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On approximation with nodes

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Introduction

1. Approximation with nodes. In this paper the properties of approximation with nodes are considered. This notion, introduced in the paper [5] [1]), will be presented in §§ 1 and 2 with the generality needed for the sequel.

$\mathcal{C}$ will stand for the set of all continuous functions in a finite interval $I = \langle a, b \rangle$ [2]).

$\mathcal{W}_n$ will denote the set of all algebraic polynomials of degree not greater than n.

T is a fixed set of m distinct points $t_1, t_2, \ldots, t_m$ (where $t_1 < t_2 < \ldots < t_m$), called the *nodes*. We suppose that

(1) $$T \subset I, \quad m < n \text{ [3])}.$$

Given a function $\xi \in \mathcal{C}$, we introduce the class

$$\mathcal{W}_n(\xi; T) = \underset{\omega \in \mathcal{W}_n}{E} \{\|\xi - \omega\|_T = 0\};$$

it is the set of all polynomials from $\mathcal{W}_n$ taking on the same values at the nodes as the function ξ. Inequality $m < n$ guarantees that this set is non-denumerable. On the space of continuous functions we define the functionals

(2) $$\varepsilon_n(\xi) = \inf_{\psi \in \mathcal{W}_n} \|\xi - \psi\|_I,$$

(3) $$\varepsilon_n(\xi; T) = \inf_{\omega \in \mathcal{W}_n(\xi; T)} \|\xi - \omega\|_I.$$

The theory of approximation (without nodes) is based on the classical theorems of Borel and Tchebyshev. These state that there exists exactly one polynomial ψ_n at which the infimum (2) is attained, *i. e.* such that

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[1]) Numbers in square brackets refer to the bibliography at the end of this paper.

[2]) All the symbols used in this paper and not belonging to the theory of approximation are listed in § 3.

[3]) Each paragraph has a separate numeration. When quoting the formulae from another §, we give first the number of that paragraph; *e. g.* the formula (2) of § 1 is quoted as (1.2).

$\|\xi-\psi_n\|_I = \varepsilon_n(\xi)$. This polynomial is called *optimal* for the function ξ in the class $\mathcal{W}_n$ and the interval I[4]). The theory of approximation deals among other problems with the following ones: 1° the properties of optimal polynomials for fixed n, 2° the limit properties of optimal polynomials as $n \to \infty$. For example, the first problem is the subject of the theorem of Tchebyshev on the alternant, the theorem of Weierstrass deals with the second.

The same problems may be considered for *approximation with nodes*, which consists in the fact that we approximate a continuous function ξ by polynomials of the class $\mathcal{W}_n(\xi; T)$ depending both on ξ and on a system T of nodes. Therefore we impose on the polynomials ω approximating ξ the conditions $\omega(t_i) = \xi(t_i)$ $(i = 1, 2, \ldots, m)$ which deteriorate the quality of the approximation (in the sense of deviation $\|\xi-\omega\|_I$), and simultaneously modify its properties in a direction which may happen advantageous, in some cases.

In paper [5] we have examined the properties of the *optimal polynomial* ω_n for the function ξ in the class $\mathcal{W}_n(\xi; T)$ and the interval I. By definition, this polynomial realizes the infimum (3), *i. e.* $\|\xi-\omega_n\|_I = \varepsilon_n(\xi; T)$ holds. We have proved that ω_n is uniquely determined, and we have given a necessary and sufficient condition for a polynomial of class $\mathcal{W}_n(\xi; T)$ to be optimal for ξ in this class. Three equivalent formulations of this condition, which will be used in chapter II, are quoted in § 2.

In the first chapter of this paper we compare the limit properties (for $n \to \infty$) of approximation with nodes with the same properties of approximation without nodes. One of the results is that, for sufficiently large n, approximation with nodes is at most two times less accurate than approximation without nodes. The error of approximation of both kinds is determined by the *deviations* (2) and (3). The case $m = 2$, in which T consists of two elements, is worked out with more detail.

In the second chapter we put forward problems which seem to be new even in the domain of approximation without nodes. We give necessary and sufficient conditions for two polynomials $\psi_n \epsilon \mathcal{W}_n$ and $\psi_{n+1} \epsilon \mathcal{W}_{n+1}$ to be optimal for the same function ξ in the classes $\mathcal{W}_n$ and $\mathcal{W}_{n+1}$ or $\mathcal{W}_n(\xi; T)$ and $\mathcal{W}_{n+1}(\xi; T)$ respectively. As a by-product we obtain some modification of a well-known theorem on Tchebyshev polynomials, dealing with their rapidity of increase outside the interval $\langle -1, 1\rangle$.

2. Characteristic property of the optimal polynomial. Let us first recall some definitions used in the theory of approximation.

The *alternant* of the polynomial ω is the set of those points t of the interval I for which $|\xi(t)-\omega(t)| = \|\xi-\omega\|_I$.

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[4]) Several authors use a longer term: *polynomial of best approximation.*

The elements of the alternant are called the (e) *points*; an (e) point t is called a $(+)$ *point* if $\xi(t)-\omega(t) \geqslant 0$; if $\xi(t)-\omega(t) \leqslant 0$, it is called a $(-)$ *point*. Two (e) points are said to be *of the same kind* if they are both $(+)$ points or both $(-)$ points, otherwise they are said to be *of different kind.*

Each of the conditions (I)-(III) given below is necessary and sufficient for the involved polynomial ω_n to be optimal for the function ξ in the class $\mathcal{W}_n(\xi; T)$ and the interval I. We disregard in these conditions the trivial case $\xi \in \mathcal{W}_n$; then $\varepsilon_n(\xi) = \varepsilon_n(\xi, T) = 0$, and the optimal polynomial for ξ in the class $\mathcal{W}_n$ and $\mathcal{W}_n(\xi, T)$ is ξ itself. Apart from this case the nodes are not (e) points of any polynomial.

(I) In the interval I there exist (e) points $p_1 < p_2 < \ldots < p_l$ such that if

$$(1) \qquad \{s_1, s_2, \ldots, s_{l+m}\} = \{p_1, p_2, \ldots, p_l\} \cup T \qquad (s_1 < s_2 < \ldots < s_{l+m}),$$

$$p_1 = s_j, \qquad \delta = \xi - \omega_n, \qquad z_0 = (-1)^j \operatorname{sign} \delta(s_j),$$

$$z_i = \operatorname{sign} \delta(s_i) + z_{i-1}(\operatorname{sign}|\delta(s_i)| - 1) \qquad (i = 1, 2, \ldots, l+m),$$

then in the sequence $z_1, z_2, \ldots, z_{l+m}$ there are at least $n+1$ changes of sign.

(II) In the interval I there are (e) points $p_1 < p_2 < \ldots < p_{n-m+2}$ such that if s_k are defined by (1) with $l = n-m+2$, then $(-1)^i \operatorname{sign}\big(\xi(s_i) - \omega_n(s_i)\big) \geqslant 0$ for $i = 1, 2, \ldots, n+2$, or $(-1)^i \operatorname{sign}\big(\xi(s_i) - \omega_n(s_i)\big) \leqslant 0$ for $i = 1, 2, \ldots, n+2$.

(III) In the interval I there exist (e) points $p_1 < p_2 < \ldots < p_{n-m+2}$ such that the number of nodes lying between p_i and p_{i+1} $(i = 1, 2, \ldots, n-m+1)$ is odd if p_i and p_{i+1} are of the same kind, and even if they are of different kinds (in particular it may be equal to 0).

Condition (I) is proved in paper [5] (theorems 2 and 5). The implication (II) $\Rightarrow$ (I) is trivial, since with (II) being satisfied the sequence $z_1, z_2, \ldots, z_{n+2}$ in (I) consists of numbers equal to $+1$ and to -1 alternately. The equivalence of (II) and (III) is also obvious. Finally, the implication (I) $\Rightarrow$ (II) results from the definition of $z_1, z_2, \ldots, z_{l+m}$.

3. Notation. In the whole paper small Roman letters denote numbers, great Roman letters — set of points on the straight line or in the plane, small Greek letters denote functions, great written Roman letters denote classes of functions. Moreover

1° $x \in X$ denotes that x belongs to X, $x \bar{\in} X$ denotes that x is not in X;

2° $\bar{X}$ is the closure of the set X;

3° $X \cap Y$ and $X \cup Y$ denote the intersection and the union of the set X and Y, respectively;

4° $\{x_1, x_2, \ldots, x_n\}$ denotes the set composed of the elements $x_1, x_2, \ldots, x_n$;

5° $\underset{t}{E}\{\ \}$ denotes the set of points t satisfying the condition in $\{\ \}$;

6° $\langle a, b\rangle$ and (a, b) stand respectively for the closed and the open interval with the bounds a and b; semiclosed intervals are denoted by $\langle a, b)$ and $(a, b\rangle$;

7° $\xi = \eta$ means that the functions ξ and η are identical;

8° the norm of the function $\xi \epsilon \mathcal{C}$ on the closed set $F \subset I$ is defined as

$$\|\xi\|_F = \max_{t \epsilon F} |\xi(t)|;$$

9° $\deg \omega$ is the degree of the polynomial ω;

10° $\max\{x, y\}$ denotes the non-smaller of the numbers x, y; $\min\{x, y\}$ denotes the non-greater of the numbers x, y;

11° $[x]$ denotes the integer part of x.

CHAPTER I

Limit properties of approximation with nodes

4. Statement of the problem. In this chapter the notation adopted in § 1 and the hypotheses (1.1) are retained.

We shall compare the properties of the sequences $\{\varepsilon_n(\xi)\}$ and $\{\varepsilon_n(\xi; T)\}$ characterizing the accuracy of approximation of the function ξ by algebraic polynomials, without and with nodes respectively. Let us first recapitulate what is known about these sequences.

Weierstrass has proved that for every function $\xi \in \mathcal{C}$ we have

$$\lim_{n\to\infty} \varepsilon_n(\xi) = 0.$$

From a theorem of Yamabe [8], dealing with a more general problem it follows that for every $\xi \in \mathcal{C}$ and every system T of nodes

$$\lim_{n\to\infty} \varepsilon_n(\xi; T) = 0$$

holds. Hence, imposing at the nodes additional conditions on the polyno, mials approximating the function, we retain the possibility of arbitrarily precise ap proximation of every function by algebraic polynomials.

Since $\mathcal{W}_n(\xi; T) \subset \mathcal{W}_n$, the definitions (1.2) and (1.3) imply that for $n > m$ the inequality $\varepsilon_n(\xi) \leqslant \varepsilon_n(\xi; T)$ is satisfied for every function $\xi \in \mathcal{C}$ and every system T of nodes. It is also known [6] that there is such a constant $s > 0$ depending only on the basic interval I and on the set T of nodes that for every continuous function ξ and every $n > m$

$$\varepsilon_n(\xi; T) \leqslant s\varepsilon_n(\xi). \tag{1}$$

The last two inequalities give the estimation from above and from below of the terms of the sequence $\{\varepsilon_n(\xi; T)\}$.

We know many theorems dealing with the kind of the convergence of $\{\varepsilon_n(\xi)\}$ for several classes of continuous functions. By (1) these results may be transferred to approximation with nodes.

In §§ 5-7 we strengthen inequality (1) by various methods and present some related results. The principal results are the theorems 5.2, 6.3, 6.4, and 7.10.

5. Qualitative results. We shall prove

THEOREM 5.1. *If the system* $T = \{t_1, t_2, \ldots, t_m\}$ *of different points is contained in* $I = \langle a, b\rangle$, *then there exists a positive integer* $\nu = \nu(T)$ *such that for arbitrary numbers* $x_1, x_2, \ldots, x_m$ *there exists a polynomial* φ *satisfying the conditions*: 1° $\varphi \in \mathfrak{W}_\nu$, 2° $\varphi(t_k) = x_k$ *for* $k = 1, 2, \ldots, m$, 3° $\|\varphi\|_I = \max_k |x_k|$.

It is essential in this theorem, connected with a result of Wolibner [7], that ν is independent of $x_1, x_2, \ldots, x_m$.

Proof. If $x_1 = x_2 = \ldots = x_m = 0$, the polynomial $\varphi = 0$ satisfies the condition 1° for every positive integer ν, and the conditions 2°, 3°. Suppose now that $x_1, x_2, \ldots, x_m$ do not vanish simultaneously. Let us suppose, for simplicity, that $a < t_1 < t_2 < \ldots < t_m < b$ (from the theorem proved for this case follows its validity for the case $t_1 = a$ or $t_m = b$). Let us write $u_1 = a$,

$$u_{2k} = \frac{2t_k + t_{k+1}}{3}, \qquad u_{2k+1} = \frac{t_k + 2t_{k+1}}{3}$$

for $k = 1, 2, \ldots, m-1$ and $u_{2m} = b$. Hence from the ordering of numbers t_k's it follows that

$$u_1 < t_1 < u_2 < u_3 < t_2 < u_4 < \ldots < u_{2m-1} < t_m < u_{2m}.$$

Let $c_{k,l}$ $(k, l = 1, 2, \ldots, m)$ be such that $c_{k,l}^2 = 1$ (the signs will be fixed latter). Let L_k be the polygonal line with vertices

$$\left(u_1, \frac{c_{k,1}}{2m}\right), (t_1, 0), \left(u_2, \frac{c_{k,1}}{3m}\right), \left(u_3, \frac{c_{k,2}}{2m}\right), (t_2, 0), \left(u_4, \frac{c_{k,2}}{3m}\right), \ldots,$$

$$\left(u_{2k-1}, \frac{c_{k,k}}{2m}\right), (t_k, 1), \left(u_{2k}, \frac{c_{k,k}}{3m}\right), \ldots, \left(u_{2m-1}, \frac{c_{k,m}}{2m}\right), (t_m, 0), \left(u_{2m}, \frac{c_{k,m}}{3m}\right),$$

and let λ_k be the function defined in the interval $\langle u_1, u_{2m}\rangle = I$, whose graph is L_k.

The adjacent vertices of L_k have different ordinates, whence we may apply to the function λ_k the theorem of Wolibner of [7]. Let $\varphi_k = \varphi_k(c_{k,1}, c_{k,2}, \ldots, c_{k,m}; t_1, t_2, \ldots, t_m)$ be the polynomial existing in virtue of this theorem; φ_k decreases and increases together with λ_k, its graph passes through the vertices of L_k. Thus

$$\varphi_k(t_l) = \delta_{kl} \tag{1}$$

and in $\langle u_{2l-1}, u_{2l}\rangle$ the derivatives of the functions $\varphi_1, \ldots, \varphi_{l-1}, \varphi_l, \varphi_{l+1}, \ldots, \varphi_m$ change the sign only at t_l from $-c_{1,l}, \ldots, -c_{l-1,l}, 1, -c_{l+1,l}, \ldots, -c_{m,l}$ to $c_{1,l}, \ldots, c_{l-1,l}, -1, c_{l+1,l}, \ldots, c_{m,l}$ respectively.

Put

$$\varphi = \sum_{k=1}^{m} x_k \varphi_k$$

where $x_1, x_2, \ldots, x_m$ do not vanish simultaneously. If $x_k = 0$ for a k, then the definition of $c_{k,1}, c_{k,2}, \ldots, c_{k,m}$ does not affect the form of φ; let us set, for example, $c_{k,1} = c_{k,2} = \ldots = c_{k,m} = 1$. Now let $x_l \neq 0$. Setting $c_{k,l} = -\operatorname{sign} x_k x_l$ for $x_k \neq 0$ (for $x_k = 0$ the number $c_{k,l}$ is already defined) we see that the derivatives

$$x_1\varphi_1', \ldots, x_{l-1}\varphi_{l-1}', x_l\varphi_l', x_{l+1}\varphi_{l+1}', \ldots, x_m\varphi_m'$$

of the terms of the sum $\sum_{k=1}^{m} x_k \varphi_k$, change the sign in $\langle u_{2l-1}, u_{2l}\rangle$ only at t_l from

$$x_1^2 \operatorname{sign} x_l, \ldots, x_{l-1}^2 \operatorname{sign} x_l, \operatorname{sign} x_l, x_{l+1}^2 \operatorname{sign} x_l, \ldots, x_m^2 \operatorname{sign} x_l$$

to

$$-x_1^2 \operatorname{sign} x_l, \ldots, -x_{l-1}^2 \operatorname{sign} x_l, -\operatorname{sign} x_l, -x_{l+1}^2 \operatorname{sign} x_l, \ldots, -x_m^2 \operatorname{sign} x_l$$

respectively. Hence the polynomial φ constructed for the formerly defined numbers $c_{k,l}$'s increases in the left-hand neighbourhood of t_l to x_l, and decreases in the right-hand neighbourhood if $x_l > 0$; if $x_l < 0$ it behaves in the opposite manner.

We shall prove now that φ satisfies all the desired conditions. Indeed, 2° follows from (1). Now we prove 1° and 3°. Let us consider φ in $\langle u_{2l-1}, u_{2l}\rangle$. As has been proved φ may attain its extrema in this interval only at the points u_{2l-1}, t_l, u_{2l}. Since

$$|\varphi(u_{2l-1})| = \Big|\sum_{k=1}^{m} x_k \frac{c_{k,l}}{2m}\Big| \leqslant \frac{1}{2m}\sum_{k=1}^{m}|x_k| < \max_k |x_k|,$$

$$|\varphi(u_{2l})| = \Big|\sum_{k=1}^{m} x_k \frac{c_{k,l}}{3m}\Big| \leqslant \frac{1}{3m}\sum_{k=1}^{m}|x_k| < \max_k |x_k|,$$

we see that for $x_l \neq 0$

$$\|\varphi\|_{\langle u_{2l-1}, u_{2l}\rangle} \leqslant \max_k |x_k|. \tag{2}$$

In $\langle u_2, u_3\rangle, \langle u_4, u_5\rangle, \ldots, \langle u_{2m-2}, u_{2m-1}\rangle$ and in those intervals $\langle u_{2l-1}, u_{2l}\rangle$ for which $x_l = 0$ we have

$$|x_k \varphi_k(t)| \leqslant \frac{1}{2m}|x_k|,$$

since $|x_l\varphi_l(t)| = 0$ and $|\varphi_k(t)| \leqslant 1/2m$ for $k \neq l$. It follows that

(3) $$|\varphi(t)| \leqslant \frac{1}{2m}\sum_{k=1}^{m}|x_k| < \max_k |x_k|$$

in these intervals. By (2) and (3), φ satisfies condition 3°. The polynomials φ_k depend on $x_1, x_2, \ldots, x_m$ only through the parametres $c_{k,1}, c_{k,2}, \ldots, c_{k,m}$, each of which can assume only two values (1 or -1). Hence for an arbitrary system $x_1, x_2, \ldots, x_m$ of values

$$\deg\varphi \leqslant \max_k \max_{c_{k,l}^2=1} \deg\varphi_k(c_{k,1}, c_{k,2}, \ldots, c_{k,m}, t_1, t_2, \ldots, t_m)$$

and the quantity on the right-hand side may be taken as $\nu(T)$. Then for every system $x_1, x_2, \ldots, x_m$ of numbers we have $\varphi \in \mathcal{W}_\nu$.

THEOREM 5.2. *Let $\nu(T)$ be the number defined by theorem 5.1, the system T of nodes being arbitrary. If $\xi \in \mathcal{C}$ and $n \geqslant \nu(T)$, then*

(4) $$\varepsilon_n(\xi; T) \leqslant 2\varepsilon_n(\xi).$$

Proof. Let ψ_n be the optimal polynomial for the function ξ in the class $\mathcal{W}_n$ and in the interval I. Thus,

(5) $$\|\xi - \psi_n\|_I = \varepsilon_n(\xi).$$

Applying theorem 5.1 to the nodes $t_1, t_2, \ldots, t_m$ and numbers

$$x_1 = \xi(t_1) - \psi_n(t_1), \quad x_2 = \xi(t_2) - \psi_n(t_2), \quad \ldots, \quad x_m = \xi(t_m) - \psi_n(t_m)$$

we see that there exists a polynomial φ such that

(6) $$\varphi(t_k) = \xi(t_k) - \psi_n(t_k) \quad \text{for} \quad k = 1, 2, \ldots, m,$$

(7) $$\|\varphi\|_I = \max_k |\xi(t_k) - \psi_n(t_k)|.$$

By (5), (7) and the first relation of (1.1)

(8) $$\|\xi - (\psi_n + \varphi)\|_I \leqslant \varepsilon_n(\xi) + \max_k |\xi(t_k) - \psi_n(t_k)| \leqslant 2\varepsilon_n(\xi).$$

Since $\deg(\psi_n + \varphi) \leqslant \max\{n, \nu(T)\}$, we have virtue of (6)

$$\psi_n + \varphi \in \mathcal{W}_n(\xi; T) \quad \text{for} \quad n \geqslant \nu(T),$$

and by formula (8)

$$\varepsilon_n(\xi; T) \leqslant \|\xi-(\psi_n+\varphi)\|_I \leqslant 2\varepsilon_n(\xi),$$

q. e. d.

6. Quantitative results. From the theorem of Wolibner, fundamental for the proof of theorem 5.2, we cannot deduce for what n the inequality (5.4) is valid, *i. e.* we cannot estimate $\nu(T)$ from above. Only for $m = 1$ (as $T = \{t_1\}$) it is obvious that (5.4) holds for every n since

$$\psi_n+\big(\xi(t_1)-\psi_n(t_1)\big)\in \mathcal{W}_n\big(\xi;\{t_1\}\big)$$

and

$$\varepsilon_n\big(\xi;\{t_1\}\big) \leqslant \big\|\xi-\psi_n-\big(\xi(t_1)-\psi_n(t_1)\big)\big\|_I \leqslant \|\xi-\psi_n\|_I+|\xi(t_1)-\psi_n(t_1)| \leqslant 2\varepsilon_n(\xi).$$

Hence we suppose in this § that T contains at least two nodes.

Theorems 6.3 and 6.4 give some quantitative results dealing, however, with inequalities weaker than (5.4): the constant 2 is replaced by a greater one.

THEOREM 6.1. *Let*

$$(1)\qquad v = \begin{vmatrix} 1 & r_{1,2} & r_{1,3} & \dots & r_{1,m} \\ r_{2,1} & 1 & r_{2,3} & \dots & r_{2,m} \\ r_{3,1} & r_{3,2} & 1 & \dots & r_{3,m} \\ \dots & \dots & \dots & \dots & \dots \\ r_{m,1} & r_{m,2} & r_{m,3} & \dots & 1 \end{vmatrix},$$

$$(2)\qquad w = \begin{vmatrix} r_{1,1} & r_{1,2} & r_{1,3} & \dots & r_{1,m} \\ r_{2,1} & 1 & r_{2,3} & \dots & r_{2,m} \\ r_{3,1} & r_{3,2} & 1 & \dots & r_{3,m} \\ \dots & \dots & \dots & \dots & \dots \\ r_{m,1} & r_{m,2} & r_{m,3} & \dots & 1 \end{vmatrix},$$

$$(3)\qquad \begin{cases} \alpha_m = f_m\binom{m}{0}r^m+f_{m-1}\binom{m}{1}r^{m-1}+\dots+f_1\binom{m}{m-1}r, \\ \beta_m = \dfrac{1}{m}\left(f_{m+1}\binom{m}{0}r^m+f_m\binom{m}{1}r^{m-1}+\dots+f_2\binom{m}{m-1}r\right), \end{cases}$$

where

$$(4)\qquad f_k = k!\left(1-\frac{1}{1!}+\frac{1}{2!}-\dots+(-1)^k\frac{1}{k!}\right) \qquad (k = 1, 2, \dots, m).$$

If $|r_{k,l}| \leqslant r$ *for* $k, l = 1, 2, \ldots, m$, *then*

(5) $$1-a_m(r) \leqslant v \leqslant 1+a_m(r), \qquad |w| \leqslant \beta_m(r)\,[5].$$

Proof. To estimate the determinant v from below we develop it into the sum of products of its elements and replace all signs of the products by $-$, except the sign $+$ of the product 1 of the principal diagonal, and we replace all $r_{k,l}$ by r. This gives $v \geqslant 1-a_m(r)$ where $1-a_m$ is the polynomial of degree m arising from the determinant v by the described procedure. Then we change in the development of v all the signs to $+$ and replace $r_{k,l}$ by r; this gives $v \leqslant 1+a_m(r)$, a_m being the same as above.

To obtain an upper bound for the determinant w we change all signs in its development to $+$ and replace $r_{k,l}$ by r. On the other hand, to obtain a lower bound for this determinant we change signs in its development to $-$ and replace $r_{k,l}$ by r. These estimations give jointly $-\beta_m(r) \leqslant w \leqslant \beta_m(r)$ where β_m is a polynomial of degree m.

We shall obtain recurrence formulae for the polynomials a_m and β_m which will enable us to prove (3); to do this we vary the procedure leading to the estimation (5). Developing v and w by the first line we get

(6) $$v = v' - (r_{1,2}w_2 + r_{1,3}w_3 + \ldots + r_{1,m}w_m),$$

(7) $$w = r_{1,1}v' - (r_{1,2}w_2 + r_{1,3}w_3 + \ldots + r_{1,m}w_m),$$

where the determinants $v', w_2, w_3, \ldots, w_m$ are of order $m-1$, v' is of the same type as v (the principal diagonal consists of 1's, other terms are the $r_{k,l}$'s), and $w_2, w_3, \ldots, w_m$ are of the same type as w (the principal diagonal, except the first element, consits of 1's, the other elements are $r_{k,l}$'s). We may obtain the estimation from above for v by the bound $1+a_m(r)$ replacing $r_{1,2}, r_{1,3}, \ldots, r_{1,m}$ in formula (6) by r, writing for v'

[5]) Several papers have appeared recently, dealing with the estimation of determinants of matrices with *dominant diagonal*. We quote them in chronological order: E. B. Price, ***Bounds for determinants with dominant principal diagonal***, Proc. Amer. Math. Soc. 2 (1951), p. 497-502, A. Ostrovsky, ***Note on bounds for determinants with dominant principal diagonal***, ibidem 3 (1952), p. 26-30, J. L. Brenner, ***Une borne pour un déterminant avec diagonale majorante***, C. R. de l'Acad. de Sci. 238 (1954), p. 555-556, J. L. Brenner, ***A bound for a determinant with dominant main diagonal***, Proc. Amer. Math. Soc. 5 (1954), p. 631-634, Д. М. Котелянский, ***Оценки для определителей матриц с преобладающей главной диагональю***, Известия Акад. Наук СССР, сер. мат. 20 (1956), p. 137-144.

The strongest estimation (of Brenner) gives for the determinant (1)

$$\prod_{k=1}^{m-1} \left(1-(m-1)kr^2\right)$$

as lower bound; our bound (5) is better.

its upper bound $1+\alpha_{m-1}(r)$ and estimating the determinants $w_2, w_3, \ldots, w_m$ by their lower bound $-\beta_{m-1}(r)$. Hence

(8) $$1+\alpha_m = 1+\alpha_{m-1}+(m-1)r\beta_{m-1}.$$

An analogous procedure for the upper bound of w, starting from formula (7), gives

(9) $$\beta_m = r\big(1+\alpha_{m-1}+(m-1)\beta_{m-1}\big).$$

Now, $m = 1$ implies $v = 1$, $w = r_{1,1}$, whence $\alpha_1 = 0$, $\beta_1 = r$ and the formulae (3) are satisfied for $m = 1$. Thus, it suffices to verify that for $m = 1, 2, \ldots$ the polynomials α_m and β_m defined by (3) satisfy the recurrence formulae resulting from (8) and (9):

(10) $$\alpha_{m+1} = \alpha_m+mr\beta_m, \qquad \beta_{m+1} = r(1+\alpha_m+m\beta_m).$$

By computation

$$\alpha_m+mr\beta_m = f_m\binom{m}{0}r^m+\ldots+f_2\binom{m}{m-2}r^2+f_1\binom{m}{m-1}r+$$
$$+f_{m+1}\binom{m}{0}r^{m+1}+f_m\binom{m}{1}r^m+\ldots+f_2\binom{m}{m-1}r^2$$
$$= f_{m+1}\binom{m+1}{0}r^{m+1}+f_m\binom{m+1}{1}r^m+\ldots+f_2\binom{m+1}{m-1}r^2+f_1\binom{m+1}{m}r = \alpha_{m+1}$$

(we have used the fact that, according to (4), $f_1 = 0$). Testing the second of the formulae (10) we get by computation

$$1+\alpha_m+m\beta_m = f_m\binom{m}{0}r^m+f_{m-1}\binom{m}{1}r^{m-1}+\ldots+f_1\binom{m}{m-1}r+1+$$
$$+f_{m+1}\binom{m}{0}r^m+f_m\binom{m}{1}r^{m-1}+\ldots+f_2\binom{m}{m-1}r+f_1\binom{m}{m}$$
$$= (f_m+f_{m+1})\binom{m}{0}r^m+(f_{m-1}+f_m)\binom{m}{1}r^{m-1}+\ldots+(f_1+f_2)\binom{m}{m-1}r+1.$$

By definition (4) we have, for $k = 1, 2, \ldots, m$,

$$f_k+f_{k+1} = k!\left(1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-1)^k\frac{1}{k!}\right)+$$
$$+(k+1)!\left(1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-1)^k\frac{1}{k!}+(-1)^{k+1}\frac{1}{(k+1)!}\right)$$

$$= \big(k!+(k+1)!\big)\left(1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-1)^{k+1}\frac{1}{(k+1)!}+(-1)^{k+2}\frac{1}{(k+2)!}\right)-$$

$$-k!\left((-1)^{k+1}\frac{1}{(k+1)!}+(-1)^{k+2}\frac{1}{(k+2)!}\right)-(k+1)!(-1)^{k+2}\frac{1}{(k+2)!}$$

$$= \frac{k!+(k+1)!}{(k+2)!}f_{k+2}+(-1)^k\left(\frac{1}{k+1}-\frac{1}{(k+1)(k+2)}-\frac{1}{k+2}\right)=\frac{f_{k+2}}{k+1}$$

and $1 = f_2$, whence using the properties of the binomial coefficients

$$r(1+a_m+m\beta_m)$$

$$= \frac{f_{m+2}}{m+1}\binom{m}{0}r^{m+1}+\frac{f_{m+1}}{m}\binom{m}{1}r^m+\ldots+\frac{f_3}{2}\binom{m}{m-1}r^2+\frac{f_2}{1}\binom{m}{m}r$$

$$= \frac{1}{m+1}\left(f_{m+2}\binom{m+1}{0}r^{m+1}+f_{m+1}\binom{m+1}{1}r^m+\ldots+f_2\binom{m+1}{m}r\right)=\beta_{m+1},$$

q. e. d.

Now we introduce new notation. $T = \{t_1, t_2, \ldots, t_m\}$ with $t_1 < t_2 < \ldots < t_m$ being a system of nodes in the interval $I = \langle a, b\rangle$, we set

$$c = \tfrac{1}{2}\min_{1\leqslant k\leqslant m-1}(t_{k+1}-t_k), \qquad d = \max_{1\leqslant k\leqslant m}\max\{t_k-a,\, b-t_k\}.$$

By r_m we denote the positive root of equation $a_m(r) = 1$, where m denotes the number of nodes of T and a_m is defined by the first formula of (3). There follows $a_m(0) = 0$, $a_m'(r) > 0$ for $r > 0$, whence r_m is uniquely determined.

THEOREM 6.2. *If*

$$n > 2\log\frac{2}{r_m}\Big/\log\frac{d+c}{d-c}, \tag{11}$$

then for every system $x_1, x_2, \ldots, x_m$ of numbers there exists in the class $\mathcal{W}_n$ a polynomial φ such that

$$\varphi(t_k) = x_k \quad \textit{for} \quad k = 1, 2, \ldots, m,$$

$$\|\varphi\|_I \leqslant \frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-a_m(r)\big)}\max_k|x_k|, \tag{12}$$

where

$$r = 2\left(\frac{d-c}{d+c}\right)^{n/2}. \tag{13}$$

Let us notice that in virtue of (3) the function

$$\frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\sigma_m(r)\big)}$$

involved in the inequality (12) is greater than 1 for $r > 0$ and decreases to 1 as $r \to 0$, *i. e.* as $n \to \infty$. This means that theorem 6.2 is weaker than 5.1, but asymptotically (as $n \to \infty$) they are of equal efficacy.

Proof. Let T_l be the lth Tchebyshev polynomial: $T_l = \cos(l \arccos u)$, $-1 \leqslant u \leqslant 1$ (to retain the traditional notation, we give up the principle, assumed in this paper, of denoting fúnctions by small Greek letters). It is known ([1], p. 303) that for $0 < z < 1$ the polynomial of degree $2l$ in u, defined as

$$\tau_l(u;z) = T_l\left(\frac{2u^2-z^2-1}{z^2-1}\right),$$

satisfies the relations

$$(14)\qquad |\tau_l(u;z)| \begin{cases} \leqslant 1 & \text{for} \quad u \in \langle -1, -z\rangle \quad \text{and for} \quad u \in \langle z, 1\rangle, \\ > 1 & \text{for} \quad u \in (-z, z), \end{cases}$$

$$(15)\qquad \max_{|u| \leqslant z} \tau_l(u;z) = \tau_l(0;z) = T_l\left(\frac{1+z^2}{1-z^2}\right).$$

As $(1+z^2)/(1-z^2) > 1$, we can determine the quantity in (15) from the formula

$$T_l(u) = \tfrac{1}{2}\big\{(u+\sqrt{u^2-1})^l + (u-\sqrt{u^2-1})^l\big\}$$
$$= \tfrac{1}{2}\big\{(u+\sqrt{u^2-1})^l + (u+\sqrt{u^2-1})^{-l}\big\}$$

([4], p. 72). We have

$$\frac{1+z^2}{1-z^2} + \sqrt{\left(\frac{1+z^2}{1-z^2}\right)^2 - 1} = \frac{1+z^2+2z}{1-z^2} = \frac{1+z}{1-z},$$

$$T_l\left(\frac{1+z^2}{1-z^2}\right) = \frac{1}{2}\left\{\left(\frac{1+z}{1-z}\right)^l + \left(\frac{1-z}{1+z}\right)^l\right\},$$

whence

$$(16)\qquad \tau_l(0;z) \geqslant \frac{1}{2}\left(\frac{1+z}{1-z}\right)^l,$$

and $\lim_{l\to\infty} \tau_l(0;z) = +\infty$ for every $z \in (0,1)$. Let

$$I_k = \langle t_k - c, t_k + c\rangle, \qquad u = \frac{t-t_k}{d}.$$

Hence $t \in I_k$ if and only if $u \in \langle -c/d,\ c/d \rangle$, and by definition of the quantity d, $t \in I = \langle a, b \rangle$ implies $|u| \leqslant 1$. Hence for the polynomial

$$\varphi_{l,k}(t) = r\tau_l\left(\frac{t-t_k}{d};\ \frac{c}{d}\right) \quad \text{with} \quad r = \left(\tau_l\left(0;\ \frac{c}{d}\right)\right)^{-1}$$

inequalities (14) give

$$|\varphi_{l,k}(t)| \leqslant r \quad \text{for} \quad t \in I - I_k, \tag{17}$$

$$r \leqslant \varphi_{l,k}(t) \leqslant 1 \quad \text{for} \quad t \in I_k. \tag{18}$$

For $x_1, x_2, \ldots, x_m$ fixed, the coefficients $y_1, y_2, \ldots, y_m$ of the linear combination

$$\varphi_l = \sum_{k=1}^{m} y_k \varphi_{l,k}$$

are defined by the equations

$$\varphi_l(t_j) = \sum_{k=1}^{m} y_k \varphi_{l,k}(t_j) = x_j \qquad (j = 1, 2, \ldots, m). \tag{19}$$

Since $\varphi_{l,k}(t_k) = 1$, the determinant of the system (19) is equal to

$$v = \begin{vmatrix} 1 & \varphi_{l,2}(t_1) & \ldots & \varphi_{l,m}(t_1) \\ \varphi_{l,1}(t_2) & 1 & \ldots & \varphi_{l,m}(t_2) \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_{l,1}(t_m) & \varphi_{l,2}(t_m) & \ldots & 1 \end{vmatrix}. \tag{20}$$

By the definition of c, in this determinant $|t_j - t_k| > c$ for $j \neq k$, whence t_j is in $I - I_k$ and the inequality (17) implies $|\varphi_{l,k}(t_j)| \leqslant r$ for $j \neq k$. By (16) and definition of r we get $r \leqslant 2\big((d-c)/(d+c)\big)^l$, whence $r \to 0$ as $l \to \infty$. Therefore we suppose that l is so large that the lower bound $1 - a_m(r)$ of the determinant v is positive, *i. e.* that $a_m(r) < 1$, or, using the notation introduced previously, that $r < r_m$. Then $v \neq 0$ and the system (19) has precisely one solution $y_1, y_2, \ldots, y_m$. Determining this solution by Cramer's formulae we obtain

$$y_k = \sum_{j=1}^{m} g_{kj} x_j \qquad (k = 1, 2, \ldots, m). \tag{21}$$

Every coefficient g_{kj} is the ratio of two determinants: the numerator is a determinant of order $m-1$, of type (1) if $k = j$ and of type (2) if $k \neq j$. The denominator is the determinant (20). The bounds of theorem 6.1 (with m replaced by $m-1$, if necessary) lead to

$$|g_{k,j}| \leqslant \frac{\beta_{m-1}(r)}{1-\alpha_m(r)} \quad \text{for} \quad k \neq j, \tag{22}$$

$$\frac{1-\alpha_{m-1}(r)}{1+\alpha_m(r)} \leqslant g_{k,k} \leqslant \frac{1+\alpha_{m-1}(r)}{1-\alpha_m(r)},$$

whence, in virtue of the inequality $|1-\alpha_{m-1}(r)| \leqslant 1+\alpha_{m-1}(r)$, we get

$$|g_{k,k}| \leqslant \frac{1+\alpha_{m-1}(r)}{1-\alpha_m(r)}. \tag{23}$$

Setting

$$x = \max_j |x_j|$$

and applying (22) and (23) we obtain for the quantity (21)

$$|y_k| \leqslant \frac{(m-1)\beta_{m-1}(r)+1+\alpha_{m-1}(r)}{1-\alpha_m(r)}\, x.$$

Formula (9) may be written in the form $(m-1)\beta_{m-1}+1+\alpha_{m-1} = \beta_m/r$, whence

$$|y_k| \leqslant \frac{\beta_m(r)}{r\big(1-\alpha_m(r)\big)}\, x. \tag{24}$$

Now we determine a bound for the function φ_l in the intervals $I_1, I_2, \ldots, I_m$, which, by the definition of number c, do not overlap. For $t \in I_j$ it follows from (18) and (17) that

$$|\varphi_l(t)| = \Big|\sum_{k=1}^{m} y_k \varphi_{l,k}(t)\Big| \leqslant |y_j| + r \sum_{\substack{k=1\\k\neq j}}^{m} |y_k|,$$

whence, by (24),

$$|\varphi_l(t)| \leqslant \frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\alpha_m(r)\big)}\, x \quad \text{for} \quad t \in \bigcup_{j=1}^{m} I_j. \tag{25}$$

In the rest of the interval I, *i. e.* in the set

$$I - \bigcup_{j=1}^{m} I_j,$$

all the functions $\varphi_{l,k}$ satisfy inequality (17), whence

$$|\varphi_{l,k}(t)| = \Big|\sum_{k=1}^{m} y_k \varphi_{l,k}(t)\Big| \leqslant r \sum_{k=1}^{m} |y_k| \leqslant \frac{m\beta_m(r)}{1-\alpha_m(r)}\, x.$$

This estimation is stronger than (25), for $r < 1$, and therefore inequality (12) follows finally from (25). This inequality is satisfied in particular when r is equal to $2\big((d-c)/(d+c)\big)^l$, *i. e.*, when it satisfies (13) where $n \doteq 2l$ is the degree of the polynomial φ_l. Condition (11) results from the previously postulated inequality $r < r_m$ and from (13). Thus we have proved that the polynomial φ_l satisfies the conditions of our theorem.

THEOREM 6.3. *If*

$$n > \left(2\log\frac{2}{r_m}\right) \Big/ \log\frac{d+c}{d-c},$$

then every function ξ of $\mathcal{C}$ satisfies the inequality

$$\varepsilon_n(\xi;T) \leqslant \left\{1+\frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\alpha_m(r)\big)}\right\}\varepsilon_n(\xi)$$

where $r = 2\big((d-c)/(d+c)\big)^{n/2}$.

Proof. Let ψ_n be the optimal polynomial for the function ξ in the class $\mathcal{W}_n$ and in the interval I, *i. e.* $\|\xi-\psi_n\|_I = \varepsilon_n(\xi)$. By theorem 6.2, for n satisfying the hypothesis of the theorem, there exists a polynomial $\varphi \in \mathcal{W}_n$ such that $\varphi(t_k) = \xi(t_k)-\psi_n(t_k)$ for $k = 1, 2, \ldots, m$ and

$$\|\varphi\|_I \leqslant \frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\alpha_m(r)\big)}\max_k|\xi(t_k)-\psi_n(t_k)| \leqslant \frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\alpha_m(r)\big)}\varepsilon_n(\xi).$$

There follows $\varphi+\psi_n \in \mathcal{W}_n(\xi;T)$,

$$\varepsilon_n(\xi;T) \leqslant \|\xi-(\varphi+\psi_n)\|_I \leqslant \varepsilon_n(\xi)+\frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\alpha_m(r)\big)}\varepsilon_n(\xi),$$

q. e. d.

THEOREM 6.4. *If* $n > p/c$ *where*

(26) $$p = \min\left\{6(b-a), \frac{m-1}{\pi}\big(2d-(m-1)\big)c\right\},$$

$$c = \tfrac{1}{2}\min_{1\leqslant k\leqslant m-1}(t_{k+1}-t_k), \qquad d = \max_{1\leqslant k\leqslant m}\max\{t_k-a, b-t_k\},$$

then for every continuous function ξ the inequality $\varepsilon_n(\xi;T) \leqslant 2\varepsilon_n(\xi)/(1-p/cn)$ holds.

Proof. The inequality to be proved is trivial when $\varepsilon_n(\xi) = 0$. If $\varepsilon_n(\xi) > 0$ let us write $s_\xi = \varepsilon_n(\xi;T)/\varepsilon_n(\xi)$. We must prove that $n > p/c$ implies

(27) $$s_\xi \leqslant \frac{2}{1-p/cn}.$$

Let ψ_n be the optimal polynomial for the function ξ in the class $\mathcal{W}_n$ and let $\delta = \xi-\psi_n$. Let λ_ξ be the function defined in $I = \langle a, b\rangle$, whose graph is the polygonal line with vertices

(28) $$\big(a, \delta(t_1)\big),\ \big(t_1, \delta(t_1)\big),\ \big(t_2, \delta(t_2)\big),\ \ldots,\ \big(t_m, \delta(t_m)\big),\ \big(b, \delta(t_m)\big).$$

In particular

(29) $$\lambda_\xi(t_k) = \delta(t_k) \qquad (k = 1, 2, \ldots, m),$$

(30) $$\|\lambda_\xi\|_I = \max_k |\delta(t_k)| \leqslant \varepsilon_n(\xi).$$

Now we estimate the deviation $\varepsilon_n(\lambda_\xi)$ in two ways.

1. By a theorem of Jackson ([3] or [4], p. 161)

$$\varepsilon_n(\lambda_\xi) \leqslant 12\omega\left(\frac{b-a}{2n}\right),$$

where ω is the modulus of continuity of λ_ξ. By its definition

$$\omega(h) \leqslant \max_{1\leqslant k\leqslant m-1} \frac{|\lambda_\xi(t_{k+1})-\lambda_\xi(t_k)|}{t_{k+1}-t_k}\, h.$$

By (29), (30), and by the definition of the quantity c

(31) $$\frac{|\lambda_\xi(t_{k+1})-\lambda_\xi(t_k)|}{t_{k+1}-t_k} = \frac{|\delta(t_{k+1})-\delta(t_k)|}{t_{k+1}-t_k} \leqslant \frac{2\varepsilon_n(\xi)}{2c},$$

whence $\omega(h) \leqslant hc^{-1}\varepsilon_n(\xi)$ and

(32) $$\varepsilon_n(\lambda_\xi) \leqslant \frac{6(b-a)}{cn}\varepsilon_n(\xi).$$

2. The function λ_ξ may be represented in the form

(33) $$\lambda_\xi = \frac{1}{2}\cdot\frac{\delta(t_2)-\delta(t_1)}{t_2-t_1}|t-t_1| +$$

$$+\frac{1}{2}\sum_{i=2}^{m-1}\left(\frac{\delta(t_{i+1})-\delta(t_i)}{t_{i+1}-t_i} - \frac{\delta(t_i)-\delta(t_{i-1})}{t_i-t_{i-1}}\right)|t-t_i| -$$

$$-\frac{1}{2}\cdot\frac{\delta(t_m)-\delta(t_{m-1})}{t_m-t_{m-1}}|t-t_m| -$$

$$-\frac{1}{2}\sum_{i=2}^{m-1}\left(\frac{\delta(t_{i+1})-\delta(t_i)}{t_{i+1}-t_i} - \frac{\delta(t_i)-\delta(t_{i-1})}{t_i-t_{i-1}}\right)|t_1-t_i| +$$

$$+\frac{1}{2}\cdot\frac{\delta(t_m)-\delta(t_{m-1})}{t_m-t_{m-1}}|t_1-t_m| + \delta(t_1).$$

Indeed, it is easy to see that 1° the graph of the function (33) is a polygonal line whose vertices have the abcissae $t_1, t_2, \ldots, t_m$, 2° $\lambda_\xi(t_1) = \delta(t_1)$ by (33), 3° the sum of the coefficients at $|t-t_1|, |t-t_2|, \ldots, |t-t_m|$ is equal to zero. From 3° it follows that for $t \leqslant t_1 < t_2 < \ldots < t_m$ and for $t \geqslant t_m > \ldots > t_2 > t_1$ the function λ_ξ defined by (33) does not depend on the variable t, whence $\lambda_\xi(a) = \lambda_\xi(t_1)$, $\lambda_\xi(b) = \lambda_\xi(t_m)$. We must verify that this function satisfies equalities (29) for $k = 2, \ldots, m$. It suffices to prove that in the interval (t_k, t_{k+1}) where $k = 1, 2, \ldots, m-1$ the derivative of function (33) is equal to $(\delta(t_{k+1})-\delta(t_k))/(t_{k+1}-t_k)$, *i. e.* equal in virtue of (28), to the derivative of λ_ξ. If $t_k < t < t_{k+1}$, then $|t-t_1|' = \ldots = |t-t_k|' = 1$, $|t-t_{k+1}|' = \ldots = |t-t_m|' = -1$, and the derivative is equal to

$$\frac{1}{2}\cdot\frac{\delta(t_2)-\delta(t_1)}{t_2-t_1}+\frac{1}{2}\sum_{i=2}^{k}\left(\frac{\delta(t_{i+1})-\delta(t_i)}{t_{i+1}-t_i}-\frac{\delta(t_i)-\delta(t_{i-1})}{t_i-t_{i-1}}\right)-$$
$$-\frac{1}{2}\sum_{i=k+1}^{m-1}\left(\frac{\delta(t_{i+1})-\delta(t_i)}{t_{i+1}-t_i}-\frac{\delta(t_i)-\delta(t_{i-1})}{t_i-t_{i-1}}\right)+\frac{1}{2}\cdot\frac{\delta(t_m)-\delta(t_{m-1})}{t_m-t_{m-1}}.$$

The sum of the terms of each line of the above expression is equal to $\big(\delta(t_{k+1})-\delta(t_k)\big)/2(t_{k+1}-t_k)$, q. e. d.

Bernstein has proved that if the interval of approximation is $\langle -1, 1\rangle$, then $\varepsilon_n(|t|) \leqslant 2/\pi n$ (see [2] or [4], p. 187); it follows that for $I = \langle a, b\rangle$ as the interval of approximation we have

$$\varepsilon_n(|t|) \leqslant \frac{2}{\pi n}\max\{|a|, |b|\}.$$

The approximation to the function $|t-t_k|$ in the interval $\langle a, b\rangle$ is equivalent to the approximation of the function $|t|$ in the interval $\langle a-t_k, b-t_k\rangle$. As $a \leqslant t_k \leqslant b$,

$$\varepsilon_n(|t-t_k|) \leqslant \frac{2}{\pi n}\max\{t_k-a, b-t_k\}. \tag{34}$$

It is known that for arbitrary continuous functions $\xi_1, \xi_2, \ldots, \xi_l$ and arbitrary values $x_1, x_2, \ldots, x_l$

$$\varepsilon_n(x_1\xi_1+x_2\xi_2+\ldots+x_l\xi_l) \leqslant |x_1|\varepsilon_n(\xi_1)+|x_2|\varepsilon_n(\xi_2)+\ldots+|x_l|\varepsilon_n(\xi_l), \tag{35}$$

and if η is constant, then $\varepsilon_n(\eta) = 0$. Taking into account (31) and (33)-(35), we get

$$\varepsilon_n(\lambda_\xi) \leqslant \frac{2\varepsilon_n(\xi)}{c\pi n}\left(\frac{1}{2}\max\{t_1-a, b-t_1\}+\sum_{i=2}^{m-1}\max\{t_i-a, b-t_i\}+\right.$$
$$\left.+\frac{1}{2}\max\{t_m-a, b-t_m\}\right).$$

We have

$$\max\{t_1-a,\, b-t_1\} \leqslant d, \qquad \max\{t_m-a,\, b-t_m\} \leqslant d,$$
$$\max\{t_i-a,\, b-t_i\} = \max\{t_m-a-(t_m-t_i),\, b-t_1-(t_i-t_1)\}$$

for $i = 2, \ldots, m-1$.

By the definition of c and d: $|t_k-t_l| \geqslant 2|k-l|c$, whence

$$t_m-a-(t_m-t_i) \leqslant d-2(m-i)c, \qquad b-t_1-(t_i-t_1) \leqslant d-2(i-1)c,$$
$$\max\{t_i-a,\, b-t_i\} \leqslant d-2c\min\{m-i,\, i-1\}.$$

Therefore

$$\varepsilon_n(\lambda_\xi) \leqslant \frac{2\varepsilon_n(\xi)}{c\pi n}\Big(d+(m-2)d-2c\sum_{i=2}^{m-1}\min\{m-i,\, i-1\}\Big).$$

If m is even, then

$$\sum_{i=2}^{m-1}\min\{m-i,\, i-1\} = 2\Big(1+2+\ldots+\frac{m-2}{2}\Big) = \frac{m(m-2)}{4} < \frac{(m-1)^2}{4}.$$

If m is odd (whence, by the hypothesis of this §, greater than 1), then

$$\sum_{i=2}^{m-1}\min\{m-i,\, i-1\} = 2\Big(1+2+\ldots+\frac{m-3}{2}\Big)+\frac{m-1}{2} = \frac{(m-1)^2}{4}.$$

In both cases

$$\varepsilon_n(\lambda_\xi) \leqslant \frac{\varepsilon_n(\xi)}{c\pi n}(m-1)\big(2d-(m-1)c\big).$$

Hence introducing the quantity p we obtain from (32)

$$\varepsilon_n(\lambda_\xi) \leqslant \frac{p}{cn}\varepsilon_n(\xi), \qquad i.\ e. \qquad \varepsilon_n(\lambda_\xi;\, T) \leqslant \frac{p}{cn}\, s_{\lambda_\xi}\varepsilon_n(\xi)$$

(since λ_ξ is not a polynomial, $\varepsilon_n(\lambda_\xi) > 0$ and the quantity s_{λ_ξ} is well defined).

If χ is the optimal polynomial for the function λ_ξ in the class $\mathcal{W}_n^{\lambda}(\lambda_\xi;\, T)$, then

$$\|\chi-\lambda_\xi\|_I \leqslant \frac{p}{cn}\, s_{\lambda_\xi}\varepsilon_n(\xi),$$

whence, by (30),

$$\|\chi\|_I \leqslant \Big(1+\frac{p}{cn}\, s_{\lambda_\xi}\Big)\varepsilon_n(\xi).$$

As $\chi(t_k) = \lambda_\xi(t_k) = \delta(t_k) = \xi(t_k)-\varphi_n(t_k)$ for $k = 1, 2, \ldots, m$, we get $\chi+\varphi_n \in \mathcal{W}_n'(\xi;\, T)$ and

$$\varepsilon_n(\xi;\, T) \leqslant \|\xi-(\chi+\varphi_n)\|_I \leqslant \|\xi-\varphi_n\|_I+\|\chi\|_I \leqslant \Big(2+\frac{p}{cn}\, s_{\lambda_\xi}\Big)\varepsilon_n(\xi),$$

i. e.

(36) $$s_\xi \leqslant 2+\frac{p}{cn}s_{\lambda_\xi};$$

we iterate this inequality putting successively $\lambda_\xi, \lambda_{\lambda_\xi}, \ldots$ for ξ, and we obtain

(37) $$s_{\lambda_\xi} \leqslant 2+\frac{p}{cn}s_{\lambda_{\lambda_\xi}}, \ \ldots$$

Since $s_\eta \leqslant s$ for every continuous function η with s depending only on the system T of the nodes and on the interval I ([6], and the estimation (4.1)), by the superposition of i of the inequalities (36), (37), ... we obtain

$$s_\xi \leqslant 2\frac{1-(p/cn)^i}{1-p/cn}+\left(\frac{p}{cn}\right)^i s.$$

If $n > p/c$ we obtain (27) as the limit-inequality letting i increase infinitely.

Theorems 6.3 and 6.4 give an upper bound for the ratio $\varepsilon_n(\xi; T)/\varepsilon_n(\xi)$, defined when $\varepsilon_n(\xi) > 0$.

Both theorems imply that

$$\overline{\lim_{n\to\infty}}\frac{\varepsilon_n(\xi; T)}{\varepsilon_n(\xi)} \leqslant 2.$$

Now we shall compare both estimations as $n \to \infty$. For theorem 6.3 this passage to the limit is equivalent to $r \to 0$. For fixed m and $r \to 0$ by (3) and (4) we see that $\alpha_m(r) = o(r)$, $\beta_m(r) = r+(m-1)r^2+ro(r)$, whence

$$1+\frac{\big(1+(m-1)r\big)\beta_m(r)}{r\big(1-\alpha_m(r)\big)} = 1+\frac{\big(1+(m-1)r\big)\big(1+(m-1)r+o(r)\big)}{1-o(r)}$$

$$= 2+2(m-1)r+o(r) = 2+4(m-1)\left(\frac{d-c}{d+c}\right)^{n/2}+o\left(\left(\frac{d-c}{d+c}\right)^{n/2}\right).$$

The quantity

$$4(m-1)\left(\frac{d-c}{d+c}\right)^{n/2}$$

tends to zero with the rapidity of the geometrical series, *i. e.* more rapidly than

$$\frac{2}{1-p/cn}-2 = \frac{2p}{cn-p}.$$

Hence the estimation of theorem 6.3 is asymptotically stronger than that of theorem 6.4. This is especially evident for $m = 2$; in this case $\alpha_2 = r^2$, $\beta_2 = r^2+r$, whence theorem 6.3 gives

$$\frac{\varepsilon_n(\xi; T)}{\varepsilon_n(\xi)} \leqslant 2 \Big/ \left(1-2\left(\frac{d-c}{d+c}\right)^{n/2}\right).$$ [6]

7. The case of two nodes. In this section we consider the case when T consist of two points t_1 and t_2; we determine a positive integer $\nu(t_1, t_2)$ such that $\varepsilon_n(\xi; \{t_1, t_2\}) \leqslant 2\varepsilon_n(\xi)$ for every $n \geqslant \nu(t_1, t_2)$ and $\xi \epsilon \mathcal{C}$. This is a complement to theorem 5.2. To find $\nu(t_1, t_2)$ we shall solve the following problem:

Given two points t_1 and $t_2 > t_1$ of I and numbers x_1 and x_2 such that $x = \max\{|x_1|, |x_2|\} > 0$, in the class $\mathcal{W}_n^0$ of polynomials $\varphi \epsilon \mathcal{W}_n$ such that

$$(1) \qquad \varphi(t_1) = x_1, \quad \varphi(t_2) = x_2,$$

we must find a polynomial φ_n such that

$$\|\varphi_n\|_I = \inf_{\varphi \epsilon \mathcal{W}_n^0} \|\varphi\|_I.$$

The existence of φ_n is almost evident; this polynomial, which is not necessarily unique, will be called *minimal* in the class $\mathcal{W}_n^0$.

By the definition of $\mathcal{W}_n^0$: $\|\varphi_2\|_I \geqslant \ldots \geqslant \|\varphi_n\|_I \geqslant \|\varphi_{n+1}\|_I \geqslant \ldots \geqslant x$, whence by theorem 5.1 there exists a positive integer $\nu(t_1, t_2)$ (the same as at the beginning of this section) independent of x_1 and x_2 such that $\|\varphi_n\|_I = x$ for $n \geqslant \nu(t_1, t_2)$.

Since we are seeking a polynomial of the best approximation to 0 in I among all polynomials of class $\mathcal{W}_n$, satisfying the conditions (1), we shall use the terminology settled in § 2: an (e) point of a polynomial φ is any point t of I such that $|\varphi(t)| = \|\varphi\|_I$; t is called the $(+)$ point if $\varphi(t) = \|\varphi\|_I$ and $(-)$ point if $\varphi(t) = -\|\varphi\|_I$. The set of the (e) points is called the alternant of φ. We also consider two functions: $a(\varphi)$ denotes the number of points of the alternant of φ, and $a_{\text{int}}(\varphi)$ denotes the number of points of the alternant lying in the interior of I. $a(\varphi)$ can be infinite only when $x_1 = x_2$; in this case the alternant of the function $\varphi = x_1$ is the entire interval I. For all other functions we have $a(\varphi) \leqslant n+1$, since if the alternant of φ contained $n+2$ points, then at least n of them would be situated in the interior of I. These points, however, are roots of the derivative of φ, which is a polynomial of degree $\leqslant n-1$.

---

[6]) In our later paper, *On the accuracy of approximation with nodes*, Bull. de l'Acad. Pol. de Sci., Cl. III, 4 (1956), p. 745-748, we prove that if $n \geqslant 14[p/c]+12$, then the inequality (5.4) for every continuous function ξ is satisfied (added in the correction).

THEOREM 7.1. *The polynomial* $\varphi \in \mathcal{W}_n^0$ *is minimal in the class* $\mathcal{W}_n^0$ *if and only if there exists no polynomial* $\chi \in \mathcal{W}_n$ *such that*

$$\chi(t_1) = \chi(t_2) = 0, \tag{2}$$

$$\operatorname{sign}\chi(t) = \operatorname{sign}\varphi(t) \quad \textit{for} \quad t \in A, \tag{3}$$

where A *is the alternant of* φ.

Proof. Suppose that a polynomial χ with the above properties exists; we shall prove that φ is not minimal. By hypothesis the points t_1 and t_2 do not belong to the alternant. Indeed, if, for example, $t_1 \in A$, then $\|\varphi\|_I = |\varphi(t_1)| = x$. By (2) and (3) $\operatorname{sign}\varphi(t_1) = \operatorname{sign}\chi(t_1) = 0$ and $x = 0$, contrary to the hypothesis.

Let

$$F_h = \underset{t \in I}{E}\{|\varphi(t)| \geqslant \|\varphi\|_I - h\},$$

where $0 < h < \|\varphi\|_I$. As $t \in A$ implies $|\varphi(t)| = \|\varphi\|_I$, we have $A \subset F_h$, and by (3) it is possible to choose $h > 0$ so small that the inequality

$$\operatorname{sign}\chi(t) = \operatorname{sign}\varphi(t) \neq 0 \tag{4}$$

will be satisfied in F_h, since it is satisfied in A. Hence for $t \in F_h$

$$|\chi(t)| \geqslant f > 0. \tag{5}$$

Let

$$c = \min\left\{\frac{\|\varphi\|_I - h}{\|\chi\|_I}, \frac{h}{1 + \|\chi\|_I}\right\} > 0. \tag{6}$$

Since $\min\limits_{t \in F_h}|\varphi(t)| = \|\varphi\|_I - h$, and by (6)

$$c \max_{t \in F_h}|\chi(t)| \leqslant c\|\chi\|_I \leqslant \|\varphi\|_I - h,$$

we see that the inequality

$$\min_{t \in F_h}|\varphi(t)| \geqslant c \max_{t \in F_h}|\chi(t)|$$

is satisfied. Hence, from (4) and (5) we obtain for $t \in F_h$

$$|\varphi(t) - c\chi(t)| = |\varphi(t)| - c|\chi(t)| \leqslant \|\varphi\|_I - cf. \tag{7}$$

On the other hand, for $t \in I - F_h$, by (6)

$$|\varphi(t) - c\chi(t)| \leqslant \|\varphi\|_{\overline{I - F_h}} + c\|\chi\|_I = \|\varphi\|_I - h + c\|\chi\|_I \leqslant \|\varphi\|_I - \frac{h}{1 + \|\chi\|_I}. \tag{8}$$

By (7) and (8) it follows that $\|\varphi - c\chi\|_I < \|\varphi\|_I$, and since, by hypothesis (2), $\varphi - c\chi \in \mathcal{W}_n^0$, φ cannot be the minimal polynomial in the class $\mathcal{W}_n^0$.

Let us suppose now that there exists no polynomial χ with the properties described in our theorem. Hence for every polynomial ω of class $\mathcal{W}_n^0$ the polynomial $\chi = \varphi - \omega$ satisfies conditions (2) and at least at one (e) point of φ have $\operatorname{sign}\chi(t) \neq \operatorname{sign}\varphi(t)$. It follows for every polynomial $\omega \epsilon \mathcal{W}_n^0$ that $\|\varphi - \chi\|_I = \|\omega\|_I \geqslant \|\varphi\|_I$, whence φ is minimal, q. e. d.

In the following theorem we use the notation introduced in this section.

THEOREM 7.2. *If $\varphi \epsilon \mathcal{W}_n^0$, $a(\varphi) < +\infty$ and if the* (e) *points of the polynomial φ, written in the natural order, can be divided into k groups of* (e) *points of the same kind, where adjacent groups contain* (e) *points of different kind, then $k \geqslant a(\varphi) + c_{\text{int}}(\varphi) - n + 1$.*

Proof. Let the groups of (e) points consist respectively of $g_1, g_2, \ldots, g_k$ points; thus $g_1 + g_2 + \ldots + g_k = a(\varphi)$. The (e) points inside I, $a_{\text{int}}(\varphi)$ in number, are roots of the derivative of φ. Moreover, between each two consecutive (e) points of the same kind there lies a root of this derivative, different from the roots distinguished hitherto. Hence the derivative of φ has $(g_1 - 1) + (g_2 - 1) + \ldots + (g_k - 1) + a_{\text{int}}(\varphi) = a(\varphi) + a_{\text{int}}(\varphi) - k \leqslant n - 1$ roots, q. e. d.

In theorems 7.3-7.5 we give the conditions for the polynomial $\varphi \epsilon \mathcal{W}_n^0$ to be minimal in the class $\mathcal{W}_n^0$. By theorem 7.1 their proof consists in testing whether or not there exists a polynomial $\chi \epsilon \mathcal{W}_n$ satisfying conditions (2) and (3).

THEOREM 7.3. *If $\varphi \epsilon \mathcal{W}_n^0$, $\|\gamma\|_I > x$, $a(\varphi) > n$, then the polynomial φ is not minimal in the class $\mathcal{W}_n^0$.*

Proof. We divide the (e) points $p_1, p_2, \ldots, p_{a(\varphi)}$ of φ into groups composed of (e) points of the same kind:

$$
\begin{aligned}
&p_1, p_2, \ldots, p_{l_1} && \text{(first group)},\\
&p_{l_1+1}, p_{l_1+2}, \ldots, p_{l_2} && \text{(second group)},\\
&\ldots\ldots\ldots\ldots &&\\
&p_{l_{k-1}+1}, \ldots, p_{l_k = a(\varphi)} && \text{(}k\text{th group)}.
\end{aligned}
\tag{9}
$$

The number of groups satisfies the inequality $k \leqslant a(\varphi) < n$, whence

$$k + 1 \leqslant n. \tag{10}$$

The construction of the polynomial χ will be carried out separately in each of the four cases which will be distinguished in the sequel — they are all the possible ones since $\|\varphi\|_I > x$, and t_1 and t_2 are not (e) points.

I. The points t_1, t_2 do not separate (e) points in the group (9).

We choose $k-1$ points $r_1, r_2, \ldots, r_{k-1}$ so that

$$p_{l_i} < r_i < p_{l_i+1} \quad (i = 1, 2, \ldots, k-1); \tag{11}$$

if one of the points t_1, t_2 is situated between the jth and the $(j+1)$th group, then as r_j we choose that point (or either of those points if both are so situated). To the set of points $r_1, r_2, \ldots, r_{k-1}$ we add those of the points t_1, t_2 which are not in this set, and moreover if both t_1 and t_2 lie between the same adjacent groups we add a point t^* lying between them. Thus we obtain a set $s_1, s_2, \ldots, s_{k'}$, of k' points where $k' \leqslant k+1$, containing t_1 and t_2 as elements.

We define the polynomial χ as

$$\chi = c(t-s_1)(t-s_2)\ldots(t-s_{k'}), \tag{12}$$

where the sign of the constant c is chosen so as to give

$$\operatorname{sign}\chi(p_1) = \operatorname{sign}\varphi(p_1). \tag{13}$$

From the construction of the points $s_1, s_2, \ldots, s_{k'}$ it follows that the polynomial χ changes the sign between two consecutive (e) points of different kind an odd number of times, and does not change the sign between (e) points of the same group (9). Thus,

$$\operatorname{sign}\chi(p_i) = \operatorname{sign}\varphi(p_i) \quad \text{for} \quad i = 1, 2, \ldots, a(\varphi). \tag{14}$$

Moreover

$$\chi(t_1) = \chi(t_2) = 0, \tag{15}$$

and, by (10), $k' = n$, whence $\chi \in \mathcal{W}_n$.

II. Exactly one of the points t_1, t_2 (for example t_1) separates the (e) points of the same group (9).

From the hypothesis it follows directly that one of the groups (9) contains at least two (e) points. Hence the number of groups $k \leqslant a(\varphi)-1$. We define the points $r_1, r_2, \ldots, r_{k-1}$ by inequalities (11); t_1 is not among those points, and t_2 is among them if $t_2 \in (p_1, p_{a(\varphi)})$. Suppose, for example, that $p_1 < t_1 < p_2$, whence $l_1 \geqslant 2$. We add the point t_2 to the set $r_1, r_2, \ldots, r_{k-1}$ if it does not lie in $(p_1, p_{a(\varphi)})$; we also add t_1 and a point t_1^* such that $t_1 < t_1^* < p_2$. Thus we obtain a set $s_1, s_2, \ldots, s_{k'}$ of points where $k' \leqslant (k-1)+1+2 = k+2 \leqslant a(\varphi)+1 \leqslant n$ and for these points we define the polynomial χ by (12). Hence $\chi \in \mathcal{W}_n$ and equalities (15) are satisfied. If c fulfils condition (13), the polynomial χ has, at the (e) points $p_1, p_2, \ldots, p_{a(\varphi)}$, the same sign as φ. Indeed, between two consecutive (e) points of different type it changes the sign precisely once (at one of the points $r_1, r_2, \ldots, r_{k-1}$) and between two consecutive (e) points of the same type it does not change the sign or changes it twice (at the points t_1 and t_1^*).

III. The points t_1, t_2 separate different pairs of (e) points in groups (9).

Hence one of the groups (9) contains three points or two groups contain two points each. Thus $k \leqslant a(\varphi)-2 \leqslant n-3$. We define the points $r_1, r_2, \ldots, r_{k-1}$ by (11).

Let $\chi = c(t-r_1)\ldots(t-r_{k-1})(t-t_1)^2(t-t_2)^2$; then $\deg\chi = (k-1)+4 \leqslant n$, whence $\chi \in \mathcal{W}_n$. Equalities (15) are also satisfied. When c satisfies condition (13), the polynomial χ has at the points (9) the same sign as φ; in particular, because of the introduction of the factor $(t-t_1)^2(t-t_2)^2$, the polynomial does not change the sign between consecutive (e) points of the same type.

IV. The points t_1, t_2 separate the same pair of (e) points lying in the same group (9).

In this case we set

$$\chi = c(t-r_1)\ldots(t-r_{k-1})(t-t_1)(t-t_2). \tag{16}$$

As in the case II: $k \leqslant a(\varphi)-1$, whence $\deg\chi = k+1 \leqslant a(\varphi) < n$ and $\chi \in \mathcal{W}_n$. If c satisfies (13), then (14) and (15) are satisfied.

THEOREM 7.4. *If* $\varphi \in \mathcal{W}_n^0$, $\|\varphi\|_I > x$, $a(\varphi) = n$, *then the polynomial* φ *is minimal in the class* $\mathcal{W}_n^0$ *if, and only if, its* (e) *points are alternately* (+) *and* (−) *points and lie all outside the interval* (t_1, t_2).

Proof. 1. We shall first prove the sufficiency, *i. e.* that if the (e) points of φ are not alternately the (+) and (−) points or if some (e) points lie in (t_1, t_2), then φ is not minimal.

Suppose the first case: there are two consecutive (e) points of the same kind. Then the number of the groups of the (e) points (9) satisfies the inequality

$$n-1 = n+(n-2)-n+1 \leqslant k \leqslant n-1,$$

whose left-hand part results from theorem 7.2 and from $a(\varphi) = n$, $a_{\text{int}}(\varphi) \geqslant n-2$, the right-hand part results from hypothesis. Thus $k = n-1$, and applying theorem 7.2 again, we obtain $a_{\text{int}}(\varphi) = n-2$ whence $p_1 = a$, $p_n = b$ (both ends of I are (e) points).

Now, as in theorem 7.3, in each of the four cases quoted in it we construct a polynomial χ satisfying (14) and (15). In case I the construction is not altered and inequality (10) resulting from $k = n-1$ is applied. In case II we define the points $r_1, r_2, \ldots, r_{k-1}$ by (11) in such a manner that t_2 is among these points ($p_1 = a$, $p_n = b$ implies that t_2 satisfies one of these inequalities). We set $\chi = c(t-r_1)\ldots(t-r_{k-1})(t-t_1)^2$; χ changes the sign once between two consecutive (e) points of different kind, and does not change the sign between two consecutive points of the same kind, whence (14) and (15) is satisfied if c satisfies (13). Case III in which $k \leqslant a(\varphi)-2 = n-2$ must be excluded, for $k = n-1$. In case IV we adopt definition (16) and use the fact that $k = n-1$.

We now pass to the second case: an (e) point of φ lies in (t_1, t_2). Since $a(\varphi) = n$ and since (e) points of φ lying in the interior of I are roots of the derivative of φ, $a_{\text{int}}(\varphi) \leqslant n-1$ and at least one bound of I is an (e) point. Hence, by the existence in (t_1, t_2) of an (e) point, it follows that one of the points t_1, t_2, say t_1, lies between such two (e) points that the second point, say t_2, does not lie between them. Thus, in case I, t_1 is among the numbers $r_1, r_2, \ldots, r_{k-1}$ of inequality (11) if we suitably define these numbers. If t_2 does not lie among the points $r_1, r_2, \ldots, r_{k-1}$, we join it to this system and obtain a system $s_1, s_2, \ldots, s_{k'}$ of points, and then adopt definition (12). Since $k' \leqslant k \leqslant a(\varphi) = n$, we have $\chi \in \mathcal{W}_n$, and c being determined in a suitable manner, (14) and (15) are satisfied. From the hypotheses of cases II-IV it follows that there exist two consecutive (e) points of the same kind. Hence we have the case of the first possibility, considered above.

2. Now, by aid of theorem 7.1, we shall prove that if the (e) points of φ are alternately (+) and (−) points and lie outside (t_1, t_2), then polynomial φ is minimal in class $\mathcal{W}_n^0$.

Suppose that φ is not minimal. Let $p_1, p_2, \ldots, p_n$ be its (e) points. Hence there exists a polynomial $\chi \in \mathcal{W}_n$ satisfying (15) and having at these points the same sign as φ. Hence, by the hypothesis that the (+) and (−) points lie alternately, it follows that χ has in each of the intervals

$$(17) \qquad (p_1, p_2),\ (p_2, p_3),\ \ldots,\ (p_{n-1}, p_n)$$

an odd number of roots (counted with their multiplicity). Two cases are possible:

I. There is a j such that t_1 and t_2 lie in (p_j, p_{j+1}). Then χ has at least three roots in (p_j, p_{j+1}), and one root in each of the intervals (p_i, p_{i+1}) for $i \neq j$ whence it has at least $n+1$ roots and $\chi = 0$.

II. The points t_1 and t_2 lie either in $\langle a, p_1)$ or in $(p_n, b\rangle$. Then χ has a root in each of the $n-1$ intervals (17) whence it has together at least $n+1$ roots and $\chi = 0$.

The relation $\chi = 0$, obtained in both cases, contradicts conditions (14) fulfilled by χ. Hence φ is minimal.

Theorem 7.5. *If $\varphi \in \mathcal{W}_n^0$, $\|\varphi\|_I > x$, $a(\varphi) = n+1$, then the polynomial φ is minimal in class $\mathcal{W}_n^0$ if, and only if, all its* (e) *points lie outside the interval* (t_1, t_2).

Proof. $a(\varphi) = n+1$ implies $a_{\text{int}}(\varphi) \geqslant n-1$. As already seen, $a_{\text{int}}(\varphi) \leqslant n-1$, whence $a_{\text{int}}(\varphi) = n-1$, and by theorem 7.2 it follows that the number of groups of (e) points of the same kind satisfies the inequality $(n+1)+(n-1)-n+1 \leqslant k \leqslant n+1$, whence $k = n+1$. This means that the (e) points $a = p_1 < p_2 < \ldots < p_{n+1} = b$ of φ are alternately $(+)$ and $(-)$ points.

We shall prove first that if there are (e) points of φ in (t_1, t_2), then φ is not minimal. We choose the points $r_1, r_2, \ldots, r_n$ so that $p_i < r_i < p_{i+1}$ for $i = 1, 2, \ldots, n$ and so that they comprise the points t_1, t_2 lying, by hypothesis, each between different consecutive (e) points. Let

$$\chi = \operatorname{sign}\varphi(p_1)\cdot(r_1-t)(r_2-t)\ldots(r_n-t).$$

Thus $\chi \in \mathcal{W}_n$, $\chi(t_1) = \chi(t_2) = 0$ and $\operatorname{sign}\chi(p_i) = \operatorname{sign}\varphi(p_i)$ for $i = 1, 2, \ldots, n+1$, and from theorem 7.1 it follows that φ is not minimal.

We shall prove now that if there are no (e) points of φ in (t_1, t_2), the polynomial φ is minimal in class $\mathcal{W}_n^0$. Suppose the contrary. Thus there exist a polynomial $\chi \in \mathcal{W}_n$ vanishing at t_1 and t_2, having at the (e) points $p_1, p_2, \ldots, p_{n+1}$ the same sign as φ. Hence in that of the intervals (p_1, p_2), $(p_2, p_3), \ldots, (p_n, p_{n+1})$ which contains t_1, t_2 there exist three roots of χ, and in the remaining $n-1$ intervals there are $n-1$ roots of this polynomial. It follows that $\chi = 0$, which proves our assumption to be false.

Taking into account theorems 7.3-7.5 and the fact that $a(\varphi) \leqslant n+1$ for $\|\varphi\|_I > x$, we can formulate the following

COROLLARY. *If $\varphi \in \mathcal{W}_n^0$, $\|\varphi\|_I > x$, then the polynomial φ is minimal in class $\mathcal{W}_n^0$ if, and only if, conditions* 1°-3° *are satisfied*:

1° $n \leqslant a(\varphi) \leqslant n+1$,

2° *the* (e) *points of φ are alternately* $(+)$ *and* $(-)$ *points*,

3° *all the* (e) *points of φ lie outside* (t_1, t_2).

For $a(\varphi) = n+1$ condition 2° was proved in theorem 7.5.

Let us introduce the following notation: the polynomial $t^n+ct^{n-1}+\ldots$, giving the best approximation of zero in the interval $\langle -1, 1\rangle$ among all polynomials of degree n with the coefficient 1 at t^n and with the coefficient c at t^{n-1}, will be denoted by $\tau_n\{c\}$. The properties of $\tau_n\{c\}$, which will be used later, are listed in the monograph [1], p. 297-301. Here let us notice that:

1° the Tchebyshev polynomial

$$\frac{1}{2^{n-1}}\cos(n\arccos t)$$

is, by definition, identical with the polynomial $\tau_n\{0\}$, for it gives the best approximation of zero among all polynomials of degree n with the coefficient 1 at t^n, and contains t only in even powers when n is even and in odd ones when n is odd;

2° the polynomial $\psi = t^n+ct^{n-1}+c_2t^{n-2}+\ldots+c_n$ has in the interval $\langle -1, 1\rangle$ n points of the alternant, which are alternately $(+)$ and $(-)$

points if, and only if, $\psi = \tau_n\{c\}$. This follows by the application of the theorems of Tchebyshev ([4], p. 51 and 56) to the approximation of the function $t^n + ct^{n-1}$ by polynomials of degree $n-2$;

3° if $c \neq 0$, the alternant of $\tau_n\{c\}$ consists of precisely n points, which are alternately $(+)$ and $(-)$ points.

THEOREM 7.6. *Let $c < c^* < 0$ or $0 < c < c^*$. If the points $p_0, p_1, \ldots, p_{n-1}$ (where $-1 \leqslant p_0 < p_1 < \ldots < p_{n-1} \leqslant 1$) and $p_0^*, p_1^*, \ldots, p_{n-1}^*$ (where $-1 \leqslant p_0^* < p_1^* < \ldots < p_{n-1}^* \leqslant 1$) are all the* (e) *points of the polynomials $\tau_n\{c\}$ and $\tau_n\{c^*\}$ respectively in $\langle -1, 1\rangle$, then $p_0 \geqslant p_0^*, p_i > p_i^*$ for $i = 1, 2, \ldots, n-2$, $p_{n-1} \geqslant p_{n-1}^*$.*

Proof. The hypotheses being symmetrical, we consider only the case $c < c^* < 0$.

I. If

$$-n\,\mathrm{tg}^2\frac{\pi}{2n} \leqslant c < c^* < 0,$$

then

$$\tau_n\{c\} = \frac{1}{2^{n-1}}\left(1-\frac{c}{n}\right)^n T_n\left(\frac{t+c/n}{1-c/n}\right) \tag{18}$$

(formula (18) results from the formula for the polynomial y_1 given in the monograph [1], p. 297, with $c = -n(\beta-1)/2$) and similarly

$$\tau_n\{c^*\} = \frac{1}{2^{n-1}}\left(1-\frac{c^*}{n}\right)^n T_n\left(\frac{t+c^*/n}{1-c^*/n}\right).$$

The Tchebyshev polynomial $T_n = \cos(n\arccos t)$ takes on in $\langle -1, 1\rangle$ its extremal values at the points $-\cos i\pi/n$ where $i = 0, 1, \ldots, n$ (written in their natural order). Hence the (e) points of $\tau_n\{c\}$ and $\tau_n\{c^*\}$ may be expressed by the formulae

$$p_i = \left(\frac{c}{n}-1\right)\cos\frac{i\pi}{n} - \frac{c}{n} \quad (i = 0, 1, \ldots, n-1), \tag{19}$$

$$p_i^* = \left(\frac{c^*}{n}-1\right)\cos\frac{i\pi}{n} - \frac{c^*}{n} \quad (i = 0, 1, \ldots, n-1)$$

(for $i = n$ we obtain the points $1-2c/n > 1$, $1-2c^*/n > 1$ lying outside $\langle -1, 1\rangle$). For $i = 0$ we have $p_0 = p_0^* = -1$ and for $i = 1, 2, \ldots, n-1$

$$p_i - p_i^* = \frac{c^*-c}{n}\left(1-\cos\frac{i\pi}{n}\right) > 0,$$

q. e. d.

II. Let us suppose now that $c < c^* \leqslant -n\,\mathrm{tg}^2\pi/2n$. Then

$$p_0 = p_0^* = -1, \quad p_{n-1} = p_{n-1}^* = 1 \tag{20}$$

([1], p. 298), the points p_i, p_i^* ($i = 0, 1, \ldots, n-1$) are of the same kind and $p_0, p_1, \ldots, p_{n-1}$ are alternately (+) and (−) points.

It is sufficient to prove that $p_i < p_i^*$ ($0 < i < n-1$) holds for every $c^* \leqslant -n\,\mathrm{tg}^2\pi/2n$ and for an arbitrarily small positive difference c^*-c. The polynomial $\tau_n\{c\}$ and its (e) points depend continuously on the parameter c; thus, let us suppose c^*-c to be so small that

$$\max\{p_{i-1}, p_{i-1}^*\} < \min\{p_i, p_i^*\} \quad (1 \leqslant i \leqslant n-1). \tag{21}$$

Let us set

$$\sigma = \frac{\tau_n\{c\}}{\|\tau_n\{c\}\|_{\langle -1,1\rangle}}, \quad \sigma^* = \frac{\tau_n\{c^*\}}{\|\tau_n\{c^*\}\|_{\langle -1,1\rangle}}, \quad \delta = \sigma - \sigma^*. \tag{22}$$

If $p_i = p_i^*$, where $0 < i < n-1$, then p_i is a double root of the polynomial δ, for

$$\delta(p_i) = \sigma(p_i) - \sigma^*(p_i^*) = 0, \quad \sigma'(p_i) = \sigma^{*\prime}(p_i^*) = 0, \quad \delta'(p_i) = 0.$$

If $p_i \neq p_i^*$, then the interval $(\min\{p_i, p_i^*\}, \max\{p_i, p_i^*\})$ contains a root of δ. Indeed, by definition (22),

$$\sigma(p_i) = \sigma^*(p_i^*), \quad |\sigma(p_i)| = 1. \tag{23}$$

On the other hand from (21) and from the fact that in $\langle -1, 1\rangle$ the functions $|\sigma|, |\sigma^*|$ assume the value 1 only at the points $p_0, p_1, \ldots, p_{n-1}$ and $p_0^*, p_1^*, \ldots, p_{n-1}^*$ respectively it follows that

$$|\sigma(p_i^*)| < 1, \quad |\sigma^*(p_i)| < 1. \tag{24}$$

By (23) and (24) $\operatorname{sign}\delta(p_i) = -\operatorname{sign}\delta(p_i^*) \neq 0$.

By (20) -1 and 1 are also roots of δ. Thus we have shown that δ has in $\langle -1, 1\rangle$ at least $n+z$ roots, z being the number of zeros in the sequence

$$p_1 - p_1^*, \; p_2 - p_2^*, \; \ldots, \; p_{n-2} - p_{n-2}^*. \tag{25}$$

Now $\deg\delta \leqslant n$ and by hypothesis $\tau_n\{c\} \neq \tau_n\{c^*\}$, $\sigma \neq \sigma^*$, whence $z = 0$, *i. e.* $p_i \neq p_i^*$ for $i = 1, 2, \ldots, n-2$. We have also proved that δ has no multiple roots, and has precisely n single ones lying, except -1 and 1, in the interval $(\min\{p_1, p_1^*\}, \max\{p_{n-2}, p_{n-2}^*\})$. Let us notice now that none of the inequalities

$$p_{i-1}^* < p_{i-1} < p_i < p_i^*, \quad p_{i-1} < p_{i-1}^* < p_i^* < p_i$$

can be satisfied. Indeed, suppose that the first of them holds. Then $\sigma(p_{i-1}) = -\sigma(p_i)$, $|\sigma(p_i)| = 1$, $|\sigma^*(p_{i-1})| < 1$, $|\sigma^*(p_i)| < 1$ whence $\operatorname{sign}\delta(p_{i-1}) = -\operatorname{sign}\delta(p_i) \neq 0$ and the interval (p_{i-1}, p_i) contains root of δ different from the n roots quoted above, which is impossible. Thus all the quantities (25) are of the same sign.

In case II each of the polynomials $\tau_n\{c\}$ and $\tau_n\{c^*\}$ has exactly $n-1$ roots in $\langle -1, 1\rangle$ (one between each two consecutive (e) points) and exactly one root greater than 1 ([1], p. 298, in particular fig. 7). Since the values of these polynomials tend to $+\infty$ as $t \to +\infty$, by definition (22) we get $\sigma(1) = \sigma^*(1) = -1$ and $\sigma(p_{n-2}) = \sigma^*(p_{n-2}^*) = 1$.

If the quantities (25) are positive, then in particular $p_{n-2}^* < p_{n-2}$; the last equality implies $\sigma(p_{n-2}) = 1$, $\sigma^*(p_{n-2}) < 1$, $\delta(p_{n-2}) > 0$, and in the entire interval $\langle p_{n-2}, 1)$ in which there are no roots of δ, the inequality $\delta(t) > 0$ is satisfied. On the other hand, if the quantities (25) are negative, then $p_{n-2} < p_{n-2}^*$ and $\delta(t) < 0$ in $\langle p_{n-2}^*, 1)$.

Thus δ has in the left-hand neighbourhood of its single root 1 the same sign as the quantities (25), and in the right-hand one the opposite sign. It should also be taken into account that if c decreases to $-\infty$, then the greatest root of $\tau_n\{c\}$ increases to $+\infty$. Since $c < c^*$, $\delta = \sigma - \sigma^*$ is negative for $t > 1$ and the quantities (25) are positive.

Thus theorem 7.6 is proved completely.

THEOREM 7.7. *If the polynomial $\varphi \in \mathfrak{W}_n$ satisfies in the interval $I = \langle a, b\rangle$ the conditions*

1° $n \leqslant a(\varphi) \leqslant n+1$,

2° *the points of the alternant of φ are alternately $(+)$ and $(-)$ points, then the polynomial*

$$\varphi\left(\frac{(b-a)t+a+b}{2}\right) \tag{26}$$

is equal either to $\tau_{n-1}(0)$ or, for a suitable c, to $\tau_n(c)$, multiplied by a constant.

As above, $a(\varphi)$ denotes the number of points in the alternant of φ.

Proof. By 1° there are at least $n-2$ (e) points of φ in the interior of I. Since they are roots of the derivative of φ, $\deg\varphi \geqslant n-1$. If $\deg\varphi = n-1$, then inside I lie exactly $n-2$ (e) points, and, since $a(\varphi) \geqslant n$, both bounds of I are (e) points. Thus by hypothesis 2° it follows that if we transform I into the interval $\langle -1, 1\rangle$, which corresponds to the introduction of the polynomial (26), and divide this polynomial by its coefficient at t^{n-1}, we shall obtain the Tchebyshev polynomial of degree $n-1$, whence

$$\frac{1}{2^{n-2}}\cos\big((n-1)\arccos t\big) = \tau_{n-1}\{0\}.$$

If $\deg\varphi = n$, we introduce again the polynomial (26) and divide by the coefficient at t^n. In virtue of the remark 2° preceeding theorem 7.6 we obtain thus the polynomial $\tau_n\{c\}$.

THEOREM 7.8. *If the polynomial* $\varphi \in \mathfrak{M}_n^2$ *satisfies in* $I = \langle a, b\rangle$ *the conditions*

1° $n \leqslant a(\varphi) \leqslant n+1$,

2° (e) *points* $p_1, p_2, \ldots, p_{a(\varphi)}$ *(where* $a \leqslant p_1 < p_2 < \ldots < p_{a(\varphi)} \leqslant b$*) are alternately* $(+)$ *points and* $(-)$ *points,*

then for $n \geqslant 5$ *the inequality*

$$(27)\qquad \max\{p_1-a, p_2-p_1, \ldots, p_{a(\varphi)}-p_{a(\varphi)-1}, b-a_{a(\varphi)}\} < \frac{7}{2}\cdot\frac{b-a}{n}$$

is satisfied.

Proof. Let d be the maximum of (27). It is sufficient to prove that if $\langle a, b\rangle = \langle -1, 1\rangle$, then, for $n \geqslant 5$, $d < 7/n$; (27) will the follow by a change of the unit and translation along the axis t.

Let us observe first that the greatest distance between the consecutive (e) points of the nth Tchebyshev polynomial, which are -1, $-\cos\pi/n, \ldots, \cos\pi/n, 1$, is equal to

$$(28)\quad \cos\frac{\left[\frac{n}{2}\right]\pi}{n} - \cos\frac{\left[\frac{n}{2}+1\right]\pi}{n} = 2\sin\frac{\left(2\left[\frac{n}{2}\right]+1\right)\pi}{2n}\sin\frac{\pi}{2n} \leqslant 2\sin\frac{\pi}{2n} < \frac{\pi}{n}.$$

I. If $\deg\varphi = n-1$, then by theorem 7.7 and (28) $d \leqslant \pi/(n-1)$. For $n \geqslant 5$ we have

$$d \leqslant \frac{\pi}{n}\cdot\frac{n}{n-1} < \frac{4}{n}\cdot\frac{5}{4} = \frac{5}{n},$$

for the expression $n/(n-1)$ is a decreasing function of n.

II. If $\deg\varphi = n$, then dividing φ by the coefficient at t^n we obtain, in virtue of theorem 7.7, the polynomial $\tau_n\{c\}$ with the same (e) points as φ. Let $c \leqslant 0$ (the case $c \geqslant 0$ is symmetrical).

II.1. Suppose first that $-n\,\mathrm{tg}^2\pi/2n \leqslant c \leqslant 0$. Then $\tau_n\{c\}$ is expressed by formula (18). The argument of the Tchebyshev polynomial involved is

$$u = \left(t+\frac{c}{n}\right)\Big/\left(1-\frac{c}{n}\right)$$

and the introduction of the argument u enlarges the interval $\langle -1, 1\rangle$ of the variable t at most $1-c/n \leqslant 1+\mathrm{tg}^2\pi/2n = 1/\cos^2\pi/2n$ times. In the same ratio the distance between two consecutive (e) points of (18) can increase as compared with (28), *i. e.* with the greatest difference of two consecutive (e) points of the Tchebyshev polynomial. For $n \geqslant 5$ this distance is not greater than

$$\frac{\pi}{n\cos^2\pi/2n} \leqslant \frac{\pi}{n\cos^2\pi/10} < \frac{4}{0{,}9n} < \frac{5}{n}.$$

By (19) $p_0 = -1$ and

$$1-p_{n-1} = 1+\left(\frac{c}{n}-1\right)\cos\frac{\pi}{n}+\frac{c}{n} \leqslant 1-\cos\frac{\pi}{n} = 2\sin^2\frac{\pi}{2n} < 2\sin\frac{\pi}{2n} < \frac{4}{n}.$$

Hence by the foregoing inequality

(29) $$d < \frac{5}{n}.$$

II.2. Let us suppose now that

(30) $$c \leqslant -n\,\mathrm{tg}^2\frac{\pi}{2n}$$

and let us denote by $p_0^0, p_1^0, \ldots, p_{n-1}^0$ the (e) points of the polynomial $\tau_n\{-n\,\mathrm{tg}^2\pi/2n\}$ defined by (19) for $c = -n\,\mathrm{tg}^2\pi/2n$. Thus

(31) $$p_i^0 = \frac{-\cos\dfrac{i\pi}{n}+\sin^2\dfrac{\pi}{2n}}{\cos^2\dfrac{\pi}{2n}} \qquad (i = 0, 1, \ldots, n-1).$$

As $c \to -\infty$, the polynomial

(32) $$\tau_n\{c\} = t^n+ct^{n-1}+c_2t^{n-2}+\ldots+c_n,$$

whose coefficients $c_2, \ldots, c_n$ depend on c, divided by c tends in $\langle -1, 1\rangle$ to the Tchebyshev polynomial T_{n-1}. Indeed, the best approximation of zero by the polynomial (32), with fixed c is equivalent to the approximation of zero by the polynomial of the form

$$\frac{\tau_n\{c\}}{c} = \frac{t^n}{c}+t^{n-1}+\frac{c_2}{c}t^{n-2}+\ldots+\frac{c_n}{c},$$

whence, as $c \to -\infty$, asymptotically equivalent to the best approximation of zero by a polynomial of degree $n-1$ with the coefficient 1 at t^{n-1}. Hence by theorem 7.6 it follows that as c decreases from $-n\,\mathrm{tg}^2\pi/2n$ to $-\infty$, the (e) points of $\tau_n\{c\}$ do not decrease and tend to the limit values

(33) $$p_i^\infty = -\cos\frac{i\pi}{n-1} \qquad (i = 0, 1, \ldots, n-1)$$

(with the same indices i) which are (e) points of T_{n-1}. If the parameter c in $\tau_n\{c\}$ satisfies (30), then its (e) points $p_0, p_1, \ldots, p_{n-1}$ are such that $p_0^0 = p_0 = p_0^\infty = -1$, $p_i^0 < p_i < p_i^\infty$ for $i = 1, 2, \ldots, n-2$, $p_{n-1}^0 = p_{n-1} = p_{n-1}^\infty = 1$.

It follows that

$$p_i - p_{i-1} \leqslant p_i^\infty - p_{i-1}^0 \quad (i = 1, 2, \ldots, n-1),$$

whence by (31) and (33) we have, for $i = 1, 2, \ldots, n-1$,

$$p_i^\infty - p_{i-1}^0 = \frac{\cos(i-1)\pi/n - \sin^2\pi/2n}{\cos^2\pi/2n} - \cos\frac{i\pi}{n-1}.$$

Since $-\cos t$ increases for $0 \leqslant t \leqslant \pi$, and

$$\frac{i}{n-1} \leqslant \frac{i+1}{n} \quad \text{for} \quad i = 1, 2, \ldots, n-1$$

we have

$$p_i^\infty - p_{i-1}^0 \leqslant \frac{\cos(i-1)\pi/n - \sin^2\pi/2n - \cos^2\pi/2n \cdot \cos(i+1)\pi/n}{\cos^2\pi/2n}$$

$$= \frac{\cos(i-1)\pi/n - \cos(i+1)\pi/n - \sin^2\pi/2n \cdot \big(1 - \cos(i+1)\pi/n\big)}{\cos^2\pi/2n}$$

$$\leqslant \frac{2\sin i\pi/n \cdot \sin\pi/n}{\cos^2\pi/2n} \leqslant \frac{2\sin\pi/n}{\cos^2\pi/2n}.$$

Taking into account that $p_0 = -1$, $p_{n-1} = 1$, we see that for every c satisfying (30) and for $n \geqslant 5$ we have

$$d = \max_i (p_i - p_{i-1}) \leqslant \frac{2\pi}{\cos^2\pi/10} \cdot \frac{1}{n} < \frac{6{,}3}{0{,}9} \cdot \frac{1}{n} = \frac{7}{n},$$

whence by (29) the theorem follows for all $c \leqslant 0$.

THEOREM 7.9. *For every interval $I = \langle a, b\rangle$, points t_1, t_2 (where $t_1 < t_2$) belonging to I, and numbers x_1, x_2 there is a polynomial φ such that*

$$\deg\varphi \leqslant \max\left\{5, \left[\frac{7(b-a)}{2(t_2-t_1)}\right]\right\},$$

(34) $$\varphi(t_1) = x_1, \quad \varphi(t_2) = x_2,$$

$$\|\varphi\|_I = \max\{|x_1|, |x_2|\}.$$

Proof. It is sufficient to consider the case when

$$x = \max\{|x_1|, |x_2|\} > 0.$$

We have shown (corollary, p. 29) that if the polynomial φ is minimal in class $\mathcal{W}_n^0$ (*i. e.* satisfies conditions (34)) and if $\|\varphi\|_I > x$, then between the points t_1, t_2 there are no (e) points of φ. By theorem 7.8, whose hypotheses are fulfilled by the minimal polynomial, this is impossible when $n \geqslant 5$ and

$$t_2 - t_1 \geqslant \frac{b-a}{2} \cdot \frac{7}{n},$$

i. e. when

$$n \geqslant \max\left\{5, \left[\frac{7(b-a)}{2(t_2-t_1)}\right]\right\},$$

for in this case $t_2 - t_1$ is greater than the length of the greatest subinterval of I, containing no (e) points. This means that for

$$n = \max\left\{5, \left[\frac{7(b-a)}{2(t_2-t_1)}\right]\right\}$$

the polynomial φ, minimal in class $\mathcal{W}_n^0$, does not satisfy the condition $\|\varphi\|_I > x$, whence it satisfies $\|\varphi\|_I = x$, q. e. d.

From theorem 7.9 we obtain a result concerning approximation with the system $T = \{t_1, t_2\}$ of nodes. The method of the proof is the same as for theorems 5.2 and 6.3.

THEOREM 7.10. *If*

$$n \geqslant \max\left\{5, \left[\frac{7(b-a)}{2(t_2-t_1)}\right]\right\},$$

then $\varepsilon_n(\xi; \{t_1, t_2\}) \leqslant 2\varepsilon_n(\xi)$ *for every function* $\xi \in \mathcal{C}$.

To end this chapter, we shall prove that, except some particular cases, inequality (5.4) is stronger than inequality (4.1), proved in [6]. In the latter

$$s = \Big\|\sum_{i=1}^{m} |\varphi_i|\Big\|_I + 1 \qquad \text{where} \qquad \varphi_i = \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{t-t_j}{t_i-t_j} \qquad (i = 1, 2, \ldots, m)$$

(see formula (3) in [6]); φ_i is a polynomial of degree $m-1$ defined by the conditions

$$\varphi_i(t_j) = \delta_{ij} \quad (j = 1, 2, \ldots, m). \tag{35}$$

The quantity

$$\Big\|\sum_{i=1}^{m} |\varphi_i|\Big\|_I$$

plays an important role in problems of interpolation (cf. [4], p. 512, where it is denoted by λ_m).

It has been proved (ibidem) that for every system T composed of m nodes

$$\Big\|\sum_{i=1}^{m}|\varphi_i|\Big\|_I > (\log m)/8\sqrt{\pi}.$$

It follows that $s > 2$ but only for $m \geqslant e^{8\sqrt{\pi}}$, *i. e.* for $m \geqslant 1439247$. However, it is easy to prove that for $m \geqslant 3$

$$\Big\|\sum_{i=1}^{m}|\varphi_i|\Big\|_I > 1, \tag{36}$$

i. e. $s > 2$. Indeed, in this case $a \leqslant t_1 < t_2 < t_m \leqslant b$ and the node t_2 is inside $I = \langle a, b\rangle$. Inequality (36) is true if $\|\varphi_2\|_I > 1$; if, however,

$$\|\varphi_2\|_I = 1, \tag{37}$$

then by (35)

$$\varphi_2'(t_2) = 0 \quad \text{and} \quad \varphi_i'(t_2) \neq 0 \quad \text{for} \quad i \neq 2, \tag{38}$$

for t_2 is a root of the polynomials φ_i $(i \neq 2)$, which have no multiple roots. The functions $|\varphi_i|$ for $i \neq 2$ decrease to zero in the left-hand neighbourhood of t_2, and increase in the right-hand neighbourhood, whence by (37) and (38) in every neighbourhood of t_2 there is a point t at which

$$\sum_{i=1}^{m}|\varphi_i(t)| > 1.$$

In particular, such a point exists in I and (36) is valid also in the case (37).

If at least one of the nodes lies in the interior of I, inequality (36) is valid also for $m = 2$, for in this case

$$\varphi_1 = \frac{t-t_2}{t_1-t_2}, \quad \varphi_2 = \frac{t-t_1}{t_2-t_1}$$

and

$$|\varphi_1(t)|+|\varphi_2(t)| = \begin{cases} \dfrac{2t-t_1-t_2}{t_1-t_2} & \text{for} \quad t < t_1, \\ 1 & \text{for} \quad t_1 \leqslant t \leqslant t_2, \\ \dfrac{2t-t_1-t_2}{t_2-t_1} & \text{for} \quad t > t_2, \end{cases}$$

and for $t < t_1$, $t > t_2$ we have $|\varphi_1(t)|+|\varphi_2(t)| > 1$.

CHAPTER II

System of optimal polynomials

The subject of this chapter was explained in § 1; the main results are formulated as corollaries at the end of §§ 8 and 10 as theorems 9.1-9.4.

8. Necessary and sufficient conditions in approximation without nodes. We shall prove

THEOREM 8.1. *If the polynomials* $\varphi_n \in \mathcal{W}_n$, $\varphi_{n+1} \in \mathcal{W}_{n+1}$ *are optimal for the function* ξ *in the interval* I *and in the classes* $\mathcal{W}_n$, $\mathcal{W}_{n+1}$ *respectively, and if* $\deg \varphi_{n+1} = n+1$, *then the equation*

(1) $$\varphi_n(t) = \varphi_{n+1}(t)$$

has exactly $n+1$ *different single roots in the interior of* I.

Proof. Let us remark first that the hypothesis $\deg \varphi_{n+1} = n+1$ is equivalent to $\varphi_n \neq \varphi_{n+1}$ and equivalent to

(2) $$\varepsilon_{n+1}(\xi) < \varepsilon_n(\xi).$$

This follows from the unicity of optimal polynomials ([4], p. 55). Hence

(3) $$\|\gamma_{n+1}\|_I < \|\gamma_n\|_I,$$

where $\gamma_n = \varphi_n - \xi$, $\gamma_{n+1} = \varphi_{n+1} - \xi$.

Let $p_1, p_2, \ldots, p_{n+2}$ be the (e) points of φ_n written in their natural order and such that

(4) $$\gamma_n(p_i) = (-1)^i c\varepsilon_n(\xi) \qquad (i = 1, 2, \ldots, n+2),$$

where $c^2 = 1$; the existence of these points follows from the theorem of Tchebyshev ([4], p. 51). By (3) and (4)

$$\operatorname{sign}\big(\gamma_n(p_i) - \gamma_{n+1}(p_i)\big) = \operatorname{sign}\gamma_n(p_i) = (-1)^i c \neq 0$$

and in each of the open intervals $(p_1, p_2), (p_2, p_3), \ldots, (p_{n+1}, p_{n+2})$ there is at least one root of $\gamma_n - \gamma_{n+1} = \varphi_n - \varphi_{n+1}$. Since $\varphi_n - \varphi_{n+1}$ is of degree $n+1$, each of these intervals contains precisely one single root of (1), q. e. d.

THEOREM 8.2. *If the polynomials $\psi_n \in \mathcal{W}_n$, $\psi_{n+1} \in \mathcal{W}_{n+1}$ are optimal for the function ξ in the interval I and the classes $\mathcal{W}_n$, $\mathcal{W}_{n+1}$ respectively and* $\deg \psi_{n+1} = n+1$, *then*

$$(5) \qquad \varepsilon_n(\xi) + \varepsilon_{n+1}(\xi) \geqslant \|\psi_n - \psi_{n+1}\|_{\langle u_1, u_{n+1}\rangle},$$

$$(6) \qquad \varepsilon_n(\xi) - \varepsilon_{n+1}(\xi) \leqslant \min_{1\leqslant i\leqslant n} \|\psi_n - \psi_{n+1}\|_{\langle u_i, u_{i+1}\rangle},$$

$$(7) \qquad \begin{aligned} \zeta_{\min}\big(\varepsilon_n(\xi) + \varepsilon_{n+1}(\xi)\big) \leqslant a \leqslant \zeta_{\min}\big(\varepsilon_n(\xi) - \varepsilon_{n+1}(\xi)\big), \\ \zeta_{\max}\big(\varepsilon_n(\xi) - \varepsilon_{n+1}(\xi)\big) \leqslant b \leqslant \zeta_{\max}\big(\varepsilon_n(\xi) + \varepsilon_{n+1}(\xi)\big), \end{aligned}$$

where $u_1, u_2, \ldots, u_{n+1}$ denote the roots of (1) written in their natural order, and $\zeta_{\min}(h)$ and $\zeta_{\max}(h)$ are, for $h > 0$, the smallest and the greatest of the roots of the equation $|\psi_n(t) - \psi_{n+1}(t)| = h$, respectively.

Proof. By hypothesis, for $t \in I$ we have $|\xi(t) - \psi_n(t)| \leqslant \varepsilon_n(\xi)$,

$$(8) \qquad |\xi(t) - \psi_{n+1}(t)| \leqslant \varepsilon_{n+1}(\xi).$$

By addition, we obtain for $t \in I$

$$(9) \qquad |\psi_n(t) - \psi_{n+1}(t)| \leqslant \varepsilon_n(\xi) + \varepsilon_{n+1}(\xi).$$

Since, by theorem 8.1, $\langle u_1, u_{n+1}\rangle \subset I$, (9) implies that

$$\|\psi_n - \psi_{n+1}\|_{\langle u_1, u_{n+1}\rangle} \leqslant \varepsilon_n(\xi) + \varepsilon_{n+1}(\xi),$$

whence (5) is satisfied.

If t is an (e) point of ψ_n, then $|\xi(t) - \psi_n(t)| = \varepsilon_n(\xi)$ and, consequently,

$$(10) \qquad |\psi_n(t) - \psi_{n+1}(t)| \geqslant |\xi(t) - \psi_n(t)| - |\xi(t) - \psi_{n+1}(t)| \geqslant \varepsilon_n(\xi) - \varepsilon_{n+1}(\xi) > 0,$$

since, as in theorem 8.1, inequality (2) holds. Hence the roots of (1) are not (e) points. On the other hand, between any two (e) points of different kind of the polynomial ψ_n is a root of the equation (1). Indeed, if, for example, $\xi(p') - \psi_n(p') = \varepsilon_n(\xi)$, $\xi(p'') - \psi_n(p'') = -\varepsilon_n(\xi)$ where $p' < p''$, then from inequality (8), valid for the points p', p'', it follows that $\psi_{n+1}(p') - \psi_n(p') > 0$, $\psi_{n+1}(p'') - \psi_n(p'') < 0$ and in (p', p'') there is a point at which $\psi_{n+1} - \psi_n$ is equal to zero.

Thus we have shown that each of the intervals $\langle a, u_1)$, (u_1, u_2), $\ldots$, (u_n, u_{n+1}), $(u_{n+1}, b\rangle$, whose bounds $u_1, u_2, \ldots, u_{n+1}$ are all roots of (1), only (e) points of the same kind ((+) points or (−) points) can lie and that in consecutive intervals they are of different kind. At the same time a theorem of Tchebyshev ([4], p. 51) states that the alternant of ψ_n contains at least $n+2$ points which are alternately (+) and (−) points. It follows that each of the intervals (u_1, u_2), (u_2, u_3), $\ldots$, (u_n, u_{n+1})

contains an (e) point of ψ_n. That point satisfies (10), whence it follows that $\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi) \leqslant \|\psi_n-\psi_{n+1}\|_{\langle u_i, u_{i+1}\rangle}$ for $i = 1, 2, \ldots, n$. Hence (6) is true.

There exist also (e) points $p_a \epsilon \langle a, v_1)$, $p_b \epsilon (v_{n+1}, b\rangle$ satisfying (10). The function $|\psi_n-\psi_{n+1}|$ decreases in $\langle a, v_1)$ and increases in $(v_{n+1}, b\rangle$. Hence using the notation introduced above, we have

$$(11) \qquad \zeta_{\min}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big) \epsilon \langle a, v_1), \qquad \zeta_{\max}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big) \epsilon (v_{n+1}, b\rangle,$$

since by (2) and by the definition of (e) points p_a, p_b

$$\zeta_{\min}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big) < v_1, \qquad \zeta_{\max}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big) > v_{n+1},$$

$$\Big|\psi_n\Big(\zeta_{\min}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big)\Big)-\psi_{n+1}\Big(\zeta_{\min}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big)\Big)\Big|$$
$$= \varepsilon_n(\xi)-\varepsilon_{n+1}(\xi) \leqslant |\psi_n(p_a)-\psi_{n+1}(p_a)|,$$

$$\Big|\psi_n\Big(\zeta_{\max}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big)\Big)-\psi_{n+1}\Big(\zeta_{\max}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big)\Big)\Big|$$
$$= \varepsilon_n(\xi)-\varepsilon_{n+1}(\xi) \leqslant |\psi_n(p_b)-\psi_{n+1}(p_b)|.$$

From (11) it follows that

$$a \leqslant \zeta_{\min}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big), \qquad b \geqslant \zeta_{\max}\big(\varepsilon_n(\xi)-\varepsilon_{n+1}(\xi)\big).$$

The rest of conditions (7) may obtained analogously from inequality (9), valid for $t = a$ and $t = b$.

In the following theorems we shall continue to use the symbols $\zeta_{\min}(h)$, $\zeta_{\max}(h)$.

THEOREM 8.3. *Let the polynomials $\psi_n \epsilon \mathcal{W}_n$, $\psi_{n+1} \epsilon \mathcal{W}_{n+1}$, the numbers e_n, e_{n+1} and the interval $I = \langle a, b\rangle$ satisfy the following conditions: $\deg \psi_{n+1} = n+1$, the equation (1) has $n+1$ different real roots $u_1 < u_2 < \ldots < u_{n+1}$ and*

$$(12) \qquad 0 \leqslant e_{n+1} < e_n,$$

$$(13) \qquad e_n + e_{n+1} \geqslant \|\psi_n-\psi_{n+1}\|_{\langle u_1, u_{n+1}\rangle},$$

$$(14) \qquad e_n - e_{n+1} \leqslant \min_{1\leqslant i\leqslant n} \|\psi_n-\psi_{n+1}\|_{\langle u_i, u_{i+1}\rangle},$$

$$(15) \qquad \begin{aligned} \zeta_{\min}(e_n+e_{n+1}) \leqslant a \leqslant \zeta_{\min}(e_n-e_{n+1}),\\ \zeta_{\max}(e_n-e_{n+1}) \leqslant b \leqslant \zeta_{\max}(e_n+e_{n+1}); \end{aligned}$$

then there exists a continuous function ξ for which the polynomials ψ_n, ψ_{n+1} are optimal in I and respectively in classes $\mathcal{W}_n, \mathcal{W}_{n+1}$, and $\varepsilon_n(\xi) = e_n$, $\varepsilon_{n+1}(\xi) = e_{n+1}$.

Proof. We define the sets

$$(16)\qquad P = \mathop{E}_{(t,x)}\{|x-\psi_n(t)| \leqslant e_n \quad \text{and} \quad |x-\psi_{n+1}(t)| \leqslant e_{n+1}\},$$

$$A_n^+ = \mathop{E}_{t}\{(t, \psi_n(t)+e_n)\in P\}, \quad A_n^- = \mathop{E}_{t}\{(t, \psi_n(t)-e_n)\in P\}, \quad A_n = A_n^+ \cup A_n^-,$$

$$(17)\quad A_{n+1}^+ = \mathop{E}_{t}\{(t, \psi_{n+1}(t)+e_{n+1})\in P\}, \quad A_{n+1}^- = \mathop{E}_{t}\{(t, \psi_{n+1}(t)-e_{n+1})\in P\}.$$

These sets have the following meaning: the graph of every continuous function for which the polynomials ψ_n, ψ_{n+1} are optimal in classes $\mathcal{W}_n$, $\mathcal{W}_{n+1}$ with the deviations e_n, e_{n+1} respectively and with an arbitrary interval of approximation lies in the set P. For every function of this type the $(+)$ points of ψ_n belong to A_n^+, the $(-)$ points to A_n^-, and all the (e) points are in A_n. Analogously the $(+)$ points of ψ_{n+1} belong to A_{n+1}^+ and the $(-)$ points to A_{n+1}^-. We shall first examine the properties of these sets and then apply them in the construction of the function ξ.

We prove first that the interval I satisfying (15) is contained in the projection P^* of P on the axis t. This is a necessary condition for the existence of the function ξ in this interval. By (16) t is in P^* if, and only if, there exists a point x_t such that $-e_n \leqslant x_t-\psi_n(t) \leqslant e_n$, $-e_{n+1} \leqslant x_t - \psi_{n+1}(t) \leqslant e_{n+1}$, *i.e.* such that $-e_n+\psi_n(t) \leqslant x_t \leqslant e_n+\psi_n(t)$, $-e_{n+1}+\psi_{n+1}(t) \leqslant x_t \leqslant e_{n+1}+\psi_{n+1}(t)$.

The number x_t satisfying these inequalities exists if, and only if, $-e_n+\psi_n(t) \leqslant e_{n+1}+\psi_{n+1}(t)$, $-e_{n+1}+\psi_{n+1}(t) \leqslant e_n+\psi_n(t)$, *i. e.* if $|\psi_n(t) - \psi_{n+1}(t)| \leqslant e_n+e_{n+1}$, whence

$$P^* = \mathop{E}_{t}\{|\psi_n(t)-\psi_{n+1}(t)| \leqslant e_n+e_{n+1}\}.$$

By the definition of the functions $\zeta_{\min}$ and $\zeta_{\max}$ and by (12) and (15)

$$(18)\qquad \begin{aligned} &\zeta_{\min}(e_n+e_{n+1}) \leqslant a \leqslant \zeta_{\min}(e_n-e_{n+1}) < v_1, \\ &v_{n+1} < \zeta_{\max}(e_n-e_{n+1}) \leqslant b \leqslant \zeta_{\max}(e_n+e_{n+1}). \end{aligned}$$

Thus, since $|\psi_n-\psi_{n+1}|$ is decreasing for $t<v_1$ and increasing for $t>u_{n+1}$, in the intervals $\langle a, u_1\rangle$, $\langle u_{n+1}, b\rangle$ the inequality $|\psi_n(t) - \psi_{n+1}(t)| \leqslant e_n+e_{n+1}$ holds, which together with (13) leads to $\|\psi_n - \psi_{n+1}\|_I \leqslant e_n+e_{n+1}$. It follows that $I\subset P^*$.

By (16) and by the definition of A_n^+ we have

$$|\psi_n(t)+e_n-\psi_n(t)| \leqslant e_n, \quad |\psi_n(t)+e_n-\psi_{n+1}(t)| \leqslant e_{n+1} \quad (t\in A_n^+).$$

The first of these inequalities is an identity, whence in virtue of the second the definition of A_n^+ may be written in the form

$$(19)\qquad A_n^+ = \mathop{E}_{t}\{|\psi_n(t)-\psi_{n+1}(t)+e_n| \leqslant e_{n+1}\}.$$

Similarly

$$A_n^- = \underset{t}{E}\{|\varphi_n(t)-\varphi_{n+1}(t)-e_n| \leqslant e_{n+1}\}. \tag{20}$$

The union of (19) and (20) is equal to

$$A_n = \underset{t}{E}\{-e_n-e_{n+1} \leqslant \varphi_n(t)-\varphi_{n+1}(t) \leqslant -e_n+e_{n+1} \quad \text{or}$$
$$e_n-e_{n+1} \leqslant \varphi_n(t)-\varphi_{n+1}(t) \leqslant e_n+e_{n+1}\},$$

whence

$$A_n = \underset{t}{E}\{e_n-e_{n+1} \leqslant |\varphi_n(t)-\varphi_{n+1}(t)| \leqslant e_n+e_{n+1}\}. \tag{21}$$

On the other hand, the intersection

$$A_n^+ \cap A_n^- = \underset{t}{E}\{-e_n-e_{n+1} \leqslant \varphi_n(t)-\varphi_{n+1}(t) \leqslant -e_n+e_{n+1} \quad \text{and}$$
$$e_n-e_{n+1} \leqslant \varphi_n(t)-\varphi_{n+1}(t) \leqslant e_n+e_{n+1}\}$$

is empty, for, by hypothesis (12), $-e_n+e_{n+1} < e_n-e_{n+1}$.

By (17)

$$A_{n+1}^+ = \underset{t}{E}\{|\varphi_{n+1}(t)-\varphi_n(t)+e_{n+1}| \leqslant e_n\}, \tag{22}$$

$$A_{n+1}^- = \underset{t}{E}\{|\varphi_{n+1}(t)-\varphi_n(t)-e_{n+1}| \leqslant e_n\}, \tag{23}$$

and by (19), (22) and (20), (23)

$$\begin{aligned} A_n^+ \cap A_{n+1}^+ &= \underset{t}{E}\{\varphi_n(t)-\varphi_{n+1}(t) = e_{n+1}-e_n\}, \\ A_n^- \cap A_{n+1}^- &= \underset{t}{E}\{\varphi_n(t)-\varphi_{n+1}(t) = e_n-e_{n+1}\}, \end{aligned} \tag{24}$$

$$A_n^+ \cup A_{n+1}^+ = A_n^- \cup A_{n+1}^- = \underset{t}{E}\{|\varphi_n(t)-\varphi_{n+1}(t)| \leqslant e_n+e_{n+1}\} = P^*, \tag{25}$$

$$A_{n+1}^+ \cap A_{n+1}^- = \underset{t}{E}\{|\varphi_n(t)-\varphi_{n+1}(t)| \leqslant e_n-e_{n+1}\}. \tag{26}$$

Let us consider the intervals

$$\langle a, u_1), \ (u_1, u_2), \ \ldots, \ (u_n, u_{n+1}), \ (u_{n+1}, b\rangle. \tag{27}$$

The first and the last are non-empty by (18). The set A_n has points in every of the intervals (27). Indeed, the point $p_k \epsilon (u_k, u_{k+1})$ where $k = 1, 2, \ldots, n$, defined by $|\varphi_n(p_k)-\varphi_{n+1}(p_k)| = e_n-e_{n+1}$ and existing because of (14), belongs to A_n according to (21). It follows from (18) that also in the intervals

$$\langle a, u_1), \ (u_{n+1}, b\rangle \tag{28}$$

there are points of A_n; in particular such points are a and b. On the other hand $u_1, u_2, \ldots, u_{n+1}$ do not belong to A_n, for, as roots of (1), they do not satisfy the inequality $|\psi_n(t)-\psi_{n+1}(t)| \geqslant e_n - e_{n+1} > 0$.

In each of the intervals (u_1, u_2), $(u_2, u_3), \ldots, (u_n, u_{n+1})$ the polynomial $\psi_n - \psi_{n+1}$ has one extremum (being of degree $n+1$ it cannot have more than one), and in the intervals (28) is a monotone function. Thus the sets $A_n \cap \langle a, u_1)$, $A_n \cap (u_k, u_{k+1})$ for $k = 1, 2, \ldots, n$, $A_n \cap$ $\cap (u_{n+1}, b\rangle$ lying in I are disjoint closed intervals, which may be reduced to single points. Their union is the set

$$A_n \cap I = \bigcup_{k=0}^{n+1} I_k$$

where

$$(29)\qquad \begin{cases} I_0 = \langle a, a_0''\rangle \subset \langle a, u_1), \\ I_k = \langle a_k', a_k''\rangle \subset (u_k, u_{k+1}) \quad (k = 1, 2, \ldots, n), \\ I_{n+1} = \langle a_{n+1}', b\rangle \subset (u_{n+1}, b\rangle, \end{cases}$$

(whence

$$(30)\qquad a \leqslant a_0'' < a_1' \leqslant a_1'' < \ldots < a_n' \leqslant a_n'' < a_{n+1}' \leqslant b).$$

The points a and b are the bounds of the intervals I_0, I_{n+1}, respectively, for, as we have observed, $a \in A_n, b \in A_n$. The intervals (29), separated from one another by the roots of (1), belong alternately to the sets A_n^+ and A_n^-, since by (19) and (20) $\operatorname{sign}\big(\psi_n(t)-\psi_{n+1}(t)\big)$ is equal to -1 when $t \in A_n^+$ and to 1 when $t \in A_n^-$. Hence

$$(31)\qquad \begin{aligned} A_n^+ \cap I &= I_0 \cup I_2 \cup \ldots = \langle a, a_0''\rangle \cup \langle a_2', a_2''\rangle \cup \ldots, \\ A_n^- \cap I &= I_1 \cup I_3 \cup \ldots = \langle a_1', a_1''\rangle \cup \langle a_3', a_3''\rangle \cup \ldots, \end{aligned}$$

or, conversely,

$$(32)\qquad \begin{aligned} A_n^+ \cap I &= I_1 \cup I_3 \cup \ldots = \langle a_1', a_1''\rangle \cup \langle a_3', a_3''\rangle \cup \ldots, \\ A_n^- \cap I &= I_0 \cup I_2 \cup \ldots = \langle a, a_0''\rangle \cup \langle a_2', a_2''\rangle \cup \ldots \end{aligned}$$

By (22) and (23) it follows that the sets A_{n+1}^+, A_{n+1}^- are composed of a finite number of intervals and from (24) we infer that intersections $A_n^+ \cap A_{n+1}^+, A_n^- \cap A_{n+1}^-$ consist of a finite number of points at which $|\psi_n(t)-\psi_{n+1}(t)| = e_n - e_{n+1}$. By (21) these points are the bounds of the intervals whose union is the set A_n. Thus, they are the points a_0'', a_1', $a_1'', \ldots, a_n', a_n'', a_{n+1}'$. Moreover, taking into account the fact that, by (25), $I \subset A_n^+ \cup A_{n+1}^+$, $I \subset A_n^- \cup A_{n+1}^-$, we get in the case (31)

$$(33)\qquad \begin{aligned} A_{n+1}^+ \cap I &= \langle a_0'', a_2'\rangle \cup \langle a_2'', a_4'\rangle \cup \ldots, \\ A_{n+1}^- \cap I &= \langle a, a_1'\rangle \cup \langle a_1'', a_3'\rangle \cup \ldots, \end{aligned}$$

and in the case (32)

$$(34)\qquad \begin{aligned} A_{n+1}^{+}\cap I &= \langle a, a_1'\rangle \cup \langle a_1'', a_3'\rangle \cup \ldots, \\ A_{n+1}^{-}\cap I &= \langle a_0'', a_2'\rangle \cup \langle a_2'', a_4'\rangle \cup \ldots \end{aligned}$$

By (30), and by (33) or (34)

$$(35)\qquad A_{n+1}^{+}\cap A_{n+1}^{-}\cap I = \langle a_0'', a_1'\rangle \cup \langle a_1'', a_2'\rangle \cup \ldots \cup \langle a_n'', a_{n+1}'\rangle .$$

Consider now the case (31), (33) (the analogous case (32), (34) will not be considered). Thus

$$(36)\qquad a_0'', a_2', a_2'', a_4', a_4'', \ldots \in A_n^{+}\cap A_{n+1}^{+},$$

$$(37)\qquad a_1', a_1'', a_3', a_3'', \ldots \in A_n^{-}\cap A_{n+1}^{-}.$$

Let us set

$$\xi(t) = \begin{cases} \psi_n(t) + e_n & \text{for } t\in A_n^{+}\cap I, \quad (38)\\ \psi_n(t) - e_n & \text{for } t\in A_n^{-}\cap I, \quad (39)\\ \psi_{n+1}(t) + \dfrac{6e_{n+1}}{a_1'-a_0''}\left(\dfrac{5a_0''+a_1'}{6} - t\right) & \text{for } t\in\left\langle a_0'', \dfrac{2a_0''+a_1'}{3}\right\rangle, \quad (40)\\ \psi_{n+1}(t) + \dfrac{6e_{n+1}}{a_1'-a_0''}\left(t - \dfrac{a_0''+a_1'}{2}\right) & \text{for } t\in\left\langle \dfrac{2a_0''+a_1'}{3}, \dfrac{a_0''+2a_1'}{3}\right\rangle, \quad (41)\\ \psi_{n+1}(t) + \dfrac{6e_{n+1}}{a_1'-a_0''}\left(\dfrac{a_0''+5a_1'}{6} - t\right) & \text{for } t\in\left\langle \dfrac{a_0''+2a_1'}{3}, a_1'\right\rangle, \quad (42)\\ \psi_{n+1}(t) + \dfrac{2(-1)^k e_{n+1}}{a_{k+1}'-a_k''}\left(\dfrac{a_k''+a_{k+1}'}{2} - t\right) & \text{for } t\in\langle a_k'', a_{k+1}'\rangle, \quad (43)\\ & k = 1, 2, \ldots, n. \end{cases}$$

Since $I = \langle a, b\rangle$ is the union of the following non-empty and disjoint intervals

$$I_0 = \langle a, a_0''\rangle,\quad (a_0'', a_1'),\quad I_1 = \langle a_1', a_1''\rangle,\quad (a_1'', a_2'),\quad \ldots,$$
$$I_n = \langle a_n', a_n''\rangle,\quad (a_n'', a_{n+1}'),\quad I_{n+1} = \langle a_{n+1}', b\rangle,$$

formulae (38)-(43) define the function ξ in the whole of I. It must be proved that the individual definitions of ξ are not contradictory with respect to one another; this will imply directly that ξ is continuous.

The function ξ is defined by two formulae only at the points

$$\frac{2a_0''+a_1'}{3},\ \frac{a_0''+2a_1'}{3},\ a_0'',\ a_1',\ a_1'',\ \ldots,\ a_n'',\ a_{n+1}'.$$

The consistency of the two definitions at the first two points follows directly from (40)-(42). Concerning the remaining points, at points (36) we obtain from (38), (40), (43)

$$\xi(t) = \psi_n(t)+e_n, \qquad \xi(t) = \psi_{n+1}(t)+e_{n+1}. \tag{44}$$

These relations are equivalent in virtue of (36) and the first of formulae (24). Analogously at points (37) we have $\xi(t) = \psi_n(t)-e_n$ and $\xi(t) = \psi_{n+1}(t)-e_{n+1}$, which are equivalent by (37) and the second of formulae (24).

Thus it is proved that ξ is continuous. We shall prove now that

$$\|\xi-\psi_n\|_I = e_n \quad \text{and} \quad \|\xi-\psi_{n+1}\|_I = e_{n+1}. \tag{45}$$

The inequality

$$|\xi(t)-\psi_n(t)| \leqslant e_n \quad \text{for} \quad t\in A_n^+\cap I \tag{46}$$

follows directly by definitions (38) and (39). Similarly, from (40)-(43) it follows that in the rest of the interval I, *i. e.* in the set (35), we have

$$|\xi(t)-\psi_{n+1}(t)| \leqslant e_{n+1}, \tag{47}$$

since the graph of $\xi-\psi_{n+1}$ over this set consists of segments of a straight line whose end-points lie on the straight lines $x = e_{n+1}$ and $x = -e_{n+1}$. Inequality (46) in the set (35) results from the addition of inequalities (47) to the inequality $|\psi_n(t)-\psi_{n+1}(t)| \leqslant e_n-e_{n+1}$ resulting from (26). Finally, inequality (47) in the set A_n^+ is a consequence of (38) and (19), and for the set A_n^- it results from (39) and (20). Equalities (45) follow from formulae (46), (47) valid in the whole of I and from the fact that at points (36) belonging to I formulae (44) are satisfied. Finally, let us notice that

1° at the $n+2$ points $a_0', a_1', a_2', \ldots, a_{n+1}'$ we have, by (36)-(39), $\xi(a_0'')-\psi_n(a_0'') = e_n$, $\xi(a_k')-\psi_n(a_k') = (-1)^k e_n$ for $k = 1, 2, \ldots, n+1$, whence ψ_n is optimal for the function ξ in class $\mathcal{W}_n^0$ in the interval I with deviation e_n;

2° at the $n+3$ points

$$a_0'', \frac{2a_0''+a_1'}{3}, \frac{a_0''+2a_1'}{3}, a_1', a_2', \ldots, a_n'$$

we have, by (40)-(43),

$$\xi(a_0'')-\psi_{n+1}(a_0'') = e_{n+1},$$

$$\xi\left(\frac{2a_0''+a_1'}{3}\right)-\psi_{n+1}\left(\frac{2a_0''+a_1'}{3}\right) = -e_{n+1},$$

$$\xi\left(\frac{a_0''+2a_1'}{3}\right)-\psi_{n+1}\left(\frac{a_0''+2a_1'}{3}\right) = e_{n+1},$$

$$\xi(a_k')-\psi_{n+1}(a_k') = (-1)^k e_{n+1} \qquad (k = 1, 2, \ldots, n),$$

whence ψ_{n+1} is optimal for ξ in class $\mathcal{W}_{n+1}$ in the interval I, with deviation e_{n+1}.

Thus theorem 8.3 is completely proved.

Theorems 8.1-8.3 enable us to formulate the following

COROLLARY. *Suppose we are given the polynomials $\psi_n \in \mathcal{W}_n$ and $\psi_{n+1} \in \mathcal{W}_{n+1} - \mathcal{W}_n$, numbers e_n and e_{n+1}, and the interval $I = \langle a, b\rangle$. Then there exists a continuous function ξ for which ψ_n and ψ_{n+1} are optimal polynomials in I, in the classes $\mathcal{W}_n$ and $\mathcal{W}_{n+1}$ respectively, and with deviations $\varepsilon_n(\xi) = e_n$, $\varepsilon_{n+1}(\xi) = e_{n+1}$, if, and only if, the equation $\psi_n(t) = \psi_{n+1}(t)$ has $n+1$ different real roots $u_1 < u_2 < \ldots < u_{n+1}$ and the following conditions are satisfied: $0 \leqslant e_{n+1} < e_n$,*

(48) $$e_n + e_{n+1} \geqslant \|\psi_n - \psi_{n+1}\|_{\langle u_1, u_{n+1}\rangle},$$

(49) $$e_n - e_{n+1} \leqslant \min_{1 \leqslant i \leqslant n} \|\psi_n - \psi_{n+1}\|_{\langle u_i, u_{i+1}\rangle},$$

$$\zeta_{\min}(e_n + e_{n+1}) \leqslant a \leqslant \zeta_{\min}(e_n - e_{n+1}), \qquad \zeta_{\max}(e_n - e_{n+1}) \leqslant b \leqslant \zeta_{\max}(e_n + e_{n+1}).$$

As above, $\zeta_{\min}(h)$ and $\zeta_{\min}(h)$ denote for $h > 0$ the smallest and the greatest root of the equation $|\psi_n(t) - \psi_{n+1}(t)| = h$.

9. Corollaries. From the corollary given above we obtain the following theorem on the relative position of the alternants of the optimal polynomials ψ_n and ψ_{n+1}.

THEOREM 9.1. *Let the polynomials ψ_n and ψ_{n+1} be optimal for the function ξ in the classes $\mathcal{W}_n$ and $\mathcal{W}_{n+1}$ respectively. If $v_0, v_1, \ldots, v_{n+1}$ ($v_0 < v_1 < \ldots < v_{n+1}$) are alternately (+) and (−) points of the polynomial ψ_n, and $w_0, w_1, \ldots, w_{n+2}$ ($w_0 < w_1 < \ldots < w_{n+2}$) are alternately (+) points and (−) points of ψ_{n+1}, then the inequalities*

(1) $$w_0 < v_0, \qquad v_{n+1} < w_{n+2}$$

cannot be satisfied simultaneously.

Proof. We shall use the facts established in proving theorem 8.3; we retain the notation used there. We know that $v_k \in I_k$ for $k = 0, 1, \ldots, n+1$, in particular from (8.29) it follows that $v_0 \in \langle a, a_0''\rangle$, $v_{n+1} \in \langle a_{n+1}', b\rangle$. If inequalities (1) are satisfied, then $w_0 < v_0 \leqslant a_0'' < a_1'$ and $w_0 \in \langle a, a_1'\rangle$. Similarly $w_{n+2} > v_{n+1} \geqslant a_{n+1}' > a_n''$ and $w_{n+2} \in \langle a_n'', b\rangle$. Since $I_0 = \langle a, a_0''\rangle$, then from (8.31) and (8.33) or from (8.32) and (8.34) it follows that $v_0 \in A_n^+$ and $w_0 \in A_{n+1}^-$ or $v_0 \in A_n^-$ and $w_0 \in A_{n+1}^+$, whence v_0 and w_0 are (e) points of different kind.

The points v_{n+1} and w_{n+2} are also of different kind. Indeed, let n be even, for example. Then in the case of (8.31) and (8.33)

$$A_n^- \cap I = \langle a_1', a_1'' \rangle \cup \ldots \cup \langle a_{n+1}', b \rangle, \quad A_{n+1}^+ \cap I = \langle a_0'', a_2' \rangle \cup \ldots \cup \langle a_n'', b \rangle.$$

From the inclusions proved for v_{n+1} and w_{n+2} it follows that $v_{n+1} \in A_n^-$, $w_{n+2} \in A_{n+1}^+$. In the case of (8.32) and (8.34) we have, however, $v_{n+1} \in A_n^+$, $w_{n+2} \in A_{n+1}^-$. This property of the points v_{n+1}, w_{n+2} is in contradiction with the alternating behaviour of the (e) points in the sequences $v_0, v_1, \ldots, v_{n+1}$ and $w_0, w_1, \ldots, w_{n+2}$: if v_0 and w_0 are (e) points of different kind, then v_{n+1} and w_{n+2}, whose indices differ by 1, must be (e) points of the same kind. Hence at least one of inequalities (1) is not satisfied.

From theorem 9.1 it follows in particular that the alternants cannot separate themselves mutually in the strict sense; we mean there by that $w_0 < v_0 < w_1 < v_1 < \ldots < w_{n+1} < v_{n+1} < w_{n+2}$.

We now give two theorems on the Tchebyshev polynomials $T_n = \cos(n \arccos t)$. Let us notice first that by their definition 1° $\|T_n\|_{\langle -1,1\rangle} = 1$, 2° at the points $p_k = -\cos k\pi/n$ we have

$$T_n(p_k) = (-1)^{n-k} \qquad (k = 0, 1, \ldots, n), \tag{2}$$

3° $-\cos\pi/2n$ and $\cos\pi/2n$ are respectively the smallest and the greatest root of the nth Tchebyshev polynomial.

Among the many properties of the Tchebyshev polynomials the following is also known ([4], p. 78): for every $\varphi \in \mathcal{W}_n$, not identically null and for every t such that $|t| \geqslant 1$ we have $|\varphi(t)|/\|\varphi\|_{\langle -1,1\rangle} \leqslant |T_n(t)|$. This means that among the polynomials whose absolute value has the maximum in $\langle -1, 1\rangle$ equal to 1 the Tchebyshev polynomial tends to ∞ the most rapidly outside this interval.

Theorem 9.2 is a modification of this theorem enabling us to estimate from above the rapidity of growth of the polynomial in a wider set than the half-lines $(-\infty, -1\rangle$ and $\langle 1, +\infty)$, the class of polynomials being at the same time restricted.

THEOREM 9.2. *If the polynomial φ satisfies the condition* $\deg\varphi = n \geqslant 2$ *and has real roots the smallest of which is equal to* $-\cos\pi/2n$ *and the greatest to* $\cos\pi/2n$, *then* $|t| \geqslant \cos\pi/2n$ *implies*

$$\frac{|\varphi(t)|}{\|\varphi\|_B} \leqslant |T_n(t)| \quad \textit{where} \quad B = \left\langle -\cos\frac{\pi}{2n}, \cos\frac{\pi}{2n} \right\rangle. \tag{3}$$

Proof. It is sufficient to prove the theorem for the polynomials φ satisfying $|\varphi|/\|\varphi\|_B \neq |T_n|$. We may suppose without loss of generality that

$$\varphi(t) > 0 \quad \text{for} \quad t > \cos\frac{\pi}{2n}. \tag{4}$$

Let us set $\delta = \|\varphi\|_B T_n - \varphi$; thus $\delta \neq 0$.

We shall show that δ has n roots in the interval B, whence it follows that it has precisely n roots. Let us consider the points $p_1, p_2, \ldots, p_{n-1}$ defined above. If

(5) $$(-1)^{n-k}\varphi(p_k) = \|\varphi\|_B,$$

then by (2): $\delta(p_k) = 0$. Also $T_n'(p_k) = 0$. Since $p_k \in \operatorname{Int} B$, the definition of the norm $\|\gamma\|_B$ implies $\varphi'(p_k) = 0$. Hence $\delta'(p_k) = 0$, and the point p_k satisfying (5) is a double root of δ.

If, for a given i

(6) $$(-1)^{n-i}\varphi(p_i) < \|\varphi\|_B, \quad (-1)^{n-i-1}\varphi(p_{i+1}) < \|\varphi\|_B,$$

then, by (2),

(7) $$\operatorname{sign}\delta(p_i) = \operatorname{sign}T_n(p_i) = (-1)^{n-i},$$

$$\operatorname{sign}\delta(p_{i+1}) = \operatorname{sign}T_n(p_{i+1}) = -\operatorname{sign}\delta(p_i),$$

and there exists a point $t \in (p_i, p_{i+1})$ at which $\delta(t) = 0$. Hence, if (5) is satisfied for exactly l values of the index k, then for $2l < n-2$ among the $n-2$ intervals $(p_1, p_2), (p_2, p_3), \ldots, (p_{n-2}, p_{n-1})$ there exist at least $n-2-2l$ intervals satisfying (6). Since

$$\delta\left(-\cos\frac{\pi}{2n}\right) = \delta\left(\cos\frac{\pi}{2n}\right) = 0, \quad -\cos\frac{\pi}{2n} < p_1 < p_{n-1} < \cos\frac{\pi}{2n},$$

we have proved the existence of $2l + (n-2-2l) + 2 = n$ real roots of δ (this is also true if $2l \geqslant n-2$).

Let us notice also that (5) is not satisfied for $k = 1$ and $k = n-1$. Indeed, in the contrary case there would exist more than $n-2-2l$ intervals (p_i, p_{i+1}) satisfying (6), and δ, having $n+1$ roots, would be identically null.

Thus we have proved that all the roots of δ, except $-\cos\pi/2n$ and $\cos\pi/2n$, lie in the open interval (p_1, p_{n-1}). At the point p_{n-1} we have $T_n(p_{n-1}) = -1$, whence by (7) for $i = n-1$ it follows that $\delta(p_{n-1}) < 0$, and for $t > \cos\pi/2n$ the opposite inequality holds: $\delta(t) > 0$. By (4) for those t we have $\|\varphi\|_B T_n(t) > \varphi(t) > 0$, whence (3) follows. If $t < -\cos\pi/2n$, we prove this inequality using the fact that $\operatorname{sign}\varphi(t) = \operatorname{sign}T_n(t)$ in this case.

The following theorem does not present such analogies with the well-known properties for the Tchebyshev polynomials. It enables us, the class of polynomials being further restricted, to give estimations from below for the rapidity od growth of those polynomials, whence it supplements theorem 9.2.

THEOREM 9.3. *Let the polynomial* φ *satisfy the condition* $\deg\varphi = n \geqslant 2$ *and let it have* n *real roots* $u_1, u_2, \ldots, u_n$ *such that*

$$-\cos\frac{\pi}{2n} = u_1 < u_2 < \ldots < u_{n-1} < u_n = \cos\frac{\pi}{2n};$$

then, for $|t| \geqslant \cos\pi/2n$,

$$\frac{|\varphi(t)|}{\min\limits_k \|\varphi\|_{\langle u_k, u_{k+1}\rangle}} \geqslant |T_n(t)|. \tag{8}$$

Proof. The theorem is satisfied by the polynomial $\varphi = T_n$ for

$$\min_k \|T_n\|_{\langle u_k, u_{k+1}\rangle} = 1,$$

where $u_1, u_2, \ldots, u_n$ are the roots of T_n. Hence let us assume that

$$|\varphi| / \min_k \|\varphi\|_{\langle u_k, u_{k+1}\rangle} \neq |T_n|$$

let us retain condition (4), and let us set

$$f = \min_k \|\varphi\|_{\langle u_k, u_{k+1}\rangle},$$

$\delta = \varphi - fT_n$. Thus $\delta \neq 0$.

We define (uniquely) the numbers $v_1, v_2, \ldots, v_{n-1}$ by the relations $v_k \in (u_k, u_{k+1})$, $|\varphi(v_k)| = \|\varphi\|_{\langle u_k, u_{k+1}\rangle}$, where $k = 1, 2, \ldots, n-1$. Hence

$$|\varphi(v_k)| \geqslant f \quad (k = 1, 2, \ldots, n-1), \tag{9}$$

$$\varphi'(v_k) = 0 \quad (k = 1, 2, \ldots, n-1). \tag{10}$$

We shall prove that in the interval (v_1, v_{n-1}) there are $n-2$ roots of δ. If

$$\varphi(v_k) = fT_n(v_k) \tag{11}$$

is satisfied, then it follows by (9) that $f|T_n(v_k)| \geqslant f$. Since $\|T_n\|_{\langle -1,1\rangle} = 1$, we obtain $|T_n(v_k)| = 1$, $T_n'(v_k) = 0$ and by (10) $\delta'(v_k) = 0$. From the last equality and from (11) it follows that if (11) holds, then v_k is a double root of δ. On the other hand, if v_i, v_{i+1} are such that

$$\varphi(v_i) \neq fT_n(v_i), \quad \varphi(v_{i+1}) \neq fT_n(v_i), \tag{12}$$

then, by (9),

$$f|T_n(v_i)| \leqslant f \leqslant |\varphi(v_i)|, \quad f|T_n(v_{i+1})| \leqslant |\varphi(v_{i+1})|. \tag{13}$$

By hypothesis (12) the sign $=$ can appear in inequalities (13) only if $\varphi(v_i) = -fT_n(v_i)$, $\varphi(v_{i+1}) = -fT_n(v_{i+1})$ respectively. Taking into account that, by hypothesis, $\varphi(v_i)\varphi(v_{i+1}) < 0$, we infer that $\operatorname{sign}\delta(v_i)$

$= \operatorname{sign}\varphi(v_i)$, $\operatorname{sign}\delta(v_{i+1}) = \operatorname{sign}\varphi(v_{i+1}) = -\operatorname{sign}\delta(v_i)$ and in (v_i, v_{i+1}) there is a root of δ. As in the foregoing theorem, it follows that this polynomial has n real roots, which, except $-\cos\pi/2n$ and $\cos\pi/2n$, lie in (v_1, v_{n-1}).

At the point v_{n-1} we have, by (4), $\varphi(v_{n-1}) < 0$, and from the first of inequalities (13), valid also when $i = n-1$, it follows that $\delta(v_{n-1}) < 0$. Thus $t > \cos\pi/2n$ implies $\delta(t) > 0$, which together with (4) leads to inequality (8), which was to be proved. Analogously one can verify its validity for $t < -\cos\pi/2n$.

It is easy to construct an example showing that the hypothesis that φ has n distinct real roots is essential to maintain theorem 9.3.

Let us denotes by $\tau_{\min}^{(n)}(h)$ and $\tau_{\max}^{(n)}(h)$ for $h > 0$ respectively the smallest and the greatest of the roots of the equation $|T_n(t)| = h$.

THEOREM 9.4. *A necessary condition for the existence of a continuous function ξ for which the polynomials ψ_n and ψ_{n+1}, such that* $\deg\psi_n \leqslant n$, $\deg\psi_{n+1} = n+1$, *are optimal in $I = \langle a, b\rangle$ and in the classes $\mathcal{W}_n$ and $\mathcal{W}_{n+1}$ respectively, with deviations $\varepsilon_n(\xi) = e_n$, $\varepsilon_{n+1}(\xi) = e_{n+1}$, is that the following inequalities be satisfied:*

$$\tau_{\min}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right) \leqslant \frac{2a - (u_1 + u_{n+1})}{u_{n+1} - u_1} \cos\frac{\pi}{2(n+1)} \leqslant \tau_{\min}^{(n+1)}\left(\frac{e_n - e_{n+1}}{g}\right),$$

$$\tau_{\max}^{(n+1)}\left(\frac{e_n - e_{n+1}}{g}\right) \leqslant \frac{2b - (u_1 + u_{n+1})}{u_{n+1} - u_1} \cos\frac{\pi}{2(n+1)} \leqslant \tau_{\max}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right),$$

where

$$f = \min_k \|\psi_n - \psi_{n+1}\|_{\langle u_k, u_{k+1}\rangle},$$

$$g = \max_k \|\psi_n - \psi_{n+1}\|_{\langle u_k, u_{k+1}\rangle} = \|\psi_n - \psi_{n+1}\|_{\langle u_1, u_{n+1}\rangle},$$

and the points $u_1, u_2, \ldots, u_{n+1}$ $(u_1 < u_2 < \ldots < u_{n+1})$ are roots of the equation $\psi_n(t) = \psi_{n+1}(t)$.

To begin with, let us notice that assuming the existence of $n+1$ different real roots of the above equation we have made use of theorem 8.1.

Proof. We shall restrict ourselves to the case

$$u_1 = -\cos\frac{\pi}{2(n+1)}, \qquad u_{n+1} = \cos\frac{\pi}{2(n+1)}. \tag{14}$$

It is to be proved that

$$\begin{gathered} \tau_{\min}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right) \leqslant a \leqslant \tau_{\min}^{(n+1)}\left(\frac{e_n - e_{n+1}}{g}\right), \\ \tau_{\max}^{(n+1)}\left(\frac{e_n - e_{n+1}}{g}\right) \leqslant b \leqslant \tau_{\max}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right). \end{gathered} \tag{15}$$

In other possible cases the theorem is proved by the transformation of the interval $\langle u_1, u_{n+1}\rangle$ into $\langle -\cos\pi/2(n+1), \cos\pi/2(n+1)\rangle$.

By theorem 8.2

$$\begin{aligned}\zeta_{\min}(e_n+e_{n+1}) &\leqslant a \leqslant \zeta_{\min}(e_n-e_{n+1}),\\ \zeta_{\max}(e_n-e_{n+1}) &\leqslant b \leqslant \zeta_{\max}(e_n+e_{n+1}),\end{aligned} \tag{16}$$

where $\zeta_{\min}(h)$ and $\zeta_{\max}(h)$ denote respectively the smallest and the greatest root of the equation $|\psi_n(t)-\psi_{n+1}(t)| = h$. From this definition it follows by (14) that for every $h > 0$

$$\zeta_{\min}(h) < -\cos\frac{\pi}{2(n+1)}, \qquad \zeta_{\max}(h) > \cos\frac{\pi}{2(n+1)}. \tag{17}$$

Applying to the polynomial $\varphi = \psi_n - \psi_{n+1}$ theorem 9.2 with n replaced by $n+1$, which satisfies the hypotheses of 9.2, and introducing the quantity g, we obtain

$$|\psi_n(t)-\psi_{n+1}(t)| \leqslant g|T_{n+1}(t)| \quad \text{for} \quad |t| \geqslant \cos\frac{\pi}{2(n+1)}.$$

By (17) we are enabled to put into this inequality $\zeta_{\min}(e_n-e_{n+1})$ and $\zeta_{\max}(e_n-e_{n+1})$ instead of t, whence

$$e_n-e_{n+1} \leqslant g\big|T_{n+1}\big(\zeta_{\min}(e_n-e_{n+1})\big)\big|,$$

$$e_n-e_{n+1} \leqslant g\big|T_{n+1}\big(\zeta_{\max}(e_n-e_{n+1})\big)\big|$$

and

$$\begin{aligned}\left|T_{n+1}\left(\tau_{\min}^{(n+1)}\left(\frac{e_n-e_{n+1}}{g}\right)\right)\right| &= \frac{e_n-e_{n+1}}{g} \leqslant \big|T_{n+1}\big(\zeta_{\min}(e_n-e_{n+1})\big)\big|,\\ \left|T_{n+1}\left(\tau_{\max}^{(n+1)}\left(\frac{e_n-e_{n+1}}{g}\right)\right)\right| &= \frac{e_n-e_{n+1}}{g} \leqslant \big|T_{n+1}\big(\zeta_{\max}(e_n-e_{n+1})\big)\big|.\end{aligned} \tag{18}$$

Since the function $|T_{n+1}|$ decreases (from $+\infty$ to 0) on the half-line $\big(-\infty, -\cos\pi/2(n+1)\big)$ and increases (from 0 to $+\infty$) on the half-line $\big(\cos\pi/2(n+1), +\infty\big)$, it follows by (17) and (18) that

$$\begin{aligned}\tau_{\min}^{(n+1)}\left(\frac{e_n-e_{n+1}}{g}\right) &\geqslant \zeta_{\min}(e_n-e_{n+1}),\\ \tau_{\max}^{(n+1)}\left(\frac{e_n-e_{n+1}}{g}\right) &\leqslant \zeta_{\max}(e_n-e_{n+1}).\end{aligned} \tag{19}$$

Analogously, we apply theorem 9.3 to the polynomial $\varphi = \psi_n - \psi_{n+1}$ and infer for $|t| \geqslant \cos\pi/2(n+1)$ that $|\psi_n(t) - \psi_{n+1}(t)| \geqslant f|T_{n+1}(t)|$. Replacing t by $\zeta_{\min}(e_n + e_{n+1})$ and $\zeta_{\max}(e_n + e_{n+1})$, we obtain

$$e_n + e_{n+1} \geqslant f\left|T_{n+1}\left(\zeta_{\min}(e_n + e_{n+1})\right)\right|, \qquad e_n + e_{n+1} \geqslant f\left|T_{n+1}\left(\zeta_{\max}(e_n + e_{n+1})\right)\right|,$$

whence

$$\left|T_{n+1}\left(\tau_{\min}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right)\right)\right| = \frac{e_n + e_{n+1}}{f} \geqslant \left|T_{n+1}\left(\zeta_{\min}(e_n + e_{n+1})\right)\right|,$$

$$\left|T_{n+1}\left(\tau_{\max}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right)\right)\right| = \frac{e_n + e_{n+1}}{f} \geqslant \left|T_{n+1}\left(\zeta_{\max}(e_n + e_{n+1})\right)\right|,$$

(20)
$$\tau_{\min}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right) \leqslant \zeta_{\min}(e_n + e_{n+1}),$$
$$\tau_{\max}^{(n+1)}\left(\frac{e_n + e_{n+1}}{f}\right) \geqslant \zeta_{\max}(e_n + e_{n+1}),$$

and the theorem is proved by (16), (19) and (20).

Let us also notice that in virtue of the definitions of f and g and by (8.48) and (8.49)

$$\frac{e_n + e_{n+1}}{f} \geqslant 1, \qquad \frac{e_n - e_{n+1}}{g} \leqslant 1,$$

which means that inequalities (15) are satisfied if we set $a = -1$, $b = 1$.

10. Necessary and sufficient conditions in approximation with nodes. Now we shall investigate pairs of optimal polynomials in the case of approximation with the system of nodes $T = \{t_1, t_2, \ldots, t_m\}$ under the hypotheses $T \subset I$ and $n > m$ (conditions (1.1)).

THEOREM 10.1. *Let the polynomials* $\omega_n \in \mathcal{W}_n(\xi; T)$, $\omega_{n+1} \in \mathcal{W}_{n+1}(\xi; T)$ *be optimal for the function* ξ *in the interval* $I = \langle a, b\rangle$ *and in the classes* $\mathcal{W}_n(\xi; T)$, $\mathcal{W}_{n+1}(\xi; T)$ *respectively. If* $\deg \omega_{n+1} = n+1$, *then* 1° *the equation*

(1) $$\omega_n(t) = \omega_{n+1}(t)$$

has in I *precisely* $n+1$ *real roots of multiplicity at most* 2, 2° *all double roots are nodes,* 3° a *and* b *are roots of this equation if, and only if, they are nodes,* 4° a, b *can only be single roots of* (1), 5° *in each of the intervals*

(2) $$(a, z_1),\ (z_1, z_2),\ \ldots,\ (z_{q-1}, z_q),\ (z_q, b),$$

where $z_1 < z_2 < \ldots < z_q$ *denote all the different roots of* (1) *which are not at the same time nodes and single roots, there exists an* (e) *point of* ω_n.

Proof. Let us write $\delta_n = \omega_n - \xi$, $\delta_{n+1} = \omega_{n+1} - \xi$. From condition (II) of § 2 it follows that there exist (e) points $p_1, p_2, \ldots, p_{n-m+2}$ of ω_n such that, retaining the notation (2.1), we have $\operatorname{sign}\delta_n(s_j) = (-1)^j c_j$ for $j = 1, 2, \ldots, n+2$ where c_j's are all either non-negative or non--positive. Moreover, if $s_j \in T$, then $c_j = 0$ and if s_j is an (e) point, then $c_j \neq 0$ (the nodes cannot be (e) points for which $|\delta_n(t)| = \varepsilon_n(\xi; T)$ because of $\varepsilon_n(\xi; T) > 0$).

Since $\deg \omega_{n+1} = n+1$, it follows that $\varepsilon_n(\xi; T) = \|\delta_n\|_I > \|\delta_{n+1}\|_I = \varepsilon_{n+1}(\xi; T)$, whence for those s_j which are (e) points

$$(3)\quad \operatorname{sign}\big(\omega_n(s_j) - \omega_{n+1}(s_j)\big) = \operatorname{sign}\big(\delta_n(s_j) - \delta_{n+1}(s_j)\big) = \operatorname{sign}\delta_n(s_j) = (-1)^j c_j.$$

We join the (e) points $p_1, p_2, \ldots, p_{n-m+2}$ of ω_n into maximal groups not separated by nodes; let these groups have $k_1, k_2, \ldots, k_r$ elements (whence $k_1 + k_2 + \ldots + k_r = n - m + 2$). Thus among the (e) points

$$p_1, p_2, \ldots, p_{k_1},$$
$$p_{k_1+1}, p_{k_1+2}, \ldots, p_{k_1+k_2},$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$p_{k_1+\ldots+k_{r-1}+1}, p_{k_1+\ldots+k_{r-1}+2}, \ldots, p_{k_1+\ldots+k_r=n-m+2},$$

only the points p_{k_1} and $p_{k_1+1}, p_{k_1+k_2}$ and $p_{k_1+k_2+1}, \ldots, p_{k_1+\ldots+k_{r-1}}$ and $p_{k_1+\ldots+k_{r-1}+1}$ are separated by nodes.

We join the nodes $t_1, t_2, \ldots, t_m$ into maximal groups not separated by (e) points, with $l_1, l_2, \ldots, l_{r'}$ elements respectively (whence $l_1 + l_2 + \ldots + l_{r'} = m$). One of the mutually exclusive inequalities $r = r'+1$, $r = r'$, $r' = r+1$ holds.

In the first case the (e) points and the nodes have the following order (we write the number of elements of a group in the place of the group itself): $k_1, l_1, k_2, \ldots, k_{r'}, l_{r'}, k_{r'+1}$; in the second case the order $l_1, k_1, l_2, \ldots, l_r, k_r$ or

$$(4)\qquad k_1, l_1, k_2, \ldots, k_r, l_r;$$

in the third $l_1, k_1, l_2, \ldots, l_r, k_r, l_{r+1}$.

We shall prove 1° - 4° only in case (4); all other cases may be treated similarly. From (3) it follows that between consecutive (e) points not separated by a node, *i. e.* consecutive in the system $s_1, s_2, \ldots, s_{n+2}$, there is at least one root of the difference $\omega_n - \omega_{n+1}$ which, by definition, is not a node. The number of these roots is not less than

$$(5)\qquad (k_1-1) + (k_2-1) + \ldots + (k_r-1) = n - m + 2 - r.$$

On the other hand, between two consecutive (e) points p_i, p_{i+1} which are separated by l nodes there are not less than $l+1$ roots (counted with their multiplicity). For, if only the nodes were the roots and singular ones at that, then the difference $\omega_n - \omega_{n+1}$ would change the sign exactly l times between p_i and p_{i+1}. This, however, contradicts formula (3) implying $\operatorname{sign}\big(\omega_n(p_{i+1}) - \omega_{n+1}(p_{i+1})\big) = (-1)^{l+1}\operatorname{sign}\big(\omega_n(p_i) - \omega_{n+1}(p_i)\big)$. Since, according to (4), to the right of the last (e) point p_{n-m+2} there are l_r nodes, we have shown that there exist at least

$$(l_1+1)+(l_2+1)+\ldots+(l_{r-1}+1)+l_r = m+r-1 \tag{6}$$

roots disregarded in formula (5). From (5) and (6) it follows that the polynomial $\omega_n - \omega_{n+1}$ has in I not less than $(n-m+2-r)+(m+r-1) = n+1$ roots, and since $\deg(\omega_n - \omega_{n+1}) = n+1$, it has exactly $n+1$ real roots. It follows that at all those places in the proof where we have said "at least" we may say "exactly". Thus all the roots of (1) are of multiplicity not greater than 2 (part 1° of the thesis), double roots may be only the nodes (2°) lying between (e) points, whence, in particular, they cannot be the bounds of I (4°). The points a and b are roots if, and only if, they are nodes (3°).

We shall now prove part 5° of the thesis. We have shown that the roots of equation (1) lying outside (p_1, p_{n-m+2}) are nodes and single roots of that equation. Since, by definition, the points z_1 and z_q are devoid of at least one of these properties, they must lie in (p_1, p_{n-m+2}), whence $p_1 \in \langle a, z_1)$, $p_{n-m+2} \in (z_q, b\rangle$. It follows that the remaining intervals (2) are all included in the interval (p_1, p_{n-m+2}), which is the union of the intervals

$$(p_1, p_{k_1}),\ (p_{k_1+1}, p_{k_1+k_2}),\ \ldots,\ (p_{k_1+\ldots+k_{r-1}+1}, p_{k_1+\ldots+k_r=n-m+2}), \tag{7}$$

of the intervals

$$(p_{k_1}, p_{k_1+1}),\ (p_{k_1+k_2}, p_{k_1+k_2+1}),\ \ldots,\ (p_{k_1+\ldots+k_{r-1}}, p_{k_1+\ldots+k_{r-1}+1}) \tag{8}$$

and of the points $p_2, \ldots, p_{n-m+1}$. As we have already remarked (p. 53), all the nodes lie in the intervals (8). No consecutive (e) points can lie in one of the intervals (8), for it would mean that in such an interval, besides the single roots at the nodes, there are two more roots of equation of (1), but as we have already shown there is only one. Thus only the following cases are possible:

I. The points z_i, z_{i+1} are in different intervals of (8); II. z_i is in one of the intervals of (7), z_{i+1} is in one of the intervals of (8), or *vice versa*; III. z_i and z_{i+1} are in different intervals of (7); IV. z_i and z_{i+1} belong to the same interval of (7).

From the definition of cases I-III it follows in each case that in (z_i, z_{i+1}) there is an (e) point of ω_n. The same holds in case IV, since, as we have proved, between two consecutive (e) points not separated by a node there is at most one root of (1). This completes the proof of theorem 10.1.

Subsequently $\mathfrak{Z}$ will stand for the class of the intervals (z_1, z_2), $(z_2, z_3), \ldots, (z_{q-1}, z_q)$ where, as before $z_1, z_2, \ldots, z_q$ are all the different roots of (1) which are not simultaneously single roots and nodes. We shall show that class $\mathfrak{Z}$ is non-empty, *i. e.* that $q \geqslant 2$. Indeed, $q = 0$ would imply that (1) has m single roots which are nodes and has no multiple roots. If $q = 1$, then either precisely one node is a double root or there exists exactly one root of (1) different from the nodes and all the roots are single. Hence for $q \leqslant 1$ equation (1) has at most $m+1 < n+1$ roots (cf. the hypotheses of this §), which contradicts theorem 10.1.

As in § 8, $\zeta_{\min}(h)$ and $\zeta_{\max}(h)$ for $h \geqslant 0$ will denote the smallest and the greatest root of the equation $|\omega_n(t) - \omega_{n+1}(t)| = h$. For $h = 0$ we use simpler symbols: $\zeta_{\min}(0) = u_{\min}$, $\zeta_{\max}(0) = u_{\max}$, denoting the smallest and the greatest of roots of the equation (1) respectively. Since $z_1, z_2, \ldots, z_q$ are different roots of this equation and $q \geqslant 2$, $u_{\min} < u_{\max}$.

THEOREM 10.2. *Let the polynomials* $\omega_n \in \mathcal{W}_n(\xi; T)$, $\omega_{n+1} \in \mathcal{W}_{n+1}(\xi; T)$ *be optimal for the function* ξ *in the interval* $I = \langle a, b\rangle$ *and in the classes* $\mathcal{W}_n(\xi; T)$, $\mathcal{W}_{n+1}(\xi; T)$ *respectively. If* $\deg \omega_{n+1} = n+1$, *then the following conditions are satisfied:*

(9) $$\varepsilon_n(\xi; T) + \varepsilon_{n+1}(\xi; T) \geqslant \|\omega_n - \omega_{n+1}\|_{\langle u_{\min}, u_{\max}\rangle},$$

(10) $$\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T) \leqslant \min_{J \in \mathfrak{Z}} \|\omega_n - \omega_{n+1}\|_J;$$

if $z_1 = u_{\min}$, *then*

(11) $$\zeta_{\min}\big(\varepsilon_n(\xi; T) + \varepsilon_{n+1}(\xi; T)\big) \leqslant a \leqslant \zeta_{\min}\big(\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T)\big);$$

if $z_1 > u_{\min}$, *then* (11) *holds or*

(12) $$\zeta_{\min}\big(\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T)\big) < a \leqslant u_{\min} \quad \textit{and}$$
$$\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T) \leqslant \|\omega_n - \omega_{n+1}\|_{\langle u_{\min}, z_1\rangle};$$

if $z_q = u_{\max}$, *then*

(13) $$\zeta_{\max}\big(\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T)\big) \leqslant b \leqslant \zeta_{\max}\big(\varepsilon_n(\xi; T) + \varepsilon_{n+1}(\xi; T)\big);$$

if $z_q < u_{\max}$, *then* (13) *holds or*

(14) $$u_{\max} \leqslant b < \zeta_{\max}\big(\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T)\big) \quad \textit{and}$$
$$\varepsilon_n(\xi; T) - \varepsilon_{n+1}(\xi; T) \leqslant \|\omega_n - \omega_{n+1}\|_{\langle z_q, u_{\max}\rangle}.$$

Proof. From the hypothesis $\|\omega_n - \xi\|_I = \varepsilon_n(\xi; T)$, $\|\omega_{n+1} - \xi\|_I = \varepsilon_{n+1}(\xi; T)$ there follows

(15) $$\|\omega_n - \omega_{n+1}\|_I \leqslant \varepsilon_n(\xi; T) + \varepsilon_{n+1}(\xi; T).$$

By theorem 10.1 all the roots of (1) lie in I, *i. e.* $\langle v_{\min}, v_{\max}\rangle \subset I$, which gives

(16) $$a \leqslant v_{\min}, \quad u_{\max} \leqslant b.$$

By the above inclusion and by (15) we obtain (9).

From part 5° of theorem 10.1 it follows that in each interval of class $\mathfrak{X}$ there exists an (e) point p of the polynomial ω_n. At that point $|\omega_n(p)-\xi(p)| = \varepsilon_n(\xi; T)$, $|\omega_{n+1}(p)-\xi(p)| \leqslant \varepsilon_{n+1}(\xi; T)$, whence

(17) $$|\omega_n(p)-\omega_{n+1}(p)| \geqslant \varepsilon_n(\xi; T)-\varepsilon_{n+1}(\xi; T).$$

Therefore if J is one of the intervals of class $\mathfrak{X}$, then $\|\omega_n-\omega_{n+1}\|_J \geqslant \varepsilon_n(\xi; T)-\varepsilon_{n+1}(\xi; T)$, *i. e.* (10) is satisfied.

The polynomial $\omega_n-\omega_{n+1}$ has only real roots, whence it is monotone on the half-lines $(-\infty, u_{\min})$, $(u_{\max}, +\infty)$. The function $|\omega_n-\omega_{n+1}|$ decreases on the half-line $(-\infty, u_{\min})$ and increases on the half-line $(u_{\max}, +\infty)$. Therefore, except the interval

$$\langle\zeta_{\min}\big(\varepsilon_n(\xi; T)+\varepsilon_{n+1}(\xi; T)\big),\ \zeta_{\max}\big(\varepsilon_n(\xi; T)+\varepsilon_{n+1}(\xi; T)\big)\rangle$$

according to the definition of its bounds, we have $|\omega_n(t)-\omega_{n+1}(t)| > \varepsilon_n(\xi; T)+\varepsilon_{n+1}(\xi; T)$, whence by (15) there follows

(18) $$\zeta_{\min}\big(\varepsilon_n(\xi; T)+\varepsilon_{n+1}(\xi; T)\big) \leqslant a,\ b \leqslant \zeta_{\max}\big(\varepsilon_n(\xi; T)+\varepsilon_{n+1}(\xi; T)\big).$$

From the definition of z_1 and $u_{\min}$ it follows that $z_1 \geqslant u_{\min}$. If $z_1 = u_{\min}$, then by part 5° of theorem 10.1 there exists in the interval $(a, u_{\min})$ an (e) point of the polynomial ω_n, *i. e.* a point $p \geqslant a$ satisfying (17). We have remarked that the function $|\omega_n-\omega_{n+1}|$ decreases on the half-line $(-\infty, u_{\min})$. Hence, by the definition of $\zeta_{\min}(h)$, $a \leqslant p \leqslant \zeta_{\min}\big(\varepsilon_n(\xi; T)-\varepsilon_{n+1}(\xi; T)\big)$, which, with (18), gives (11) in the case where $z_1 = u_{\min}$.

On the other hand, in $z_1 > u_{\min}$, the conditions to be proved may be formulated as follows: either inequality (11) holds and one of the mutually exclusive inequalities

(19) $$\varepsilon_n(\xi; T)-\varepsilon_{n+1}(\xi; T) \leqslant \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, z_1\rangle},$$

$$\varepsilon_n(\xi; T)-\varepsilon_{n+1}(\xi; T) > \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, z_1\rangle}$$

is satisfied, or (11) is not satisfied but (12) holds. If we join case (12) with the case where (11) and (19) are satisfied, we must prove that either the inequalities

(20) $$\varepsilon_n(\xi; T)-\varepsilon_{n+1}(\xi; T) \leqslant \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, z_1\rangle},$$

(21) $$\zeta_{\min}\big(\varepsilon_n(\xi; T)+\varepsilon_{n+1}(\xi; T)\big) \leqslant a \leqslant u_{\min}$$

are satisfied, or

$$(22)\qquad \varepsilon_n(\xi;T)-\varepsilon_{n+1}(\xi;T) > \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, z_1\rangle},$$

$$(23)\qquad \zeta_{\min}\big(\varepsilon_n(\xi;T)+\varepsilon_{n+1}(\xi;T)\big) \leqslant a \leqslant \zeta_{\min}\big(\varepsilon_n(\xi;T)-\varepsilon_{n+1}(\xi;T)\big)$$

hold. By (16) and (18), (21) is satisfied independently of (20). Now let us suppose (22) to be satisfied. It follows from theorem 10.1 that in the interval

$$(24)\qquad \langle a, z_1\rangle = \langle a, u_{\min}\rangle \cup \langle u_{\min}, z_1\rangle$$

there is an (e) point p of ω_n satisfying (17). Therefore $\|\omega_n-\omega_{n+1}\|_{\langle a,z_1\rangle} \geqslant \varepsilon_n(\xi;T)-\varepsilon_{n+1}(\xi;T)$, whence by (22) and (24) it follows that $\|\omega_n-\omega_{n+1}\|_{\langle a,u_{\min}\rangle} \geqslant \varepsilon_n(\xi;T)-\varepsilon_{n+1}(\xi;T)$. Since the function $|\omega_n-\omega_{n+1}|$ decreases in $\langle a, u_{\min}\rangle$, $|\omega_n(a)-\omega_{n+1}(a)| \geqslant \varepsilon_n(\xi;T)-\varepsilon_{n+1}(\xi;T)$ and by the definition of $\zeta_{\min}$ we have $a \leqslant \zeta_{\min}\big(\varepsilon_n(\xi;T)-\varepsilon_{n+1}(\xi;T)\big)$, which, together with (18), leads to (23).

Thus we have proved (11) and (12). An analogous argument in the case $z_q = v_{\max}$ or $z_q < u_{\max}$ gives (13) and (14).

THEOREM 10.3. *Let the polynomials* $\omega_n \in \mathcal{W}_n$, $\omega_{n+1} \in \mathcal{W}_{n+1}$, *the numbers* e_n, e_{n+1} *and the interval* $I = \langle a, b\rangle$ *satisfy the following conditions*: 1° $\deg\omega_{n+1} = n+1$, 2° *the equation*

$$(25)\qquad \omega_n(t) = \omega_{n+1}(t)$$

has only real roots with multiplicity not greater than 2, 3° *all nodes are roots of* (25), 4° *the double roots of* (25) *are nodes, and, retaining the notation used above, we have,* $0 \leqslant e_{n+1} < e_n$

$$(26)\qquad e_n+e_{n+1} \geqslant \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, u_{\max}\rangle},$$

$$(27)\qquad e_n-e_{n+1} \leqslant \min_{J\in\mathcal{Z}} \|\omega_n-\omega_{n+1}\|_J,$$

if $z_1 = v_{\min}$, *then*

$$(28)\qquad \zeta_{\min}(e_n+e_{n+1}) \leqslant a \leqslant \zeta_{\min}(e_n-e_{n+1}),$$

if $z_1 > u_{\min}$, *then* (28) *holds or*

$$(29)\qquad \zeta_{\min}(e_n-e_{n+1}) < a \leqslant v_{\min} \quad \textit{and} \quad e_n-e_{n+1} \leqslant \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, z_1\rangle},$$

if $z_q = u_{\max}$, *then*

$$(30)\qquad \zeta_{\max}(e_n-e_{n+1}) \leqslant b \leqslant \zeta_{\max}(e_n+e_{n+1}),$$

if $z_q < v_{\max}$, *then* (30) *holds or*

$$(31)\qquad u_{\max} \leqslant b < \zeta_{\max}(e_n-e_{n+1}) \quad \textit{and} \quad e_n-e_{n+1} \leqslant \|\omega_n-\omega_{n+1}\|_{\langle z_q, u_{\max}\rangle}.$$

Under these hypotheses there exists a continuous function ξ for which the polynomials ω_n, ω_{n+1} are optimal in I and in the classes $\mathcal{W}_n(\xi; T)$, $\mathcal{W}_{n+1}(\xi; T)$ respectively; moreover $\varepsilon_n(\xi; T) = e_n$, $\varepsilon_{n+1}(\xi; T) = e_{n+1}$.

Proof. For the proof we construct the function ξ with the desired properties; the construction does not differ essentially from the one used in the proof of theorem 8.3. In the case we are considering now ξ must satisfy the additional condition

$$\xi(t_i) = \omega_n(t_i) = \omega_{n+1}(t_i) \quad (i = 1, 2, \ldots, m), \tag{32}$$

necessary in order that $\omega_n \in \mathcal{W}_n(\xi; T)$ and $\omega_{n+1} \in \mathcal{W}_{n+1}(\xi; T)$.

As in the proof of theorem 8.3, we introduce the sets

$$A_n^+ = \mathop{E}_t\{|\omega_n(t) - \omega_{n+1}(t) + e_n| \leqslant e_{n+1}\}, \tag{33}$$

$$A_n^- = \mathop{E}_t\{|\omega_n(t) - \omega_{n+1}(t) - e_n| \leqslant e_{n+1}\}, \tag{34}$$

$$A_n = A_n^+ \cup A_n^- = \mathop{E}_t\{e_n - e_{n+1} \leqslant |\omega_n(t) - \omega_{n+1}(t)| \leqslant e_n + e_{n+1}\}. \tag{35}$$

Since $\omega_n - \omega_{n+1}$ is a polynomial, the sets $A_n \cap I$ and $\overline{I - A_n}$ are closed and consist of a finite number of closed intervals. Let us therefore write

$$A_n \cap I = \bigcup_{i=0}^{n'} \langle a_i', a_i'' \rangle$$

where $a \leqslant a_i' \leqslant a_1'' < a_2' \leqslant a_2'' < \ldots < a_{n'}' \leqslant a_{n'}'' \leqslant b$. Each of the intervals $\langle a_i', a_i'' \rangle$ is included either in A_n^+ or in A_n^-, for by (33) and (34) we have

$$\begin{aligned} \omega_n(t) - \omega_{n+1}(t) &\leqslant e_{n+1} - e_n < 0 \quad \text{for} \quad t \in A_n^+, \\ \omega_n(t) - \omega_{n+1}(t) &\geqslant e_n - e_{n+1} > 0 \quad \text{for} \quad t \in A_n^-. \end{aligned} \tag{36}$$

Let $u_{\min} = u_1 < u_2 < \ldots < u_r = u_{\max}$ be all the different roots of (25); they do not lie in $A_n \cap I$, for $e_n - e_{n+1} > 0$ and, by hypothesis, they are in I. These roots separate each pair of intervals

$$\langle a_i', a_i'' \rangle, \quad \langle a_{i+1}', a_{i+1}'' \rangle. \tag{37}$$

Indeed, if one of these intervals belongs to A_n^+ and the other to A_n^-, then by (36) there exists a root of (25) lying between them. If both intervals (37) belong to the same set among (33), (34), then in each of the intervals (a_i', a_i''), (a_i'', a_{i+1}'), (a_{i+1}', a_{i+1}'') there is an extremum of $\omega_n - \omega_{n+1}$. This polynomial has only real roots not lying in the intervals (37), whence in (a_i'', a_{i+1}') it has a root.

Moreover, if $a < a_1'$, i. e. $a \bar{\in} A_n$, then a root of (25) lies in $\langle a, a_1')$. Indeed, by (29) we have in this case $e_n - e_{n+1} \leqslant \|\omega_n - \omega_{n+1}\|_{\langle u_{\min}, z_1 \rangle}$, and by the definition of A_n we have $a_1' \in (u_{\min}, z_1)$, whence $a \leqslant u_{\min} < a_1'$. Similarly, if $a_{n'}'' < b$, i. e. $b \bar{\in} A_n$, then in $(a_{n'}'', b\rangle$ there exists a root of (25).

Therefore let

$$a \leqslant u_1 < \ldots < u_{k_1} < a_1',$$

$$a_i'' < u_{k_i+1} < u_{k_i+2} < \ldots < u_{k_{i+1}} < a_{i+1}' \qquad (i = 1, 2, \ldots, n'-1),$$

$$a_{n'}'' < u_{k_{n'}+1} < \ldots < u_r \leqslant b,$$

where $k_1+1 \leqslant k_2, \ldots, k_{n'-1}+1 \leqslant k_{n'}$. We introduce the first and the third of these definitions only in the case when $a < a_1'$, $a_{n'}'' < b$ respectively; in the contrary case $k_1 = 0$, $k_{n'} = r$.

Let us set

$$(38) \qquad \xi(t) = \begin{cases} \omega_n(t)+e_n & \text{for} \quad t \in A_n^+ \cap I, \\ \omega_n(t)-e_n & \text{for} \quad t \in A_n^- \cap I, \\ \omega_{n+1}(t) & \text{for} \quad t \in \langle a, u_{k_1}\rangle \cup \langle u_{k_1+1}, u_{k_2}\rangle \cup \ldots \\ & \qquad \cup \langle u_{k_{n'-1}+1}, u_{k_{n'}}\rangle \cup \langle u_{k_{n'}+1}, b\rangle\,^{7)}, \\ \omega_{n+1}(t) + \dfrac{e_{n+1}\operatorname{sign}\big(\omega_{n+1}(t)-\omega_n(t)\big)}{u_{k_i+1}-a_i''}(u_{k_i+1}-t) & \text{for} \\ & \qquad t \in \langle a_i'', u_{k_i+1}) \quad (i = 2, \ldots, n'), \\ \omega_{n+1}(t) + \dfrac{e_{n+1}\operatorname{sign}\big(\omega_{n+1}(t)-\omega_n(t)\big)}{a_i'-u_{k_i}}(t-u_{k_i}) & \text{for} \\ & \qquad t \in (u_{k_i}, a_i\rangle \quad (i = 2, \ldots, n'), \\ \omega_{n+1}(t) + \dfrac{2(n+3)(-1)^j e_{n+1}\operatorname{sign}\big(\omega_{n+1}(t)-\omega_n(t)\big)}{u_{k_1+1}-a_1''} \times & \\ \times\left(a_1'' + \dfrac{2j+1}{2}\cdot\dfrac{u_{k_1+1}-a_1''}{n+3} - t\right) & \text{for} \quad t \in \left\langle a_1'' + j\dfrac{u_{k_1+1}-a_1''}{n+3},\right. \\ & \qquad \left. a_1'' + (j+1)\dfrac{u_{k_1+1}-a_1''}{n+3}\right\rangle \quad (j = 0, 1, \ldots, n+1), \\ \omega_{n+1}(t) + \dfrac{(n+3)(-1)^{n+2} e_{n+1}\operatorname{sign}\big(\omega_{n+1}(t)-\omega_n(t)\big)}{u_{k_1+1}-a_1''}(u_{k_1+1}-t) & \\ & \text{for} \quad t \in \left\langle u_{k_1+1} - \dfrac{u_{k_1+1}-a_1''}{n+3}, u_{k_1+1}\right). \end{cases}$$

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[7]) We adopt this definition for $t \in \langle a, u_{k_1}\rangle$ and $t \in \langle u_{k_{n'}+1}, b\rangle$ only in the case when $a < a_1'$ and $a_{n'}'' < b$ respectively.

As in theorem 8.3, one can prove that the function ξ is defined in a consistent manner in the whole interval $I = \langle a, b\rangle$, that it is continuous in this interval and that $\|\xi-\omega_n\|_I = e_n$ and $\|\xi-\omega_{n+1}\|_I = e_{n+1}$. We shall remark only that between consecutive roots of (25) the function $\operatorname{sign}(\omega_{n+1}(t)-\omega_n(t))$ is continuous and that from (38) it follows directly that (32) is satisfied.

It is also obvious that the polynomial ω_{n+1} is optimal for ξ in class $\mathcal{W}^2_{n+1}(\xi; T)$ with the deviation e_{n+1}. Indeed, from the last but one definition of ξ it follows that

$$\xi\left(a_1''+j\frac{u_{k_1+1}-a_1''}{n+3}\right) = \omega_{n+1}\left(a_1''+j\frac{u_{k_1+1}-a_1''}{n+3}\right)+$$
$$+(-1)^j e_{n+1}\operatorname{sign}\left(\omega_{n+1}(a_1'')-\omega_n(a_1'')\right) \qquad (j = 0, 1, \ldots, n+2),$$

thus ω_{n+1} is optimal for ξ in class $\mathcal{W}^2_{n+1}$, whence *a fortiori*, in each subclass of $\mathcal{W}^2_{n+1}$ to which ω_{n+1} belongs.

It is to be proved that the polynomial ω_n is optimal for ξ in class $\mathcal{W}^2_n(\xi; T)$.

From (26) and (27) it follows that in each interval of class $\mathfrak{X}$ there exist points of A_n; points of A_n are also in $\langle a, z_1)$. Indeed, if $z_1 = u_{\min}$ this follows from (28) and from the definition of $\zeta_{\min}$; if $z_1 > u_{\min}$ and

$$e_n - e_{n+1} \leqslant \|\omega_n-\omega_{n+1}\|_{\langle u_{\min}, z_1\rangle}, \tag{39}$$

then by (35) and the first of inequalities (29) there exist in the interval $\langle u_{\min}, z_1) \subset \langle a, z_1)$ points of A_n. If $z_1 > u_{\min}$ and (39) is not satisfied, then (28) holds and according to (32) in the interval $\langle a, u_{\min}) \subset \langle a, z_1)$ there exist points of A_n (for example, $a \in A_n$). Similarly, by aid of (30) and (31) we may show that $(z_q, b\rangle$ contains points of A_n.

Since ξ is defined in $A_n \cap I$ so that in each interval composing this set there exists an (e) point of ω_n at which $|\xi(t)-\omega_n(t)| = e_n$, we infer that (e) points of ω_n exist in each of the intervals $\langle a, z_1), (z_1, z_2), \ldots, (z_{q-1}, z_q), (z_q, b\rangle$. The number of these intervals is equal to $n-m+2$, whence it is possible to select the (e) points

$$p_1, p_2, \ldots, p_{n-m+2} \tag{40}$$

lying exactly one in each interval. Indeed, equation (25) has r different roots, and, their multiplicity being counted, $n+1$ roots; all the $n+1-r$ double roots are nodes. Consequently, (25) has $m-(n+1-r) = m-n-1+r$ different roots, which are simultaneously nodes and single roots, and $q = r-(m-n-1+r) = n-m+1$ roots which do not satisfy this hypothesis, which is what we wished to prove.

Now we shall prove that if consecutive points in (40) are of the same kind, then there is an odd number of nodes between them, and if those points are of different kind, the number of nodes between them is even. By condition (III) of § 2 this implies that ω_n is optimal for ξ in $\mathcal{W}_n(\xi; T)$.

We shall use the equality $\operatorname{sign}(\xi(t)-\omega_n(t)) = \operatorname{sign}(\omega_{n+1}(t)-\omega_n(t))$ valid at the points (40), for at those points $|\xi(t)-\omega_n(t)| = e_n$ and $|\xi(t)-\omega_{n+1}(t)| \leqslant e_{n+1} < e_n$. Each pair p_i, p_{i+1} $(i = 1, 2, \ldots, n-m+1)$ is separated by a point z_i which is not simultaneously a node and a single root of (25). Another roots of this equation, contained in the interval (p_i, p_{i+1}), are single roots and nodes. Hence the number of the roots of (25) in (p_i, p_{i+1}), counted with their multiplicity, exceeds by an odd quantity the number of nodes lying there. Thus, if that number is even, then $\omega_{n+1}-\omega_n$ changes the sign in (p_i, p_{i+1}) an odd number of times, and p_i, p_{i+1} are (e) points of different kind; if the number of nodes in (p_i, p_{i+1}) is odd, then $\omega_{n+1}-\omega_n$ changes the sign in this interval an even number of times and p_i, p_{i+1} are (e) points of the same kind.

Hence theorem 10.3 is proved.

Theorems 10.1-10.3 enable us to state the following corollary (the former notation being retained):

COROLLARY. *Suppose we are given polynomials* $\omega_n \in \mathcal{W}_n$ *and* ω_{n+1} *such that* $\deg \omega_{n+1} = n+1$, *numbers* e_n *and* e_{n+1} *and an interval* $I = \langle a, b\rangle$. *A continuous function* ξ *for which the polynomials* ω_n *and* ω_{n+1} *are optimal in* I *and in the classes* $\mathcal{W}_n(\xi; T)$ *and* $\mathcal{W}_{n+1}(\xi; T)$ *respectively with deviations* $\varepsilon_n(\xi; T) = e_n$, $\varepsilon_{n+1}(\xi; T) = e_{n+1}$ *exists if and only if* 1° *the equation*

$$\omega_n(t) = \omega_{n+1}(t) \tag{41}$$

has only real roots of multiplicity not greater than 2, 2° *all nodes are roots of* (41), 3° *double roots of* (41) *are nodes, and if the following inequalities are satisfied*: $0 \leqslant e_{n+1} < e_n$,

$$e_n + e_{n+1} \geqslant \|\omega_n - \omega_{n+1}\|_{\langle u_{\min}, u_{\max}\rangle}, \qquad e_n - e_{n+1} \leqslant \min_{J \in \mathfrak{Z}} \|\omega_n - \omega_{n+1}\|_J;$$

if $z_1 = u_{\min}$, *then*

$$\zeta_{\min}(e_n + e_{n+1}) \leqslant a \leqslant \zeta_{\min}(e_n - e_{n+1}); \tag{42}$$

if $z_1 > u_{\min}$ *then* (42) *holds or*

$$\zeta_{\min}(e_n - e_{n+1}) < a \leqslant u_{\min} \qquad \text{and} \qquad e_n - e_{n+1} \leqslant \|\omega_n - \omega_{n+1}\|_{\langle u_{\min}, z_1\rangle};$$

if $z_q = u_{\max}$, *then*

$$\zeta_{\max}(e_n - e_{n+1}) \leqslant b \leqslant \zeta_{\max}(e_n + e_{n+1}); \tag{43}$$

if $z_q < u_{\max}$, *then* (43) *holds or*

$$u_{\max} \leqslant b < \zeta_{\max}(e_n - e_{n+1}) \qquad \text{and} \qquad e_n - e_{n+1} \leqslant \|\omega_n - \omega_{n+1}\|_{\langle z_q, u_{\max}\rangle}.$$

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