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SOME ASPECTS OF HOLOMORPHIC APPROXIMATIONS FROM THE POINT OF VIEW OF PDEs

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§ 1. Introduction

One of the main problems in complex analysis is the problem of holomorphic approximation. There are of course many variations of this problem, for instance

1° approximation of holomorphic functions by functions which are holomorphic in essentially larger sets;

2° holomorphic approximation of functions from wider classes of functions, for example: continuous, smooth, CR, continuous and holomorphic in the interior;

3° holomorphic approximation by functions from restricted classes, for example: exponential polynomials, functions of restricted growth;

4° holomorphic approximation in different senses, for example: uniform, L^p, weighted, and in other norms or distances.

There is a natural generalization of the holomorphic approximation problem in the direction of partial differential equations. Namely, instead of holomorphic functions we may consider solutions of one equation or of a system of partial differential equations. Mainly elliptic equations, elliptic systems or homogeneous first order systems are considered. There is a vast literature on this subject; see for example Lax [12], Malgrange [13],

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Browder [3], Narasimhan [15], Polking [17], Modica [14], Hörmander [11], Dufresnoy, Gauthier and Ow [4], Baouendi and Treves [2].

In many cases, holomorphic functions are used for approximation of functions which satisfy some system of PDEs (CR functions, RC functions), weaker than the full Cauchy–Riemann system. The problem is then to approximate functions satisfying one system of equations by functions satisfying a larger system of equations.

The importance of such problems is obvious, not only for complex analysis, but also for the theory of partial differential equations and the geometry of submanifolds (Gaier [8], Gauthier and Hengartner [9], Wells [22]).

The purpose of this paper is to give a short survey of results concerning Runge type theorems for solutions of one PDE or a system of PDEs (§ 2, § 3). The survey does not pretend to be full; we formulate only a classical Runge type theorem for solutions of PDEs and also give a few fairly recent results in this direction. For a more ample description of this topic we refer to the books of Hörmander [11], Treves [18], [19], [20]. In § 4 we give a PDE point of view of the authors' result concerning global holomorphic approximation of CR functions.

§ 2. Approximation theorems for linear PDEs

Let P be a polynomial in $\boldsymbol{R}^n$ with constant complex coefficients

$$P(x) = \sum_{|\alpha| \leqslant m} a_\alpha x^\alpha, \qquad x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n,\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}^n_+,$$

where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We obtain the corresponding operator $P(D)$ by replacing x_α by $i\partial/\partial x_\alpha$.

For such operators we formulate a Runge type theorem due to Malgrange [13]. Denote by $M^* = M \cup \{*\}$ the one point compactification of M.

THEOREM (see [11, Th. 44.5]). *Assume that P has a fundamental solution E which is real analytic in $\boldsymbol{R}^n \setminus \{0\}$, and let $U \subset M \subset \boldsymbol{R}^n$ be open sets such that $M^* \setminus U$ is connected. Every solution $u \in C^\infty(U)$ of the equation $Pu = 0$ is then a limit in $C^\infty(U)$ of restrictions to U of solutions of the same equation in M.*

Of course for the Cauchy–Riemann and the Laplace operators we have fundamental solutions which are real analytic in $\boldsymbol{R}^n \setminus \{0\}$; therefore the above theorem is applicable to these operators.

We remark that the condition that $M^* \setminus U$ be connected in the above theorem just says that $M \setminus U$ has no compact components. Also, this is equivalent to saying that $M \setminus U$ cannot be written as the union of a nonempty compact set and a set which is closed in M. The equivalence can be seen by an exhaustion argument. It also follows from Proposition 3.10.4 in [15].

The above theorem can be extended to more general differential operators with constant coefficients and sufficiently wide spaces of distributions. However, the formulation of the extended version needs many special definitions, and therefore we refer the reader to Chapter X, Theorem 10.5.2 in the book of Hörmander [11].

In the next section, we shall formulate some theorems of Runge type for systems of operators with nonconstant coefficients. Of course, such theorems also apply to a single operator.

For operators with constant coefficients we formulate also some results obtained by Dufresnoy, Gauthier and Ow [4] on uniform approximations on closed subsets (not necessarily compact).

Let M be an open set in $\boldsymbol{R}^n$ and $P(D)$ a linear elliptic differential operator with constant coefficients. For an arbitrary subset F of M denote by $\mathscr{H}(F)$ the space of smooth solutions u of the equation $P(D)u = 0$. By $\mathscr{M}(M)$ we denote the space of solutions u of the same equation which are smooth on M except for isolated singularities where the singular part of u is a linear combination of a fundamental solution and its derivatives. Finally, for an arbitrary subset F of M let $\mathscr{M}_F(M)$ denote the space of those elements from $\mathscr{M}(M)$ which have no singularities on F.

THEOREM ([4]). *Let M be an open set in $\boldsymbol{R}^n$ and suppose one of the following*:

(a) *M is bounded.*

(b) *$P(D)$ has a fundamental solution which is uniformly continuous away from zero.*

(c) *$P(D)$ is homogeneous.*

Then, for each (relatively) closed subset F of M, every element of the space $\mathscr{H}(F)$ can be approximated uniformly on F by elements of $\mathscr{M}_F(M)$.

THEOREM ([4]). *Let F be a (relatively) closed subset of M. If $M^* \setminus F$ is connected and locally connected and $P(D)$ is homogeneous, then every element of the space $\mathscr{H}(F)$ can be uniformly approximated on F by elements from the space $\mathscr{H}(M)$.*

An interesting result was given by Polking [17] who showed that certain approximation problems can be reformulated in a way which is independent of the particular differential operator $P(D)$, even with variable coefficients, and also independent of the open set Ω on which it is defined. We shall now formulate his result.

Let $P(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha$ be an elliptic differential operator of order m with infinitely differentiable coefficients defined in an open subset M of $\boldsymbol{R}^n$. For a closed subset $F \subset M$, let as before $\mathscr{H}(F)$ denote the set of all smooth solutions u of the equation $P(x, D)u = 0$ in a neighborhood of F. For $1 < p < \infty$ we set $\mathscr{H}L^p(F) = L^p(F) \cap \mathscr{H}(\mathring{F})$, the set of functions $u \in L^p(F)$ which satisfy the equation $P(x, D)u = 0$ in the interior $\mathring{F}$ of F.

The operator $P(x, D)$ is said to have a *biregular fundamental solution* $E(x, y)$ on M if $E \in L^1_{loc}(\Omega \times \Omega)$ is infinitely differentiable off the diagonal in $M \times M$ and satisfies the equations

$$P(x, D)E(x, y) = \delta_y, \quad {}^tP(y, D)E(x, y) = \delta_x.$$

THEOREM ([17]). *Suppose $P(x, D)$ is an elliptic differential operator of order m with infinitely differentiable coefficients defined in an open subset M of $\boldsymbol{R}^n$ and suppose that $P(x, D)$ has a biregular fundamental solution in M. Then if $K \subset M$ is compact and $1 < p < \infty$, the following statements are equivalent:*

1) *$\mathscr{H}(K)$ is dense in $\mathscr{H}L^p(K)$.*

2) *$C_0^\infty(\mathring{K})$ is dense in the Sobolev space $W_m^q(K)$, $1/p + 1/q = 1$.*

3) *$C_0(\boldsymbol{R}^n \backslash K)$ is dense in $W_{-m}^p(\boldsymbol{R}^n \backslash \mathring{K})$.*

§ 3. Approximation theorems for systems of PDEs

In this section we formulate the classical approximation theorem of Malgrange and Lax (see [12], [13]), and also we give another recent result concerning this type of problems.

Let M be an oriented real analytic manifold and ξ, η be real analytic vector bundles on M whose fibres are complex vector spaces. Assume moreover that the complex dimensions of the fibres of these vector bundles are the same, i.e. $\operatorname{rank}(\xi) = \operatorname{rank}(\eta)$.

THEOREM ([15, Th. 3.10.7]). *Under the above notation let P be an elliptic operator of order m from ξ to η with analytic coefficients. Let U be an open subset of M such that $M^* \backslash U$ is connected. Then any smooth C^∞-section of the bundle ξ over U such that $Pu = 0$ on U is the limit together with all its partial derivatives, uniformly on compact subsets of U, of smooth sections u_ν of ξ over M ($\nu = 1, 2, \dots$) with $Pu_\nu = 0$ on M.*

Next, we shall formulate a Runge type theorem for overdetermined systems with constant coefficients due to Modica [14].

Let $M(D)$ be a linear partial differential system with constant coefficients, that is, $M(D)u = 0$ is the short form of the system of r equations in s unknowns $u_1, \dots, u_s$

$$\sum_{j=1}^{s} M_{ij}(D)u_j = 0, \quad i = 1, \dots, r,$$

where $M_{ij}(D)$ are differential polynomials in n variables with constant coefficients.

We say that a square system $Q(D)$ is *elliptic* if $(\det Q)_m(\xi) \neq 0$, $\xi \in \boldsymbol{R}^n \backslash \{0\}$, where m is the degree of the polynomial $\det Q$ and $(\det Q)_m$ is the homogeneous part of $\det Q$ of degree m exactly.

We now proceed to a formulation of the theorem.

Let $P(D)$, $Q(D)$ be two square $(r \times r)$ systems of partial differential equations (in n variables) with constant coefficients, such that $PQ = QP$ and Q is elliptic.

THEOREM [14, Th. 4.1]. *Let M, U be two connected open subsets of $\mathbf{R}^n$, with $M \supset U$, such that $\mathbf{R}^{n*} \setminus M$ and $M^* \setminus U$ are both connected. Suppose that, for every function f of class C^∞ on M such that $Q(D)f = 0$, the overdetermined system $P(D)u = f$, $Q(D)u = 0$ has a solution u of class C^∞ on U. Then, if $M_\nu = \{x \in M;\ \operatorname{dist}(x, \partial M) > \nu\} \subset U$ for some $\nu > 0$, each solution of class C^∞ on M_ν of the homogeneous system $P(D)u = 0$, $Q(D)u = 0$ can be approximated by a sequence of solutions of class C^∞ on U of the same system uniformly with all their derivatives on the compact subsets of U_ν.*

As a corollary of the above theorem, it is easy to obtain, roughly speaking, the following result:

If the system $P(D)u = f$, $Q(D)u = g$ is "solvable" on each element of an increasing sequence of open subsets, then the system is solvable on the union of this sequence.

An exact formulation however would require that we introduce some additional notions, so we refer the reader to the paper of Modica [14], Section 5, Theorem 5.2.

§ 4. Approximation of solutions of a system of PDEs by solutions of a larger one

(a) Definition of RC and CR manifolds

First we introduce the notion of RC manifolds, where RC is the abbreviation of real/complex. RC structures were introduced by Treves [21].

Let M denote a smooth manifold of real dimension $m \geqslant 1$, TM its tangent bundle and T^*M its cotangent bundle, CTM and CT^*M their complexifications. Take an l-dimensional complex subbundle H of CTM and let $H^\perp$ be the subbundle in CT^*M orthogonal to H with respect to the duality between tangent and cotangent vectors. We shall use the subbundles H and $H^\perp$ interchangeably, depending on the context.

We say that a subbundle H of CTM is *involutive* if it is closed with respect to the Poisson bracket, i.e. $[P, Q]$ is a section of H whenever P and Q are sections of H.

We say that H (or $H^\perp$) is *locally integrable* if any point $p_0 \in M$ has a neighborhood U in which there exist smooth functions $\xi_1, \ldots, \xi_{m-l}$ such that the differentials $d\xi_1, \ldots, d\xi_{m-l}$ are linearly independent and span the subbundle $H^\perp | U$.

Following Treves [21], we say that the subbundle H defines an *almost*

RC structure on M if H is involutive and we refer to the pair (M, H) as an *almost RC manifold*. Any function whose differential is a section of $H^{\perp}$ will be called an *RC function*. When H is locally integrable we say that it defines an *RC structure* and (M, H) is an *RC manifold*.

By an *RC submanifold* (N, H_N) of the RC manifold (M, H) we mean an embedded smooth submanifold N of M together with the structure bundle $H_N = H|_N \cap CTN$, provided H_N is a bundle and gives an RC structure on N.

The most important subclass of RC manifolds is the class of the so-called Cauchy–Riemann (CR) manifolds. Recall that by a *CR manifold* we mean an RC manifold M with an RC structure bundle H which satisfies $H \cap \bar{H} = 0$, where 0 is the zero section and the bar denotes the complex conjugation. By the *RC* (or *CR*) *dimension* of an RC (or CR) manifold we mean the fibre dimension of the bundle H. Of course CR functions on a CR manifold are defined in the same way as RC functions.

It is worth mentioning that a smooth submanifold M of C^n is said to be a CR submanifold if the dimension of the holomorphic tangent space to M at a point $p \in M$ does not depend on p, i.e. the function

$$l(p) = \dim_C H_p(M)$$

is constant, where

$$H_p(M) = \left\{X = a_1 \frac{\partial}{\partial z_1} + \dots + a_n \frac{\partial}{\partial z_n};\ X \in CT_p M\right\}.$$

For a more ample description of CR submanifolds see for example Wells [22], Treves [21].

We shall not go into definitions of other subclasses of RC manifolds, for example hypo-analytic manifolds or hypo-complex manifolds. For a description and study of the mentioned classes see for example the papers of Baouendi, Chang and Treves [1], Hanges and Treves [10], Treves [21].

Notice that from the definition of an RC function on an almost RC manifold it follows that such a function is annihilated by all vector fields which are sections of the subbundle H; consequently the function satisfies locally a system of linear first order partial differential equations.

(b) Approximation by solutions of extended systems of PDEs

We can formulate the main problem in the following way.

Let there be given an almost RC manifold (M, H) and an almost RC submanifold (N, H_N). Assume that an RC function f is given on N (or on an open subset V of N). Under what conditions can such a function be approximated on N (or V) by RC functions defined on neighborhoods of N (or V) in M?

The problem formulated, in such a general version, seems to be very difficult. Even in particular cases (holomorphic or CR), the solutions are far from satisfactory, particularly in the global situations. For a review of approximation problems in complex analysis and CR theory, see for example papers [5] and [22].

Notice also that, roughly speaking, we want to approximate functions satisfying some system of PDEs on a submanifold by functions which satisfy a larger system of PDEs on an ambient manifold.

First we give an example, from the PDEs point of view, that such a problem is not an artificial one.

EXAMPLE. Let M be a CR submanifold of C^n of real dimension k ($k \geqslant n$), and CR dimension $k-n$. In this example we consider the problem only locally. Take an arbitrary point $p_0 \in M$. There exists a neighborhood U of p_0 in C^n and a parametrization of U, say

$$R^{2n} \supset V \ni (x, y) \xrightarrow{\varphi} \varphi(x, y) \in U \subset C^n, \quad (x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_{2n-k}),$$

such that $\varphi(0) = p_0$ and the submanifold M in the coordinates in V is given by the equations $y_1 = \ldots = y_{2n-k} = 0$, i.e.

$$\varphi(\{(x, y) \in V;\ y = 0\}) = M \cap U.$$

Denote the set standing under φ above by N. We can pull back to V and N the antiholomorphic tangent bundles to U and M, respectively. So we have the following bundles over V and N:

$$\varphi_*^{-1}\big(\overline{H(C^n)}|_U\big) \quad \text{and} \quad \varphi_*^{-1}\big(\overline{H(M)}|_{U \cap M}\big). \tag{1}$$

Decreasing U if necessary, we can assume that there are global vector fields on V which span the above bundles. We can choose the complex vector fields, i.e. with complex coefficients, of the form

$$\begin{cases} L_1 = a_{11}(x, y)\dfrac{\partial}{\partial x_1} + \ldots + a_{1k}(x, y)\dfrac{\partial}{\partial x_k}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ L_m = a_{m1}(x, y)\dfrac{\partial}{\partial x_1} + \ldots + a_{mk}(x, y)\dfrac{\partial}{\partial x_k}, \end{cases} \tag{2}$$

$$\begin{cases} L_{m+1} = a_{m+1,1}(x, y)\dfrac{\partial}{\partial x_1} + \ldots + a_{m+1,k}(x, y)\dfrac{\partial}{\partial x_k} \\ \qquad + a_{m+1,k+1}(x, y)\dfrac{\partial}{\partial y_1} + \ldots + a_{m+1,2n}(x, y)\dfrac{\partial}{\partial y_{2n-k}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ L_n = a_{n1}(x, y)\dfrac{\partial}{\partial x_1} + \ldots + a_{nk}(x, y)\dfrac{\partial}{\partial x_k} \\ \qquad + a_{n,k+1}(x, y)\dfrac{\partial}{\partial y_1} + \ldots + a_{n,2n}(x, y)\dfrac{\partial}{\partial y_{2n-k}}, \end{cases} \tag{3}$$

where the vector fields $L_1, \ldots, L_n$ span the first bundle of (1) and $L_1, \ldots, L_m$ restricted to the points $(x, 0)$ span the second bundle of (1).

The functions on N which correspond to CR functions on $M \cap U$ are the functions u which satisfy the system of equations

$$L_1 u = \ldots = L_m u = 0 \quad \text{on } N. \tag{4}$$

The functions on V which correspond to holomorphic functions on U are the functions v which satisfy the system of equations

$$L_1 v = \ldots = L_n v = 0 \quad \text{on } V. \tag{5}$$

From the Baouendi–Treves theorem [2] about holomorphic approximation of CR functions (see also the paper of the authors [6]) it follows that we can uniformly approximate solutions of the system (4) by solutions of the system (5).

Now we formulate some global result about holomorphic approximation of CR functions but from the point of view of PDEs.

Let U be an open subset of $\boldsymbol{R}^{2n}$ and H a complex subbundle of CTU such that (U, H) forms a complex manifold. Take the intersection $V = U \cap (\boldsymbol{R}^k \times \{0\})$, where $k \geqslant n$. Take the subbundle $H_V = H \cap CTV$, and assume that (V, H_V) is a CR submanifold of (U, H) of CR dimension $m = k - n$. Assume moreover that the CR submanifold V satisfies the following condition (R).

(R) There exist global holomorphic coordinates $(z_1, \ldots, z_n)$ on U, a global CR matrix-valued function on V, A: $V \to \mathrm{GL}(n, \boldsymbol{C})$, and a smooth n-real-dimensional distribution L: $V \to TV$ such that for each vector $\xi \in A(p) L_p$, $p \in V$,

$$|\mathrm{Im}\,\xi| < |\mathrm{Re}\,\xi|,$$

where the coordinates of the vector ξ are induced by the coordinates $(z_1, \ldots, z_n)$.

THEOREM ([7]). *Let (U, H), (V, H_V) be complex and CR submanifolds respectively as above and assume that V satisfies the condition* (R). *Then there exists a neighborhood Ω of V in U with the following property:*

If in U there exists a strongly pseudoconvex domain G with smooth boundary ∂G, then an arbitrary CR function on V can be uniformly approximated on $\bar{G} \cap V$ by holomorphic functions defined in a neighborhood of $\bar{G}$.

Roughly speaking, the above theorem gives a sufficient condition for approximating, on the compact set $\bar{G} \cap V$, solutions of a system of equations (CR functions on V) by solutions of a larger system of equations (holomorphic functions in a neighborhood of $\bar{G}$). More precisely: assume that there are global sections of the bundle H over U and the bundle H_V over V. Then we can take the sections of the same form as in (2) and (3), where $x=(x_1, \dots, x_{n+m})$, $y=(y_1, \dots, y_{n-m})$, and all coefficients $a_{\alpha\beta}$ are smooth complex-valued functions defined on V or on U. Moreover, the vector fields $L_1, \dots, L_m$ span the vector bundle H_V over V and all vector fields $L_1, \dots, L_n$ span the vector bundle H over U. The theorem then says that under certain assumptions, any solution $f=f(x)$ of the system

$$L_1 f = 0, \quad \dots, \quad L_m f = 0 \quad \text{on } V$$

can be uniformly approximated on $\bar{G} \cap V$ by solutions $g=g(x, y)$ of the system $L_1 g = 0, \dots, L_n g = 0$ on a neighborhood of $\bar{G}$.

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