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TAME ORDERS

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This note gives an overview on the main results in the subject of tame orders, ranging from oldest, by now classical results, up to most recent, yet unpublished developments. It stresses historical aspects and puts particular emphasis on reductions, induced problems, the classifications and their different nature.

The following topics are being presented: Nazarova's classification of the indecomposable 2-adic representations of Klein's four group; a general reduction principle and its application to Bäckström orders, leading to a characterization of tame Bäckström orders; Yakovlev's classification of the indecomposable 2-adic representations of the cyclic group of order eight; classification of tame local group rings over complete discrete valuation rings; complete local rings of curve singularities viewed as orders, and recent progress in the problem of classifying all tame curve singularities.

1. Introduction

The following framework will be valid throughout this note. We always have a ground ring R which, by assumption, is a complete discrete valuation ring. By an *R-order* we mean an R-algebra which is finitely generated free as an R-module. With any R-order Λ there is associated the category $\operatorname{latt}\Lambda$ of Λ-lattices which is, by definition, the full subcategory of the category of all left Λ-modules, given by those modules which are finitely generated free over R.

Any Λ-lattice decomposes into a finite direct sum of indecomposable Λ-lattices. Since our ground ring is complete, this decomposition is unique, in the sense that each lattice uniquely determines the isomorphism types and multiplicities of its indecomposable direct summands. Therefore many ques-

This paper is in final form and no version of it will be submitted for publication elsewhere.

tions on the lattice category lattΛ are reduced to questions on its full subcategory indΛ, given by all indecomposable objects.

A problem of interest which arises here is the problem of classifying all indecomposable Λ-lattices, up to isomorphism, the so-called classification problem of Λ. In this note, we are concerned exclusively with orders whose classification problem is of infinite type, but nevertheless is solved. In other words, there exist infinitely many isomorphism classes of indecomposable lattices, and these are completely classified. It turns out that these classifications share the common phenomenon that for each $d \in \mathbf{N}$, almost all indecomposable lattices of rank d occur in a finite collection of one-parameter families of indecomposable lattices of rank d, up to isomorphism. Orders whose lattice categories show this phenomenon are called *tame orders.*

This description needs some supplementary explanation. Denote by $[\mathrm{ind}\Lambda]$ the set of all isomorphism classes of indΛ. By a *classification* of indΛ I mean a set of invariants I, usually consisting of a discrete and a continuous part, together with a bijective mapping ζ: $I \to [\mathrm{ind}\Lambda]$ which has to be constructive in both directions. Moreover, let k be the residue class field of R, let $\mathscr{I}$ be the set of all normed irreducible polynomials in $k[X]$, and set $\mathscr{I}_\infty = \mathscr{I} \cup \{\infty\}$. Then, by a *one-parameter family* of indecomposable lattices of rank d I mean an $\mathscr{I}_\infty$-family $(M_p)_{p \in \mathscr{I}_\infty}$ of pairwise nonisomorphic indecomposable lattices, such that $\mathrm{rank}(M_p) = d \cdot \deg(p)$.

Our knowledge of tame orders is still very restricted. For example, classifications of tame orders exist only with reference to very special classes of orders, such as group rings, Bäckström orders, or curve singularities. Actually, the class of known tame orders is rather small, and I conjecture that the multitude of those tame orders which are not yet understood exceeds by far the collection of those for which we have some insight. Accordingly, there is not much of a theory on the subject of tame orders. Instead, there are some scattered results, some stray approaches towards the classification problems, and certain typical phenomena showing up repeatedly and in various disguise.

In this vein I will present some of the main results. Hopefully they will become transparent for typical ideas and characteristic phenomena, among which I count the reductions, induced problems, the classifications and their different nature.

The following notation will be valid throughout: π is a chosen parameter of R, and $k = R/\pi$. More generally, $R_l = R/\pi^l$, where $l \in \mathbf{N}$. Moreover, $K = \mathrm{fract}(R)$, and $A = K \otimes_R \Lambda$ is the finite-dimensional K-algebra generated by Λ. In most of the cases to be considered, A will be separable. We denote by $\mathscr{I}$ the set of all normed irreducible polynomials in $k[X]$, and we set $\mathscr{I}_\infty = \mathscr{I} \cup \{\infty\}$. As to k-matrices, Φ_f denotes the Frobenius matrix of a polynomial $f \in k[X]$, and E is the identity matrix. If C is an object in a category $\mathscr{C}$, then $[C]$ denotes the isomorphism class of C, and $[\mathscr{C}]$ denotes the collection of all isomorphism classes in $\mathscr{C}$.

The present note is an elaborate version of the series of talks which I gave at the Banach international mathematical centre in Warsaw during the last week of the semester on representations of algebras, in May 1988.

2. The group ring $\Lambda = \hat{\mathbf{Z}}_2(C_2 \times C_2)$

Historically, this group ring is the first example of a tame order. Its classification problem was solved by Nazarova in 1961. Here I will present two solutions. First, I sketch Nazarova's approach [Na61], [Na67].

First solution of the classification problem of Λ

Consider for the moment an arbitrary complete discrete valuation ring. We define a particular R-order Γ upon setting

$$\Gamma = \left\{(r_1, \ldots, r_4) \in \prod_{i=1}^{4} R_i \,\middle|\, r_1 \equiv \ldots \equiv r_4 \pmod{\pi}\right\}.$$

In the language of the sixties, this order is called the *tetrad*. Analogously one defines the dyad, triad, quintad,... by taking a different number of copies of R.

Moreover, I quote a result due to Faddeev [Fa65] and Roiter [Ro66] which is as useful here, as it is of independent interest.

THEOREM. *If Λ is a local Gorenstein order which is not maximal, then there exists a uniquely determined minimal overorder Λ_0 of Λ, and* $[\operatorname{ind}\Lambda] = [\operatorname{ind}\Lambda_0] \dot{\cup} [\Lambda]$.

The existence of a minimal overorder is a triviality. The point is its uniqueness, together with the fact that the corresponding lattice categories are as close to each other as they possibly can be.

Now we turn to the group ring $\Lambda = \hat{\mathbf{Z}}_2(C_2 \times C_2)$. Being a group ring, Λ is Gorenstein, and the group being of prime power order, Λ is local. Hence we may apply the above theorem, and it turns out that the unique minimal overorder Λ_0 of Λ is the tetrad (with ground ring $\hat{\mathbf{Z}}_2$): $\Lambda_0 = \Gamma \subset \prod_{i=1}^{4} \hat{\mathbf{Z}}_2 e_i$, where $e_1, \ldots, e_4$ are the primitive orthogonal idempotents of $A = \hat{\mathbf{Q}}_2(C_2 \times C_2)$. Thus, for classifying $\operatorname{ind}\Lambda$ it is sufficient to classify $\operatorname{ind}\Gamma$.

We proceed towards the classification problem of the tetrad (for arbitrary ground ring R), now forgetting about the group ring Λ. From the definition of Γ it is clear that $e_i \notin \Gamma$, but $\pi e_i \in \Gamma$, for all i. Further, it is easy to see that

$$\Gamma/\pi(e_1 + e_2) \cong \Delta \cong \Gamma/\pi(e_3 + e_4),$$

where Δ denotes the dyad. With any $M \in \operatorname{latt}\Gamma$ we associate $L = \{m \in M \mid \pi(e_1 + e_2)m = 0\}$ and $N = M/L$. Both L and N are in $\operatorname{latt}\Delta$. Thus we obtain a functor $\operatorname{latt}\Gamma \to \{(L, N, \varepsilon) \mid L, N \in \operatorname{latt}\Delta,\ \varepsilon \in \operatorname{Ext}^1_\Gamma(N, L)\}$, given on objects by $M \mapsto [\varepsilon: 0 \to L \to M \to N \to 0]$, which induces a bijection between the isomorphism classes of indecomposable objects of the related categories. The dyad is well known to have only three indecomposable lattices, say S_1, S_2, S_3. Hence

$$\mathrm{Ext}^1_\Gamma(N, L) \cong \mathrm{Ext}^1_\Gamma\Big(\bigoplus_{i=1}^{3} S_i^{m_i}, \bigoplus_{j=1}^{3} S_j^{l_j}\Big) \cong k^{l\times m},$$

where $l = \sum_{j=1}^{3} l_j$ and $m = \sum_{i=1}^{3} m_i$. Under this isomorphism, the action of $\mathrm{Aut}_\Delta L \times \mathrm{Aut}_\Delta N$ on $\mathrm{Ext}^1_\Gamma(N, L)$ corresponds to a system of admissible transformations on matrices in $k^{l\times m}$:

(a) arbitrary nonsingular transformations on each row strip and on each column strip, and

(b) addition of arbitrary linear combinations of rows from the first and second row strips to the third, and addition of arbitrary linear combinations of columns from the first and second column strips to the third.

We indicate this in Fig. 1. In the combinatorial language introduced by

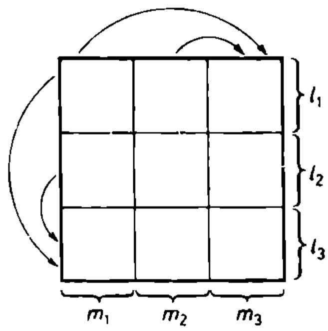

Fig. 1

Nazarova and Roiter [NaRo73a], [Na78], this is the matrix problem which is given by the pair of posets of Fig. 2. On setting $l_3 = m_3 = 0$, one obtains as a subproblem the matrix problem (o o, o o). This is the base-dependent formulation for the classification problem of the quiver of Fig. 3 of extended

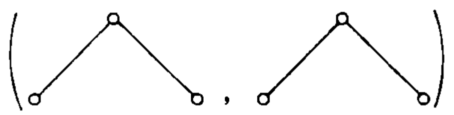

Fig. 2

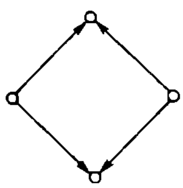

Fig. 3

Dynkin type $\tilde{A}_{22}$. Nazarova solves this subproblem in a first step, then inserts the obtained normal forms in place of the submatrix, and accomplishes the solution of the full-sized matrix problem in a second step. The resulting classification is the following.

With any matrix $M \in k^{l \times m}$ we associate its "rank" $r(M) = l+m$. It is in fact the rank of the Λ-lattice corresponding to M. Then:

(i) For each $n \in \mathbf{N}$ there is a one-parameter family of indecomposable matrices of rank $4n$, given by $\left[\begin{array}{c|c} \Phi_{\chi^n} & E \\ \hline E & E \end{array}\right]$, $\left[\begin{array}{c|c} E & \Phi_{p^n} \\ \hline E & E \end{array}\right]$, $p \in \mathscr{I}$.

(ii) Let $\nu: \mathbf{N} \to \mathbf{N} \cup \{0, \infty\}$ be the function which counts the isomorphism classes of indecomposable matrices of a given rank which do not occur in (i). Then $4 \leqslant \nu(r) \leqslant 8$, for all $r \in \mathbf{N}$. Normal forms are also given for these indecomposable matrices (see [Na67]).

Let me summarize. In a first step, the classification problem of Λ was reduced to the classification problem of Γ, by application of the Faddeev–Roiter theorem. In a second step, the classification problem of Γ was reduced to a matrix problem over k. This second reduction was based on the observation that each Γ-lattice is the middle term of a short exact sequence in $\operatorname{latt}\Gamma$, whose end terms are lattices over a factororder Δ. Moreover, the classification of $\operatorname{ind}\Delta$ is known. The automorphism groups of the end terms L and N operate on $\operatorname{Ext}^1_\Gamma(N, L)$, and the isomorphism classes of Γ-lattices are in bijective correspondence with the orbits of $\operatorname{Ext}^1_\Gamma(N, L)$ under this operation, where (N, L) ranges through a complete set of representatives of $[\operatorname{latt}\Delta] \times [\operatorname{latt}\Delta]$.

This basic idea for the second reduction is as old as integral representation theory of finite groups. It has been used already in Diederichsen's classification of $\operatorname{ind}\hat{\mathbf{Z}}_p C_p$ [Di40], as well as by various authors during the fifties. It has been made explicit by Heller and Reiner [HeRe62]. Nazarova's achievement is, having used this tool successfully for the first time for the solution of a tame classification problem.

The fact that, in our case, $\operatorname{Ext}^1_\Gamma(N, L)$ is a k-vector space, is very special and is due to the inclusions $\pi\Omega \subset \Gamma \subset \Omega$, where $\Omega = \prod_{i=1}^4 R_i$ is hereditary. Note the general statement which does not depend on this special hypothesis: If Λ is an R-order such that $K \otimes_R \Lambda$ is semisimple, then $\operatorname{Ext}^1_\Lambda(N, L)$ is an R-module of finite length, for all N, $L \in \operatorname{latt}\Lambda$ [Au84].

In the classification we encounter the basic behaviour of tame type: a complete classification of all indecomposable lattices is given by a sequence of one-parameter families, one for each rank $r \in 4\mathbf{N}$, together with an additional sequence of finite collections of indecomposable lattices, one for each rank $r \in \mathbf{N}$, with cardinality varying between 4 and 8. All of these indecomposable lattices are given explicitly, in terms of normal forms of k-matrices.

Second solution of the classification problem of Λ

Having sketched Nazarova's original solution of the classification problem of Λ, I will now outline a second way of solving this problem. It carries a more modern flavour.

Again, let Γ be the tetrad over an arbitrary ground ring R. Let $\Omega = \prod_{i=1}^{4} R_i$ and $\mathscr{J} = \pi\Omega$. Then Ω is the unique maximal order in $A = K \otimes_R \Lambda$, and $\mathscr{J} = \operatorname{rad}\Omega = \operatorname{rad}\Gamma$. Moreover, let Q be the quiver of Fig. 4 of

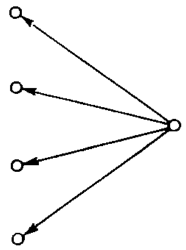

Fig. 4

extended Dynkin type $\tilde{\mathbf{D}}_4$, and denote by $\operatorname{rep}'_k Q$ the category of all those k-representations of Q which contain no simple direct summand. Any Γ-lattice M generates the Ω-lattice ΩM, where the extended scalar multiplication by Ω is defined inside the A-module $K \otimes_R M$. The filtration $\mathscr{J}M \subset M \subset \Omega M$ gives rise to a functor $\operatorname{latt}\Gamma \to \operatorname{rep}'_k Q$, defined on objects by $M \mapsto [M/\mathscr{J}M \subsetneq \Omega M/\mathscr{J}M]$, which induces a bijection between the isomorphism classes of indecomposable objects of the related categories. The representation theory of Q is well understood. In particular, the classification of its indecomposable representations [GePo70] entails the classification of the indecomposable lattices over the tetrad, respectively over $\Lambda = \hat{\mathbf{Z}}_2(C_2 \times C_2)$.

Let us review this section. The two solutions of the classification problem of Γ presented above are based on two different reductions which lead to two different matrix problems over k (see Fig. 5). This illustrates a general fact.

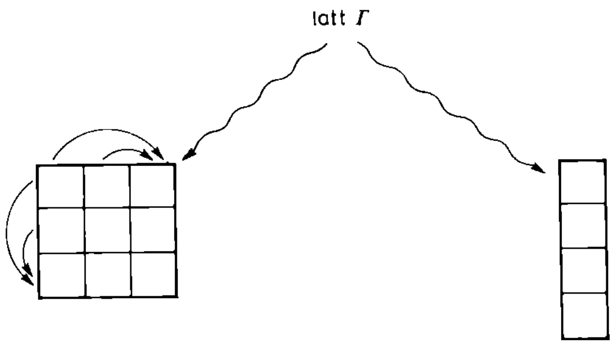

Fig. 5

Given an R-order Λ, there is usually more than one possibility to translate its classification problem to a problem over k. Among these various reductions there is no canonical choice. This fact can be uncomfortable as long as the classification problem of Λ is unsolved. Namely, a chosen reduction may be badly adapted to the nature of the particular lattice category and may contain unnecessary complications. On the other hand, this fact has an interesting consequence as soon as the classification problem of Λ is solved. Because then all the other classification problems which result from other reductions are solved as well!

Note that at the time when Nazarova solved the classification problem of $\hat{\mathbf{Z}}_2(C_2 \times C_2)$, the representation theory of the quiver of type $\tilde{\mathbf{D}}_4$ was not yet known. This leads to a curious historical remark: The 4-subspace problem was solved originally in disguise of $\mathrm{latt}\,\hat{\mathbf{Z}}_2(C_2 \times C_2)$, respectively in disguise of the matrix problem of Fig. 2 over $\mathbf{F}_2$!

3. Bäckström orders

Having discussed the classification problem of a particular order, the tetrad, in detail, let us now turn to a basic principle which allows one to reduce problems on lattices to problems in finite length categories, and which works in great generality. We begin with a definition.

An *admissible triple* (I, Λ, Γ) consists, by definition, of the following data:

(a) two R-orders Λ and Γ, generating the same K-algebra A, such that $\Lambda \subsetneqq \Gamma$, and

(b) a two-sided Γ-ideal I such that $I \subset \mathrm{rad}\,\Lambda$.

Thus, with any admissible tripe (I, Λ, Γ) we are given the following context of rings and ideals:

$$\begin{tikzcd}
 & & & & A \arrow[dd, dash] \\
 & & & \Gamma \arrow[ur, dash] & \\
 & & \Lambda \arrow[ur, dash] \arrow[dd, dash] & & K \\
 & \mathrm{rad}\,\Lambda \arrow[ur, dash] & & & \\
 I \arrow[ur, dash] & & R \arrow[uurr, dash] & &
\end{tikzcd}$$

Associated with an admissible triple (I, Λ, Γ) there is a category $\mathscr{C} = \mathscr{C}_{(I,\Lambda,\Gamma)}$ and a reduction functor $\mathscr{F} = \mathscr{F}_{(I,\Lambda,\Gamma)}$: $\mathrm{latt}\,\Lambda \to \mathscr{C}$ as follows. Consider the functor $\mathscr{S} := \Gamma/I \otimes_\Gamma ?$: $\mathrm{latt}\,\Gamma \to \mathrm{mod}\,\Lambda/I$, let $\underline{\mathrm{latt}}\,\Gamma := \mathrm{latt}\,\Gamma/\ker \mathscr{S}$, and let $\underline{\mathscr{S}}$: $\underline{\mathrm{latt}}\,\Gamma \to \mathrm{mod}\,\Lambda/I$ be the faithful functor induced by $\mathscr{S}$. Note that $\mathscr{S}$, and therefore $\underline{\mathscr{S}}$, factors through $\mathrm{mod}\,\Gamma/I$. Hence we have the following commutative diagram of categories and functors:

$$\begin{tikzcd}
\mathrm{latt}\,\Gamma \arrow[rr, "\mathscr{S}"] \arrow[dr] \arrow[ddr] & & \mathrm{mod}\,\Lambda/I \\
 & \mathrm{mod}\,\Gamma/I \arrow[ur] & \\
 & \underline{\mathrm{latt}}\,\Gamma \arrow[u] \arrow[uur, "\underline{\mathscr{S}}"'] &
\end{tikzcd}$$

To the faithful functor $\underline{\mathscr{S}}$ we attach its *subspace category* $\operatorname{sub}\underline{\mathscr{S}}$. This has to be understood in the sense of the obvious generalization of the classical notion of subspace category (as introduced in [NaRo73b]) which one gets upon replacing $\operatorname{mod}k$ by $\operatorname{mod}\Lambda/I$. (In the important special case where Λ is local and $I = \operatorname{rad}\Lambda$, $\operatorname{sub}\underline{\mathscr{S}}$ really is a subspace category in the classical sense.) We are in fact not interested in the subspace category $\operatorname{sub}\underline{\mathscr{S}}$ itself, but in the full subcategory $\operatorname{sub}'\underline{\mathscr{S}} \subset \operatorname{sub}\underline{\mathscr{S}}$ which is given by the following condition. An object (U, N, α) in $\operatorname{sub}\underline{\mathscr{S}}$ is, by definition, in $\operatorname{sub}'\underline{\mathscr{S}}$ if and only if it satisfies $\Gamma(\operatorname{im}\alpha) = \underline{\mathscr{S}}(N)$, At last we define $\mathscr{C}$ to be $\operatorname{sub}'\underline{\mathscr{S}}$, and we define $\mathscr{F}$: $\operatorname{latt}\Lambda \to \mathscr{C}$ canonically by $\mathscr{F}(M) = (M/IM, \Gamma M, \alpha_M\colon M/IM \subsetneq \Gamma M/IM)$. The significance of all this is contained in the following proposition.

PROPOSITION. (i) *For each admissible triple (I, Λ, Γ), the functor $\mathscr{F}$: $\operatorname{latt}\Lambda \to \mathscr{C}$ is an epivalence.*

(ii) *For each nonhereditary order Λ in a separable K-algebra A, the triple $(\operatorname{rad}\Lambda, \Lambda, \mathscr{O}(\operatorname{rad}\Lambda))$ is admissible.*

Here, by an *epivalence* we mean an additive functor which is dense, full and isomorphism-reflecting. (The term "epivalence" has been suggested by P. Gabriel. In literature, such functors are commonly called "representation equivalence". Yet the latter term is misleading as it suggests a symmetry, while in general "representation equivalences" exist only in one direction.)

The proof of (i) is straighforward. Special cases of (i) are studied in [GrRe78] and [RiRo79]. Moreover, by $\mathscr{O}(\operatorname{rad}\Lambda)$ we mean the ring of *two-sided multipliers* of $\operatorname{rad}\Lambda$, i.e. $\mathscr{O}(\operatorname{rad}\Lambda) = \{a \in A \mid a\cdot\operatorname{rad}\Lambda + \operatorname{rad}\Lambda\cdot a \subset \operatorname{rad}\Lambda\}$, which is by construction an overorder of Λ containing $\operatorname{rad}\Lambda$ as a two-sided ideal. Thus (ii) amounts to saying that $\mathscr{O}(\operatorname{rad}\Lambda)$ contains Λ properly, which is known to be true [Ja71].

Of course, being an epivalence, $\mathscr{F}$ induces a bijection between the isomorphism classes of indecomposable objects of the related categories $\operatorname{latt}\Lambda$ and $\mathscr{C}$. Moreover, if I is full in Λ, then Λ/I and Γ/I are Artin algebras. In this sense, (i) asserts that $\mathscr{F}$ is a good reduction functor, while (ii) indicates the degree of generality to which it exists.

If we want to use this reduction, then we must continue working with $\mathscr{C}$. In general, this category is not at all easy to handle. (Even, maybe, in representation-finite cases.) However, under suitable additional assumptions on the admissible triple, there may be surprisingly neat interpretations of $\mathscr{C}$. In this respect, the simplest situation one may consider is when the admissible triple (I, Λ, Γ) satisfies the following condition:

(B) $\qquad I = \operatorname{rad}\Lambda = \operatorname{rad}\Gamma$, and Γ is hereditary.

Admissible triples which satisfy condition (B) I call *Bäckström triples*, and R-orders Λ which occur in a Bäckström triple are called *Bäckström orders*

[RiRo79]. It follows from (B) that a Bäckström order uniquely determines its Bäckström triple.

Suppose now that (I, Λ, Γ) is a Bäckström triple, with associated reduction functor $\mathscr{F}$: latt$\Lambda \to \mathscr{C}$. Then $\mathscr{C}$ has the following convenient interpretation in representation-theoretical terms. First of all, the functor $\underline{\mathscr{S}}$: $\underline{\text{latt}}\,\Gamma \to \text{mod}\,\Gamma/I$ is now an equivalence. (It is faithful by construction, full because every Γ-lattice is projective, and dense because Γ/I is semisimple and idempotents can be lifted from Γ/I to Γ.) Consequently, the forgetful functor on $\mathscr{C}$ which associates with each object (U, N, α) its linear map $\alpha\colon U \hookrightarrow \underline{\mathscr{S}}(N)$ also is an equivalence. Finally, this linear map can be viewed as a representation of a species. This is most easily perceived in case $\Lambda/I \cong \prod_{i=1}^{r} k_i$ and $\Gamma/I \cong \prod_{j=1}^{s} k_j$. Here, evidently

$$\begin{array}{ccc} U & \xhookrightarrow{\ \alpha\ } & \underline{\mathscr{S}}(N) \\ \| & & \| \\ \bigoplus_{i=1}^{r} U_i & \xhookrightarrow{(\alpha_{ji})} & \bigoplus_{j=1}^{s} V_j \end{array}$$

is a representation of the quiver

$$\begin{array}{ccc} a_1 & & b_1 \\ a_2 & \to & b_2 \\ \vdots & & \vdots \\ a_r & & b_s \end{array}$$

where the arrows are defined by the rule $a_i \xrightarrow{\exists} b_j :\Leftrightarrow k_i \otimes_\Lambda k_j \neq 0$. The general case is covered by replacing the quiver by a suitable species, upon attaching skew fields to the points and bimodules to the arrows (see [RiRo79]).

Altogether we have, for Bäckström orders Λ, a sequence of two functors latt$\Lambda \xrightarrow{\mathscr{F}} \mathscr{C} \xrightarrow{\mathscr{G}} \text{rep}'_k \boldsymbol{S}$, where $\boldsymbol{S}$ is the k-species associated with Λ, $\text{rep}'_k \boldsymbol{S}$ is the category of all those representations of $\boldsymbol{S}$ which contain no simple direct summand, and $\mathscr{G}$ is the equivalence just indicated, while $\mathscr{F}$ is the epivalence defined above. In connection with the well-known classification of tame species [DlRi76] we obtain the following main result for Bäckström orders [RiRo79].

THEOREM. *Let Λ be a Bäckström order, with associated species $\boldsymbol{S}$ and underlying valued graph $\boldsymbol{G}$. Then*:

(i) *There is an epivalence* latt$\Lambda \to \text{rep}'_k \boldsymbol{S}$.

(ii) *Λ is tame and connected if and only if $\boldsymbol{G}$ is an extended Dynkin diagram.*

Here, some overviewing remarks are due.

(1) The second approach to the classification problem of the tetrad, presented in Section 2, appears now as a special instance of the above theorem. It can be rephrased tersely as follows: The tetrad Γ is a Bäckström order, with Bäckström triple $(\mathscr{J}, \Gamma, \Omega)$ and associated valued graph as in Fig. 4.

(2) Hence so far we have considered Bäckström orders only, essentially. We have encountered a phenomenon which is typical to them, namely that reductions lead directly to matrix problems over k. In general, a given order Λ will lie deeper below some hereditary overorder, and reductions will lead to matrix problems over $R_l = R/\pi^l$, for some $l \in \mathbf{N}$. Therefore the general strategy for solving the classification problem of Λ consists of three steps:

Step 1. Reduce the classification problem of Λ to a classification problem over R_l, for some $l \in \mathbf{N}$.

Step 2. Reduce the classification problem over R_l to a classification problem over k.

Step 3. Solve the classification problem over k.

While for Bäckström orders the second step in this program is empty, it appears to be the most involved one for other types of orders. In this respect I cite the rather complicated reductions which lead to the solution of the classification problem of $\Lambda = \hat{\mathbf{Z}}_2 C_8$ [Ya72] and of $\Lambda = \hat{\mathbf{Z}}_3[\sqrt[4]{3}]C_3$ [Di85b]. We will come to Yakovlev's reduction in the next section.

(3) The reduction functor $\mathscr{F}$: latt$\Lambda \to \mathscr{C}$ appears implicitly in several articles around the late sixties, e.g. [Ja67], [DrRo67], [Bä72]. It has been made explicit in [GrRe78] and in [RiRo79]. It is used in [GrRe78] for reproving Jacobinski's and Drozd–Roiter's characterization of representation-finite commutative orders. Here, the handling of $\mathscr{C}$ is cumbersome. It is used in [RiRo79] for the investigation of Bäckström orders, as we have seen. Apart from these instances there seem to be only two cases where a conceptual investigation of $\mathscr{C}$ has been attempted, namely

a) generalized Bäckström orders, and
b) complete local rings of curve singularities with large conductor.

While generalized Bäckström orders are closely related to Bäckström orders (here, $\mathscr{C}$ corresponds to the category of socle-projective representations of the species associated with the order [Ro83], [Kö88]), quite different phenomena occur in lattice categories of orders which arise from curve singularities [Di88]. We will come to this in the last section.

4. The group ring $\Lambda = \hat{\mathbf{Z}}_2 C_8$

Historically, this was the second instance where an order of tame type appeared. Its classification problem was solved by Yakovlev in 1972 [Ya72], [Ya80]. We will see that the nature of this tame problem, as well as the reduction developed by Yakovlev, is far away from what we met before. I proceed by giving an outline of Yakovlev's reduction and solution of the problem.

Step 1. Choose a generator λ for the cyclic group C_8. Then

$$\Lambda = R[\lambda] \cong R[X]/(X^8-1) = R[X]/(X^2+1)(X+1)(X-1)(X^4+1),$$

and therefore in Λ the identity

$$(*) \qquad (\lambda^2+1)(\lambda+1)(\lambda-1)(\lambda^4+1) = 0$$

holds. Keeping the indicated order of the cyclotomic factors, there are three possible ways of putting brackets such that the left-hand side of $(*)$ appears as a product of two elements. Accordingly, for each Λ-lattice M, $(*)$ can be interpreted in three different ways as a pair of endomorphisms of M with composition zero.

Moreover, let $\mathscr{M}$ be the category of all those Λ-module representations of the quiver $\overset{1}{\circ} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \overset{2}{\circ} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \overset{3}{\circ}$ which satisfy the conditions $(\lambda^2+1)M_1 = 0$, $\beta_1\alpha_1 = (\lambda+1)|_{M_1}$, $\alpha_1\beta_1 = (\lambda+1)|_{M_2}$, $\beta_2\alpha_2 = (\lambda-1)|_{M_2}$, $\alpha_2\beta_2 = (\lambda-1)|_{M_3}$, $(\lambda^4+1)M_3 = 0$.

There is a functor $\operatorname{latt}\Lambda \to \mathscr{M}$, assigning to each Λ-lattice M the homologies of the three pairs of endomorphisms with composition zero arising from $(*)$, together with the Λ-linear maps α_i, β_i induced by multiplication with one of the cyclotomic factors, respectively by inclusion. This functor induces a bijection

$$[\operatorname{ind}\Lambda]\backslash[3 \text{ indecomposables}] \xrightarrow{\sim} [\operatorname{ind}\mathscr{M}],$$

according to [Ya70, Theorem 3].

Step 2. Now the Λ-ideal $I := ((\lambda^2+1)(\lambda+1), \pi(\lambda-1))$ plays an important rôle. By definition it is generated by two elements, each a product of two factors. Accordingly for each $N \in \operatorname{mod}\Lambda/I$, multiplication by λ^2+1 and by $\lambda+1$, respectively by π and by $\lambda-1$, in the indicated and reverse ordering, gives four pairs of endomorphisms on N, each with composition zero. The homologies of these four pairs are all in $\operatorname{mod}\Lambda/(\pi, \lambda-1) \cong \operatorname{mod}k$. Thus we obtain four homology functors $\mathscr{H}_i$: $\operatorname{mod}\Lambda/I \to \operatorname{mod}k$.

Consider the subspace category associated with this quadruple of homology functors. $\mathscr{K} := \operatorname{sub}(\mathscr{H}_1, \ldots, \mathscr{H}_4) = \{(N: \alpha_1, \ldots, \alpha_4) \mid N \in \operatorname{mod}\Lambda/I,$ $\alpha_i \in \operatorname{Hom}_k(U_i, \mathscr{H}_i(N)), \ker\alpha_i = 0\}$. There is a functor $\mathscr{M} \to \mathscr{K}$, assigning to each representation $(M_i; \alpha_j, \beta_j) \in \mathscr{M}$ the Λ-module M_2 (which is in fact annihilated by I), together with the subspaces of $\mathscr{H}_i(M_2)$ which are induced by $\ker\beta_1$, $\operatorname{im}\alpha_1$, $\ker\alpha_2$, $\operatorname{im}\beta_2$. This functor induces a bijection

$$[\operatorname{ind}\mathscr{M}]\backslash[2 \text{ indecomposables}] \xrightarrow{\sim} [\operatorname{ind}\mathscr{K}],$$

according to [Ya72, Theorem 1].

So far, the classification problem of Λ is reduced to the classification problem for $\mathscr{K}$. But still, $\mathscr{K}$ is given only in abstract terms. In order to solve its classification problem, the next aim is to get at a more concrete description of $\mathscr{K}$. In this vein, the following steps are concerned with

3) the classification of $\operatorname{ind} \Lambda/I$,
4) the evaluation of $\mathscr{H}_i$ on objects, for all $i = 1, \dots, 4$, and
5) the evaluation of $\mathscr{H}_i$ on morphisms, for all $i = 1, \dots, 4$.

As a result, we will be in a position to reformulate the classification problem for $\mathscr{K}$ as a matrix problem over k.

Step 3. Because I contains $I_1 := (\pi^3, \pi(\lambda - 1))$, $\operatorname{mod} \Lambda/I$ is canonically embedded into $\operatorname{mod} \Lambda/I_1$ as a full subcategory. In order to classify $\operatorname{ind} \Lambda/I$, we first classify $\operatorname{ind} \Lambda/I_1$ and then distinguish $\operatorname{ind} \Lambda/I$ among $\operatorname{ind} \Lambda/I_1$.

Clearly, $\operatorname{ind} \Lambda/I_1$ is equivalent to the category of all those R_3-module representations $(N, \lambda - 1)$ of the quiver $\circ \circlearrowleft$ which satisfy the condition $\pi(\lambda - 1) = 0$. Due to this condition, the R_3-linear endomorphism $\lambda - 1$ of N induces a k-linear map $\overline{\lambda - 1}$: $\operatorname{top} N \to \operatorname{soc} N$, such that the diagram

$$\begin{tikzcd} N \arrow[r, "\lambda-1"] \arrow[d, two heads] & N \\ \operatorname{top} N \arrow[r, "\overline{\lambda-1}"] & \operatorname{soc} N \arrow[u, hook] \end{tikzcd}$$

commutes. The choice of an R_3-basis for N (i.e. a system of generators for the cyclic direct summands of N) implies the simultaneous choice of k-bases for $\operatorname{top} N$ and $\operatorname{soc} N$, and yields a square k-matrix corresponding to $\overline{\lambda - 1}$. Changes of the R_3-basis correspond to admissible transformations on the k-matrix. Their type is indicated in Fig. 6. In the combinatorial language of Nazarova

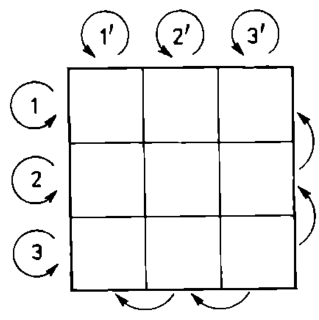

Fig. 6

and Roiter, this matrix problem is given by a pair of posets together with a binary relation on the set of points (see Fig. 7). The solution of this matrix problem is known [NaRo69]. It corresponds to a classification of $\operatorname{ind} \Lambda/I_1$. Picking among $\operatorname{ind} \Lambda/I_1$ those objects which are in $\operatorname{ind} \Lambda/I$, we obtain the following classification of $\operatorname{ind} \Lambda/I$.

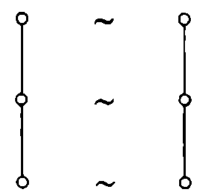

Fig. 7

(i) The indecomposable projective module Λ/I.

(ii) *Band modules*: The invariants are pairs (f, α), where $f = p^n$, $p \in \mathscr{I}\backslash\{X\}$, $n \in \mathbf{N}$, and $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a finite sequence in $\{1, 2\}$ modulo cyclic permutation which is not constant (except for $l = 1$) and which has no repetition of 1 (under cyclic reading). The corresponding band module $N_f(\alpha)$ is given by

$$\begin{tikzcd} R^d_{\alpha_1} \arrow[r, "\tilde{E}"] & R^d_{\alpha_2} \arrow[r, "\tilde{E}"] & \cdots \arrow[r, "\tilde{E}"] & R^d_{\alpha_{l-1}} \arrow[r, "\tilde{E}"] & R^d_{\alpha_l} \arrow[llll, bend left, "\tilde{\Phi}_f"] \end{tikzcd}$$

where $R_{\alpha_i} = R/\pi^{\alpha_i}$, $d = \deg(f)$, and where $\tilde{M}$ denotes the R_3-linear map determined by the k-matrix M via the commutative diagram

$$\begin{tikzcd} R^d_{\alpha_i} \arrow[r, "\tilde{M}"] \arrow[d, two heads] & R^d_{\alpha_j} \\ k^d \arrow[r, "M"] & k^d \arrow[u, hook] \end{tikzcd}$$

(iii) *String modules*: The invariants are finite sequences $\alpha = (\alpha_1, \ldots, \alpha_l)$ in $\{1, 2\}$ which have no repetition of 1 inside $(\alpha_2, \ldots, \alpha_{l-1})$. The corresponding string module $N(\alpha)$ is given by

$$R^d_{\alpha_1} \xrightarrow{\tilde{E}} R^d_{\alpha_2} \xrightarrow{\tilde{E}} \ldots \xrightarrow{\tilde{E}} R^d_{\alpha_{l-1}} \xrightarrow{\tilde{E}} R^d_{\alpha_l}.$$

(Note that all indecomposable nonprojective Λ/I-modules are already annihilated by π^2, while this is not the case with Λ/I.)

Step 4. Since now we know the objects of $\operatorname{ind}\Lambda/I$ explicitly, the values of $\mathscr{H}_i$ on them are easily calculated. It turns out that for all $i = 1, \ldots, 4$ and for all $N \in \operatorname{ind}\Lambda/I$, $\mathscr{H}_i(N) \cong 0$ or k. The vanishing or nonvanishing of $\mathscr{H}_i$ on N gives rise to a partition

$$\operatorname{ind}\Lambda/I = \mathscr{N}_1 \dot{\cup} \mathscr{N}_2 \dot{\cup} \mathscr{N}_3 \dot{\cup} \mathscr{N}_4$$

of $\operatorname{ind}\Lambda/I$ into four pairwise disjoint module classes, as follows:

	$\mathscr{H}_1$	$\mathscr{H}_2$	$\mathscr{H}_3$	$\mathscr{H}_4$
$\mathscr{N}_1$	0	0	0	0
$\mathscr{N}_2$	k	k	0	0
$\mathscr{N}_3$	0	0	k	k
$\mathscr{N}_4$	k	k	k	k

$$\mathscr{N}_1 = \{N_f(\alpha) \mid l \geqslant 3 \text{ or } f \neq (X-1)^n\} \dot\cup \{\Lambda/I\},$$

$$\mathscr{N}_2 = \{N_f(\alpha) \mid l \leqslant 2 \text{ and } f = (X-1)^n\},$$

$$\mathscr{N}_3 = \{N(\alpha) \mid \alpha \neq (1, 2, \ldots, 2, 1)\},$$

$$\mathscr{N}_4 = \{N(\alpha) \mid \alpha = (1, 2, \ldots, 2, 1)\}.$$

Step 5. Recall that we are concerned with the classification problem of $\mathscr{K} = \operatorname{sub}(\mathscr{H}_1, \ldots, \mathscr{H}_4)$, where all $\mathscr{H}_i$ are functors in $\operatorname{Fun}(\operatorname{mod}\Lambda/I, \operatorname{mod}k)$. Let $z = (N; \alpha_i)$ be an arbitrary indecomposable object in $\mathscr{K}$. If N contains a direct summand $N_1 \in \mathscr{N}_1$, then $z = (N_1; 0, 0, 0, 0)$, because all $\mathscr{H}_i$ vanish on $\mathscr{N}_1$. If N contains a direct summand $N_2 \in \mathscr{N}_2$, then $z = (N_2; \varepsilon_1, \varepsilon_2, 0, 0)$, with $\varepsilon_1, \varepsilon_2$ either 0 or id_k. This follows from the analysis of the admissible transformations linked with $\mathscr{N}_2$, together with the vanishing of $\mathscr{H}_3$ and $\mathscr{H}_4$ and $\mathscr{N}_2$. Similarly, if N contains a direct summand $N_3 \in \mathscr{N}_3$, then $z = (N_3; 0, 0, \varepsilon_3, \varepsilon_4)$, with $\varepsilon_3, \varepsilon_4$ either 0 or id_k.

Thus we have listed all those $z \in \operatorname{ind}\mathscr{K}$ whose support is not contained in $\operatorname{add}\mathscr{N}_4$. This reduces the classification problem of $\mathscr{K}$ to the classification problem of its full subcategory $\mathscr{K}_{\mathscr{N}_4} := \{(N; \alpha_i) \in \mathscr{K} \mid N \in \operatorname{add}\mathscr{N}_4\}$. The latter is the true core of the classification problem of Λ, which was posed at the outset. In order to derive its base-dependent formulation, as a matrix problem over k, it remains to calculate the admissible transformations by evaluating $\mathscr{H}_i$ on $\operatorname{Hom}_{\Lambda/I}(X, Y)$, for all $i = 1, \ldots, 4$ and for all $X, Y \in \mathscr{N}_4$.

This matrix problem turns out to belong to a general class of matrix problems which is defined as follows. Suppose the given data are an arbitrary field k, an index set I, and two total orderings $<$, $\lesssim$ on I. Then the matrix problem over k associated with $(I; <, \lesssim)$ has as objects k-matrices M, endowed with a block partition into four row strips and into a finite number of column strips indexed by elements of I:

$$M = \begin{array}{|c|} \hline A_1 \\ \hline B_1 \\ \hline\hline A_2 \\ \hline B_2 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & \cdots & & & \\ \hline & & & \cdots & & & \\ \hline\hline & & & \cdots & & & \\ \hline & & & \cdots & & & \\ \hline \multicolumn{1}{c}{i_1} & \multicolumn{1}{c}{i_2} & \multicolumn{1}{c}{i_3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{i_{n-2}} & \multicolumn{1}{c}{i_{n-1}} & \multicolumn{1}{c}{i_n} \end{array}$$

The admissible transformations on M are:

(a) arbitrary nonsingular transformations on each row strip and on each column strip,

(b) addition of linear combinations of columns from $\left[\begin{array}{c} A_{1i} \\ \hline B_{1i} \end{array}\right]$ to $\left[\begin{array}{c} A_{1j} \\ \hline B_{1j} \end{array}\right]$, whenever $i < j$,

(c) addition of linear combinations of columns from $\left[\begin{array}{c} A_{2i} \\ \hline B_{2i} \end{array}\right]$ to $\left[\begin{array}{c} A_{2j} \\ \hline B_{2j} \end{array}\right]$, whenever $i \precsim j$.

In these terms, the classification problem of $\mathscr{K}'_{(\ )}$ is the matrix problem which is given by the following specialization of k and $(I;\ <,\ \precsim)$: $k = \hat{\mathbf{Z}}_2/2\hat{\mathbf{Z}}_2 \cong \mathbf{F}_2$, and $I = \{\text{invariants for } \mathscr{N}_4\} = \{\text{finite sequences } (\alpha_1, \ldots, \alpha_l) \text{ in } \{1, 2\} \mid \alpha_1 = \alpha_l = 1, \text{ and no repetition of } 1\}$. Moreover, if $\alpha = (\alpha_1, \ldots, \alpha_l)$ and $\beta = (\beta_1, \ldots, \beta_m)$ are arbitrary elements in I, then

$$\alpha > \beta \Leftrightarrow \begin{cases} \text{in the sequence } \alpha_l - \beta_m, \alpha_{l-1} - \beta_{m-1}, \ldots \text{ either the first nonzero} \\ \text{term is negative, or all terms are zero and } l > m; \end{cases}$$

$$\alpha \succsim \beta \Leftrightarrow \begin{cases} \text{in the sequence } \alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots \text{ either the first nonzero} \\ \text{term is positive, or all terms are zero and } l < m. \end{cases}$$

This finishes the entire reduction of the classification problem of Λ to a matrix problem over k. It remains to solve this matrix problem!

Step 6. Yakovlev solves at once all the matrix problems given by triples $(I;\ <,\ \precsim)$, over arbitrary fields k. I give a brief qualitative description of his method.

Usually we have to deal with block matrices. A *cell* is a nonzero block, and the *cell structure* of a matrix is understood to be the location of its cells. At any stage of the process of transforming a given matrix $M = \left[\begin{array}{c} A_1 \\ \hline B_1 \\ \hline\hline A_2 \\ \hline B_2 \end{array}\right]$ into some matrix $M' = \left[\begin{array}{c} A'_1 \\ \hline B'_1 \\ \hline\hline A'_2 \\ \hline B'_2 \end{array}\right]$, we simply call $\left[\begin{array}{c} A'_1 \\ \hline B'_1 \end{array}\right]$ the "upper matrix" and $\left[\begin{array}{c} A'_2 \\ \hline B'_2 \end{array}\right]$ the "lower matrix" of M'. Now if M is an arbitrary k-matrix in the matrix problem given by a triple $(I;\ <,\ \precsim)$, then we proceed as follows.

1) Solve the problem on the upper matrix. The resulting normal form defines a cell structure on the upper matrix and a refinement of column strips on the lower matrix.

2) The induced problem on the lower matrix has as object the lower matrix, endowed with the refined column partition, and as admissible transformations those which stabilize the cell structure of the upper matrix (although they may spoil the normal forms of the cells of the upper matrix).

3) Solve the induced problem on the lower matrix. The resulting normal form defines a cell structure on the lower matrix and a refinement of column strips on the upper matrix.

4) The induced problem on the upper matrix has as object the upper matrix, endowed with the refined column partition, and as admissible transformations those which stabilize the cell structure of the lower matrix.

1′) Solve the induced problem on the upper matrix... and continue as in 1).

This procedure is algorithmic because all the problems induced on the upper or lower matrices are of the same type, namely they are given by the pair of posets ($\substack{\circ\\\circ}$, $\circ - \circ - \ldots - \circ - \circ$). For any given M, the refinement of column strips will become improper after a finite number of steps, and so the algorithm terminates.

Carrying out this program, Yakovlev obtains a complete list of normal forms for the indecomposable matrices M. I sketch his classification in three steps.

(i) There are *combinatorial invariants* $(\Delta, \delta_0, \delta_1)$, consisting of the following data: Δ is one of the graphs $\mathbf{A}_n$, $\mathbf{D}_n$, $\tilde{\mathbf{A}}_n$, $\tilde{\mathbf{D}}_n$; δ_0: $\Delta_0 \to I$ is a map, subject to certain restrictive conditions; δ_1: $\Delta_1 \to \{u, l\}$ is an alternating mapping. (As usual, Δ_0 denotes the set of points and Δ_1 denotes the set of edges of Δ.)

(ii) These combinatorial invariants define the cell structure of M as follows. By *special points* of Δ we mean the encircled points in Fig. 8 in case

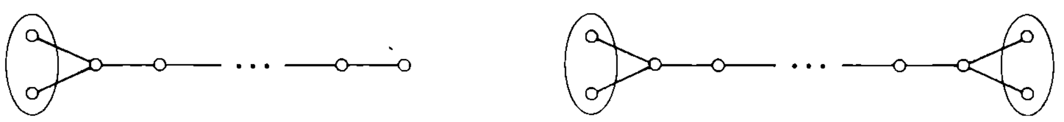

Fig. 8

$\Delta = \mathbf{D}_n$ or $\tilde{\mathbf{D}}_n$. To each point in Δ_0 assign a refined column strip of M such that outside the special points the assignment is injective, while on the special points the circles indicate the fibres. Any refined column strip x is contained in the unrefined column strip $\delta_0(x)$. If $x, y \in \Delta_0$ are connected in Δ by an edge $x \stackrel{e}{-} y$, then the refined column strips x, y are connected in M by a submatrix of type

$$\begin{array}{cc} \left[\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right] \\ \;\;x \quad\; y \end{array} \text{ or } \begin{array}{cc} \left[\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right] \\ \;\;x \quad\; y \end{array},$$

where the blocks “$*$” are cells. This connecting submatrix is contained in the upper matrix of M if $\delta_1(e) = u$, respectively in the lower matrix of M if $\delta_1(e) = l$. In the refined column strip given by two special points there occurs a pair of *special blocks* S, T having the property that $\left[\begin{array}{c} S \\ \hline T \end{array}\right]$ is a nonsingular square matrix. One out of the pair of special blocks may have zero rows. If this is not the case, then clearly it is a cell. The pair of special blocks is in the upper matrix, respectively in the lower matrix of M, if δ_1 takes value l, respectively u, on the edges adjacent to the special points. All blocks in M which are neither special nor contained in a connecting submatrix are zero blocks.

(iii) In the classification of the indecomposable matrices M with given cell structure $(\Delta, \delta_0, \delta_1)$ we encounter four different types of behaviour, corresponding to the type of Δ.

Strings ($\Delta = \mathbf{A}_n$): For each invariant $(\mathbf{A}_n, \delta_0, \delta_1)$ there is one equivalence class of indecomposable matrices of cell structure $(\mathbf{A}_n, \delta_0, \delta_1)$. The normal form is given by setting each cell to equal (1).

Modified strings ($\Delta = \mathbf{D}_n$): For each invariant $(\mathbf{D}_n, \delta_0, \delta_1)$ there are two equivalence classes of indecomposable matrices of cell structure $(\mathbf{D}_n, \delta_0, \delta_1)$, depending on which of the pair of special blocks is a cell. The normal form is given by setting each cell to equal (1).

Bands ($\Delta = \tilde{\mathbf{A}}_n$): For each invariant $(\tilde{\mathbf{A}}_n, \delta_0, \delta_1)$, the equivalence classes of indecomposable matrices of cell structure $(\tilde{\mathbf{A}}_n, \delta_0, \delta_1)$ are parametrized by p^ν, where $p \in \mathscr{I} \setminus \{X\}$ and $\nu \in \mathbf{N}$. The normal form of an indecomposable matrix with parameter p^ν is given by setting one chosen cell to equal Φ_{p^ν}, and all other cells to equal E.

Modified bands ($\Delta = \tilde{\mathbf{D}}_n$): For each invariant $(\tilde{\mathbf{D}}_n, \delta_0, \delta_1)$, the equivalence classes of indecomposable matrices of cell structure $(\tilde{\mathbf{D}}_n, \delta_0, \delta_1)$ correspond bijectively to the equivalence classes of those indecomposable quadruples of matrices $\left[\begin{array}{c} \Phi_1 \\ \hline \Phi_2 \\ \hline \Phi_3 \\ \hline \Phi_4 \end{array}\right]$ which have the property that both $\left[\begin{array}{c} \Phi_1 \\ \hline \Phi_2 \end{array}\right]$ and $\left[\begin{array}{c} \Phi_3 \\ \hline \Phi_4 \end{array}\right]$ are nonsingular square matrices. The normal form of an indecomposable matrix with corresponding quadruple (Φ_i) is given by setting the two pairs of special blocks to equal $\left[\begin{array}{c} \Phi_1 \\ \hline \Phi_2 \end{array}\right]$ and $\left[\begin{array}{c} \Phi_3 \\ \hline \Phi_4 \end{array}\right]$, while setting all other cells to equal E.

Here we are content with this sketch of a classification. The reader who wants it perfectly accomplished yet has to settle the following points:

a) Precise description of the condition imposed on δ_0: $\Delta_0 \to I$.

b) Location of the zero block in the connecting matrices.

c) Classification of the indecomposable quadruples which parametrize the modified bands.

For a) and b) one has to immerse in [Ya72, §4], while c) is easily derived from the well-known representation theory of the quiver of Fig. 4.

In conclusion to this report on Yakovlev's work on latt$\hat{\mathbf{Z}}_2 C_8$, I want to remark that, historically, this is the first instance where a matrix problem of infinite growth, with "modified strings" and "modified bands", has been solved. Later, such matrix problems also appear in Nazarova and Roiter's solution of a problem of Gelfand [NaRo73a]. However, this is marred by mistakes on the combinatorial level. Recently, the investigation of problems of this type has been resumed by Crawley-Boevey, using a functorial approach [Cr88a], [Cr88b].

5. Local group rings

Our present state of knowledge is still very far away from a complete answer to the problem of characterizing, or even classifying, tame orders in general. The partial answers which are known by now all refer to restricted classes of orders. In this respect, the first example presented here was the characterization of tame Bäckström orders, given in Section 3. I am now going to present the second example, restricting our attention to local group rings. It turns out that among them, apart from the group rings $\hat{\mathbf{Z}}_2(C_2 \times C_2)$ and $\hat{\mathbf{Z}}_2 C_8$ which we studied in Sections 2 and 4, there exist only two additional tame cases. The precise result is the following.

THEOREM. *Let $\Lambda = RG$ be the group ring of a finite p-group G over a complete discrete valuation ring R, and assume that Λ is not hereditary and $K \otimes_R \Lambda$ is semisimple. Let v: $R \to \mathbf{N} \cup \{0, \infty\}$ be the valuation of R. Then Λ is tame in each of the following cases*:

(i) $G = C_2 \times C_2$ *and* $v(2) = 1$,
(ii) $G = C_8$ *and* $v(2) = 1$,
(iii) $G = C_4$ *and* $v(2) = 2$,
(iv) $G = C_3$ *and* $v(3) = 4$.

In all other cases, Λ is either representation-finite or wild.

The assumption imposed on Λ is no restriction of generality. I briefly recall what happens in case the assumption is not satisfied. For any group ring $\Lambda = RG$ of a finite p-group G over a complete discrete valuation ring R, the following assertions are equivalent: a) Λ is hereditary, b) Λ is maximal, c) $v(p) = 0$, d) each Λ-lattice is projective. In particular, if Λ is hereditary, then Λ is representation-finite. On the other hand, Maschke's theorem asserts that $K \otimes_R \Lambda$ is not semisimple if and only if $\operatorname{char} K = p$ (or, in other terms,

$v(p) = \infty$). If $K \otimes_R \Lambda$ is not semisimple and $|G| > 2$, then Λ is wild [Gu78], [Di81]. If $K \otimes_R \Lambda$ is not semisimple and $|G| = 2$, then a complete list of indecomposable matrix representations of G in R is given by (1), $\begin{bmatrix} 1 & \pi^n \\ 0 & 1 \end{bmatrix}$, $n \in \mathbf{N} \cup \{0\}$ [Gu71], [Di80], [Di81]. Observe that the latter group ring is neither representation-finite nor wild, nor is it tame in the sense declared in the introduction (because there is no one-parameter family of indecomposable lattices in rank 2)! Also, it is a counterexample to Brauer–Thrall I (because it is representation-infinite, although the ranks of the indecomposable lattices are bounded by 2)! This indicates that general phenomena to which we are commonly well acquainted in representation theory, such as the distinction of representation types into finite, tame and wild, or the validity of the Brauer–Thrall conjectures, no longer need to be true as soon as we deal with algebras which are *nonisolated singularities*, in the sense of Auslander [Au84]. For further examples of algebras showing such a degenerate behaviour and for a more geometric than representation-theoretic discussion of them I refer to the $\mathbf{A}_\infty$- and $\mathbf{D}_\infty$-singularities studied in [BGS87], [Schr85].

I now return to the above theorem. The classification problem of the group ring (iii) has been solved by Kopelevich in case K is a splitting field for C_4 [Ko75], respectively by Bondarenko in case K is not a splitting fiield for C_4 (not yet published). The classification problem of the group ring (iv) has been solved by myself [Di84], [Di85b]. Here, the first reduction step is of the type which already occurred in Section 2. Namely, the classification problem of Λ is translated to the problem of classifying the orbits of Ext-groups under the action of the automorphism groups of the variables entering into the bifunctor $\mathrm{Ext}^1_\Lambda(?, ?)$. Moreover, these variables are lattices over representation-finite factororders of Λ, with known classification of their indecomposable lattices. (For a precise functorial formulation of this first reduction, see [Di85a].) However, due to the hypothesis $v(3) = 4$, and in contrast to the situation which arose in Section 2, here the reduced problem is defined over R_4. The main difficulty lies in the second reduction which eventually leads to a matrix problem over k. Essentially, this is given by one of the two patterns in Fig. 9. Here, the pattern on the left-hand side is incident to the case where K is a splitting field for C_3, while the one on the right is incident to the case where

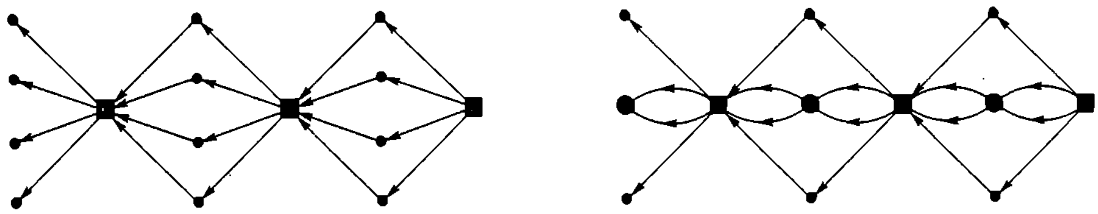

Fig. 9

K is not a splitting field for C_3. In turn, this matrix problem over k is closely related to the classification problem of a *tubular k-algebra* of tubular type $\tilde{\mathbf{D}}_4$, respectively $\widetilde{\mathbf{CD}}_3$. The latter problems have been solved by Ringel [Ri84]. Translating his solution back, along the performed reduction, one obtains the following classification for $\operatorname{ind}\Lambda$.

(i) First we have to define the set of *exceptional parameters* $\mathscr{E} \subset \mathscr{I}_\infty$. Choose a parameter element $\pi \in R$. Then the element $d := -3/\pi^4 + \pi R \in k\setminus\{0\}$ is uniquely determined. Consider the quadratic polynomial $\delta := X^2 - d$ in $k[X]$. If K is a splitting field for C_3, then $\delta = (X-\zeta)(X+\zeta)$ in $k[X]$, and we set $\mathscr{E} := \{\infty, X, X-\zeta, X+\zeta\}$. If K is not a splitting field for C_3, then δ is irreducible in $k[X]$, and we set $\mathscr{E} := \{\infty, X, \delta\}$.

(ii) Define $\tilde{\mathscr{I}}_\infty := \mathscr{I}_\infty \dot{\cup} \mathscr{E}$. Then the set of invariants is $\mathbf{P}^1\mathbf{Q} \times \tilde{\mathscr{I}}_\infty \times \mathbf{N}$.

(iii) There is a bijection β: $\mathbf{P}^1\mathbf{Q} \times \tilde{\mathscr{I}}_\infty \times \mathbf{N} \xrightarrow{\sim} [\operatorname{ind}\Lambda]\setminus[\Lambda]$ which is constructive in both directions. That is to say, there is an effective algorithm for the construction of $\beta(i) \in \operatorname{ind}\Lambda$, for any given invariant i, and also there is an effective algorithm for the determination of the invariant $\beta^{-1}(M)$ corresponding to M, for any given nonprojective $M \in \operatorname{ind}\Lambda$.

Historically, the group ring (iv) is the first example of an order which is nondomestic tame of *finite growth*. Up to now, only one additional order of this type has occurred, namely the complete local ring of the simple elliptic plane curve singularity of type $\tilde{\mathbf{E}}_7$, which will be mentioned, among others, in the next section.

The last statement of the above theorem, concerning the representation-finite or wild cases, emerged gradually as the result of a rather long process to which numerous mathematicians contributed. The respective historical references can be found in [Di80] and [Di84]. A unified proof can be found in [Di83a] for the representation-finite cases, and in [Di83b] for the wild cases.

6. Curve singularities

Throughout this last section, k is an algebraically closed field of characteristic 0, $R = k[[\pi]]$, and Λ is an R-order which is commutative, local, and isolated singular. (Here, the term "isolated singular" is used in the sense of commutative algebra, meaning that the singular locus of Λ contains $\operatorname{rad}\Lambda$ as its only element. However, this is equivalent to the requirement that Λ is not hereditary and $K \otimes_R \Lambda$ is semisimple. Hence we may interpret the term "isolated singular" as well in the sense of Auslander [Au84].) I call such R-orders *curve singularities*, and I denote the class of all such R-orders by $\mathscr{C}$. The term "curve singularity" is justified by the following fact.

PROPOSITION. *If $C \subset \mathbf{A}^n k$ is an affine-algebraic curve with singular point $O \in C$, then the complete local ring $\hat{\mathcal{O}}_{C,O}$ of C at O is an R-order in $\mathscr{C}$. Conversely, if Λ is an R-order in $\mathscr{C}$, then there exists an affine-algebraic curve $C \subset \mathbf{A}^n k$ with singular point $O \in C$ such that $\hat{\mathcal{O}}_{C,O} \cong \Lambda$.*

This follows from standard connections between algebraic geometry and commutative algebra. If (C, O) is an affine-algebraic curve singularity, then $\hat{\mathcal{O}}_{C,O}$ is reduced, and so there exists a non-zero-divisor ϱ in $\operatorname{rad}\hat{\mathcal{O}}_{C,O}$. The assignment $\pi \mapsto \varrho$ defines a ring monomorphism $R \to \hat{\mathcal{O}}_{C,O}$ with respect to which $\hat{\mathcal{O}}_{C,O}$ is an R-order in $\mathscr{C}$. Conversely, if Λ is an R-order in $\mathscr{C}$, then $\Lambda \cong k[[X_1, \ldots, X_n]]/I$, and the ideal I is generated by polynomials. These define an affine-algebraic curve singularity (C, O) such that $\hat{\mathcal{O}}_{C,O} \cong \Lambda$.

Being concerned with tame orders, my interest in curve singularities is guided by the following general problems.

(A) Find an effective criterion for deciding tame type, for any given curve singularity.

(B) Classify all tame curve singularities, up to analytical isomorphism.

(C) Find a general strategy for classifying $\operatorname{ind}\Lambda$, in case Λ is a tame curve singularity.

It is remarkable that, when rephrasing these problems for representation-finite instead of tame curve singularities, satisfactory solutions are actually known for all of them. I recall the respective results.

THEOREM. *For any curve singularity Λ, the following assertions are equivalent*:

(i) *Λ is representation-finite.*

(ii) *$\mu_\Lambda(\Omega/\Lambda) \leqslant 2$ and $\mu_\Lambda(\operatorname{rad}(\Omega/\Lambda)) \leqslant 1$, where Ω is the maximal order in $K \otimes_R \Lambda$.*

(iii) *Λ dominates the complete local ring of a simple plane curve singularity (C_s, O).*

Here, $\mu_\Lambda(M)$ denotes the minimal number of generators of a Λ-module M. One says that Λ *dominates* an order Σ if $\Sigma \subset \Lambda \subset K \otimes_R \Sigma$. The equivalence (i)⇔(ii) is the classical result of Jacobinski, Drozd and Roiter [Ja67], [DrRo67], specialized to curve singularities. It solves problem (A). The equivalence (i)⇔(iii) has been observed by Greuel and Knörrer [GrKn85]. Since the simple plane curve singularities are all classified and because each of them admits only a finite number of overorders, it solves problem (B). Finally, as a solution of problem (C), there are techniques for calculating the Auslander–Reiten quiver of Λ, or for solving the matrix problem to which the classification problem of Λ can be reduced.

The equivalence (i)⇔(iii) has an obvious analogue for tame curve singularities which as yet is mostly conjectural. I add it as a fourth guiding problem which may serve as yardstick for measuring future progress.

(D) CONJECTURE. A curve singularity Λ is tame if and only if it dominates the complete local ring of a unimodular plane curve singularity (C_u, O).

The unimodular hypersurface singularities have been classified by Arnol'd [Ar74]. For unimodular plane curve singularities, see also [Scha85a].

Having formulated main problems, I proceed to give a brief account on the present state of knowledge concerning tame curve singularities. Essentially only five curve singularities are known to be tame (apart from those which occur in the most recent theorem which will be stated below). These are the following complete local rings $\Sigma_1, \ldots, \Sigma_5$:

$$\Sigma_1 = k[[X, Y, Z]]/(X(Y-Z), (X-Y)Z),$$

$$\Sigma_2 = k[[X, Y]]/(XY(Y-X^4)(X+Y^2)),$$

$$\Sigma_3 = k[[X, Y]]/(XY(Y-X)(Y-X^2)),$$

$$\Sigma_4 = k[[X, Y]]/(Y(Y-X^2)(Y-aX^2), \quad a \in k \setminus \{0, 1\},$$

$$\Sigma_5 = k[[X, Y]]/(XY(Y-X)(Y-aX)), \quad a \in k \setminus \{0, 1\}.$$

Some overviewing remarks on this list are due.

(1) If Σ is a tame curve singularity and Σ' is a representation-infinite curve singularity dominating Σ, then clearly Σ' is also tame. The term "essentially" used above refers both to this fact and to the fact that Σ_4 and Σ_5 contain a parameter ranging through $k \setminus \{0, 1\}$.

(2) Originally, no one of the curve singularities $\Sigma_1, \ldots, \Sigma_5$ has been proved to be tame within our present set-up. Instead, they are "curve analogues" of orders which were investigated in other mathematical contexts.

(3) In the case of $\Sigma_1, \ldots, \Sigma_4$ these are the tame group rings $\Lambda_1, \ldots, \Lambda_4$ listed in Section 5. If Λ is one of these group rings, let s be the number of simple components of $K \otimes_R \Lambda$, and consider the system of congruences which defines the embedding of Λ into its maximal order. Then by the *curve analogue* Σ corresponding to Λ we mean the suborder $\Sigma \subset \prod_{i=1}^{s} R_i$ which is defined by a system of congruences analogous to the above one. With this construction it turns out that all the proofs solving the classification problems of tame group rings Λ go through in a parallel way for their curve analogues Σ. (This is well-known folklore among a few specialists. But one must admit that it is nowhere written up, and that it has not been checked through in every detail.) Summarizing we maintain that the passage from the tame group rings $\Lambda_1, \ldots, \Lambda_4$ to their curve analogues $\Sigma_1, \ldots, \Sigma_4$ only means a change of view. It does not provide us with substantially new examples of tame orders.

(4) Recently, C. Kahn has shown how Atiyah's classification of vector bundles over elliptic curves [At57] can be used for classifying the indecom-

posable reflexive modules over simple elliptic surface singularities [Ka87], [Ka88]. These include the surface singularities given by the polynomials

$$Y(Y-X^2)(Y-aX^2)+Z^2, \quad a\in k\setminus\{0, 1\} \quad (\text{type } \tilde{\mathbf{E}}_8),$$

$$XY(Y-X)(Y-aX)+Z^2, \quad a\in k\setminus\{0, 1\} \quad (\text{type } \tilde{\mathbf{E}}_7),$$

On the other hand, Knörrer has described the connection between the category of maximal Cohen–Macaulay modules over a hypersurface singularity given by a polynomial $f(X_0, \ldots, X_d)$ and the category of maximal Cohen–Macaulay modules over the associated hypersurface singularity given by $f(X_0, \ldots, X_d)+Z^2$ [Kn87]. Applying this to the simple elliptic surface singularities of type $\tilde{\mathbf{E}}_8$ and $\tilde{\mathbf{E}}_7$, one obtains from Kahn's result a new solution for the classification problem of Σ_4, and an original solution for the classification problem of Σ_5. Both curve singularities are nondomestic tame of finite growth, and their Auslander–Reiten quivers have tubular structure of tubular type $\tilde{\mathbf{D}}_4$. Within our context of tame orders, Σ_5 is the first example which does not originate from integral representation theory of finite groups.

(5) It is typical for the historical development of the subject of tame orders that for $\Sigma_1, \ldots, \Sigma_5$ tame type has been proved by five mathematicians, using five different approaches. Accordingly, these cases are stray examples which tell us nothing with respect to problems (A), (B) and (C). Knowing the efforts which had to be undergone to reach this state, this seems to be a very discouraging result.

(6) Finally, concerning conjecture (D) it is known that a curve singularity is wild whenever it does not dominate a unimodular plane curve singularity [Scha85b], [Kn85]. Moreover, the cases $\Sigma_1, \ldots, \Sigma_5$ are consistent with the conjecture. But still, there is an abundance of curve singularities dominating a unimodular plane curve singularity and not being isomorphic to $\Sigma_1, \ldots, \Sigma_5$.

Motivated by this background, and as an attempt for better insight into the subject of tame curve singularities, I have begun to study systematically a particular class of curve singularities $\mathscr{C}_0$. It consists, by definition, of all those $\Lambda\in\mathscr{C}$ which satisfy the following two conditions:

(a) Λ has 4 branches, and

(b) the conductor of Λ contains the radical-squared of the normalization of Λ.

A more algebraic definition of $\mathscr{C}_0$ can be given by means of the following data. Let $R_i = k[[\pi_i]]$, $\Omega = \prod_{i=1}^4 R_i$, $\mathscr{J} = \prod_{i=1}^4 \pi_i R_i$, and $\Gamma = \mathscr{J} \oplus k\cdot 1_\Omega$. Note that Γ is the tetrad (over $R = k[[\pi]]$), and $\mathscr{J} = \operatorname{rad}\Gamma = \operatorname{rad}\Omega$. In these terms, $\mathscr{C}_0 = \{\Lambda\in\mathscr{C} \mid \mathscr{J}^2 \subset \Lambda \subset \Gamma\}$.

As before, let Q be the quiver of Fig. 4. Denote by $\mathscr{T}$ the class of all those k-representations $T=(T_i, \alpha_j)$ of Q which contain no simple injective direct summand and have dimension type $\underline{\dim}\, T = \,\vdots\, r$.

There is a mapping $\mathscr{C}_0 \to \mathscr{T}$ and a construction $\mathscr{T} \to \mathscr{C}_0$ as follows. For any $\Lambda \in \mathscr{C}_0$, set $J = \operatorname{rad} \Lambda$. The filtration of ideals $\mathscr{J}^2 \subset J \subset \mathscr{J}$ gives rise to the object $J/\mathscr{J}^2 \subset \mathscr{J}/\mathscr{J}^2$ which, in view of $\mathscr{J}/\mathscr{J}^2 \cong \bigoplus_{i=1}^4 k\pi_i$, is in $\mathscr{T}$. Conversely, for any $T \in \mathscr{T}$ the choice of basis elements $t_i \in T_i$, $i = 1, \ldots, 4$, defines an isomorphism η: $\bigoplus_{i=1}^4 T_i \xrightarrow{\sim} \bigoplus_{i=1}^4 k\pi_i$. Set

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{pmatrix} : T_0 \hookrightarrow \bigoplus_{i=1}^{4} T_i, \qquad U = \operatorname{im}(\eta\alpha), \qquad \Lambda = \mathscr{J}^2 \oplus U \oplus k \cdot 1_\Omega.$$

Then Λ is in $\mathscr{C}_0$.

These constructions induce a bijection between the analytical isomorphism classes (i.e. k-algebra isomorphism classes) of $\mathscr{C}_0$ and the equivalence classes of $\mathscr{T}$ under the equivalence relation which is generated by isomorphisms in $\operatorname{rep}_k Q$ and by permutation of the k-linear maps $\alpha_1, \ldots, \alpha_4$.

The main result, concerning tame curve singularities in $\mathscr{C}_0$, is the following.

THEOREM. (i) *A complete system of representatives for the analytical isomorphism classes of $\mathscr{C}_0$ consists of a list of 16 objects, together with an infinite series of objects which is parametrized by $k \backslash \{0, 1\}$.*

(ii) *Among these, 10 curve singularities are wild, and the remaining $6 + 1 \cdot \infty$ curve singularities are tame.*

(iii) *The tame ones are the singularities $\Lambda_0, \ldots, \Lambda_6$ given in the following table.*

Λ_0	T_0	$\bigoplus_{i=1}^{4} I_i$	Fig. 10(a)
Λ_1	T_1	τI_0	Fig. 10(b)
Λ_2	T_2	$I_1 \oplus \tau I_1$	Fig. 10(c)
$\Lambda_{3,a}$	$T_{3,a}$	$R_1(a)$	Fig. 11
Λ_4	T_4	$I_1 \oplus I_2 \oplus R_1(e)$	Fig. 12(a)
Λ_5	T_5	$R_1(e) \oplus \tau R_1(e)$	Fig. 12(b)
Λ_6	T_6	$R_2(e)$	Fig. 12(c)

(iv) *Among these tame curve singularities, Λ_0, Λ_1, Λ_2 are domestic, $\Lambda_{3,a}$ are nondomestic of finite growth for all $a \in k \backslash \{0, 1\}$, and Λ_4, Λ_5, Λ_6 are of infinite growth.*

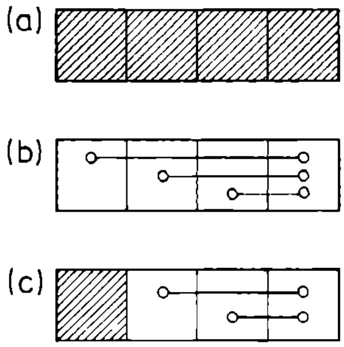

Fig. 10

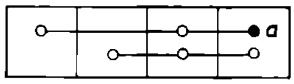

Fig. 11

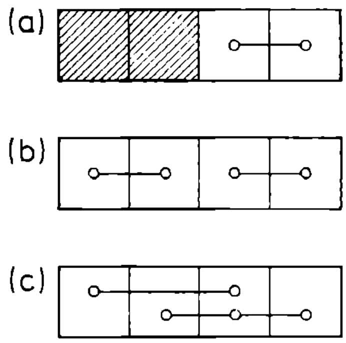

Fig. 12

In conclusion I add a few explanatory remarks to this theorem.

(1) In the table I have indicated the curve singularities Λ by means of their corresponding representations T. In the third column, the representations T are specified as objects in $\mathrm{rep}_k Q$, using the following notation: I_i is the indecomposable injective representation corresponding to $i \in \{0, \ldots, 4\}$, and $R_l(\lambda)$ is the indecomposable regular representation of regular length l and corresponding to $\lambda \in \mathbf{P}^1(k)$. Further, $\tau = D\mathrm{Tr}$, $a \in k \setminus \{0, 1\}$, and $e \in \{\infty, 0, 1\}$. In the fourth column, T is specified in base-dependent terms, by defining a subspace $U \subset \bigoplus_{i=1}^{4} k\pi_i$. Here, the meaning of the pictures should be self-explanatory. For example, the picture corresponding to T_2 indicates the 3-dimensional subspace of $\bigoplus_{i=1}^{4} k\pi_i$ which is spanned by the rows of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(2) Some of the tame curve singularities $\Lambda_0, \ldots, \Lambda_6$ are old acquaintances of ours, reappearing in the context of $\mathscr{C}_0$. Namely, Λ_0 is the tetrad Γ, and also it is the unique minimal overorder of Λ_1; Λ_1 is Σ_1, the curve analogue of $\hat{\mathbf{Z}}_2(C_2 \times C_2)$; $\Lambda_{3,a}$ is $\Sigma_{5,a}$, the simple elliptic plane curve singularity of type $\tilde{\mathbf{E}}_7$. The remaining four singularities $\Lambda_2, \Lambda_4, \Lambda_5, \Lambda_6$ have not occurred so far. They are substantially new examples of tame orders.

(3) I also include a few remarks concerning the proof.

(i) As mentioned above, the classification of $\mathscr{C}_0$ up to analytical isomorphism amounts to the classification of $\mathscr{T}$ up to equivalence. The latter is a matrix problem which is easily solved.

(ii) For any $\Lambda \in \mathscr{C}_0$, the triple (J, Λ, Γ) is admissible. According to Section 3, there is an epivalence $\mathscr{F}_\Lambda$: $\mathrm{latt}\,\Lambda \to \mathrm{sub}'\underline{\mathscr{S}}$ associated with it. On the other hand, consider the functor $\mathscr{R}$: $\mathrm{latt}\,\Gamma \to \mathrm{mod}\,k$ given by $\mathscr{R}(N) = \mathscr{J}N/JN$. It is a subfunctor of $\mathscr{S}$, and it factors through the epivalence $\mathscr{F}_\Gamma$: $\mathrm{latt}\,\Gamma \to \mathrm{rep}'_k Q$ associated with the Bäckström triple $(\mathscr{J}, \Gamma, \Omega)$. Indeed, there is a functorial isomorphism

$$\mathscr{R}(N) \simeq \mathrm{Hom}_Q(\mathscr{F}_\Gamma(N), \tilde{T})^*,$$

where * denotes vector space duality and $\tilde{T} := S_0^+(T^*)$, with S_0^+ being the reflection functor. For any object $(U, N, \alpha) \in \mathrm{sub}'\underline{\mathscr{S}}$ there exist bases of U and $\mathscr{S}(N)$, the latter adapted to the subspace $\mathscr{R}(N) \subset \mathscr{S}(N)$, such that the k-matrix representing α is of the form $\left[\begin{array}{c|c} E & Y \\ 0 & X \end{array}\right]$. The condition $Y = 0$ defines a full subcategory $\mathscr{U}$ of $\mathrm{sub}'\underline{\mathscr{S}}$, and there is a sequence of epivalences $\mathscr{U} \to \mathrm{sub}\,\mathscr{R} \to \mathrm{sub}(\mathrm{Hom}_Q(?, \tilde{T})^*)$ starting with $\mathscr{U}$. Now suppose that $\Lambda \in \mathscr{C}_0$ belongs to the singularities which are claimed to be wild. Then the associated representation T contains a preprojective direct summand, and therefore $\tilde{T}$ contains a preinjective direct summand. Hence clearly $\mathrm{sub}(\mathrm{Hom}_Q(?, \tilde{T})^*)$ is wild, and therefore so is Λ.

(iii) Suppose $\Lambda \in C_0$ belongs to the singularities which are claimed to be tame, and let T be its associated object in $\mathrm{rep}_k Q$. The key to the solution of the classification problem of Λ lies in the exhibition of an epivalence $\mathrm{sub}'\underline{\mathscr{S}} \to \mathscr{E}_T$, where $\mathscr{E}_T$ is the following new category. Objects in $\mathscr{E}_T$ are pairs (X, ξ), where X is in $\mathrm{rep}_k Q$ and without simple projective direct summand, and $\xi \in \mathrm{Ext}_Q^1(X, T \otimes X)$. Morphisms μ: $(X, \xi) \to (Y, \eta)$ in $\mathscr{E}_T$ are those morphisms $\mu \in \mathrm{Hom}_Q(X, Y)$ which satisfy the equation $(\mathrm{id}_T \otimes \mu)\xi = \eta\mu$. Specifying T to any T_i out of the table, the classification problem for $\mathscr{E}_T$ can easily be rephrased as a matrix problem over k, with explicitly given admissible transformation on row strips and on column strips. This is accessible to representation-theoretical techniques. For details I must refer to [Di88].

(4) Finally, reflecting the above theorem with respect to problems (A), ..., (D), we see that indeed it contains solutions to all of them, when restricted to the range of $\mathscr{C}_0$. First of all, the theorem has the following obvious consequence.

COROLLARY. *For any curve singularity $\Lambda \in \mathscr{C}_0$, the following assertions are equivalent*:

(i) *Λ is tame.*

(ii) *The object $T \in \mathrm{rep}_k Q$ associated with Λ contains no preprojective direct summand.*

This is a satisfactory answer to problem (A). Problem (B) is solved in the table. Problem (C) has an answer in the sequence of epivalences $\mathrm{latt}\,\Lambda \to \mathrm{sub}'\underline{\mathscr{S}} \to \mathscr{E}_T$ which provides a unified approach to the classification of $\mathrm{ind}\,\Lambda$, for all tame $\Lambda \in \mathscr{C}_0$. Finally, the list of tame curve singularities in $\mathscr{C}_0$ is consistent with conjecture (D).

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