SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 20
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1988

EXPONENT OF GROWTH OF POLYNOMIAL MAPPINGS OF C^2 INTO C^2

JACEK CHĄDZYŃSKI and TADEUSZ KRASIŃSKI

Institute of Mathematics, University of Łódź, Łódź, Poland

1. Introduction

Let $H = (f, g)$: $C^2 \to C^2$ be a polynomial mapping satisfying the condition $\# H^{-1}(0) < \infty$. We shall show that, for such a mapping, the set $N(H) = \{v \in \mathbf{R}\colon \exists A > 0,\ \exists B > 0,\ \forall |z| > B,\ A|z|^v \leqslant |H(z)|\}$ is nonempty and bounded from above. In this case, by the *exponent of growth of the mapping* H we mean the number $\chi(H) = \sup N(H)$.

It can be easily seen that if $\chi(H) > 0$, then the mapping H is proper. The converse is also true, i.e. if H is proper, then $\chi(H) > 0$ ([10], Ch. 9, § 6, cf. [5]). So, the condition $\chi(H) > 0$ is equivalent to saying that H is proper. Properness plays an important part in the theory of polynomial mappings. For instance, the famous Keller's conjecture posed in 1938 (see [6]) can be formulated as follows: "Is a mapping H with a constant nonzero Jacobian proper?".

From the point of view of properness, even to estimate $\chi(H)$ from below is interesting. In [2] such an estimate was given in terms of the geometric degree of H and the degrees of its components. Next, Płoski in [8] generalized this result to the multidimensional case.

The aim of the present paper is to show that in the two-dimensional case one can give an exact formula for the exponent of growth of H. Namely, $\chi(H)$ is equal to the minimal order of growth of H on the branches of the curves $f = 0$ and $g = 0$ in a neighbourhood of infinity. From this fact we immediately obtain the above-mentioned results and an effective way of calculating $\chi(H)$. An especially interesting corollary from the basic result is the assertion that H is proper if $H^{-1}(0)$ is not empty and each of the curves $f = 0$, $g = 0$ has only one branch at infinity.

The proof of the basic result will be carried out according to the "horn neighbourhoods" method used by Kuo and Lu (see [7], cf. [3]).

2. Notation and basic definitions

If $z = (x, y) \in C^2$, then $|z| = \max(|x|, |y|)$.

By a *neighbourhood of infinity* in C^2 we mean the complement of a compact set in C^2.

Let $X = \{t \in C: |t| > E\}$. A function $h: X \to C$ is called *meromorphic at infinity* (*meromorphic at* ∞) if it can be represented as a Laurent series of the form

$$h(y) = c_k y^k + \ldots + c_0 + c_{-1}(1/y) + \ldots$$

convergent in X. If $h \neq 0$, then the greatest index i such that $c_i \neq 0$ is called the *degree of* h and denoted by $\deg h$. If $h = 0$, we put $\deg h = -\infty$.

Let $\Psi = (\psi_1, \psi_2)$ be a pair of functions defined, as above, in X and meromorphic at ∞. Then the mapping Ψ is called *meromorphic at infinity*. By the *degree of* Ψ we mean the number $\max(\deg\psi_1, \deg\psi_2)$ and we denote it by $\deg \Psi$. A mapping Ψ meromorphic at ∞ is said to *have a pole at* ∞ if at least one of the functions ψ_1, ψ_2 has a pole at ∞.

Let $H = (f, g): C^2 \to C^2$ be a polynomial mapping and let $H^{-1}(0) = \{z_1, \ldots, z_n\}$. Denote by $\mu(f, g; z_i)$ the intersection multiplicity of the curves $f = 0$, $g = 0$ at the point z_i, $i = 1, \ldots, n$ (see e.g. [11], Ch. 4, § 5). The number $\mu(f, g; z_1) + \ldots + \mu(f, g; z_n)$ is called the *geometric degree of* H and denoted by $\sigma(H)$. If the curves $f = 0$, $g = 0$ have no zeros in common, we put additionally $\sigma(H) = 0$.

3. The basic results

PROPOSITION 3.1. *Let $h: C^2 \to C$ be a nonconstant polynomial function and $\Gamma = \{(x, y) \in C^2: h(x, y) = 0\}$. Then there exists a neighbourhood Y of infinity in C^2 such that*:

(i) *the portion of Γ in Y is the union of r components, each of them being* homeomorphic to some set $X = \{t \in C: |t| > g\}$; *the homeomorphism is defined by a mapping $\Psi: X \to Y$ meromorphic at ∞ and having a pole there*;

(ii) *if $Y' \subset Y$ is a neighbourhood of infinity in C^2 and the portion of Γ in Y' is the union of r' components having the same properties as in* (i), *then $r = r'$ and, for the corresponding components of the sets $\Gamma \cap Y'$ and $\Gamma \cap Y$ and the homeomorphisms Ψ', Ψ associated with them, the mapping $\Psi^{-1} \circ \Psi'$ is conformal and has a simple pole at ∞.*

Furthermore, if $(h_1, h_2): C^2 \to C^2$ is a polynomial mapping, and h_1, h_2 are not constant, then the neighbourhood Y can be chosen for both h_1 and h_2.

The components of the set $\Gamma \cap Y'$ will be called *branches of the curve* $h = 0$ *in the neighbourhood* Y' *of infinity*, (X', Ψ', Y') *being their parametrizations*. The number r is called the *number of branches of the curve* $h = 0$ *at infinity*.

Let $H = (f, g)$: $C^2 \to C^2$ be a polynomial mapping satisfying two conditions: 1° f and g are not constant, 2° $\# H^{-1}(0) < \infty$. Let Γ_i, $i = 1, \ldots, r$, be the branches of the curve $f = 0$ in a neighbourhood Y of infinity and, respectively, let (X_i, Ψ_i, Y), $i = 1, \ldots, r$, be their parametrizations. Analogously, let Γ_i, $i = r+1, \ldots, r+s$, be the branches of the curve $g = 0$ in Y, and let (X_i, Ψ_i, Y), $i = r+1, \ldots, r+s$, be their parametrizations.

MAIN THEOREM. *Under the above assumptions,*

(i) $\chi(H) = \min_{i=1}^{r+s} (\deg H \circ \Psi_i / \deg \Psi_i)$;

(ii) $\chi(H) \in N(H)$.

The number $\deg H \circ \Psi_i / \deg \Psi_i$ is called the *order of growth of the mapping H on the branch Γ_i in Y.* It is easy to see that it depends only on Γ_i, and that

$$|H(z)| \sim |z|^{\deg H \circ \Psi_i / \deg \Psi_i} \quad \text{when } |z| \to \infty,\ z \in \Gamma_i.$$

From the above theorem we easily obtain subsequent corollaries and propositions.

COROLLARY 3.2 (see [4], cf. [8]). *If H: $C^2 \to C^2$ is a polynomial mapping and $\# H^{-1}(0) < \infty$, then $\chi(H)$ is a rational number.*

COROLLARY 3.3 (cf. [10], Ch. 9, § 6). *A polynomial mapping H: $C^2 \to C^2$ is proper if and only if there exist constants A, B, $v > 0$ such that*

$$|H(z)| \geqslant A|z|^v \quad \textit{for } |z| > B.$$

PROPOSITION 3.4 (see [2]). *If H: $C^2 \to C^2$ is a polynomial mapping and $\# H^{-1}(0) < \infty$, then*

(i) $\chi(H) \geqslant \min(\deg f, \deg g) + \sigma(H) - \deg f \deg g$;

(ii) $\chi(H) \leqslant \sigma(H)/\max(\deg f, \deg g)$.

PROPOSITION 3.5. *If $H = (f, g)$: $C^2 \to C^2$ is a polynomial mapping, $\# H^{-1}(0) < \infty$ and each of the curves $f = 0$, $g = 0$ has only one branch at infinity, then*

$$\chi(H) = \sigma(H)/\max(\deg f, \deg g).$$

If, additionally, $\# H^{-1}(0) > 0$, then the mapping H is proper.

COROLLARY 3.6 (see [8]). *If H is a polynomial automorphism of C^2, then $\chi(H) = 1/\max(\deg f, \deg g)$.*

The proof of Proposition 3.1 will be given in the next section, that of the Main Theorem in Section 5 and those of the remaining propositions in Section 7.

4. Auxiliary results

By $\mathcal{M}$ we denote the field of germs of meromorphic functions of one complex variable at ∞, i.e. the field of Laurent series centred at ∞, with finite principal part, convergent in a neighbourhood of infinity. If h is a meromorphic function at ∞, then we denote by $\hat{h}$ the germ from $\mathcal{M}$ generated by h.

By $\mathcal{M}[x]$ we denote the ring of polynomials in a variable x with coefficients in $\mathcal{M}$. It is obvious that $\mathcal{M}[x]$ is a unique factorization domain. If h is a polynomial in x with coefficients defined in a neighbourhood Δ of infinity in $\boldsymbol{C}$ and meromorphic at ∞, then we denote by $\hat{h}$ the corresponding element in $\mathcal{M}[x]$. If the polynomial h is of the form

$$h(x, y) = x^m + a_1(y)x^{m-1} + \ldots + a_m(y), \qquad m \geqslant 1,$$

we say that h is a *monic polynomial with respect to* x. The number m is called the *degree of* h *with respect to* x and denoted by $\deg_x h$. The number $\max_{i=0}^{m}(m-i+\deg a_i)$, where $a_0 = 1$, is called the *degree of* h and denoted by $\deg h$. If h is a polynomial in two variables, then $\deg h$ is identical with the usual degree of h.

Let now $h\colon \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial function. Assume that h is monic with respect to x and $\deg_x h = \deg h$. Furthermore, let $\hat{h} = \hat{h}_1^{\gamma_1} \ldots \hat{h}_r^{\gamma_r}$ be a factorization of $\hat{h}$ into irreducible factors in $\mathcal{M}[x]$, and let $h = h_1^{\gamma_1} \ldots h_r^{\gamma_r}$ in $\{(x, y) \in \boldsymbol{C}^2\colon |y| > E\}$.

Let us introduce the notation: $\Gamma = \{(x, y) \in \boldsymbol{C}^2\colon\ h(x, y) = 0\}$, $\Gamma^\varrho = \Gamma \cap \{(x, y) \in \boldsymbol{C}^2\colon\ |y| > \varrho\}$, $\Gamma_i^\varrho = \{(x, y) \in \boldsymbol{C}^2\colon\ |y| > \varrho,\ \ h_i(x, y) = 0\}$, $m_i = \deg_x h_i$, $U_i^\varrho = \{t \in \boldsymbol{C}\colon |t| > \varrho^{1/m_i}\}$, $U_i^* = U_i^{\varrho^*}$ for $i = 1, \ldots, r$, $\varrho, \varrho^* \geqslant E$.

LEMMA 4.1. (a) *There exists a number* $\varrho^* \geqslant E$ *such that, for each* $i \in \{1, \ldots, r\}$, *there exists a mapping* $\Phi_i^*\colon U_i^* \ni t \mapsto (\varphi_i^*(t), t^{m_i}) \in \boldsymbol{C}^2$ *meromorphic at* ∞ *and such that* $\deg \Phi_i^* = m_i$. *Moreover, for any* $\varrho > \varrho^*$, *the sets* $\Gamma_1^\varrho, \ldots, \Gamma_r^\varrho$ *are components of the set* Γ^ϱ, $\Gamma^\varrho = \Gamma_1^\varrho \cup \ldots \cup \Gamma_r^\varrho$ *and the mappings* $\Phi_i^\varrho = \Phi_i^* | U_i^\varrho$ *are homeomorphisms of* U_i^ϱ *onto* Γ_i^ϱ. *Furthermore, there exists a number* $\omega \geqslant \varrho$, *depending only on* ϱ, *such that in the neighbourhood of infinity* $W^\varrho = \{(x, y) \in \boldsymbol{C}^2\colon |x| > \omega \text{ or } |y| > \varrho\}$ *we have* $\Gamma^\varrho = \Gamma \cap W^\varrho$.

(b) *Suppose that* Y' *is a neighbourhood of infinity in* $\boldsymbol{C}^2$ *such that* $\Gamma \cap Y' = \Gamma'_1 \cup \ldots \cup \Gamma'_{r'}$ *where* $\Gamma'_1, \ldots, \Gamma'_{r'}$ *are components of this set, and for each* $i = 1, \ldots, r'$, *let* $\Psi'_i\colon X'_i = \{t \in \boldsymbol{C}\colon |t| > \varrho'_i\} \to Y'$ *be a mapping meromorphic at infinity, having a pole there, which is a homeomorphism of* X'_i *onto* Γ'_i; *additionally, suppose that* $Y' \subset W^\varrho$ *for some* $\varrho > \varrho^*$. *Then* $r = r'$ *and there exists a permutation* ξ *of* $\{1, \ldots, r\}$ *such that, for each* $i \in \{1, \ldots, r\}$, *we have* $\Gamma'_i = \Gamma^\varrho_{\xi(i)} \cap Y'$ *and the composition* $\tau_i = (\Phi^\varrho_{\xi(i)})^{-1} \circ \Psi'_i$ *is a conformal mapping having a simple pole at* ∞.

Proof. (a) Since $\hat{h}_1, \ldots, \hat{h}_r$ are irreducible and relatively prime in $\mathcal{M}[x]$, we can choose a sufficiently large $\varrho^* \geq E$ such that in the set $\{y \in C: |y| > \varrho^*\}$: 1° the polynomials h_i have no multiple roots, 2° the polynomials h_i, h_j, $i \neq j$, have no common zeros (see [9], Ch. 6, § 13).

Let $\mathfrak{M}_i^*$ be the analytic function (cf. [9], Ch. 6) in the set $\{y \in C: |y| > \varrho^*\}$, satisfying the equation $h_i(x, y) = 0$. It follows from 1° that $\mathfrak{M}_i^*$ is arbitrarily continuable and strictly m_i-valued in this set, so (see [9], Ch. 6, § 9), there exists a function φ_i^* defined in U_i^* and meromorphic at ∞, such that $\varphi_i^*(\sqrt[m_i]{y}) = \mathfrak{M}_i^*$. It is easily seen that the mapping Φ_i^*: $U_i^* \ni t \mapsto (\varphi_i^*(t), t^{m_i}) \in C^2$ is injective. Moreover, from the fact that $\deg_x h_i = \deg h_i$, which is easy to check, we get $\deg \varphi_i^* \leq m_i$.

From 2° it follows that, for any $\varrho > \varrho^*$, the sets $\Gamma_1^\varrho, \ldots, \Gamma_r^\varrho$ are nonvoid pairwise disjoint and $\Gamma^\varrho = \Gamma_1^\varrho \cup \ldots \cup \Gamma_r^\varrho$. Moreover, the set equality $\Gamma_i^\varrho = \Phi_i^\varrho(U_i^\varrho)$ implies the connectedness of Γ_i^ϱ. We easily verify that $(\Phi_i^\varrho)^{-1}$: $\Gamma_i^\varrho \to U_i^\varrho$ is a continuous function. Then Φ_i^ϱ: $U_i^\varrho \to \Gamma_i^\varrho$ is a homeomorphism.

Since h is monic, therefore, for any $\varrho > \varrho^*$, there exists $v \geq \varrho$ such that $h(x, y) \neq 0$ in the set $\{(x, y) \in C^2: |x| > v, |y| \leq \varrho\}$. Denote by ω the infimum of the set of numbers v with this property and put $W^\varrho = \{(x, y) \in C^2: |x| > \omega$ or $|y| > \varrho\}$. Then $\Gamma^\varrho = \Gamma \cap W^\varrho$.

(b) By the assumption, there exists $\varrho > \varrho^*$ such that $Y' \subset W^\varrho$. Hence $\Gamma \cap Y' = (\Gamma_1^\varrho \cap Y') \cup \ldots \cup (\Gamma_r^\varrho \cap Y') = \Gamma_1' \cup \ldots \cup \Gamma_{r'}'$ where the sets $\Gamma_1^\varrho \cap Y', \ldots, \Gamma_r^\varrho \cap Y'$ are nonvoid and pairwise disjoint. So, $r' \geq r$. Analogously, choosing $\varrho' > \varrho^*$ such that $W^{\varrho'} \subset Y'$, we show that $r' \leq r$. Thus $r = r'$. In consequence, there exists a permutation ξ of the set $\{1, \ldots, r\}$ such that $\Gamma_i' = \Gamma_{\xi(i)}^\varrho \cap Y'$. The function τ_i, by (a), is continuous and

$$\tau_i^{m_i} = (\pi_2 \circ \Phi_{\xi(i)}^\varrho) \circ \tau_i = \pi_2 \circ \Psi_i' \quad \text{on } X_i',$$

where π_2 is the second projection in C^2. Hence τ_i is a branch of $\sqrt[m_i]{\pi_2 \circ \Psi_i'}$ on X_i', and thus a holomorphic function. So, τ_i is a conformal mapping. Since $\tau_i(t) \to \infty$ as $t \to \infty$, τ_i has a pole at ∞, of course, a simple one.

This ends the proof of the lemma. □

The parametrizations $(U_i^\varrho, \Phi_i^\varrho, W^\varrho)$, $i = 1, \ldots, r$, of Γ_i^ϱ, the branches of the curve $h = 0$ in the neighbourhood of infinity W^ϱ, $\varrho > \varrho^*$, will be called *canonical.*

Proof of Proposition 3.1. Let us first suppose that h satisfies the assumptions of Lemma 4.1. In the notation introduced in the lemma let us take an arbitrary $\varrho > \varrho^*$. Put $Y = W^\varrho$. Then it is seen that conditions (i), (ii) of the proposition are in this case a simple consequence of Lemma 4.1.

Given two functions h_1, h_2, we find ϱ_1, ϱ_2, respectively. Put $\varrho = \max(\varrho_1, \varrho_2)$. Next, we choose ω_1, ω_2 to h_1, h_2 and we put $\omega = \max(\omega_1, \omega_2)$, $W^\varrho = \{(x, y) \in C^2: |x| > \omega$ or $|y| > \varrho\}$. Then $Y = W^\varrho$ is appropriate for both h_1 and h_2.

It is easy to notice that Proposition 3.1 is invariant with respect to linear automorphisms of C^2. So, if h, h_1, h_2 are polynomial functions as in Proposition 3.1, then there exist a linear automorphism $L\colon C^2 \to C^2$ and constants $c_1, c_2 \neq 0$, such that $h \circ L$, $c_1 h_1 \circ L$, $c_2 h_2 \circ L$ satisfy the assumptions of Lemma 4.1. Hence, in view of the fact that h_1 and $c_1 h_1$ have identical zeros (the same concerns h_2 and $c_2 h_2$), we get the proposition in the general case.

This ends the proof of Proposition 3.1. □

Let $H = (f, g)$ be a polynomial mapping satisfying the assumptions: 1° f, g are not constant, 2° $\# H^{-1}(0) < \infty$, 3° f, g are monic polynomials with respect to x, 4° $\deg_x f = \deg f$, $\deg_x g = \deg g$.

Let $\hat{f} = \hat{f}_1^{\gamma_1} \ldots \hat{f}_r^{\gamma_r}$, $\hat{g} = \hat{g}_1^{\delta_1} \ldots \hat{g}_s^{\delta_s}$ be a factorization of $\hat{f}$, $\hat{g}$ into irreducible factors in $\mathcal{M}[x]$, and let $f = f_1^{\gamma_1} \ldots f_r^{\gamma_r}$, $g = g_1^{\delta_1} \ldots g_s^{\delta_s}$ in $\{(x, y) \in C^2\colon |y| > E\}$. Moreover, according to Lemma 4.1, let $(U_i^{\varrho,f}, \Phi_i^{\varrho,f}, W^\varrho)$, $i = 1, \ldots, r$, be canonical parametrizations of the branches of the curve $f = 0$ in W^ϱ, and let $(U_j^{\varrho,g}, \Phi_j^{\varrho,g}, W^\varrho)$, $j = 1, \ldots, s$, be canonical parametrizations of the branches of the curve $g = 0$ in W^ϱ. Let us fix ϱ for further considerations and write, for simplicity, $W = W^\varrho$, $U_i^* = U_i^{\varrho,f}$, $\Phi_i^* = \Phi_i^{\varrho,f}$, $i = 1, \ldots, r$ and $U_j^{**} = U_j^{\varrho,g}$, $\Phi_j^{**} = \Phi_j^{\varrho,g}$, $j = 1, \ldots, s$. Further, write $m = \deg f$, $n = \deg g$, $m_i = \deg f_i$, $n_j = \deg g_j$, $i = 1, \ldots, r$, $j = 1, \ldots, s$. Let D be the least common multiple of $m_1, \ldots, m_r, n_1, \ldots, n_s$.

LEMMA 4.2 (cf. [1], Ch. 2, § 5). *Under the above assumptions, we have the following statements:*

(a) *in the set* $Q = \{(x, t) \in C^2\colon |t| > \varrho^{1/D}\}$

$$(4.1)\qquad \begin{aligned} f(x, t^D) &= \prod_{k=1}^{m} (x - \alpha_k(t)), \\ g(x, t^D) &= \prod_{l=1}^{n} (x - \beta_l(t)) \end{aligned}$$

where α_k, β_l *are functions defined in* $\{t \in C\colon |t| > \varrho^{1/D}\}$, *meromorphic at* ∞, *and* $\deg \alpha_k \leqslant D$, $\deg \beta_l \leqslant D$;

(b) *for any* k, *there exists* i *such that*

$$(4.2)\qquad \sum_{j=1}^{n} \deg(\alpha_k - \beta_j) = D \deg g \circ \Phi_i^* / \deg \Phi_i^*$$

and, for any i, *there exists* k *such that* (4.2) *holds;*

(c) *for any* l, *there exists* j *such that*

$$(4.3)\qquad \sum_{i=1}^{m} \deg(\beta_l - \alpha_i) = D \deg f \circ \Phi_j^{**} / \deg \Phi_j^{**}$$

and, for any j, *there exists* l *such that* (4.3) *holds.*

Proof. It suffices to notice that

$$f_i(x, t^{m_i}) = \prod_{p=1}^{m_i} (x - \varphi_i^*(\varepsilon_i^p t)) \quad \text{for } |t| > \varrho^{1/m_i},$$

$$g_j(x, t^{n_j}) = \prod_{q=1}^{n_j} (x - \varphi_j^{**}(\eta_j^q t)) \quad \text{for } |t| > \varrho^{1/n_j},$$

where ε_i, η_j are the m_i-th and n_j-th primitive roots of unity, respectively. Hence

$$(4.4)\qquad \begin{aligned} f(x, t^D) &= \prod_{i=1}^{r} \Big(\prod_{p=1}^{m_i} (x - \varphi_i^*(\varepsilon_i^p t^{D/m_i}))\Big)^{\gamma_i}, \\ g(x, t^D) &= \prod_{j=1}^{s} \Big(\prod_{q=1}^{n_j} (x - \varphi_j^{**}(\eta_j^q t^{D/n_j}))\Big)^{\delta_j}, \end{aligned}$$

which gives (4.1). Lemma 4.1(a) implies $\deg \varphi_i^* \leqslant m_i$, $\deg \varphi_j^{**} \leqslant n_j$. Hence $\deg \alpha_k \leqslant D$, $\deg \beta_l \leqslant D$. From (4.1) and (4.4) it follows that, for any k, there exist i, p such that $\alpha_k(t) = \varphi_i^*(\varepsilon_i^p t^{D/m_i})$ and vice versa. Hence we get (b). (c) is proved in an analogous way.

Now we formulate and prove a lemma playing a key part in the proof of the Main Theorem. The proof will be carried out by means of the "horn neighbourhoods" method.

Suppose that the assumptions of Lemma 4.2 are satisfied and that (4.1) holds in Q. Enlarging ϱ, if necessary, we can assume that there exist numbers c, $d > 0$ such that in the set $\{t \in \mathbf{C}: |t| > \varrho^{1/D}\}$ we have

$$(4.5)\qquad c\,|t|^{\deg(\alpha_i - \beta_j)} \leqslant |\alpha_i(t) - \beta_j(t)| \leqslant d\,|t|^{\deg(\alpha_i - \beta_j)}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and

$$(4.6)\qquad |\beta_l(t) - \beta_j(t)| \leqslant d\,|t|^{\deg(\beta_l - \beta_j)}$$

for $l, j = 1, \ldots, n$ such that $\beta_l \neq \beta_j$. Let w be a positive number with $w < c$. Again enlarging ϱ, if necessary, we can assume that $\varrho^{1/D} > 2d/w$. As in Lemma 4.2, let (U_i^*, Φ_i^*, W), $i = 1, \ldots, r$, and $(U_j^{**}, \Phi_j^{**}, W)$, $j = 1, \ldots, s$, be canonical parametrizations of the branches of the curves $f = 0$ and $g = 0$ in W, respectively.

Let us denote

$$v = \min_{i=1}^{r} \min_{j=1}^{s} (\deg g \circ \Phi_i^* / \deg \Phi_i^*,\ \deg f \circ \Phi_j^{**} / \deg \Phi_j^{**}).$$

LEMMA 4.3. *There exists a constant $A_1 > 0$ such that*

$$(4.7)\qquad |H(x, t^D)| \geqslant A_1 |t|^{vD} \quad \textit{for } (x, t) \in Q.$$

Proof. We first show that (4.7) holds in the horn neighbourhood F_k of the curve $x = \alpha_k(t)$, $|t| > \varrho^{1/D}$, where

$$F_k = \{(x, t) \in Q\colon |x - \alpha_k(t)| \leqslant w|t|^{\varkappa_k}\}$$

with $\varkappa_k = \min_j \deg(\alpha_k - \beta_j)$ for $k = 1, \ldots, m$. From the definition of F_k and from (4.5) we have

$$|x - \beta_j(t)| \geqslant c|t|^{\deg(\alpha_k - \beta_j)} - w|t|^{\varkappa_k} \geqslant (c-w)|t|^{\deg(\alpha_k - \beta_j)}.$$

Hence and from (4.1) we get

$$|g(x, t^D)| \geqslant (c-w)^m |t|^{\sum_{j=1}^{n} \deg(\alpha_k - \beta_j)}.$$

By Lemma 4.2(b), there exists i such that

$$|g(x, t^D)| \geqslant (c-w)^m |t|^{D \deg g \circ \Phi_i^* / \deg \Phi_i^{\circ}}.$$

Hence and from the definition of v we get (4.7) in F_k, where $A_1 = (c-w)^m$.

For any $l, q \in \{1, \ldots, n\}$ such that $\beta_l \neq \beta_q$, we now put

$$F_{l,q} = \{(x, t) \in Q\colon |x - \beta_l(t)| \leqslant w|t|^{\deg(\beta_l - \beta_q)}\}$$

and

$$\tilde{F}_{l,q} = F_{l,q} - \bigcup_{k=1}^{m} F_k - \bigcup_{p,k} F_{p,k}$$

where p, k run through all indices such that $F_{p,k}$ is a proper subset of $F_{l,q}$. We now show that (4.7) holds in any $\tilde{F}_{l,q}$. Fix $i \in \{1, \ldots, m\}$. Three cases can occur. In the first one, if

$$\deg(\beta_l - \alpha_i) = \varkappa_i,$$

we have the inequality

$$|x - \alpha_i(t)| \geqslant w|t|^{\deg(\beta_l - \alpha_i)}. \tag{4.8}$$

In the second case, if

$$\deg(\beta_l - \alpha_i) \geqslant \deg(\beta_l - \beta_q),$$

from the definition of $\tilde{F}_{l,q}$ and from (4.5) we get

$$\begin{aligned} |x - \alpha_i(t)| &\geqslant c|t|^{\deg(\alpha_i - \beta_l)} - w|t|^{\deg(\beta_l - \beta_q)} \\ &\geqslant (c-w)|t|^{\deg(\beta_l - \alpha_i)}. \end{aligned} \tag{4.9}$$

In the third case, if

$$\varkappa_i = \deg(\beta_p - \alpha_i) < \deg(\beta_l - \alpha_i) < \deg(\beta_l - \beta_q),$$

we easily check that $F_{p,l} \subset F_{l,q}$. In fact, if $(x, t) \in F_{p,l}$, then

$$|x-\beta_p(t)| \leqslant w|t|^{\deg(\beta_p-\beta_l)}$$

Hence and from (4.6) we have in this case

$$|x-\beta_l(t)| \leqslant (w+d)|t|^{\deg(\beta_p-\beta_l)} \leqslant w|t|^{\deg(\beta_l-\beta_q)},$$

that is to say, $(x, t) \in F_{l,q}$. In consequence, from (4.5) we have

$$(4.10)\qquad |x-\alpha_i(t)| \geqslant w|t|^{\deg(\beta_p-\beta_l)} - d|t|^{\deg(\beta_p-\alpha_i)} \geqslant (w/2)|t|^{\deg(\beta_l-\alpha_i)}.$$

Combining (4.8), (4.9) and (4.10), we find that, for $(x, t) \in \tilde{F}_{l,q}$,

$$|x-\alpha_i(t)| \geqslant A_2|t|^{\deg(\beta_l-\alpha_i)}, \qquad A_2 = \min(w/2, c-w),$$

and so,

$$|f(x, t^D)| \geqslant A_2^m|t|^{\sum_{i=1}^m \deg(\beta_l-\alpha_i)}.$$

By Lemma 4.2(c), there exists j such that

$$|f(x, t^D)| \geqslant A_2^m|t|^{D\deg f \circ \Phi_j^{**}/\deg \Phi_j^{**}}.$$

Hence and from the definition of v we get (4.7) in $\tilde{F}_{l,q}$, where $A_1 = A_2^m$.

The fact that the estimate (4.7) is true in $\tilde{F}_{l,q}$ for any l, q implies that it is also satisfied in $\bigcup_{l,q} F_{l,q}$.

To complete the proof, it suffices to prove (4.7) in the complement of $\bigcup_k F_k \cup \bigcup_{l,q} F_{l,q}$. Take $i \in \{1, \ldots, m\}$. We distinguish two cases. In the first one, if

$$\varkappa_i = \min_l \deg(\beta_l-\alpha_i) = \max_l \deg(\beta_l-\alpha_i),$$

we have

$$(4.11)\qquad |x-\alpha_i(t)| \geqslant w|t|^{\varkappa_i} = w|t|^{\max_l \deg(\beta_l-\alpha_i)}.$$

In the second case, if

$$\varkappa_i = \deg(\beta_p-\alpha_i) < \max_l \deg(\beta_l-\alpha_i) = \deg(\beta_q-\alpha_i),$$

we have

$$(4.12)\qquad |x-\alpha_i(t)| \geqslant w|t|^{\deg(\beta_p-\beta_q)} - d|t|^{\deg(\beta_p-\alpha_i)} \geqslant (w/2)|t|^{\max_l \deg(\beta_l-\alpha_i)}.$$

From (4.11) and (4.12) we get in the general case

$$|x-\alpha_i(t)| \geqslant (w/2)\,|t|^{\max_l \deg(\beta_l-\alpha_i)},$$

and so,

$$|f(x, t^D) \geqslant (w/2)^m\,|t|^{\sum_{i=1}^{m} \max_l \deg(\beta_l-\alpha_i)}$$

$$\geqslant (w/2)^m\,|t|^{\max_l \sum_{i=1}^{m} \deg(\beta_l-\alpha_i)}.$$

Further, from Lemma 4.2(c) we have

$$|f(x, t^D)| \geqslant (w/2)^m\,|t|^{\max_j D\deg f\circ\Phi_j^{**}/\deg\Phi_j^{**}}.$$

Hence and from the definition of v we get (4.7) also in this case, where $A_1 = (w/2)^m$.

This concludes the proof of the lemma. □

5. Proof of the Main Theorem

It is easy to see that if $\tilde{H} = (cf, dg)$ where $c, d \in \boldsymbol{C}-\{0\}$, then $\chi(H) = \chi(\tilde{H})$. Moreover, the exponent of growth $\chi(H)$, the numbers r, s and the ratio $\deg H\circ\Psi_i/\deg\Psi_i$, $i = 1, \ldots, r+s$, are invariants of linear automorphisms of $\boldsymbol{C}^2$. Therefore, without loss of generality, we may assume that f and g are monic polynomials with respect to x, and that $\deg_x f = \deg f$, $\deg_x g = \deg g$. So, the mapping $H = (f, g)$ satisfies the assumptions of Lemma 4.2. From condition (a) of that lemma it follows that there exists a constant $e > 0$ such that, for any $i \in \{1, \ldots, m\}$,

(5.1) $$|\alpha_i(t)| \leqslant e\,|t|^D \quad \text{for } |t| > \varrho^{1/D}.$$

Let us consider the mapping H in the set $P = \{(x, y) \in \boldsymbol{C}^2\colon |y| > \varrho\}$. Two cases can now occur. In the first one, if $|x| < (e+1)\,|y|$, then by putting $t^D = y$ in Lemma 4.3 we obtain

(5.2) $$|H(z)| \geqslant A\,|z|^v$$

where $A = A_1/(e+1)^v$. In the second case, if $|x| \geqslant (e+1)\,|y|$, we put $y = t^D$ and then, by Lemma 4.2 and (5.1), we get

$$|f(x, t^D)| \geqslant \big(|x|/(e+1)\big)^m \geqslant \big(|x|/(e+1)\big)^v.$$

Hence, in this case we also obtain (5.2), where $A = 1/(e+1)^v$. To sum up, there exists $A > 0$ such that (5.2) holds in P.

Since f is a monic polynomial, there exist numbers $A_2 > 0$ and $v \geqslant \varrho$

such that $|f(x, y)| \geqslant A_2|x|^m$ in the set $T = \{(x, y) \in C^2: |x| > v, |y| \leqslant \varrho\}$. Hence we also obtain (5.2) in T, where $A = A_2$.

Consequently, there exist a constant A and a neighbourhood V of infinity in C^2 of the form $V = \{(x, y) \in C^2: |x| > v \text{ or } |y| > \varrho\}$ in which (5.2) holds. Hence $\nu \leqslant \chi(H)$.

Let $W \subset V$ and let (U_i, Φ_i, W) be canonical parametrizations of the branches Γ_i of the curve $f = 0$ in W, $i = 1, \ldots, r$, and (U_i, Φ_i, W) be canonical parametrizations of the branches Γ_i of the curve $g = 0$ in W, $i = r+1, \ldots, r+s$. It is easy to see that, for $i = 1, \ldots, r$, we have $\deg H \circ \Phi_i = \deg g \circ \Phi_i$ and, similarly, for $i = r+1, \ldots, r+s$, we have $\deg H \circ \Phi_i = \deg f \circ \Phi_i$. Then there exists $j \in \{1, \ldots, r+s\}$ such that $\nu = \deg H \circ \Phi_j / \deg \Phi_j$. Enlarging ϱ, if necessary, we may assume that there exist positive constants c_1, c_2 such that

$$|\Phi_j(t)| \geqslant c_1 |t|^{\deg \Phi_j},$$
$$|H \circ \Phi_j(t)| \leqslant c_2 |t|^{\deg H \circ \Phi_j}$$

for $|t| > \varrho$. Hence

$$|H(z)| \leqslant c|z|^{\nu} \quad \text{for } z \in \Gamma_j, \tag{5.3}$$

where $c = c_1/c_2$. From Lemma 4.1 it follows that the set Γ_i has points in common with any neighbourhood of infinity in C^2. Hence, from the definition of $\chi(H)$ and (5.3) we get $\chi(H) \leqslant \nu$. In consequence, $\chi(H) = \nu$, and the proof of (i) is completed.

Condition (ii) is obvious.

This concludes the proof of the Main Theorem. □

6. Properties of the geometric degree

Let h: $C^2 \to C$ be a nonconstant polynomial function, Y a neighbourhood of infinity in C^2 and (X_i, Ψ_i, Y), $i = 1, \ldots, r$, parametrizations of the branches Γ_i, $i = 1, \ldots, r$, of the curve $h = 0$ in Y. Then, for any $i \in \{1, \ldots, r\}$, there exists a unique (up to a constant factor) linear function λ_i: $C^2 \to C$ such that $\deg \lambda_i \circ \Psi_i < \deg \Psi_i$. The line $\lambda_i = 0$ is called the *asymptote of the branch* Γ_i. If Y' is any other neighbourhood of infinity in C^2 and (X_i', Ψ_i', Y'), $i = 1, \ldots, r$, are parametrizations of the branches Γ_i' of $h = 0$ in Y', $i = 1, \ldots, r$, then by Proposition 3.1 there exists a permutation ξ of the set $\{1, \ldots, r\}$ such that $\lambda_{\xi(1)} = 0, \ldots, \lambda_{\xi(r)} = 0$ are the asymptotes of these branches, respectively. The lines $\lambda_1 = 0, \ldots, \lambda_r = 0$ are called the *asymptotes of the curve* $h = 0$.

Remark. Let h^* be the homogenization of the polynomial h and let $\lambda_i(x, y) = a_i x + b_i y$. Then, to any asymptote $\lambda_i = 0$ of the curve $h = 0$ there corresponds a point $(a_i, b_i, 0)$ of the projective space P^2. It is easy to see that $h^*(a_i, b_i, 0) = 0$. So, the points $(a_i, b_i, 0)$, $i = 1, \ldots, r$, are zeros of the curve $h^* = 0$ on the line at infinity.

PROPERTY 6.1. *Let* $H = (f, g)$: $C^2 \to C^2$ *be a polynomial mapping and* $\# H^{-1}(0) < \infty$. *Then*:

(a) $\sigma(H) \leqslant \deg f \deg g$;

(b) *equality in* (a) *holds if and only if the curves* $f = 0$, $g = 0$ *have no asymptotes in common.*

Proof. This property is a simple consequence of the Bezout Theorem (cf. [11], Ch. 4, § 5) and the remark given above. □

PROPERTY 6.2. *Under the same assumptions and notation as in Lemma* 4.2, *we have*

$$m = \sum_{i=1}^{r} \gamma_i \deg \Phi_i^*, \qquad n = \sum_{j=1}^{s} \delta_j \deg \Phi_j^{**} \tag{6.1}$$

and

$$\sigma(H) = \sum_{i=1}^{r} \gamma_i \deg g \circ \Phi_i^* = \sum_{j=1}^{s} \delta_j \deg f \circ \Phi_j^{**}. \tag{6.2}$$

Proof. From Lemma 4.1 we have $\deg \Phi_i^* = m_i$, $\deg \Phi_j^{**} = n_j$. Hence and from the form of the factorizations of f and g we get (6.1).

Let

$$f(x, y) = x^m + a_1(y) x^{m-1} + \ldots + a_m(y),$$
$$g(x, y) = x^n + b_1(y) x^{n-1} + \ldots + b_n(y),$$

and let $R(y)$ be the resultant of these polynomials. It is known (see [11], Ch. 4, § 5) that

$$\deg R = \sigma(H). \tag{6.3}$$

On the other hand, from a property of the resultant and from Lemma 4.2(a) we have, for $|t| > \varrho^{1/D}$,

$$R(t^D) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i(t) - \beta_j(t)).$$

Hence and from Lemma 4.2(b), (c) we obtain

$$D \deg R = D \sum_{i=1}^{r} \gamma_i \deg g \circ \Phi_i^* = D \sum_{j=1}^{s} \delta_j \deg f \circ \Phi_j^{**}. \tag{6.4}$$

By (6.3) and (6.4), we get (6.2).

This ends the proof. □

7. Proofs of the corollaries from the Main Theorem

Corollaries 3.2 and 3.3 are obvious.

Proof of Proposition 3.4. Since $\chi(H)$, $\sigma(H)$, $\deg f$, $\deg g$ are invariants of linear automorphisms L: $C^2 \to C^2$, $L(0) = 0$, we may assume, without loss of generality, that the assumptions of Lemma 4.2 are satisfied.

(i) Let us first assume that $\chi(H) = \deg g\circ\Phi_p^*/\deg\Phi_p^*$. Then, with the same notation as in Lemma 4.2, we have

$$\begin{aligned}\chi(H) - \min(m, n) &\geq (\deg g\circ\Phi_p^*/\deg\Phi_p^*) - n \\ &\geq \deg g\circ\Phi_p^* - n\deg\Phi_p^* \\ &\geq \sum_{i=1}^{r} (\gamma_i \deg g\circ\Phi_i^* - n\gamma_i \deg\Phi_i^*).\end{aligned}$$

Hence and from Property 6.2 we get (i). We show (i) in an analogous way if $\chi(H) = \deg f\circ\Phi_q^{**}/\deg\Phi_q^{**}$.

(ii) Let $m = \max(m, n)$. From Property 6.2 we have

$$\chi(H) \leq \sum_{i=1}^{r} \gamma_i(\deg g\circ\Phi_i^*/m) = \sigma(H)/m.$$

This concludes the proof of the proposition. □

From Proposition 3.4 and Property 6.1 we easily obtain

COROLLARY 7.1. *Let* $H = (f, g)\colon C^2 \to C^2$ *be a polynomial mapping and* $\#H^{-1}(0) < \infty$. *Then*:

(a) $\chi(H) \leq \min(\deg f, \deg g)$;

(b) *equality in* (a) *holds if and only if the curves* $f = 0$, $g = 0$ *have no asymptotes in common.*

Proof of Proposition 3.5. Since $\chi(H)$, $\sigma(H)$, $\deg f$, $\deg g$ and the number of the branches of the curves $f = 0$, $g = 0$ at infinity are invariants of linear automorphisms, we may suppose, as before, that the assumptions of Lemma 4.2 are satisfied with $r = s = 1$. Then, by Property 6.2, we obtain

$$\sigma(H)/m = \deg g\circ\Phi_1^*/\deg\Phi_1^*$$

and

$$\sigma(H)/n = \deg f\circ\Phi_1^{**}/\deg\Phi_1^{**}.$$

So, from the Main Theorem we have

$$\chi(H) = \sigma(H)/\max(m, n).$$

If, additionally, $\#H^{-1}(0) > 0$, then $\sigma(H) > 0$ and, by the above, $\chi(H) > 0$. Hence, by Corollary 3.4, the mapping H is proper.

This ends the proof of the proposition. □

8. Examples

We shall give here two simple examples of polynomial mappings in order to illustrate the methods for calculating the exponent of growth and deciding whether a mapping is proper.

Let $W = \{z\in C^2\colon |z| > 2\}$, $U = \{t\in C\colon |t| > 2\}$, $H = (f, g)$.

EXAMPLE 8.1. Let $H(x, y) = (xy+x-y,\ x^2-y)$. It is easy to see that the curve $f = 0$ has two branches in W and the curve $g = 0$ has one branch in W. These branches have parametrizations in W of the form (U, Ψ_i, W), $i = 1, 2, 3$. For $f = 0$, we have $\Psi_1(t) = (1-(1/t)+(1/t)^2- \ldots, t)$, $\Psi_2(t) = (t, -1-(1/t)-(1/t)^2- \ldots)$. For $g = 0$, we have $\Psi_3(t) = (t, t^2)$. Hence we easily calculate that $\deg H\circ\Psi_1/\deg\Psi_1 = 1$, $\deg H\circ\Psi_2/\deg\Psi_2 = 2$, $\deg H\circ\Psi_3/\deg\Psi_3 = 3/2$. Then $\chi(H) = 1$ and, in consequence, the mapping H is proper.

EXAMPLE 8.2. Let $H(x, y) = (xy+x-y,\ x^2)$. The curve $f = 0$ is the same as in the preceding example, so it has the same branches in W. The curve $g = 0$ has one branch in W whose parametrization is of the form (U, Ψ_3, W) where $\Psi_3(t) = (0, t)$. We easily verify that $\deg H\circ\Psi_1/\deg\Psi_1 = 0$, $\deg H\circ\Psi_2/\deg\Psi_2 = 2$, $\deg H\circ\Psi_3/\deg\Psi_3 = 1$. Hence $\chi(H) = 0$ and, consequently, the mapping H is not proper.

Acknowledgement. We thank Mr A. Płoski for many helpful discussions.

References

[1] S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry*, Tata Institute of Fundamental Research, Bombay 1977.

[2] J. Chądzyński, *On proper polynomial mappings*, Bull. Pol. Acad. Sci. Math. 31 (1983), 115–120.

[3] J. Chądzyński and T. Krasiński, *The Łojasiewicz exponent of an analytic mapping of two complex variables at an isolated zero*, this volume, 133–146.

[4] E. A. Gorin, *Asymptotic properties of polynomials and algebraic functions of several variables*, Uspekhi Mat. Nauk 16 (1961), 91–118.

[5] L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat. 3 (1958), 555–568.

[6] O. H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. 47 (1939), 299–306.

[7] T. C. Kuo and Y. C. Lu, *On analytic function germs of two complex variables*, Topology 16 (1977), 299–310.

[8] A. Płoski, *On the growth of proper polynomial mappings*, Ann. Polon. Math. 55 (1985), 297–309.

[9] S. Saks and A. Zygmund, *Analytic Functions*, PWN–Polish Scientific Publishers, Warsaw 1965.

[10] J. C. Tougeron, *Idéaux de fonctions différentiables*, Springer-Verlag, Berlin–Heidelberg–New York 1972.

[11] R. J. Walker, *Algebraic Curves*, Springer-Verlag, New York–Heidelberg–Berlin 1978.

Presented to the semester
Singularities
15 February–15 June, 1985