}(y)} u(x), \tag{4.5}$$

from Harnack's inequality ($B_\varrho(y) = \{x\colon |x-y| < \varrho\}$). Moreover, K can be estimated (see [9]):

$$K = \exp\{K_0(1 + \varrho \sup_{x \in B_\varrho(y)} \sqrt{d(x)})\}, \qquad K_0 = K_0(N). \tag{4.6}$$

As u satisfies (4.3), K is bounded independently of y. With a chain argument we deduce that there exists $C = C(q, N)$ such that

$$u(y) \leqslant C u(y'), \tag{4.7}$$

for any y, y' such that $|y| = |y'| > 0$.

Step 2: Assume $N \geqslant 3$ ($N = 2$ is a bit different, see [22]) and $|x|^{N-2} u(x)$ is not bounded in any neighbourhood of 0. We let (r, σ) be the spherical coordinates in $\boldsymbol{R}^N$ and set

$$s = \frac{r^{N-2}}{N-2}, \qquad u(r, \sigma) = r^{2-N} v(s, \sigma); \tag{4.8}$$

so v satisfies

$$s^2 \frac{\partial^2 v}{\partial s^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} v = (N-2)^{(4-N)/(N-2)-q} s^{N/(N-2)-q} v^q \tag{4.9}$$

on $(0, 1] \times S^{N-1}$. We let $\bar{v}(s)$ be the average of $v(s, \cdot)$ on S^{N-1}. As v is nonnegative, $\bar{v}$ is convex. From hypothesis, v is unbounded near 0 so there exists a sequence $\{s_n\}$ such that

$$\lim_{n \to +\infty} \big(\sup_{\sigma \in S^{N-1}} v(s_n, \sigma) \big) = +\infty. \tag{4.10}$$

From (4.7),

$$\lim_{n \to \infty} \big(\inf_{\sigma \in S^{N-1}} v(s_n, \sigma) \big) = +\infty,$$

so $\lim_{n\to\infty}\bar{v}(s_n) = +\infty$. As $\bar{v}$ is convex, $\lim_{s\to 0}\bar{v}(s) = +\infty$. Using again (4.7) implies $\lim_{s\to 0} v(s, \sigma) = +\infty$, uniformly on S^{N-1}.

For $\alpha > 0$ we let y_α be the solution of

$$(4.11)\qquad \begin{gathered} s^2 \frac{d^2}{ds^2} y_\alpha = (N-2)^{(4-N)/(N-2)-q} s^{N/(N-2)-q} y_\alpha^q \quad \text{in } (0, 1], \\ \lim_{s\to 0} y_\alpha(s) = \alpha, \qquad y_\alpha(1) = \inf_{\sigma\in S^{N-1}} v(1, \sigma). \end{gathered}$$

The existence of y_α comes from Theorem 2.1. As $(y_\alpha - v)^+$ has a compact support in $(0, 1]\times S^{N-1}$ we deduce from comparison principles that it is identically 0, so $0 \leqslant y_\alpha \leqslant v$. By the same argument $\alpha \mapsto y_\alpha$ is increasing, so y_α converges to some function y_∞ as α tends to $+\infty$ and y_∞ satisfies the equation (4.11) with α replaced by ∞. Moreover, $0 \leqslant y_\infty \leqslant v$. If we set

$$u_\infty(r) = r^{2-N} y_\infty\left(\frac{r^{N-2}}{N-2}\right)$$

then u_∞ satisfies (4.2) and $\lim_{r\to 0} r^{N-2} u_\infty(r) = +\infty$. Hence

$$\lim_{r\to 0} r^{2/(q-1)} u_\infty(r) = l_{q,N}.$$

Using (4.3) we deduce

$$(4.12)\qquad \lim_{r\to 0} r^{2/(q-1)} u(r, \sigma) = l_{q,N},$$

uniformly on S^{N-1}.

Step 3: Assume $u(x)/\mu(x)$ is bounded in $\bar{B}_2$. From Theorem 1.1 there exists $c \geqslant 0$ such that $\lim_{x\to 0} u(x)/\mu(x) = c$. If $c \neq 0$ then u admits a weak singularity at 0. If $c = 0$ then u must be bounded (comparison with $\varepsilon\mu + K$) and by Serrin's result [13] it can be extended to Ω as a C^2 solution of (4.1) in Ω.

Remark 4.1. As $g(r) = r\,|r|^{q-1}$ satisfies (2.1) it is not difficult to see that if $\lim_{x\to 0}|x|^{N-2} u(x) = c$ then u satisfies

$$(4.13)\qquad -\Delta u + u^q = (N-2)\,|S^{N-1}|\, c\delta_0$$

in $\mathscr{D}'(\Omega)$ (and this is general under the hypothesis (2.1)). Moreover, a crucial fact about strong singularities for solutions of (4.1) is that they are always obtained as the upper limit of solutions with a weak singularity (and fixed boundary value for example).

As we have seen in the introduction it is clear that all singularities are not necessarily isotropic when $1 < q < (N+1)/(N-1)$; but when $(N+1)/(N-1) \leqslant q < N/(N-2)$ they are isotropic and we prove it:

THEOREM 4.2. *Suppose*

$$\frac{N+1}{N-1} \leqslant q < \frac{N}{N-2}, \qquad N \geqslant 2,$$

and $u \in C^2(\Omega')$ *is a solution of* (4.1) *in* Ω'. *Then we have the following as* x *tends to* 0:

(i) *either* $|x|^{2/(q-1)}u(x)$ *converges to* $\pm l_{q,N}$,

(ii) *or* $u(x)/\mu(x)$ *converges to some real number* c *which can take any nonzero value,*

(iii) *or* $u(x)$ *admits a finite limit and* u *can be extended as a* C^2 *solution of* (4.1) *in* Ω.

Proof. Step 1: We claim that

$$\lim_{r\to 0} r^{N-1}\|u(r,\cdot)-\bar{u}(r)\|_{L^2(S^{N-1})} = 0, \tag{4.14}$$

with the usual notation. For that we consider the following change of variables and the unknown function:

$$\beta = 2\frac{q+1}{q-1} - N, \quad s = \frac{r^\beta}{\beta}, \quad v(s,\sigma) = \left(\frac{r}{\beta}\right)^{2/(q-1)} u(r,\sigma). \tag{4.15}$$

The function v is bounded from Lemma 4.2 and satisfies

$$s^2\frac{\partial^2 v}{\partial s^2} + \frac{l_{q,N}^{q-1}}{\beta^2} + \frac{1}{\beta^2}\Delta_{S^{N-1}} v = v|v|^{q-1} \tag{4.16}$$

in $(0, \beta^{-1}] \times S^{N-1}$. If we set $w = v - \bar{v}$, we have as in Theorem 1.1

$$s^2 \int_{S^{N-1}} w\frac{\partial^2 w}{\partial s^2}\,d\sigma + \frac{1}{\beta^2}(l_{q,N}^{q-1} - N + 1)\int_{S^{N-1}} w^2\,d\sigma \geqslant \int_{S^{N-1}} w(v|v|^{q-1} - \overline{v|v|^{q-1}})\,d\sigma. \tag{4.17}$$

As $q \geqslant (N+1)/(N-1)$, $l_{q,N}^{q-1} \leqslant N-1$. Moreover,

$$\int_{S^{N-1}} w(v|v|^{q-1} - \overline{v|v|^{q-1}})\,d\sigma = \int_{S^{N-1}} w(v|v|^{q-1} - \bar{v}|\bar{v}|^{q-1})\,d\sigma \geqslant 2^{-q}\int_{S^{N-1}} |w|^{q+1}\,d\sigma,$$

so we get

$$\frac{d^2}{ds^2}\int_{S^{N-1}} w^2(s,\cdot)\,d\sigma \geqslant s^{-2}C(q,N)\Big(\int_{S^{N-1}} w^2(s,\cdot)\,d\sigma\Big)^{(q+1)/2}. \tag{4.18}$$

Hence the function $s \mapsto \int_{S^{N-1}} w^2(s,\cdot)\,d\sigma$ is convex. As it is bounded it admits a limit as s tends to 0. Integrating (4.18) twice shows that the only admissible limit is 0 and we get (4.14).

Step 2: *End of the proof.* From Theorem 1.1, $u(x)/\mu(x)$ admits a limit in $\boldsymbol{R} \cup \{-\infty, +\infty\}$. If the limit is $+\infty$ (or $-\infty$ in the same way), it shows that $u(x)$ has a constant sign near 0; we apply Theorem 4.1 and we get (i). If the limit is finite we end the proof as in Step 3 of Theorem 4.1 and we get (ii) or (iii).

Remark 4.2. We can estimate the speed of convergence of $|x|^{2/(q-1)}u(x)$ to $l_{q,N}$ when $1 < q < N/(N-2)$ and u is a solution of (4.1) with a strong positive singularity at 0. In [22] it is proved that

$$\limsup_{x \to 0} |x|^{-\tau} \left| l_{q,N} - |x|^{2/(q-1)} u(x) \right| < +\infty \tag{4.19}$$

where τ is defined in Lemma 4.2. This number τ (which is positive) is the generalized Sommerfeld exponent. It was discovered by Sommerfeld [15] in studying radial solutions of the Thomas–Fermi equation ($N = 3$, $q = 3/2$, $l_{q,N} = 144$, $2/(q-1) = 4$, $\tau = \frac{1}{2}(\sqrt{73}+7)$). This exponent τ is optimal (Hille [10]).

A nice extension of the classification results is done by Yarur [28] for singular solutions of nonlinear stationary Schrödinger equations of the type

$$\Delta u = |x|^{\theta} u |u|^{q-1}; \tag{4.20}$$

her paper also contains many other results concerning symmetry, asymptotic behaviour and estimates in the linear case.

Remark 4.3. An interesting result would be the classification of the isotropic singularities of the following weakly superlinear equation:

$$\Delta u = u(\operatorname{Log} u)^{\alpha}, \tag{4.21}$$

$\alpha > 0$, which can be written as

$$\Delta v + |Dv|^2 = v^{\alpha} \tag{4.22}$$

on setting $v = \operatorname{Log} u$. A critical case should be $\alpha = 2$.

5. Nonisotropic singularities in the power case

As is seen in the introduction, nonisotropic singularities of solutions of (4.1) do exist when $1 < q < (N+1)/(N-1)$, since they can be obtained from formula (0.13). In this section we prove that if $N = 2$ all the nonisotropic singularities are of the form (0.13) up to a rotation. The methods used to get this result are those of dynamical systems theory and are based on a complete study of the set $\mathscr{E}$ of solutions of the following equation on S^1:

$$-\frac{d^2\omega}{d\theta^2} + \omega|\omega|^{q-1} = \left(\frac{2}{q-1}\right)^2 \omega \tag{5.1}$$

(see Appendix). We prove the following

THEOREM 5.1. *Assume $\Omega \subset \boldsymbol{R}^2$, $q > 1$ and $u \in C^1(\Omega')$ is a solution of* (4.1) *in Ω'. Then there exists a solution ω of* (5.1) *and an application α: $(0, 1] \to \mathrm{SO}(2)$ such that*

$$\lim_{x \to 0} \left(|x|^{2/(q-1)} u(x) - \omega\left(\alpha(|x|)(x/|x|)\right)\right) = 0. \tag{5.2}$$

We assume again $\Omega \supset \bar{B}_1$.

LEMMA 5.1. *There exists a constant $C = C(q) > 0$ such that for any x, $0 < |x| \leqslant 1/2$, we have*

$$\left|\frac{\partial^{\alpha+\beta} u}{\partial x_i^\alpha \, \partial x_j^\beta}(x)\right| \leqslant C|x|^{-2/(q-1)-\alpha-\beta}, \tag{5.3}$$

for $\alpha \geqslant 0$, $\beta \geqslant 0$, $0 \leqslant \alpha+\beta \leqslant 3$, $1 \leqslant i, j \leqslant 2$.

Proof. When $\alpha+\beta = 0$, (5.3) reduces to

$$|u(x)| \leqslant C|x|^{-2/(q-1)}, \tag{5.4}$$

which is known (see Lemma 4.2). Set $\Gamma = \{x\colon 1 < |x| < 4\}$ and let $\Phi \in C^2(\Gamma) \cap C^0(\bar{\Gamma})$ be a function satisfying

$$-\Delta\Phi + \Phi|\Phi|^{q-1} = 0 \tag{5.5}$$

in Γ; Φ belongs to $C^3(\Gamma)$ and from (5.4) there exists $C = C(q)$ such that

$$\left|\frac{\partial^{\alpha+\beta} \Phi}{\partial x_i^\alpha \, \partial x_j^\beta}(x)\right| \leqslant C, \tag{5.6}$$

for $1 \leqslant i, j \leqslant 2$, $1 \leqslant \alpha+\beta \leqslant 3$ and $2 \leqslant |x| \leqslant 3$. Fix x_0, $0 < |x_0| < 1/2$. There exists $\beta \leqslant 1/6$ such that $2\beta \leqslant |x_0| \leqslant 3\beta$. For $y \in \Gamma$ set $\Phi_\beta(y) = \beta^{2/(q-1)} u(\beta y)$; Φ_β satisfies (5.5) in Γ so $|D\Phi_\beta(y)| \leqslant C$ for $2 \leqslant |y| \leqslant 3$. But

$$D\Phi_\beta(y) = \beta^{(q+1)/(q-1)} Du(\beta y),$$

so we get

$$|Du(x_0)| = \beta^{-(q+1)/(q-1)} \left|D\Phi_\beta\left(\frac{x_0}{\beta}\right)\right| \leqslant \left(\frac{3}{|x_0|}\right)^{(q+1)/(q-1)} \left|D\Phi_\beta\left(\frac{x_0}{\beta}\right)\right|, \tag{5.7}$$

which implies (5.3) for $\alpha+\beta = 1$. We do the same for $\alpha+\beta = 2$ or 3.

Using polar coordinates $(r, \theta) \in (0, 1] \times S^1$ we set

$$t = \operatorname{Log}\left(\frac{4r^{4/(q-1)}}{q-1}\right), \quad v(t, \theta) = r^{2/(q-1)} u(r, \theta). \tag{5.8}$$

The function v is bounded and satisfies the autonomous equation

$$\left(\frac{4}{q-1}\right)^2 \left(\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t}\right) + \left(\frac{2}{q-1}\right)^2 v + \frac{\partial^2 v}{\partial \theta^2} = v|v|^{q-1} \tag{5.9}$$

in $[a, +\infty) \times S^1$, $a = \operatorname{Log}(4/(q-1))$. Moreover, thanks to Lemma 5.1 the sets of functions $\{v(t, \cdot)\}_{t \geq a}$, $\{(\partial v/\partial t)(t, \cdot)\}_{t \geq a}$, $\{(\partial^2 v/\partial t^2)(t, \cdot)\}_{t \geq a}$ are bounded in $C^3(S^1)$, $C^2(S^1)$ and $C^1(S^1)$ respectively.

LEMMA 5.2. *The function $\partial v/\partial t$ belongs to $H^2(a, +\infty; L^2(S^1))$.*

Proof. We get from (5.9)

$$\left(\frac{4}{q-1}\right)^2 \left(\int_{S^1} \left(\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t}\right) \frac{\partial v}{\partial t} d\theta\right) + \left(\frac{2}{q-1}\right)^2 \int_{S^1} v \frac{\partial v}{\partial t} d\theta + \int_{S^1} \frac{\partial^2 v}{\partial \theta^2} \frac{\partial v}{\partial t} d\theta = \int_{S^1} v |v|^{q-1} \frac{\partial v}{\partial t} d\theta.$$

If we set

(5.10)
$$L(v(t, \cdot)) = \int_{S^1} \left\{\frac{1}{2}\left(\frac{\partial v}{\partial \theta}\right)^2 + \frac{1}{q+1}|v|^{q+1} - \frac{2}{(q-1)^2} v^2 - \frac{8}{(q-1)^2}\left(\frac{\partial v}{\partial t}\right)^2\right\} d\theta,$$

then we have

(5.11)
$$\frac{d}{dt} L(v(t, \cdot)) = \left(\frac{4}{q-1}\right)^2 \int_{S^1} \left(\frac{\partial v}{\partial t}\right)^2 d\theta.$$

As v, $\partial v/\partial t$ and $\partial v/\partial \theta$ belong to $L^\infty((a, +\infty) \times S^1)$ we first get

(5.12)
$$\int_a^{+\infty} \int_{S^1} \left(\frac{\partial v}{\partial t}\right)^2 d\theta\, dt < +\infty.$$

We also get from (5.9), on setting $w = \partial v/\partial t$,

(5.13)
$$\left(\frac{4}{q-1}\right)^2 \left(\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t}\right) + \left(\frac{2}{q-1}\right)^2 w + \frac{\partial^2 w}{\partial \theta^2} = q w |v|^{q-1},$$

which implies

$$\left(\frac{4}{q-1}\right)^2 \int_{S^1} \left(\frac{\partial w}{\partial t}\right)^2 d\theta - q \int_{S^1} \frac{\partial w}{\partial t} w |v|^{q-1} d\theta = \frac{d}{dt} \tilde{L}(v(t, \cdot)),$$

where

$$\tilde{L}(v(t, \cdot)) = \int_{S^1} \left\{\frac{1}{2}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{2}{(q-1)^2} w^2 - \frac{8}{(q-1)^2}\left(\frac{\partial w}{\partial t}\right)^2\right\} d\theta.$$

Moreover, for any $\eta > 0$

$$\left|\int_{S^1} \frac{\partial w}{\partial t} w |v|^{q-1} d\theta\right| \leq \frac{1}{2\eta} \int_{S^1} \left(\frac{\partial w}{\partial t}\right)^2 d\theta + \frac{\eta}{2} \|v\|^{2q-2}_{L^\infty(S^1)} \int_{S^1} w^2 d\theta.$$

As $\tilde{L}(v(t, \cdot))$ remains bounded we deduce for η large enough

(5.14)
$$\int_a^{+\infty} \int_{S^1} \left(\frac{\partial^2 v}{\partial t^2}\right)^2 d\theta\, dt < +\infty.$$

From (5.13) we also get

$$\left(\frac{4}{q-1}\right)^2 \int_{S^1} \left(\frac{\partial^2 w}{\partial t^2}\right)^2 d\theta + \frac{8}{(q-1)^2} \frac{d}{dt} \int_{S^1} \left(\frac{\partial w}{\partial t}\right)^2 d\theta + \left(\frac{2}{q-1}\right)^2 \int_{S^1} w \frac{\partial^2 w}{\partial t^2} d\theta$$
$$- \int_{S^1} \frac{\partial w}{\partial \theta} \frac{\partial^3 w}{\partial \theta\, \partial t^2} d\theta = q \int_{S^1} w \frac{\partial^2 w}{\partial t^2} |v|^{q-1} d\theta.$$

By the Cauchy–Schwarz inequality we have for any $T > a$

$$\left(\frac{4}{q-1}\right)^2 \int_a^T \int_{S^1} \left(\frac{\partial^2 w}{\partial t^2}\right)^2 d\theta\, dt + \int_a^T \int_{S^1} \left(\frac{\partial^2 w}{\partial t\, \partial \theta}\right)^2 d\theta\, dt + \frac{8}{(q-1)^2} \left[\int_{S^1} \left(\frac{\partial w}{\partial t}\right)^2 d\theta\right]_a^T$$
$$-\left(\frac{2}{q-1}\right)^2 \int_a^T \int_{S^1} \left(\frac{\partial w}{\partial t}\right)^2 d\theta\, dt + \left(\frac{2}{q-1}\right)^2 \left[\int_{S^1} w \frac{\partial w}{\partial t} d\theta\right]_a^T - \left[\int_{S^1} \frac{\partial w}{\partial \theta} \frac{\partial^2 w}{\partial t\, \partial \theta} d\theta\right]_a^T$$
$$\leqslant \frac{8}{(q-1)^2} \int_a^T \int_{S^1} \left(\frac{\partial^2 w}{\partial t^2}\right)^2 d\theta\, dt + C \int_a^T \int_{S^1} w^2\, d\theta\, dt.$$

Hence we deduce

$$\int_a^{+\infty} \int_{S^1} \left(\frac{\partial^2 w}{\partial t^2}\right)^2 d\theta\, dt < +\infty, \tag{5.15}$$

from (5.12) and (5.14).

Proof of Theorem 5.1. The set of functions $\{v(t, \cdot)\}_{t \geqslant a}$ is bounded in $C^3(S^1)$ hence relatively compact in $C^2(S^1)$. So there exist $\omega \in C^2(S^1)$ and a sequence $\{t_n\}$ going to $+\infty$ such that

$$\lim_{t_n \to \infty} v(t_n, \cdot) = \omega(\cdot)$$

in the topology of $C^2(S^1)$. From Lemma 5.2

$$\lim_{t \to \infty} \frac{\partial v}{\partial t}(t, \cdot) = \lim_{t \to +\infty} \frac{\partial^2 v}{\partial t^2}(t, \cdot) = 0 \tag{5.16}$$

in $L^2(S^1)$. Going to the limit in (5.9) we deduce that ω belongs to the set $\mathscr{E}$ of solutions of (5.1). Moreover, from (5.11) the function L is nondecreasing and bounded so there exists l such that

$$l = L(\omega) = \lim_{t \to \infty} L(v(t, \cdot)), \tag{5.17}$$

and for any other convergent subsequence $\{v(t'_n, \cdot)\}$ with limit ω' we have $L(\omega) = L(\omega')$. Moreover, a direct computation gives

$$L(\omega) = \frac{1-q}{2(q+1)} \int_{S^1} |\omega|^{q+1} d\theta. \tag{5.18}$$

Applying our quantification-exclusion principle (see Appendix, Proposition 7.2) we deduce that ω and ω' must belong to the same connected component of $\mathscr{E}$ and as any nonconstant connected component $\mathscr{E}_k$ of $\mathscr{E}$ is generated by one element ω_k as follows (see Appendix):

$$\mathscr{E}_k = \{\omega_k \circ \tau_\alpha : \omega_k \circ \tau_\alpha(\theta) = \omega_k(\theta+\alpha),\ \theta, \alpha \in S^1\}, \tag{5.19}$$

we deduce the theorem.

Remark 5.1. When $q \geqslant 3$, $\mathscr{E}$ reduces to $\{0, (2/(q-1))^2, -(2/(q-1))^2\}$ and the action of SO(2) on each component of $\mathscr{E}$ is trivial.

If $\Omega \subset \boldsymbol{R}^N$, $N \geqslant 3$, a great part of the previous computations can be done and we know that $|x|^{2/(q-1)} u(x)$ approaches the set $\mathscr{E}_N$ of solutions of

$$-\Delta_{S^{N-1}} \omega + \omega|\omega|^{q-1} = l_{q,N}^{q-1} \omega. \tag{5.20}$$

What is missing is an exclusion principle as in Proposition 7.2. Moreover, the description of $\mathscr{E}_N$ should be of great interest in itself. Let us mention now two very natural and interesting open problems concerning Theorem 5.1:

(i) What happens when $|x|^{2/(q-1)} u(x)$ goes to 0? weak singularity?

(ii) Does any solution of (4.1) converge really to an element of $\mathscr{E}$ (or $\mathscr{E}_N$)?

6. Symmetry and broken symmetry of singular solutions

Some important and deep results concerning symmetries of positive solutions of semilinear elliptic equations in a ball have been given by Gidas, Ni and Nirenberg in their celebrated paper [7] and extended to singular solutions in $\boldsymbol{R}^N$ in [8]. Our goal here is to give another approach to the symmetry problem concerning the singular solutions of

$$\Delta u = g(u) \quad \text{in } B_1 - \{0\}, \tag{6.1}$$

when g is a nondecreasing function. Letting (r, σ) be the spherical coordinates in B_1, $r \in [0, 1)$, $\sigma \in S^{N-1}$, $u(x) = u(r, \sigma)$ and $\bar{u}(r)$ the average of $u(r, \cdot)$ on S^{N-1}, our main result is the following (Véron [25]):

THEOREM 6.1. *Assume g is a continuous nondecreasing function and $u \in C^2(B_1 - \{0\}) \cap C^0(\bar{B}_1 - \{0\})$ is a solution of* (6.1) *in $B_1 - \{0\}$. Then u is radially symmetric if and only if u is constant on ∂B_1 and there exists a sequence $\{r_n\}$ converging to 0 such that*

$$\lim_{r_n \to 0} r_n^{N-1} \|u(r_n, \cdot) - \bar{u}(r_n)\|_{L^2(S^{N-1})} = 0. \tag{6.2}$$

Proof. In one direction this is obvious. Now we set $w(r) = \|u(r, \cdot)$

$-\bar{u}(r)\|_{L^2(S^{N-1})}$ and

$$(6.3)\qquad \int_{S^{N-1}} w\frac{\partial^2 w}{\partial r^2}\,d\sigma+\frac{N-1}{r}\int_{S^{N-1}} w\frac{\partial w}{\partial r}\,d\sigma-\frac{N-1}{r^2}\int_{S^{N-1}} w^2\,d\sigma \geqslant 0.$$

If $X(r)=\|w(r,\cdot)\|_{L^2(S^{N-1})}$ and $I=\{r\in(0,1]\colon X(r)>0\}$ then

$$(6.4)\qquad \frac{d^2X}{dr^2}+\frac{N-1}{r}\frac{dX}{dr}-\frac{N-1}{r^2}X\geqslant 0$$

holds on I. Moreover, $X(1)=0$ and if $r_n^{N-1}X(r_n)=\varepsilon_n$, then $\lim_{n\to\infty}\varepsilon_n=0$. Comparing X with $\varepsilon_n(r_n/r)^{N-1}$ which satisfies the differential equation associated to (6.4) on $[r_n, 1]$, we get $X(r)\leqslant\varepsilon_n(r_n/r)^{N-1}$, which implies $X(r)=0$ for $r>0$ as n tends to $+\infty$ and $u(r,\cdot)=\bar{u}(r)$.

Remark 6.1. The conclusion of Theorem 6.1 is valid for any solution of (6.1) in $\boldsymbol{R}^N-\{0\}$ provided we have for some sequence $\{r_n\}$ converging to $+\infty$

$$(6.5)\qquad \lim_{n\to+\infty} r_n^{-1}\|u(r_n,\cdot)-\bar{u}(r_n)\|_{L^2(S^{N-1})}=0.$$

As a consequence, for regular solutions of (6.1) in $\boldsymbol{R}^N$ we have the following Liouville–Hadamard type result: *if* $u\in C^2(\boldsymbol{R}^N)$ *satisfies* (6.1) *and*

$$\lim_{|x|\to+\infty} u(x)/|x|=0$$

then u is constant.

We can now give applications of Theorem 6.1 when g has a power-like growth.

COROLLARY 6.1. *Assume g is a nondecreasing continuous function such that*

$$(6.6)\qquad \lim_{|r|\to+\infty} |g(r)|/|r|^{(N+1)/(N-1)}=c\in(0,+\infty]$$

and $u\in C^2(B_1-\{0\})\cap C^0(\bar{B}_1-\{0\})$ *is a solution of* (6.1) *which is constant on* ∂B_1. *Then u is radially symmetric.*

Proof. We first recall the following a priori estimate for any function φ belonging to $C^2(\boldsymbol{R}^N-\{0\})$ satisfying

$$(6.7)\qquad -\Delta\varphi+A\varphi^q\geqslant B\quad\text{a.e. on }\{x\colon \varphi(x)>0\}$$

in $\boldsymbol{R}^N-\{0\}$ where $q>1$, $A>0$ and $B\geqslant 0$: for $|x|>0$ we have

$$(6.8)\qquad \varphi(x)\leqslant(\alpha/A)^{1/(q-1)}|x|^{-2/(q-1)}+(B/A)^{1/q},$$

where $\alpha=\alpha(N,q)$ (see [25, Lemma 2.1]); and if (6.7) is satisfied in $B_1-\{0\}$ then (6.8) still holds provided $0<|x|\leqslant\frac{1}{2}$. As a consequence, if g satisfies

$$(6.9)\qquad \lim_{|r|\to+\infty} |g(r)|/|r|^{(N+1)/(N-1)}=+\infty,$$

we deduce $\lim_{x\to 0}|x|^{N-1}u(x) = 0$ which implies (6.2) and $u = \bar{u}$. So we are left with the case

$$\lim_{r\to+\infty} g(r)/r^{(N+1)/(N-1)} = c = -\lim_{r\to-\infty} g(r)/|r|^{(N+1)/(N-1)}, \tag{6.10}$$

and $0 < c < +\infty$. Moreover, there is no loss of generality in supposing $g(0) = 0$.

Step 1: Assume $v\in C^2(\boldsymbol{R}^N - \{0\})$ is a solution of

$$\Delta v = v|v|^{2/(N-1)} \tag{6.11}$$

in $\boldsymbol{R}^N - \{0\}$. We then claim that v *is radial*. From Theorem 4.2 we know that either $|x|^{N-1}v(x)$ converges to $\pm(N-1)^{(N-1)/2}$ or $v(x)/\mu(x)$ converges to some real number c, but in any case

$$\lim_{r\to 0} r^{N-1}\,\|v(r,\cdot)-\bar{v}(r)\|_{L^2(S^{N-1})} = 0. \tag{6.12}$$

From estimate (6.8) with $A = 1$, $B = 0$, $q = (N+1)/(N-1)$ we get $|v(x)| \leqslant (\alpha^2/|x|)^{2/(q-1)}$ for $x > 0$, which implies (6.5) and $v = \bar{v}$.

Step 2: *A scaling method*. Set $\varepsilon_n = 1/n$ and $v_n(x) = \varepsilon_n^{N-1}u(\varepsilon_n x)$. From estimate (6.8) we have

$$|v_n(x)| \leqslant K|x|^{1-N}, \tag{6.13}$$

for $0 < |x| < 1/(2\varepsilon_n)$. Moreover, v_n satisfies

$$\Delta v_n = \varepsilon_n^{N+1} g(\varepsilon_n^{1-N}v_n) \tag{6.14}$$

in $\mathscr{B}_n - \{0\} = \{x\in\boldsymbol{R}^N\colon 0 < |x| < 1/\varepsilon_n\}$. From (6.10) we have

$$|g(r)| \leqslant 2c|r|^{(N+1)/(N-1)} + d \qquad (d \text{ constant}),$$

so

$$\varepsilon_n^{N+1}|g(\varepsilon_n^{1-N}v_n)| \leqslant 2CK^{(N+1)/(N-1)}|x|^{1-N} + d. \tag{6.15}$$

From the Rellich and Ascoli–Arzelà theorems there exist a subsequence $\{\varepsilon_{n_k}\}$ and a function $v\in C^1(\boldsymbol{R}^N - \{0\})$ with Δv locally bounded in $\boldsymbol{R}^N - \{0\}$ such that $\{v_{n_k}\}$ converges to v in the C^1 topology on any compact subset of $\boldsymbol{R}^N - \{0\}$, and also a function ψ measurable and locally bounded in $\boldsymbol{R}^N - \{0\}$ such that for any $\varphi\in\mathscr{D}(\boldsymbol{R}^N - \{0\})$

$$\lim_{n_k\to+\infty} \int_{\boldsymbol{R}^N} \varepsilon_k^{N+1} g(\varepsilon_{n_k}^{1-N}v_n)\varphi\,dx = \int_{\boldsymbol{R}^N} \psi\varphi\,dx. \tag{6.16}$$

Now we define

$$E^+ = \{x\in\boldsymbol{R}^N\colon v(x) > 0\}, \quad E^- = \{x\in\boldsymbol{R}^N\colon v(x) < 0\}, \tag{6.17}$$

and $E^0 = \boldsymbol{R}^N - \{E^+ \cup E^-\}$. If $x \in E^+$ (resp. E^-) then

$$\lim_{n_k \to \infty} \varepsilon_{n_k}^{1-N} v_{n_k}(x) = +\infty \quad \text{(resp. } -\infty\text{)},$$

$$\lim_{n_k \to +\infty} \varepsilon_{n_k}^{N+1} g(\varepsilon_{n_k}^{1-N} v_{n_k}(x)) = c\big(v(x)\big)^{(N+1)/(N-1)} \quad \text{(resp. } -c\,|v(x)|^{(N+1)/(N-1)}\text{)}.$$

So v satisfies

$$\Delta v = v\,|v|^{2/(N-1)} \chi_{E^+ \cup E^-} + \psi \chi_{E^0} \tag{6.18}$$

in $\mathscr{D}'(\boldsymbol{R}^N - \{0\})$, where χ_Ω is the characteristic function of a measurable set Ω. But by $g(0) = 0$ and (6.10), for any $\varepsilon > 0$ there exists $C(\varepsilon)$ such that

$$|g(r)| \leqslant C(\varepsilon)\,|r|^{(N+1)/(N-1)} + \varepsilon, \tag{6.19}$$

for any r, which implies $\varepsilon_n^{N+1}\,|g(\varepsilon_n^{1-N} v_n)| \leqslant C(\varepsilon)\,|v_n|^{(N+1)/(N-1)} + \varepsilon$. As

$$\lim_{n_k \to +\infty} |v_{n_k}(x)|^{(N+1)/(N-1)} \chi_{E^0}(x) = 0 \quad \text{a.e.},$$

we deduce that for any $\varphi \in \mathscr{D}'(\boldsymbol{R}^N - \{0\})$ the following inequality holds:

$$\lim_{n_k \to +\infty} \int_{E^0} \varepsilon_{n_k}^{N+1}\,|g(\varepsilon_{n_k}^{1-N} v_{n_k})|\,\varphi\,dx \leqslant \varepsilon \int_{E^0} \varphi\,dx. \tag{6.20}$$

So $\int |\psi \chi_{E^0}|\,\varphi\,dx \leqslant \varepsilon \int_{E^0} \varphi\,dx$ which implies $\|\psi \chi_{E^0}\|_{L^\infty(\boldsymbol{R}^N)} \leqslant \varepsilon$. As ε is arbitrary, $\psi \chi_{E^0} = 0$ a.e. and v satisfies

$$\Delta v = v\,|v|^{2/(N-1)} \tag{6.21}$$

in $\mathscr{D}'(\boldsymbol{R}^N - \{0\})$; moreover, $v \in C^2(\boldsymbol{R}^N - \{0\})$ from elliptic equations theory.

Step 3: *End of the proof.* From Step 2, for any compact subset K of $\boldsymbol{R}^N - \{0\}$ we have

$$\lim_{n_k \to +\infty} \|v_{n_k} - v\|_{L^\infty(K)} = 0.$$

Taking $K = S^{N-1}$ we have

$$\lim_{n_k \to +\infty} \big(\varepsilon_{n_k}^{N-1} u(\varepsilon_{n_k} x) - v(x)\big) = 0$$

uniformly on S^{N-1}. From Step 1, v is radial, $v(x) = \tilde{v}(1)$. So in polar coordinates we have

$$\lim_{n_k \to +\infty} \|\varepsilon_{n_k}^{N-1} u(\varepsilon_{n_k}, \cdot) - \tilde{v}(1)\|_{L^\infty(S^{N-1})} = 0, \tag{6.22}$$

which implies

$$\lim_{n_k \to +\infty} \varepsilon_{n_k}^{N-1} \|u(\varepsilon_{n_k}, \cdot) - \bar{u}(\varepsilon_{n_k})\|_{L^\infty(S^{N-1})} = 0$$

and $u = \bar{u}$ from Theorem 6.1.

COROLLARY 6.2. *Assume* $1 < q < (N+1)/(N-1)$ *and* u *is a nonnegative solution belonging to* $C^2(B_1 - \{0\}) \cap C^0(\bar{B}_1 - \{0\})$ *of*

$$\Delta u = u^q. \tag{6.23}$$

If u *is constant on* ∂B_1 *then* u *is radially symmetric.*

Proof. According to Theorem 4.1, either u has a strong singularity at 0 or $\lim_{x \to 0} u(x)/\mu(x)$ exists in $\boldsymbol{R}$. In the second case (6.2) is satisfied and u is radial. So we are left with the strongly singular case. Thanks to Remark 4.2 we can estimate the speed of convergence of $|x|^{2/(q-1)} u(x)$ to $l_{q,N}$ (see (4.19)) and we have

$$r^{N-1} \|u(r, \cdot) - \bar{u}(r)\|_{L^\infty(S^{N-1})} \leqslant C |x|^{\tau - 2/(q-1) + N - 1}. \tag{6.24}$$

Computing $\tau - 2/(q-1) + N - 1 = \sigma$ yields

$$\sigma = \frac{N}{2} + \frac{1}{2} \sqrt{\left(N - 2\frac{q+1}{q-1}\right)^2 + 8\left(\frac{2q}{q-1} - N\right)}.$$

So $\sigma > 0$ and (6.2) is satisfied.

Remark 6.2. Corollary 6.2 can be extended to a more general nonlinearity by supposing that u is a nonnegative solution of (6.1) and g satisfies

$$g(r) = cr^q + o(r^{(q-1)(N+1)/2}) \tag{6.25}$$

with $c > 0$ and $1 < q < (N+1)/(N-1)$ (see [25]). However, in the linear case ($g = 0$ or 1) all nonnegative solutions of (6.1) in $B_1 - \{0\}$ which are constant on ∂B_1 are radial. It is an open question whether that result still holds when g is a nonzero nondecreasing continuous function. In [25] we also give uniqueness results concerning singular solutions of (6.1) and (6.24) in $B_1 - \{0\}$ or $\boldsymbol{R}^N - \{0\}$.

In the next result we prove the existence of solutions of (4.1) with a breaking of symmetry. For simplicity we consider the 2-dimensional case.

THEOREM 6.2. *Assume* $N = 2$ *and* $1 < q < 3$. *Then their exists a solution* $u \in C^2(\bar{B}_1 - \{0\})$ *of* (4.1) *in* $B_1 - \{0\}$ *vanishing on* ∂B_1 *which is not radially symmetric.*

Proof. Let ω be a nonconstant solution of (5.1) with period $2\pi/k$ and anti-period π/k ($\omega(\theta) = -\omega(2\pi/k - \theta)$, $\theta \in [0, 2\pi]$), so ω vanishes for $\theta = 0, \pi/k, 2\pi/k, \ldots, (2k-1)\pi/k$. For $\varepsilon > 0$ we let u_ε be the solution of

$$\begin{aligned} \Delta u_\varepsilon &= u_\varepsilon |u_\varepsilon|^{q-1} && \text{in } \varepsilon < |x| < 1, \\ u_\varepsilon(x) &= 0 && \text{for } |x| = 1, \\ u_\varepsilon(x) &= \varepsilon^{-2/(q-1)} \omega(x/|x|) && \text{for } |x| = \varepsilon. \end{aligned} \tag{6.26}$$

The function $u_\varepsilon(r, \theta)$ vanishes for $\theta = 0, \pi/k, \ldots, (2k-1)\pi/k$. For $0 < \eta < \varepsilon$

we can compare u_η and u_ε in $\{(r,\theta)\colon \varepsilon < r < 1,\ 0<\theta<\pi/k\}$ supposing for example $\omega(\theta)>0$ for $0<\theta<\pi/k$. We first notice that $u_\omega(x) = |x|^{-2/(q-1)}\omega(x/|x|)$ satisfies (4.1) in $\{(r,\theta)\colon \eta<r<1,\ 0<\theta<\pi/k\}$; moreover, it vanishes for $\theta=0$ and $\theta=\pi/k$ and is positive on the two other boundaries $\{(r,\theta)\colon 0<\theta<\pi/k,\ r=1\}$ and $\{(r,\theta)\colon 0<\theta<\pi/k,\ r=\varepsilon\}$. As a consequence, $u_\eta(r,\theta)\leqslant u_\omega(r,\theta)$ in that sector and in particular

$$u_\eta(\varepsilon,\theta)\leqslant u_\omega(\varepsilon,\theta)=\varepsilon^{-2/(q-1)}\omega(\theta).$$

Comparing u_ε and u_η in the sector $\{(r,\theta)\colon \varepsilon<r<1,\ 0<\theta<\pi/k\}$ we get $u_\eta\leqslant u_\varepsilon$. As ε goes to zero $|u_\varepsilon(r,\theta)|$ decreases so u_ε converges to some $u\in C^2(\bar B_1-\{0\})$ and u is a solution of (4.1) in $B_1-\{0\}$ which vanishes for $|x| = 1$ and for $\theta=0,\ \pi/k,\ \ldots,\ (2k-1)\pi/k$. The only problem is to show that u is not identically zero. If we set

$$s=\frac{r^{4/(q-1)}}{4/(q-1)},\qquad v_\varepsilon(s,\theta)=r^{2/(q-1)}u_\varepsilon(r,\theta), \tag{6.27}$$

then v_ε satisfies

$$\left(\frac{4}{q-1}\right)^2 s^2\frac{\partial^2 v_\varepsilon}{\partial s^2}+\left(\frac{2}{q-1}\right)^2 v_\varepsilon+\frac{\partial^2 v_\varepsilon}{\partial\theta^2}=v_\varepsilon|v_\varepsilon|^{q-1} \tag{6.28}$$

in $(s_\varepsilon,(q-1)/4]\times S^1$, with $s_\varepsilon=\varepsilon^{4/(q-1)}/[4/(q-1)]$. Moreover, $v_\varepsilon(s,\theta)$ vanishes for $s=(q-1)/4$ and coincides with $\omega(\theta)$ for $s=s_\varepsilon$. Moreover, $0<v_\varepsilon(r,\theta) <\omega(\theta)$ for $s_\varepsilon<s<(q-1)/4$, $0<\theta<\pi/k$. We set

$$X_\varepsilon(s)=\int_0^{\pi/k} v_\varepsilon(s,\theta)\,\omega(\theta)\,d\theta$$

and we have

$$\left(\frac{4}{q-1}\right)^2 s^2\frac{d^2X_\varepsilon}{ds^2}+\left(\frac{2}{q-1}\right)^2 X_\varepsilon+\int_0^{\pi/k} v_\varepsilon\frac{d^2\omega}{d\theta^2}\,d\theta=\int_0^{\pi/k}\omega v_\varepsilon|v_\varepsilon|^{q-1}\,d\theta.$$

Using the fact that ω satisfies (5.1) we finally get

$$\left(\frac{4}{q-1}\right)^2 s^2\frac{d^2X_\varepsilon}{ds^2}=\int_0^{\pi/k}\omega v_\varepsilon\left(|v_\varepsilon|^{q-1}-|\omega|^{q-1}\right)d\theta. \tag{6.29}$$

Hence X_ε is concave on $[s_\varepsilon,(q-1)/4]$ and $X_\varepsilon(s_\varepsilon)=\int_0^{\pi/k}\omega^2\,d\theta$, $X_\varepsilon((q-1)/4)=0$. As a consequence,

$$\lim_{\varepsilon\to 0}X_\varepsilon(s)=X(s)=\int_0^{\pi/k} v(s,\theta)\,\omega\,d\theta\geqslant\left(1-\frac{4s}{q-1}\right)\int_0^{\pi/k}\omega^2\,d\theta.$$

and u is not the zero function.

7. Appendix. A nonlinear eigenvalue problem on S^1

A keystone tool for proving Theorem 5.1 is the description of the set $\mathcal{E}$ of solutions of (5.1) and the exclusion principle corresponding to the energy function associated to $\mathcal{E}$. We extend our study to the following more general eigenvalue problem on S^1 (see Chen, Matano and Véron [30]):

$$-\frac{d^2\omega}{d\theta^2}+g(\omega) = \lambda\omega, \tag{7.1}$$

where g satisfies:

(i) g is a C^1 odd function,

(ii) dg/dr is increasing on $[0, +\infty)$ and vanishes only at 0,

(iii) $\lim\limits_{r\to+\infty} (g(r)/r) = +\infty$.

We write $G(r) = \int\limits_0^r g(s)\,ds$, and h is the inverse function of the restriction of $s \mapsto g(s)/s$ to $[0, +\infty)$. Let $\mathcal{E}_\lambda$ be the set of solutions of (7.1).

PROPOSITION 7.1. *If $\lambda \leqslant 0$, $\mathcal{E}_\lambda$ reduces to the zero function. If $0 < \lambda \leqslant 1$, $\mathcal{E}_\lambda$ only contains the three constant functions 0 and $\pm h(\lambda)$. If $\lambda > 1$, $\mathcal{E}_\lambda$ has $F(\sqrt{\lambda})+3$ connected components S_k (where $F(r)$ is the greatest integer strictly less than r):*

(i) *S_0 reduces to the zero function,*

(ii) *S_1 (resp. S_2) reduces to the constant function $h(\lambda)$ (resp. $-h(\lambda)$),*

(iii) *for $3 \leqslant k \leqslant F(\sqrt{\lambda})+2$, S_k is the closed curve generated by a particular solution ω_k of (7.1), with minimal period $2\pi/(k-2)$, in the following way: $S_k = \{\omega_k(\cdot+\alpha),\ \alpha \in [0, 2\pi/(k-2))\}$.*

Sketch of the proof. Step 1: $\lambda \leqslant 1$. Set $\bar{\omega} = (2\pi)^{-1}\int_{S^1}\omega(\theta)\,d\theta$. We have from (7.1)

$$\int\limits_{S^1}(\omega-\bar{\omega})^2 + \int\limits_{S^1}\big(g(\omega)-\overline{g(\omega)}\big)(\omega-\bar{\omega})\,d\theta \leqslant \lambda\int\limits_{S^1}(\omega-\bar{\omega})^2\,d\theta.$$

As $\int_{S^1}(\omega-\bar{\omega})\big(g(\omega)-g(\bar{\omega})\big)d\theta \geqslant 0$, ω must be constant. If $\lambda \leqslant 0$ the only admissible constant is 0 and for $0 < \lambda \leqslant 1$ we obtain S_0, S_1 and S_2.

So we now suppose $\lambda > 1$. It is moreover clear from periodicity condition that any nonconstant solution of (7.1) must vanish somewhere, say at 0, where the derivative is not zero. Let ω be a nonconstant solution of (7.1) such that

$$\omega(0) = 0, \quad \frac{d\omega}{d\theta}(0) = \alpha > 0. \tag{7.2}$$

Step 2. For some interval $(0, \varepsilon)$, $\varepsilon > 0$, ω is increasing and

$$\frac{1}{2}\frac{d}{d\theta}\left(\frac{d\omega}{d\theta}\right)^2 = \frac{d}{d\theta}\left(G(\omega) - \frac{\lambda}{2}\omega^2\right), \tag{7.3}$$

so $d\omega/d\theta = \sqrt{\alpha^2 + 2G(\omega) - \lambda\omega^2}$ and

$$\theta = \int_0^{\omega(\theta)} \frac{ds}{\sqrt{\alpha^2 + 2G(\omega) - \lambda\omega^2}}. \tag{7.4}$$

Moreover, formula (7.4) remains valid as long as $\omega(\theta)$ remains smaller than the first zero of the function $r \mapsto \psi(\alpha, r) = \alpha^2 + 2G(r) - \lambda r^2$. As $\psi(\alpha, \cdot)$ strictly decreases on $[0, h(\lambda)]$ we have three possibilities:

(i) $\alpha^2 > \lambda h^2(\lambda) - 2G(h(\lambda))$,

(ii) $\alpha^2 = \lambda h^2(\lambda) - 2G(h(\lambda))$,

(iii) $\alpha^2 < \lambda h^2(\lambda) - 2G(h(\lambda))$.

In cases (i) and (ii), ω could not be 2π-periodic so we are left with (iii) where the function $r \mapsto \psi(\alpha, r)$ admits a simple zero in $(0, h(\lambda))$, say $s(\alpha)$. As $(\partial\psi/\partial s)(\alpha, s(\alpha)) \neq 0$ we define a finite number $\theta(\alpha)$ by

$$\theta(\alpha) = \int_0^{s(\alpha)} \frac{ds}{\sqrt{\psi(\alpha, s)}}. \tag{7.5}$$

Morevoer, $\omega(\theta(\alpha)) = s(\alpha)$ and $(d\omega/d\theta)(\theta(\alpha)) = 0$. We continue our integration procedure on $(\theta(\alpha), \theta(\alpha) + \varepsilon)$ where ω decreases and

$$\frac{d\omega}{d\theta} = -\sqrt{\alpha^2 + 2G(\omega) - \lambda\omega^2}, \tag{7.6}$$

since $\alpha^2 = \lambda s^2(\alpha) - 2G(s(\alpha))$, which yields

$$\theta - \theta(\alpha) = -\int_{\omega(\theta)}^{s(\alpha)} \frac{ds}{\sqrt{\psi(\alpha, s)}}. \tag{7.7}$$

If

$$\theta_1 = \int_0^{\omega(\theta_1)} \frac{ds}{\sqrt{\psi(\alpha, s)}}, \qquad \theta_2 = 2\theta(\alpha) - \theta_1, \qquad \theta_2 - \theta(\alpha) = \int_{\omega(\theta_2)}^{s(\alpha)} \frac{ds}{\sqrt{\psi(\alpha, s)}}$$

then

$$\theta(\alpha) = \int_0^{\omega(\theta_1)} \frac{ds}{\sqrt{\psi(\alpha, s)}} + \int_{\omega(\theta_2)}^{s(\alpha)} \frac{ds}{\sqrt{\psi(\alpha, s)}} = \int_0^{s(\alpha)} \frac{ds}{\sqrt{\psi(\alpha, s)}}$$

and $\omega(\theta_1) = \omega(\theta_2)$. Hence ω admits $2\theta(\alpha)$ as its smallest anti-period and $4\theta(\alpha)$ as its smallest period. *The necessary and sufficient condition for* 2π-*periodicity is that* $\pi/(2\theta(\alpha))$ *must be a positive integer*. We are now left with the study of the function $\alpha \mapsto \theta(\alpha)$ under the condition

$$(7.8) \qquad 0<\alpha^2<\lambda h^2(\lambda)-2G(h(\lambda)) = T^2(\lambda).$$

Step 3: We claim that $\alpha \mapsto s(\alpha)$ *is increasing and convex on* $[0, T(\lambda))$. From the implicit function theorem $s(\alpha)$ is C^2. Moreover,

$$\frac{d}{d\alpha}(\psi(\alpha, s(\alpha))) = \frac{\partial\psi}{\partial\alpha}(\alpha, s(\alpha))+\frac{ds}{d\alpha}(\alpha)\frac{\partial\psi}{\partial s}(\alpha, s(\alpha)) = 0$$

so

$$(7.9) \qquad \frac{ds}{d\alpha}(\alpha) = \frac{\alpha}{\lambda s(\alpha)-g(s(\alpha))}.$$

As $s(\alpha)<h(\lambda)$, $\alpha \mapsto s(\alpha)$ is increasing. Moreover, a straightforward computation with the use of (7.9) yields

$$(7.10) \qquad \frac{d^2 s}{d\alpha^2}(\alpha) = \frac{(\lambda s(\alpha)-g(s(\alpha)))^2-\alpha^2\left(\lambda-\dfrac{dg}{dr}(s(\alpha))\right)}{(\lambda s(\alpha)-g(s(\alpha)))^3},$$

and the sign of $d^2 s/d\alpha^2$ is the same as that of

$$\tau(s) = (\lambda s-g(s))^2-(\lambda s^2-2G(s))\left(\lambda-\frac{dg}{ds}(s)\right), \qquad \text{where } s = s(\alpha).$$

A miraculous computation gives

$$\frac{d\tau}{dt}(s(\alpha)) = \alpha^2\frac{d^2 g}{ds^2}(s(\alpha)).$$

As $\tau(0) = 0$, τ is positive on $[0, T(\lambda))$.

Step 4: $\alpha \mapsto \theta(\alpha)$ *is continuous and strictly increasing on* $[0, T(\lambda))$. We recall that

$$\theta(\alpha) = \int_0^s \frac{ds}{\sqrt{\alpha^2+2G(s)-\lambda s^2}}.$$

For $u\in[0, \alpha]$, the function $u\mapsto\psi(u, s)$ admits a first positive zero at $s(u)$ and $\psi(\alpha, s(u)) = \alpha^2-u^2$. As a consequence,

$$(7.11) \qquad \theta(\alpha) = \int_0^\alpha \frac{ds}{du}(u)\frac{du}{\sqrt{\alpha^2-u^2}} = \int_0^1 \frac{ds}{du}(\alpha\sigma)\frac{d\sigma}{\sqrt{1-\sigma^2}}.$$

As ds/du is an increasing C^1 function the same is true for θ.

Step 5: *End of the proof.* As $\lim_{\alpha\downarrow 0} s(\alpha) = 0$, we have

$$\lim_{\alpha\downarrow 0}\frac{G(s(\alpha))}{s^2(\alpha)} = 0 \quad \text{and} \quad \lim_{\alpha\downarrow 0}\frac{s(\alpha)}{\alpha} = \frac{1}{\sqrt{\lambda}}$$

(by (7.9)) and

$$\lim_{\alpha \downarrow 0} \theta(\alpha) = \pi/\sqrt{\lambda}. \tag{7.12}$$

Moreover, $\lim_{\alpha \uparrow T(\lambda)} s(\alpha) = s(T(\lambda)) = h(\lambda)$. As $h(\lambda)$ is a zero of $\psi(T(\lambda), s)$ of order at least two there exists a continuous bounded function Q such that

$$T^2(\lambda) - \lambda s^2 + 2G(s) = \big(s(T(\lambda)) - s\big)^2 Q(s), \tag{7.13}$$

$$\lim_{\alpha \uparrow T(\lambda)} \theta(\alpha) \geqslant \int_0^{s(T(\lambda))} \frac{ds}{\big(s(T(\lambda)) - s\big)\sqrt{Q(s)}} = +\infty. \tag{7.14}$$

Hence the function θ is a continuous bijection of $(0, T(\lambda))$ onto $\big(\pi/(2\sqrt{\lambda}), +\infty\big)$ and $\alpha \mapsto \pi/(2\theta(\alpha))$ is a bijection of $(0, T(\lambda))$ onto $(0, \sqrt{\lambda})$. Moreover, in the interval $(0, \sqrt{\lambda})$ there exist $F(\sqrt{\lambda})$ nonzero integers k, $0 < k \leqslant F(\sqrt{\lambda})$, which all generate a solution ω_k of (7.1) with minimal period $2\pi/k$ and such that $\omega_k(0) = 0$, $(d\omega_k/d\theta)(0) = \alpha_k$.

We can now prove the following *exclusion principle*:

PROPOSITION 7.2. *Let f be a continuous even function increasing on $[0, +\infty)$ and ω_1 and ω_2 two solutions of* (7.1) *which do not belong to the same connected component of $\mathscr{E}_\lambda$, one of them being different from $\pm h(\lambda)$. Then*

$$\int_{S^1} f(\omega_1(\theta))\,d\theta \neq \int_{S^1} f(\omega_2(\theta))\,d\theta. \tag{7.15}$$

Proof. If ω is any nonconstant solution of (7.1), then $|\omega(\theta)| < h(\lambda)$ on S^1 and

$$2\pi f(0) < \int_{S^1} f(\omega(\theta))\,d\theta < 2\pi f\big(h(\lambda)\big). \tag{7.16}$$

Moreover, $\omega \mapsto \int_{S^1} f(\omega(\theta))\,d\theta$ is constant on each connected component of $\mathscr{E}_\lambda$. As ω_1 and ω_2 do not belong to the same connected component and are not constant we can assume $\omega_1(0) = \omega_2(0) = 0$, $(d\omega_1/d\theta)(0) = \alpha$, $(d\omega_2/d\theta)(0) = \beta$ and $0 < \alpha < \beta$; ω_1 is $4\theta(\alpha)$-periodic, ω_2 is $4\theta(\beta)$-periodic and we have $0 < \theta(\alpha) < \theta(\beta)$ and

$$\frac{\pi}{2\theta(\alpha)} = k_1, \quad \frac{\pi}{2\theta(\beta)} = k_2, \quad k_1, k_2 \in N, \quad k_1 > k_2 > 0. \tag{7.17}$$

Step 1: We claim that $0 < \omega_1(\theta) < \omega_2(\theta)$ holds on $(0, \theta(\alpha)]$. The relation is true for θ small enough. Suppose now that there exists $\theta_0 \in (0, \theta(\alpha))$ such that $\omega_1(\theta_0) = \omega_2(\theta_0) = \mu$. Then $(d\omega_1/d\theta)(\theta_0) \geqslant (d\omega_2/d\theta)(\theta_0)$ or

$$\sqrt{\alpha^2 + 2G(\mu) - \lambda\mu^2} \geqslant \sqrt{\beta^2 + 2G(\mu) - \lambda\mu^2}, \tag{7.18}$$

which is a contradiction.

Step 2: *End of the proof.* We have $0 < \omega_1(\theta) < \omega_2(\theta')$ for $\theta < \theta'$, $0 < \theta \leqslant \theta(\alpha)$, $0 < \theta' \leqslant \theta(\beta)$. Let p be the lowest common multiple of k_1 and k_2. There exist integers n_1, n_2 such that $n_1 k_1 = n_2 k_2 = p$ and $n_1/\theta(\alpha) = n_2/\theta(\beta)$ so $0 < n_1 < n_2$. We also have

$$(7.19)\qquad \begin{aligned} &\text{(i)} \quad \int_{S^1} f(\omega_1(\theta))\,d\theta = \frac{2\pi}{\theta(\alpha)} \int_0^{\theta(\alpha)} f(\omega_1(\theta))\,d\theta,\\ &\text{(ii)} \quad \int_{S^1} f(\omega_2(\theta))\,d\theta = \frac{2\pi}{\theta(\beta)} \int_0^{\theta(\beta)} f(\omega_2(\theta))\,d\theta. \end{aligned}$$

Set $T = n_2\,\theta(\alpha) = n_1\,\theta(\beta)$; then

$$\frac{2\pi}{\theta(\alpha)} \int_0^{\theta(\alpha)} f(\omega_1(\theta))\,d\theta = \frac{2\pi}{T}\int_0^T f(\omega_1(\sigma/n_2))\,d\sigma,$$

$$\frac{2\pi}{\theta(\beta)} \int_0^{\theta(\beta)} f(\omega_2(\theta))\,d\theta = \frac{2\pi}{T}\int_0^T f(\omega_2(\sigma/n_1))\,d\sigma.$$

As $\sigma/n_1 > \sigma/n_2$, we have $\omega_1(\sigma/n_2) < \omega_2(\sigma/n_1)$ which yields

$$(7.20)\qquad \int_{S^1} f(\omega_1(\theta))\,d\theta < \int_{S^1} f(\omega_2(\theta))\,d\theta.$$

Remark 7.1. As a consequence, the function defined on $H^1(S^1)$ by

$$(7.21)\qquad J(\gamma) = \tfrac{1}{2}\int_{S^1} (d\gamma/d\theta)^2\,d\theta + \int_{S^1} G(\gamma)\,d\theta - \tfrac{1}{2}\lambda \int_{S^1} \gamma^2\,d\theta,$$

whose critical points are the solutions of (7.1), admits $F(\sqrt{\lambda})+2$ different critical values: if $\omega \in \mathscr{E}_\lambda$ we have

$$(7.22)\qquad J(\omega) = \int_{S^1} \big(G(\omega) - \tfrac{1}{2}\omega g(\omega)\big)\,d\theta,$$

and the function $r \mapsto \frac{1}{2} r g(r) - G(r)$ satisfies the hypotheses of Proposition 7.2.

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Added in proof (August 1987). The problem raised in Remark 4.3 has been completely solved in

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The problem raised in Remark 5.1 has been solved in $\mathbf{R}^2$ in

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