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Fourier approximation and embeddings of Sobolev spaces

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Introduction

The reasons underlying the dominant role played by Sobolev spaces in the general theory of partial differential equations are not hard to find: these spaces provide a natural setting for the formulation of many of the problems which arise, and the variety of embedding theorems which relate appropriate pairs of Sobolev spaces often makes it possible to establish in a simple way results of great usefulness about the particular equation under study. Of especial importance among such embedding theorems are those which, like the celebrated theorem of Rellich, assert that a certain Sobolev space is compactly embedded in another such space. Indeed, as is well known, Rellich's theorem enables the Riesz–Schauder theory of compact linear operators in a Banach space to be applied to the Dirichlet problem for linear, uniformly elliptic equations in a bounded domain. It is therefore not surprising that considerable attention has been paid, over the years, to the problem of classifying, by some useful means, these compact embedding maps. The literature on this topic is quite extensive, ranging from the early work on the situations when these maps are of Hilbert–Schmidt or nuclear type (see, for example, [6] and [12]) to the later and more detailed results contained in [1], [14]–[21].

The object of this paper is to investigate the Fourier approximation of functions in Sobolev spaces in a way which throws fresh light on the embeddings mentioned above. To explain the results, let m be a positive integer, let $1 \leqslant p < \infty$, and let Ω be a bounded open subset of R^n $(n \geqslant 1)$. Define $W^{m,p}(\Omega)$ to be the Sobolev space of functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for all α with $|\alpha| \leqslant m$ (standard notation being used), endowed with the norm

$$\|u\|_{m,p,\Omega} = \Big(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{p,\Omega}^p\Big)^{1/p},$$

where $\|v\|_{p,\Omega} = \big(\int_\Omega |v(x)|^p\,dx\big)^{1/p}$. The space $W^{m,\infty}(\Omega)$ is defined by the natural adaptation of this definition, and for $1 \leqslant p \leqslant \infty$, $W_0^{m,p}(\Omega)$ is taken to be the closure in $W^{m,p}(\Omega)$ of the set of all infinitely differentiable functions with compact support in Ω.

Now let $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$, $m \geqslant 1$, $\frac{m}{n} > \frac{1}{p} - \frac{1}{s}$, and suppose the boundary of Ω is minimally smooth in the sense of Stein. Under these conditions it is known ([7], p. 107) that $W^{m,p}(\Omega)$ is compactly embedded in $L^s(\Omega)$. Our main result is that any u in $W^{m,p}(\Omega)$ can be approximated by a sum $S_r u$ of not more than r terms of Fourier type in such a way that as $r \to \infty$,

$$\text{(A)} \qquad \|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega}\, o(r^{-h+\varepsilon})$$

for all $\varepsilon > 0$, where $h = \frac{m}{n} - \max\left(\frac{1}{p} - \frac{1}{s},\ 0\right)$. The same holds for elements of $W_0^{m,p}(\Omega)$ with no hypotheses on the boundary of Ω.

We also need to discuss Orlicz spaces. Given a non-negative convex function φ on $[0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} t^{-1}\varphi(t) = \infty$, let $L^\varphi(\Omega)$ denote the Banach space of all measurable functions u on Ω such tha

$$\int_\Omega \varphi(\lambda^{-1}|u(x)|)\,dx < \infty$$

for some $\lambda > 0$, with norm

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0:\ \int_\Omega \varphi\left(\lambda^{-1}|u(x)|\right)dx \leqslant 1\right\}.$$

We study the particular Orlicz space $L^{\varphi_\nu}(\Omega)$, where $\varphi_\nu(t) = \exp(t^\nu) - 1$ for all $t \geqslant 0$, and $\nu \geqslant 1$. It is known from the work of Trudinger [11] that if Ω is of class C^1, $1 \leqslant m < n$ and $1 \leqslant \nu < \frac{n}{n-1}$, then $W_{(0)}^{m,n/m}(\Omega)$ is compactly embedded in $L^{\varphi_\nu}(\Omega)$. We are able to show that if $1 \leqslant \nu \leqslant \frac{n}{n-m}$, then for all $u \in W_{(0)}^{m,n/m}(\Omega)$ we have as $r \to \infty$,

$$\|u - S_r u\|_{\varphi_\nu,\Omega} = \|u\|_{m,n/m,\Omega}\, O\left\{(\log r)^{1-\frac{m}{n}-\frac{1}{\nu}}\right\}.$$

As a consequence of these results we can give a fairly complete analysis of compact embeddings of Sobolev spaces, using the notion of the *type* of a map due to Pietsch. To explain this let X and Y be Banach spaces, let $T: X \to Y$ be bounded and linear, and define the r-th approximation number $a_r(T)$ of T to be $\inf\|T - F\|$, where the infimum is taken over all linear maps $F: X \to Y$ with range at most r-dimensional. The map T is said to be of type l^q (where $0 < q < \infty$), or to belong to $l^q(X, Y)$, if

$$\sum_{r=0}^{\infty} (a_r(T))^q < \infty;$$

it is of type c_0 if

$$\lim_{r\to\infty} a_r(T) = 0.$$

Every map of type l^q or c_0 is compact, while if X and Y are Hilbert spaces, $l^1(X, Y)$ and $l^2(X, Y)$ coincide with the spaces of nuclear and Hilbert–Schmidt maps respectively.

The approximation results given earlier now immediately show that if $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$, $m \geqslant 1, \frac{1}{p} - \frac{1}{s} < \frac{m}{n}$, $n \geqslant 1$, $q > 0$, then the r-th approximation number of the embedding map $I\colon W^{m,p}_{(0)}(\Omega) \to L^s(\Omega)$ (that is, $W^{m,p}(\Omega)$ or $W^{m,p}_0(\Omega)$) satisfies, as $r\to\infty$,

(B) $$a_r(I) \leqslant \text{const}\, r^{-h+\varepsilon}$$

for all $\varepsilon > 0$, and accordingly I is of type l^q if $\frac{m}{n} > \frac{1}{q} + \max\left(\frac{1}{p} - \frac{1}{s}, 0\right)$. If $n \geqslant 1$, $1 \leqslant m < n$, $1 \leqslant \nu < n/(n-m)$, we also have that the embedding map $W^{m,n/m}_{(0)}(\Omega) \to L^{\varphi_\nu}(\Omega)$ is of type c_0, which tells us nothing more than that the map is compact. We thus simply recover Trudinger's result, but under the better conditions on ν which are indicated by work of Edmunds and Evans [2]. Estimates of the form of (B) have been known for a number of years, and appear to be due to Birman and Solomjak [14], [15], who used completely different methods and dealt with the case in which Ω was a cube and $p < \infty$. They were able to establish the even stronger inequality

$$a_r(I) \leqslant \text{const}\, r^{-h},$$

and for certain special cases managed to obtain similar lower bounds for $a_r(I)$. Other results in the same direction as (B) are to be found in [16]–[21], although it should be remarked that the extreme cases $p = 1$ and $p = \infty$ are not always dealt with, and that often the discussion is carried out not in terms of approximation numbers but by means of the related notions of the Kolmogorov diameter or the entropy of a set. So far as we are aware there has been no previous work of this kind on the embeddings in Orlicz spaces. Apart from this matter of Orlicz spaces, the essentially new contribution of this paper is thus not given by the results about the types of the various embeddings, but is rather provided by our estimate of the closeness of Fourier approximations contained in (A), together with our treatment of the extreme cases $p = 1$ and $p = \infty$. At this point it is convenient to remark that it is our use of Fourier approximations rather than any other kind of approximation which is responsible for the appearance of the ε in (A) and (B): while for certain values of the various parameters ε can be set equal to zero, its presence in general is essential and is related to factors involving $\log r$ which turn up in the detailed calculations.

In a forthcoming paper [4] we shall show how the notion of approximation numbers may be applied to obtain the asymptotic distribution of the eigenvalues of elliptic operators.

It is a pleasure to record our thanks to L. E. Fraenkel for numerous stimulating conversations, and above all for supplying the crucial estimates of Section 7.

1. Preliminaries

We denote by R^n the n-dimensional, real Euclidean space with general point $x = (x_1, \ldots, x_n)$. By Ω we denote, once and for all, a bounded open set in R^n and by $C_0^\infty(\Omega)$ the linear space of infinitely differentiable, complex-valued functions on R^n whose support is contained in Ω. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $k = (k_1, \ldots, k_n)$ be multi-indices, i.e. vectors in R^n with integral components, the components of α always being non-negative. We set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad |k| = \sum_{i=1}^{n} |k_i|, \quad k^\alpha = \prod_{i=1}^{n} k_i^{\alpha_i}, \quad k^{-\alpha} = (k^\alpha)^{-1},$$

this convention for multi-indices being maintained throughout the paper, and for $u \in C_0^\infty(\Omega)$ we also set

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Given the special importance that we shall attach to open sets with minimally smooth boundary in the sense of Stein ([10], pp. 181 and 189), we record here their definition.

An open set $D \subset R^n$ is called a *special Lipschitz domain* if there exists a function $\varphi\colon R^{n-1} \to R^1$ satisfying the Lipschitz condition

$$(1) \qquad |\varphi(y) - \varphi(y')| \leqslant M \Big\{ \sum_{j=1}^{n-1} (y_j - y_j')^2 \Big\}^{1/2} \quad \text{for all } y, y' \in R^{n-1},$$

such that D is a rotation of the set

$$\{x \in R^n\colon x = (y, z),\ y \in R^{n-1},\ z \in R^1,\ z > \varphi(y)\}.$$

The smallest M for which (1) holds is called the *bound* of the special Lipschitz domain D.

Now let D be an open set in R^n and let ∂D be its boundary. We say that ∂D is *minimally smooth* if there exists an $\varepsilon > 0$, an integer N, an $M > 0$, and a sequence (U_i) of open sets so that:

(i) If $x \in \partial D$, then the ball $B(x, \varepsilon)$ of centre x and radius ε is contained in some U_i.

(ii) No point of R^n is contained in more than N of the U_i's.

(iii) For each i there exists a special Lipschitz domain D_i, whose bound does not exceed M, so that

$$U_i \cap D = U_i \cap D_i.$$

Examples of open sets with minimally smooth boundary are exhibited in R^n by any bounded open set which is either convex or whose boundary is C^1-embedded in R^n, and in R^1 by any finite collection of disjoint open intervals.

Returning to our bounded open subset Ω of R^n, we shall consider the following Sobolev spaces supported by it.

(i) $W^{m,p}(\Omega)$: This is the subspace of $L^p(\Omega)$ of all complex-valued functions u with generalized derivatives up to the order m in $L^p(\Omega)$, under the norm

$$\|u\|_{m,p,\Omega} = \Big(\sum_{|\alpha| \leqslant m} \int_\Omega |D^\alpha u|^p\, dx\Big)^{1/p} \tag{2}$$

for $1 \leqslant p < \infty$, and

$$\|u\|_{m,\infty,\Omega} = \sum_{|\alpha| \leqslant m} \sup_{x \in \Omega} |D^\alpha u(x)|, \tag{3}$$

for $p = \infty$, "sup" denoting the essential supremum. The space $W^{m,p}(R^n)$ is defined similarly.

(ii) $H^{m,p}(\Omega)$: Denote by $C^\infty(\bar{\Omega})$ the space of all infinitely differentiable, complex-valued functions on Ω whose derivatives of all orders have continuous extensions to $\bar{\Omega}$. Then $H^{m,p}(\Omega)$ is defined as the completion of $C^\infty(\bar{\Omega})$ for the norm (2) if $1 \leqslant p < \infty$ and for the norm (3) if $p = \infty$.

(iii) $W_0^{m,p}(\Omega)$ and $H_0^{m,p}(\Omega)$: These spaces are the closures of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ and $H^{m,p}(\Omega)$ respectively, endowed with the induced norms.

A few remarks on the comparison of these spaces are in order.

(a) $W_0^{m,p}(\Omega) = H_0^{m,p}(\Omega)$ for $1 \leqslant p \leqslant \infty$ and any Ω.

(b) $W^{m,p}(\Omega) \supset H^{m,p}(\Omega)$, but if $1 \leqslant p < \infty$ and $\partial\Omega$ is minimally smooth, then $W^{m,p}(\Omega) = H^{m,p}(\Omega)$.

(c) $H^{m,\infty}(\Omega) = C^m(\bar{\Omega})$ is always a proper subspace of $W^{m,\infty}(\Omega)$.

When $m = 0$ then clearly $W^{m,p}(\Omega) = L^p(\Omega)$ and we simply write $\|u\|_{p,\Omega}$ instead of $\|u\|_{0,p,\Omega}$ for $u \in L^p(\Omega)$.

Finally, for $1 \leqslant p \leqslant \infty$ let p' be the complementary Lebesgue exponent of p defined by $1/p + 1/p' = 1$. If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ we set

$$(u, v) = \int_\Omega u(x)\overline{v(x)}\, dx,$$

where $\overline{v(x)}$ stands for the complex conjugate of $v(x)$.

In this paper we investigate the Fourier approximation in $L^s(\Omega)$ of functions $u \in W^{m,p}(\Omega)$, where $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$. In view of the above remarks the results obtained will hold, a fortiori, also for functions in $W_0^{m,p}(\Omega)$ or $H_{(0)}^{m,p}(\Omega)$.

2. Embedding $W^{m,p}(\Omega)$ into $L^s(\Omega)$ $(n > 1)$

Let Ω be a bounded open subset of R^n whose boundary is minimally smooth in the sense of the previous section. Without loss of generality we may assume that $\overline{\Omega}$ is contained in the n-dimensional open cube $Q = (-\pi, \pi)^n$. Then any function $u \in W^{m,p}(\Omega)$ can be extended to a function $\overline{u} \in W^{m,p}(Q)$ which is periodic in each coordinate, the extension operator $u \to \overline{u}$ being an isomorphism of $W^{m,p}(\Omega)$ onto a closed subspace of $W^{m,p}(Q)$ with norm independent of p. Moreover, for $1 \leqslant p < \infty$ the range of the extension operator is contained in $W_0^{m,p}(Q)$. In fact, if $u \in W^{m,p}(\Omega)$, then by [10], Theorem 5, p. 181, u can be extended to a function $v \in W^{m,p}(R^n)$ such that

$$\|v\|_{m,p,R^n} \leqslant c_0 \|u\|_{m,p,\Omega}, \tag{4}$$

where c_0 depends only on m, n and Ω. Let $\varphi \in C_0^\infty(R^n)$ with $\operatorname{supp}\varphi \subset Q$ and $\varphi(x) = 1$ for all $x \in \overline{\Omega}$. Then $\overline{u} = \varphi v \in W_0^{m,p}(Q)$ for $1 \leqslant p < \infty$ and $\overline{u} \in W^{m,p}(Q)$ for $p = \infty$, but in any case $\overline{u}$ can be continued periodically outside Q. Moreover, since $\overline{u}(x) = u(x)$ for $x \in \Omega$, we have from (4)

$$\|u\|_{m,p,\Omega} \leqslant \|\overline{u}\|_{m,p,Q} \leqslant c_1 \|v\|_{m,p,R^n} \leqslant c \|u\|_{m,p,\Omega}, \tag{5}$$

with c independent of p.

Let now $k = (k_1, \ldots, k_n)$ be a vector in R^n with integral components and consider the trigonometric system $\{\varphi_k\}$ restricted to Q, where

$$\varphi_k(x) = (2\pi)^{-n/2} e^{ik\cdot x}$$

and $k \cdot x$ denotes the inner product of k and x in R^n. It is well known that this system is contained in $L^\infty(Q)$, is a basis in $L^s(Q)$ for $1 < s < \infty$ and its linear span is dense in $L^1(Q)$. We restrict our attention to Fourier approximation by "square" partial sums; our estimates will then automatically hold also for "spherical" partial sums and it will be clear how to interpret our results in the case of general partial sums. In order to construct such square partial sums we proceed as follows.

It will be convenient to introduce the following notation: if $f(x)$ and $g(x)$ are two real-valued functions, we write

$$f(x) \approx g(x) \quad \text{as } x \to \infty$$

to mean that there exist two positive constants c_1 and c_2 such that

$$c_1 f(x) \leqslant g(x) \leqslant c_2 f(x) \qquad \text{for large enough } x.$$

Throughout this paper we denote by r a large positive integer. If, for every positive integer j, $N(j)$ is the number of functions φ_k with $|k| \leqslant j$, then $N(j) \approx j^n$ and hence, if we require that $N(j) \leqslant r$, we must take $j \approx r^{1/n}$. Now let $k(r)$ be a multi-index whose components $k_i(r)$ are all positive and such that

$$k_i(r) \approx |k(r)| \approx r^{1/n} \qquad (1 \leqslant i \leqslant n). \tag{6}$$

For $u \in L^s(Q)$ we consider the partial sum of order $k(r)$ of the Fourier series of u,

$$T_{k(r)} u = \sum_{\substack{|k_i| \leqslant k_i(r) \\ 1 \leqslant i \leqslant n}} (u, \varphi_k)\varphi_k, \tag{7}$$

and set, for every $u \in W^{m,p}(\Omega)$ (here we assume m, p and s such that $W^{m,p}(\Omega) \subset L^s(\Omega)$)

$$S_r u = \chi_\Omega T_{k(r)} \bar{u}, \tag{8}$$

where χ_Ω is the characteristic function of Ω and $\bar{u}$ is the extension of u to Q constructed above and satisfying (5). Moreover, $\bar{u}$ is always assumed continued periodically outside Q whenever necessary. Since $u = \chi_\Omega \bar{u}$, we clearly have for the *Fourier remainder map* $I - S_r$,

$$\|u - S_r u\|_{s,\Omega} \leqslant \|\bar{u} - T_{k(r)} \bar{u}\|_{s,Q}. \tag{9}$$

Under the condition $\bar{\Omega} \subset Q$ we shall obtain estimates of the form

$$\|u - S_r u\|_{s,\Omega} \leqslant c(p, s) f(r) \|u\|_{m,p,\Omega} \qquad (u \in W^{m,p}(\Omega)),$$

where $f(r) = r^{-h}(\log r)^\sigma$ and h is given by (19) below. The case of general Ω can be reduced to the above as follows. Let d be any real number such that

$$\operatorname{diam}(\Omega) < d < 2\operatorname{diam}(\Omega)$$

(the second inequality is just to fix the ideas, since any $d > \operatorname{diam}(\Omega)$ and independent of r, p, s, m and n will do). Choose an open cube Q_d of side d and with sides parallel to the coordinate axes such that $\bar{\Omega} \subset Q_d$. Denoting by ψ the obvious diffeomorphism of Q onto Q_d, we set $\Omega_0 = \psi^{-1}(\Omega)$; then $\partial\Omega_0 = \psi^{-1}(\partial\Omega)$ is minimally smooth and $\bar{\Omega}_0 \subset Q$. If now $u \in W^{m,p}(\Omega)$, then $u \circ \psi \in W^{m,p}(\Omega_0)$ since $\psi \in C^\infty(Q, Q_d)$, and we have, with c_1, c_2 independent of r, p and s,

$$\begin{aligned}\big\|u - \big(S_r(u \circ \psi)\big) \circ \psi^{-1}\big\|_{s,\Omega} &\leqslant c_1 \|u \circ \psi - S_r(u \circ \psi)\|_{s,\Omega_0} \\ &\leqslant c_1 c(p, s) f(r) \|u \circ \psi\|_{m,p,\Omega_0} \leqslant c_1 c_2 c(p, s) f(r) \|u\|_{m,p,\Omega}.\end{aligned}$$

After these introductory remarks, we come to far more important matters in the form of a fundamental interpolation lemma for Sobolev spaces, of which we shall make extensive use. Quite possibly this lemma is already known, but we are unable to provide a satisfactory reference.

LEMMA 1. *For $i = 1, 2$ let p_i, s_i be such that $1 < p_i < \infty$ and $1 \leqslant s_i \leqslant \infty$. Let D be any open set in R^n and let T be a linear operator mapping $W^{m,p_i}(D)$ into $L^{s_i}(D)$ such that*

$$\|Tu\|_{s_i,D} \leqslant c_i \|u\|_{m,p_i,D} \tag{10}$$

for all $u \in W^{m,p_i}(D)$ $(i = 1, 2)$. For $0 < t < 1$ set

$$\frac{1}{s} = \frac{t}{s_1} + \frac{1-t}{s_2}, \qquad \frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}. \tag{11}$$

Then T maps $W_0^{m,p}(D)$ into $L^s(D)$ and there exists a positive constant c, independent of t, such that

$$\|Tu\|_{s,D} \leqslant c c_1^t c_2^{1-t} \|u\|_{m,p,D} \tag{12}$$

for all $u \in W_0^{m,p}(D)$.

Moreover, if ∂D is minimally smooth, then T also maps $W^{m,p}(D)$ into $L^s(D)$ so that (12) *holds for every $u \in W^{m,p}(D)$.*

Proof. For the moment let $1 < p < \infty$. Following Stein ([10], Ch. V, § 3) we denote by $\mathscr{L}_m^p(R^n)$ the potential spaces corresponding to the Bessel potential $\mathscr{J}_m$ and note that, by [10], Theorem 3, p. 135, $\mathscr{L}_m^p(R^n)$ is isomorphic to $W^{m,p}(R^n)$. Now let χ_D be the characteristic function of D and set

$$Rv = \chi_D v \quad \text{for} \quad v \in \mathscr{L}_m^p(R^n).$$

Then $Rv \in W^{m,p}(D)$ and the restriction operator R is bounded from $\mathscr{L}_m^p(R^n)$ to $W^{m,p}(D)$, since

$$\|Rv\|_{m,p,D} \leqslant \|v\|_{m,p,R^n} \leqslant \gamma_p \|v\|_{\mathscr{L}_m^p(R^n)}. \tag{13}$$

From (13) and (10) we have, for $v \in \mathscr{L}_m^{p_i}(R^n)$ and $i = 1, 2$,

$$\|TRv\|_{s_i,D} \leqslant c_i \|Rv\|_{m,p_i,D} \leqslant \gamma_{p_i} c_i \|v\|_{\mathscr{L}_m^{p_i}(R^n)} = \gamma_{p_i} c_i \|\mathscr{J}_m^{-1} v\|_{p_i,R^n}. \tag{14}$$

Since $\mathscr{J}_m$ is an isometry of $L^p(R^n)$ onto $\mathscr{L}_m^p(R^n)$ for $1 \leqslant p \leqslant \infty$, (14) shows that the operator $A = TR\mathscr{J}_m$ is bounded from $L^{p_i}(R^n)$ to $L^{s_i}(D)$ $(i = 1, 2)$. Thus by the Riesz–Thorin Interpolation theorem ([13], Vol. II, Theorem (1.11), p. 95) A is also bounded from $L^p(R^n)$ to $L^s(D)$ with p and s given by (11), and we have

$$\|Aw\|_{s,D} \leqslant (\gamma_{p_1} c_1)^t (\gamma_{p_2} c_2)^{1-t} \|w\|_{p,R^n}$$

for all $w \in L^p(R^n)$, i.e.,

$$(15) \qquad \|TRv\|_{s,D} \leqslant (\gamma_{p_1} c_1)^t (\gamma_{p_2} c_2)^{1-t} \|v\|_{\mathscr{L}^p_m(R^n)}$$

for all $v \in \mathscr{L}^p_m(R^n)$.

Let now $u \in W^{m,p}_0(D)$; extending u by 0 outside D we obtain a function $\bar{u} \in \mathscr{L}^p_m(R^n)$ such that

$$(16) \qquad \|\bar{u}\|_{\mathscr{L}^p_m(R^n)} \leqslant c_p \|u\|_{m,p,D}.$$

Since $R\bar{u} = u$, (15) and (16) give

$$(17) \qquad \|Tu\|_{s,D} = \|TR\bar{u}\|_{s,D} \leqslant c_p (\gamma_{p_1} c_1)^t (\gamma_{p_2} c_2)^{1-t} \|u\|_{m,p,D},$$

which yields (12) with $c = \sup\limits_{0<t<1} c_p \gamma^t_{p_1} \gamma^{1-t}_{p_2}$.

The proof of the second assertion is similar. If ∂D is minimally smooth and p and s are given by (11), then by [10], Theorem 5, p. 181, a function $u \in W^{m,p}(D)$ can be extended to a function $\bar{u} \in W^{m,p}(R^n) = \mathscr{L}^p_m(R^n)$ satisfying (16), from which (17), whence (12), follows.

Remark 1. If D is minimally smooth, then $W^{m,p}(D)$ $(1 < p < \infty)$ is isomorphic to the space $R[\mathscr{L}^p_m(R^n)]$ of restrictions to D of elements of $\mathscr{L}^p_m(R^n)$ under the quotient norm. Hence, giving $W^{m,p}(D)$ the norm of $R[\mathscr{L}^p_m(R^n)]$ and $W^{m,p}_0(D)$ the induced norm, we see that in this case (12) holds with $c = 1$.

Armed with Lemma 1 we come now to our basic result which we state in a rather coarse form, finer estimates being given in the course of the proof.

THEOREM 1. *Let Ω be a bounded open subset of R^n with minimally smooth boundary. Suppose that $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$ and that m is a positive integer such that $\dfrac{1}{p} - \dfrac{1}{s} < \dfrac{m}{n} < 1$. Then for every $u \in W^{m,p}(\Omega)$ we have, as $r \to \infty$,*

$$(18) \qquad \|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega}\, o(r^{-h+\varepsilon}) \quad \textit{for every } \varepsilon > 0,$$

where

$$(19) \qquad h = \frac{m}{n} - \max\left(\frac{1}{p} - \frac{1}{s}, 0\right).$$

Proof. In order to establish (18) and (19) we divide the proof of the theorem in several steps. In each of these steps we shall obtain order estimates of the form

$$\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega}\, O\{r^{-h} (\log r)^\sigma\},$$

i.e.,

$$\|u - S_r u\|_{s,\Omega} \leqslant c \|u\|_{m,p,\Omega}\, r^{-h} (\log r)^\sigma,$$

where h is given by (19), g is an appropriate function of m, n, p and s, and the constant c can be taken to be independent of p and s except when stated. To help the reader follow the proof, we have drawn the enclosed Fig. 1 showing the values of the function $f = r^{-h}(\log r)^g$ in the various regions in which the set

$$(20)\qquad S = \left\{\left(\frac{1}{p}, \frac{1}{s}\right) : 1 \leqslant p \leqslant \infty, 1 \leqslant s \leqslant \infty, \frac{1}{p} - \frac{1}{s} < \frac{m}{n}\right\}$$

(for which the embeddings considered exist and are compact) is decomposed

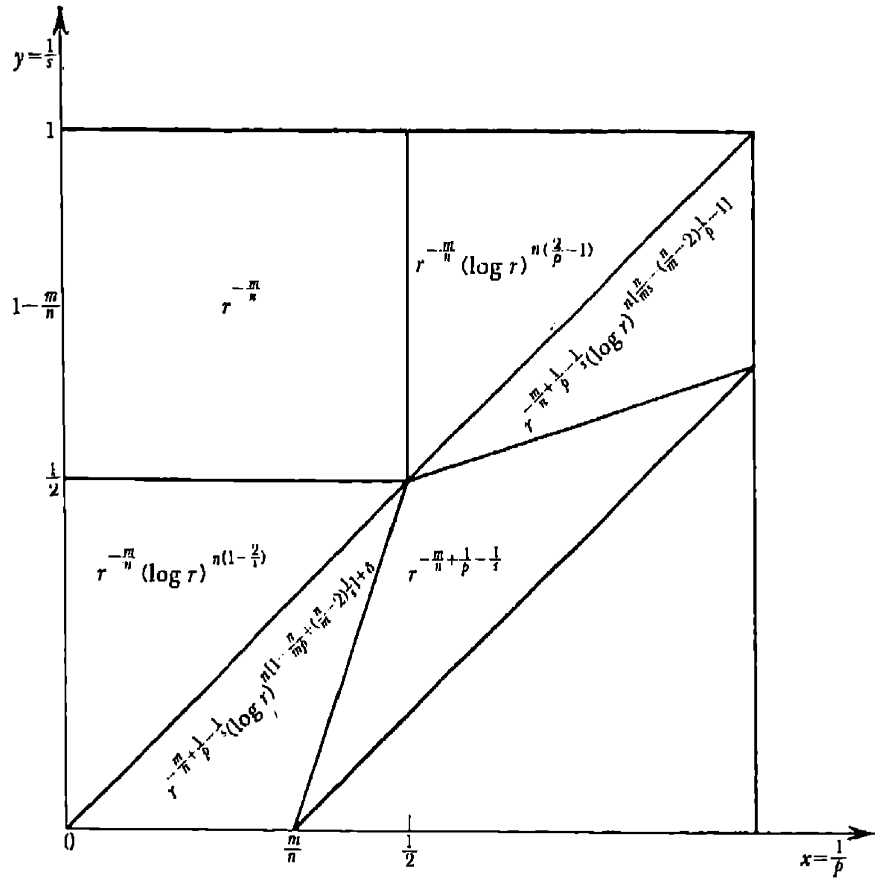

Fig. 1. Values of the function $f(r, p, s, m, n)$ estimating the norm of the remainder map of the Fourier series for $m < n$

in the course of the proof. To fix the ideas, we have assumed $2m < n$ in Fig. 1.

(i) $p = s = 2$. The system $\{\varphi_k\}$ is orthonormal and complete in $L^2(Q)$, and hence for every $u \in L^2(Q)$ we have

$$(21)\qquad u = \sum_k (u, \varphi_k)\varphi_k,$$

the series being convergent for the norm of $L^2(Q)$. Now for $u \in W_0^{m,2}(Q)$ and for every k with $|k| \geqslant m$ there exists an $\alpha(k) = (\alpha_1(k), \ldots, \alpha_n(k))$, with $|\alpha(k)| = m$, such that

$$(22) \qquad (u, \varphi_k) = (-i)^m k^{-\alpha(k)} (D^{\alpha(k)} u, \varphi_k).$$

Set $k_0(r) = \min\{k_i(r) \colon 1 \leqslant i \leqslant n\}$; then $k_0(r) \approx r^{1/n}$ by (6). By (21) and (22) we have, for $u \in W_0^{m,2}(Q)$,

$$(23) \qquad \begin{aligned} \|u - T_{k(r)} u\|_{2,Q} &= \Big\| \sum_k (u, \varphi_k)\varphi_k - \sum_{\substack{|k_i| \leqslant k_i(r) \\ 1 \leqslant i \leqslant n}} (u, \varphi_k)\varphi_k \Big\|_{2,Q} \\ &\leqslant \Big\{ \sum_{|k| > k_0(r)} |(D^{\alpha(k)} u, \varphi_k)|^2 k^{-2\alpha(k)} \Big\}^{1/2} \\ &\leqslant \Big\{ \sum_{|k| > k_0(r)} |(D^{\alpha(k)} u, \varphi_k)|^2 \Big\}^{1/2} \{ \sup_{|k| > k_0(r)} k^{-2\alpha(k)} \}^{1/2} \\ &= \Big\{ \sum_{|\alpha| \leqslant m} \sum_k |(D^\alpha u, \varphi_k)|^2 \Big\}^{1/2} O\{k_0(r)^{-m}\} \\ &= \Big(\sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{2,Q}^2 \Big)^{1/2} O(r^{-m/n}) = \|u\|_{m,2,Q} O(r^{-m/n}). \end{aligned}$$

Taking now $u \in W^{m,2}(\Omega)$, (9), (23) and (5) yield

$$(24) \qquad \|u - S_r u\|_{2,\Omega} = \|u\|_{m,2,\Omega} O(r^{-m/n}).$$

(ii) $1 \leqslant s \leqslant 2 \leqslant p \leqslant \infty$. Since the embeddings $W^{m,p}(\Omega) \to W^{m,2}(\Omega)$ and $L^2(\Omega) \to L^s(\Omega)$ are bounded, we have from (24)

$$(25) \qquad \|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} O(r^{-m/n}) \qquad (u \in W^{m,p}(\Omega)).$$

(iii) $(1/p, 1/s)$ *in the open triangle with vertices* $(1/2, 1/2)$, $(m/n, 0)$ *and* $(1, 1-m/n)$. Let p, s be given in the specified region and, with reference to the figure, let (x_0, y_0) be the intersection of the straight line $y = x - m/n$ with the line passing through the points $(1/2, 1/2)$ and $(1/p, 1/s)$. We set

$$p_0 = \frac{1}{x_0}, \qquad s_0 = \frac{1}{y_0}.$$

By [7], Théorème 3.6, p. 72, $W^{m,p_0}(\Omega) \subset L^{s_0}(\Omega)$, the embedding $W^{m,p_0}(\Omega) \to L^{s_0}(\Omega)$ being bounded. This implies the existence of a constant $c_1(p_0)$ such that $\|u\|_{s_0,\Omega} \leqslant c_1(p_0) \|u\|_{m,p_0,\Omega}$ and it is known that $c_1(p_0) \to \infty$ as $p_0 \to n/m$. Since $1 < s_0 < \infty$, the maps $T_{k(r)}\colon L^{s_0}(Q) \to L^{s_0}(Q)$ given by (7) are uniformly bounded, whence so are the operators S_r of (8). It follows that

$$(26) \qquad \|u - S_r u\|_{s_0,\Omega} \leqslant c_2(p_0) \|u\|_{m,p_0,\Omega},$$

where also $c_2(p_0) \to \infty$ as $p_0 \to n/m$.

Applying Lemma 1 to the operator $I-S_r$, with $D = \Omega$, $p_1 = 2$, $s_1 = 2$, $p_2 = p_0$ and $s_2 = s_0$, we have from (24) and (26),

$$\|u-S_r u\|_{s,\Omega} \leqslant c_3(p_0)^{1-t} r^{-t\frac{m}{n}} \|u\|_{m,p,\Omega}$$

for all $u \in W^{m,p}(\Omega)$, where $0 < t < 1$ and

$$\frac{1}{p} = \frac{t}{2} + \frac{1-t}{p_0}, \qquad \frac{1}{s} = \frac{t}{2} + \frac{1-t}{s_0}.$$

Hence, substituting for t,

$$(27) \qquad \|u-S_r u\|_{s,\Omega} \leqslant c(p,s) r^{-\frac{m}{n}+\frac{1}{p}-\frac{1}{s}} \|u\|_{m,p,\Omega},$$

where $c(p,s) \to \infty$ as the point $(1/p, 1/s)$ approaches any point on the open segment with end-points $(m/n, 0)$ and $(1/2, 1/2)$.

(iv) $p = s = \infty$. Let $u \in W^{m,\infty}(Q)$, u periodic. We have ([13], Vol. II, p. 302):

$$(T_{k(r)}u)(x) = \pi^{-n} \int_Q dt D_{k(r)}(x-t) u(t),$$

where $D_{k(r)}(x)$ is the Dirichlet kernel given by

$$D_{k(r)}(x) = \prod_{i=1}^{n} D_{k_i(r)}(x_i),$$

with

$$(28) \qquad D_{k_i(r)}(x_i) = \frac{\sin\{(k_i(r)+\frac{1}{2})x_i\}}{2\sin(\frac{1}{2}x_i)} \quad (1 \leqslant i \leqslant n)$$

(see [13], Vol. I, p. 49).

We can now write

$$(29) \qquad u(x) - (T_{k(r)}u)(x) = \sum_{i=1}^{n} (R_i u)(x)$$

where

$$(30) \quad (R_1 u)(x) = u(x_1, \ldots, x_n) - \pi^{-1} \int_{-\pi}^{\pi} dt_1 D_{k_1(r)}(x_1-t_1) u(t_1, x_2, \ldots, x_n)$$

and, for $2 \leqslant i \leqslant n$,

$$(31) \quad (R_i u)(x) = \pi^{-i+1} \int_{-\pi}^{\pi} dt_1 D_{k_1(r)}(x_1-t_1) \ldots \int_{-\pi}^{\pi} dt_{i-1} D_{k_{i-1}(r)}(x_{i-1}-t_{i-1}) \times$$

$$\times \Big\{u(t_1, \ldots, t_{i-1}, x_i, \ldots, x_n) - \pi^{-1} \int_{-\pi}^{\pi} dt_i D_{k_i(r)}(x_i-t_i) u(t_1, \ldots, t_i, x_{i+1}, \ldots, x_n)\Big\}.$$

Since $u \in W^{m,\infty}(Q)$, for any given i $(1 \leqslant i \leqslant n)$ and for a fixed $(n-1)$-tuple $(x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_{i+1}^{(0)}, \ldots, x_n^{(0)})$, the restriction of u to the line segment

$$\{x_1 = x_1^{(0)}, \ldots, x_{i-1} = x_{i-1}^{(0)}, x_{i+1} = x_{i+1}^{(0)}, \ldots, x_n = x_n^{(0)}, |x_i| < \pi\}$$

belongs to $W^{m,\infty}(J)$, where $J = (-\pi, \pi)$, for almost all $(n-1)$-tuples $(x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_{i+1}^{(0)}, \ldots, x_n^{(0)})$ corresponding to points in Q. Since every function in $W^{m,\infty}(J)$ is continuous on $\bar{J}$, we have, by [13], Vol. I, Theorem (13.6), p. 115 and Remark (e), p. 120,

$$(32) \quad \sup_{|x_i|<\pi} \Big| u(t_1, \ldots, t_{i-1}, x_i, \ldots, x_n) - \pi^{-1} \int_{-\pi}^{\pi} dt_i D_{k_i(r)}(x_i - t_i) u(t_1, \ldots, t_i, x_{i+1}, \ldots, x_n) \Big| \leqslant c_i k_i(r)^{-m} \log k_i(r) \sup_{|x_i|<\pi} \Big| \frac{\partial^m}{\partial x_i^m} u(t_1, \ldots, t_{i-1}, x_i, \ldots, x_n) \Big|.$$

Hence, for $i = 1$,

$$(33) \quad \|R_1 u\|_{\infty,Q} \leqslant c_1 k_1(r)^{-m} \log k_1(r) \left\| \frac{\partial^m u}{\partial x_1^m} \right\|_{\infty,Q} = \left\| \frac{\partial^m u}{\partial x_1^m} \right\|_{\infty,Q} O(r^{-m/n} \log r).$$

Moreover, since for every integer $j > 1$,

$$(34) \quad \int_{-\pi}^{\pi} d\xi\, |D_j(\xi)| = O(\log j)$$

(cf. [13], Vol. I, p. 67), we have from (32) and (34), for $2 \leqslant i \leqslant n$,

$$(35) \quad \|R_i u\|_{\infty,Q} \leqslant \pi^{-i+1} \sup_{x \in Q} \int_{-\pi}^{\pi} dt_1 |D_{k_1(r)}(t_1)| \ldots \int_{-\pi}^{\pi} dt_{i-1} |D_{k_{i-1}(r)}(t_{i-1})| \times$$
$$\times \Big| u(t_1 + x_1, \ldots, t_{i-1} + x_{i-1}, x_i, \ldots, x_n) - \pi^{-1} \int_{-\pi}^{\pi} dt_i D_{k_i(r)}(t_i - x_i) u(t_1 + x_1, \ldots, t_{i-1} + x_{i-1}, t_i, x_{i+1}, \ldots, x_n) \Big|$$
$$\leqslant \pi^{-i+1} c_i k_i(r)^{-m} \log k_i(r) \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{\infty,Q} \int_{-\pi}^{\pi} dt_1 |D_{k_1(r)}(t_1)| \quad \int_{-\pi}^{\pi} dt_{i-1} |D_{k_{i-1}(r)}(t_{i-1})|$$
$$= \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{\infty,Q} O\Big\{ k_i(r)^{-m} \prod_{j=1}^{i} \log k_j(r) \Big\} = \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{\infty,Q} O\{r^{-m/n} (\log r)^i\}.$$

Thus from (29), (33) and (35) we obtain

$$\|u - T_{k(r)} u\|_{\infty,Q} \leqslant \sum_{i=1}^{n} \|R_i u\|_{\infty,Q} \leqslant \sum_{i=1}^{n} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{\infty,Q} O\{r^{-m/n} (\log r)^i\}$$
$$= \|u\|_{m,\infty,Q}\, O\{r^{-m/n} (\log r)^n\},$$

whence by (7) and (5), if $u \in W^{m,\infty}(\Omega)$,

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,\infty,\Omega} O\{r^{-m/n}(\log r)^n\}. \tag{36}$$

(v) $s = \infty$, $n/m < p < \infty$. Assume first that $(m-1)/n < 1/p < m/n$ and let $\mu = m - n/p$; then $W_0^{m,p}(Q) \subset C^{(\mu)}(\bar{Q})$ by [7], Théorème 3.8, p. 72, $C^{(\mu)}(\bar{Q})$ denoting the space of Hölder continuous functions on $\bar{Q}$ of order μ. We can now use [13], Theorem (13.6), p. 115 and Remark (e), p. 120, and a proof similar to that of (36) to obtain

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{\infty,\Omega} O\{r^{-\mu/n}(\log r)^n\} \leqslant \|u\|_{m,p,\Omega} O\{r^{-\frac{m}{n}+\frac{1}{p}}(\log r)^n\},$$

for all $u \in W^{m,p}(\Omega)$. (Here the O symbol hides a dependence of the constant on p as $p \to n/m$; however it is not really necessary to make this dependence explicit, since the fact that for $u \in W^{m,p}(\Omega)$, $\|u\|_{\infty,\Omega}$ explodes as $p \to n/m$ is well expressed by the logarithmic factor.)

Suppose now that $m > 1$ and let $0 < 1/p < 1/n$, $\mu = 1 - n/p$. Then for every α with $|\alpha| = m-1$, $D^\alpha u \in W_0^{1,p}(Q)$ for $u \in W_0^{m,p}(Q)$. Thus, again by [7], Théorème 3.8, p. 72, $D^\alpha u \in C^{(\mu)}(\bar{Q})$ and hence, combining once more [13], loc. cit., with the proof of (36) we arrive at the following estimate for $u \in W^{m,p}(\Omega)$:

$$\begin{aligned} \|u - S_r u\|_{\infty,\Omega} &= \|u\|_{m-1,\infty,\Omega} O\{r^{-\frac{m-1}{n}-\frac{\mu}{n}}(\log r)^n\} \\ &\leqslant \|u\|_{m,p,\Omega} O\{r^{-\frac{m}{n}+\frac{1}{p}}(\log r)^n\}. \end{aligned} \tag{37}$$

Finally, the reader can easily satisfy himself that the above estimates also hold for $1/n \leqslant 1/p \leqslant (m-1)/n$ by interpolating, according to Lemma 1, between (p_1, ∞) and (p_2, ∞) with $0 < 1/p_1 < 1/n$ and $(m-1)/n < 1/p_2 < m/n$. Thus, for all p with $n/m < p < \infty$, we have

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-\frac{m}{n}+\frac{1}{p}}(\log r)^n\} \qquad (u \in W^{m,p}(\Omega)). \tag{38}$$

(vi) $2 < p = s < \infty$. Choose δ such that $0 < \delta < 1/n$ and define s_δ by $1/s_\delta = 1/2 + \delta$. With reference to Fig. 1, the straight line through $(1/2, 1/s_\delta)$ and $(1/p, 1/p)$ intersects the x axis in the point $(1/p_\delta, 0)$. Let t be such that $1/p = t/s_\delta$; then naturally $1/p = t/2 + (1-t)/p_\delta$. Using (25) and (38), interpolation between $(2, s_\delta)$ and (p_δ, ∞) (Lemma 1) gives

$$\|u - S_r u\|_{p,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-t\frac{m}{n}-(1-t)\frac{m}{n}+\frac{1-t}{p_\delta}}(\log r)^{n(1-t)}\}, \tag{39}$$

for every $u \in W^{m,p}(\Omega)$. Now note that as $\delta \to 0$, $p_\delta \to \infty$ and $t = t(\delta) \to 2/p$; hence, choosing δ so that

$$\frac{1-t}{p_\delta} \leqslant (\log r + 2n \log\log r)^{-1},$$

we can write (39) as follows:

$$\|u-S_r u\|_{p,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-m/n}(\log r)^{n(1-2/p)}\}. \tag{40}$$

(vii) $2<s<p<\infty$. Since the embedding $W^{m,p}(\Omega)\to W^{m,s}(\Omega)$ is bounded, it follows from (40) that

$$\|u-S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-m/n}(\log r)^{n(1-2/s)}\} \tag{41}$$

for all $u\in W^{m,p}(\Omega)$.

(viii) $(1/p, 1/s)$ *in the open triangle with vertices* $(0,0)$, $(1/2, 1/2)$ *and* $(m/n, 0)$. With reference to the figure, for p and s given in the above region, consider the open segment σ whose end points are

$$\left(\frac{1}{2}, \frac{1}{2}\right) \quad\text{and}\quad \left(\frac{m}{n}+\varepsilon, \varepsilon\right), \quad\text{where}\quad 0<\varepsilon<\min\left\{\left|\frac{1}{2}-\frac{m}{n}\right|, 1-\frac{m}{n}\right\}.$$

Each point $(x, y)\in\sigma$ lies inside the region in (iii) so that (27) holds with $p=1/x$ and $s=1/y$ whenever $u\in W^{m,1/x}(\Omega)$. Let (x_1, y_1) be the intersection of the general straight line through $(1/p, 1/s)$,

$$y-\frac{1}{s} = \lambda\left(x-\frac{1}{p}\right), \tag{42}$$

with the line

$$y = \frac{(1-2\varepsilon)x-m/n}{1-2m/n-2\varepsilon}$$

to which σ belongs. It is easily found that

$$x_1 = \frac{\dfrac{m}{n}+\left(\dfrac{\lambda}{p}-\dfrac{1}{s}\right)\left(\dfrac{2m}{n}-1+2\varepsilon\right)}{1+\lambda\left(\dfrac{2m}{n}-1+2\varepsilon\right)-2\varepsilon}, \quad y_1 = \frac{\dfrac{\lambda m}{n}+(1-2\varepsilon)\left(\dfrac{1}{s}-\dfrac{\lambda}{p}\right)}{\dfrac{2\lambda m}{n}+(1-2\varepsilon)(1-\lambda)}.$$

Moreover, for the intersection (x_2, y_2) of (42) with the line $y=0$ we obtain

$$x_2 = \frac{1}{p}-\frac{1}{\lambda s}, \quad y_2 = 0.$$

We impose the natural conditions

$$0<y_1\leqslant 1/2, \quad 0<x_2<m/n.$$

Now let t be such that

$$1/s = ty_1, \quad 1/p = tx_1+(1-t)x_2. \tag{43}$$

Then using Lemma 1 with $D = \Omega$, $p_1 = 1/x_1$, $s_1 = 1/y_1$, $p_2 = 1/x_2$ and $s_2 = 1/y_2$ we obtain from (43), (27) and (38),

$$(44)\qquad \|u - S_r u\|_{s,\Omega} \leqslant c(x_1, y_1)\|u\|_{m,n,\Omega} r^{-t\left(\frac{m}{n} - x_1 + y_1\right) - (1-t)\left(\frac{m}{n} - x_2\right)} (\log r)^{n(1-t)}$$
$$= c(x_1, y_1)\|u\|_{m,p,\Omega} r^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{s}} (\log r)^{n(1-t)}$$

for all $u \in W^{m,p}(\Omega)$, where, as in (27), $c(x_1, y_1) \to \infty$ as $\varepsilon \to 0$. Since

$$1 - t = \frac{1 - \dfrac{n}{mp} + \left(\dfrac{n}{m} - 2\right)\dfrac{1}{s} + \dfrac{2\varepsilon n}{m}\left(\dfrac{1}{p} - \dfrac{1}{s}\right)}{1 - (1 - 2\varepsilon)\dfrac{n x_2}{m}}$$

we see that $1 - t$ is an increasing function of x_2 (or λ). Set

$$\theta = \frac{\dfrac{1-2\varepsilon}{n}\left[1 - \dfrac{n}{mp} + \left(\dfrac{n}{m} - 2\right)\dfrac{1}{s}\right] x_2}{\dfrac{m}{n} - (1-2\varepsilon)x_2}, \qquad \delta = \frac{\dfrac{2\varepsilon}{n}\left(\dfrac{1}{p} - \dfrac{1}{s}\right)}{\dfrac{m}{n} - (1-2\varepsilon)x_2}.$$

Then since

$$n(1-t) = n\left[1 - \frac{n}{mp} + \left(\frac{n}{m} - 2\right)\frac{1}{s}\right] + \theta + \delta,$$

choosing x_2 (i.e., λ) so that $\theta \leqslant (\log\log r)^{-1}$, (44) becomes

$$(45)\qquad \|u - S_r u\|_{s,\Omega} \leqslant c(\delta)\|u\|_{m,p,\Omega} r^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{s}} (\log r)^{n\left[1 - \frac{n}{mp} + \left(\frac{n}{m} - 2\right)\frac{1}{s}\right] + \delta}$$

with $c(\delta) \to \infty$ as $\delta \to 0$.

(ix) $p = s = 1$. For $|\xi| \leqslant 2\pi$ denote by $\delta(\xi)$ the Dirac measure concentrated at 0 and set, for any positive integer j (cf. (28)),

$$(46)\qquad B_j(\xi) = D_j(\xi) - \pi\delta(\xi).$$

Then for $u \in C_0^\infty(Q)$, (30) and (31) can be written

$$(47)\qquad (R_1 u)(x) = -\pi^{-1} \int_{-\pi}^{\pi} dt_1 B_{k_1(r)}(x_1 - t_1) u(t_1, x_2, \ldots, x_n),$$

$$(48)\qquad (R_i u)(x) = -\pi^{-i} \int_{-\pi}^{\pi} dt_1 D_{k_1(r)}(x_1 - t_1) \ldots \int_{-\pi}^{\pi} dt_{i-1} D_{k_{i-1}(r)}(x_{i-1} - t_{i-1}) \times$$
$$\times \int_{-\pi}^{\pi} dt_i B_{k_i(r)}(x_i - t_i) u(t_1, \ldots, t_i, x_{i+1}, \ldots, x_n) \qquad (2 \leqslant i \leqslant n).$$

Note that each $(R_i u)(x)\ (1\leqslant i\leqslant n)$ is well defined, since $u\in C_0^\infty(Q)$ and the expressions

$$B_{k_1(r)}(x_1-t_1) \quad\text{and}\quad B_{k_i(r)}(x_i-t_i)\prod_{j=1}^{i-1} D_{k_j(r)}(x_j-t_j) \qquad (2\leqslant i\leqslant n)$$

generate measures, whence distributions, on Q [9].

Setting $\xi=2\eta$ and $2j+1=l$, (46) becomes

$$B_j(\xi)=\tfrac{1}{2}A_l(\eta) \qquad (|\eta|\leqslant\pi),$$

where the right-hand side is defined according to (93) of Section 7, Appendix A. Thus by Proposition 1 of the same section, $B_j(\xi)$ has an antiderivative of order m (in the sense of distributions) $B_j^{(m)}(\xi)$ such that

$$(49)\qquad |B_j^{(m)}(\xi)| = 2^{m-1}|A_l^{(m)}(\eta)| \leqslant cl^{-m+1}\left\{\frac{1}{1+l|\eta|}+\frac{1}{1+l(\pi-\eta)}+\frac{1}{1+l(\pi+\eta)}\right\},$$

with c a positive constant depending only on m. Hence, performing m integrations by parts with respect to the variable $t_i\ (1\leqslant i\leqslant n)$ in the right-hand sides of (47) and (48), we obtain

$$(50)\qquad (R_1u)(x) = -\pi^{-1}\int_{-\pi}^{\pi} dt_1 B_{k_1(r)}^{(m)}(x_1-t_1)\frac{\partial^m}{\partial t_1^m}u(t_1,x_2,\dots,x_n),$$

$$(51)\quad (R_iu)(x) = -\pi^{-i}\int_{-\pi}^{\pi} dt_1 D_{k_1(r)}(x_1-t_1)\dots\int_{-\pi}^{\pi} dt_{i-1}D_{k_{i-1}}(r)(x_{i-1}-t_{i-1})\times$$
$$\times\int_{-\pi}^{\pi} dt_i B_{k_i(r)}^{(m)}(x_i-t_i)\frac{\partial^m}{\partial t_i^m}u(t_1,\dots,t_i,x_{i+1},\dots,x_n) \qquad (2\leqslant i\leqslant n),$$

where for each $B_{k_i(r)}^{(m)}(x_i-t_i)$ an estimate of the type of (49) holds, with l and η in the right-hand side being replaced by $l_i(r)=2k_i(r)+1$ and $y_i=(x_i-t_i)/2$ respectively. Now (49) yields

$$(52)\qquad \int_{-2\pi}^{2\pi} d\xi\,|B_j^{(m)}(\xi)| = 2^m\int_{-\pi}^{\pi} d\eta\,|A_l^{(m)}(\eta)| = O(l^{-m}\log l),$$

and hence it follows from (51), in virtue of the estimates (34) and (52), that

$$\|R_iu\|_{1,Q}\leqslant \pi^{-i}\int_Q dt\left|\frac{\partial^m}{\partial t_i^m}u(t)\right|\int_{-\pi}^{\pi} dx_i\,|B_{k_i(r)}^{(m)}(x_i-t_i)|\prod_{j=1}^{i-1}\int_{-\pi}^{\pi} dx_j\,|D_{k_j(r)}(x_j-t_j)|$$
$$=\left\|\frac{\partial^m u}{\partial x_i^m}\right\|_{1,Q} O\{l_i(r)^{-m}\log l_i(r)\prod_{j=1}^{i-1}\log k_j(r)\}$$

$$= \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{1,Q} O\{r^{-m/n}(\log r)^i\} \quad (2 \leqslant i \leqslant n).$$

Similarly, from (50),

$$\|R_1 u\|_{1,Q} = \left\| \frac{\partial^m u}{\partial x_1^m} \right\|_{1,Q} O(r^{-m/n} \log r).$$

Thus from (29) we have

$$\|u - T_{k(r)} u\|_{1,Q} \leqslant \sum_{i=1}^{n} \|R_i u\|_{1,Q} = \|u\|_{m,1,Q} O\{r^{-m/n}(\log r)^n\} \tag{53}$$

for $u \in C_0^\infty(Q)$, whence for $u \in W_0^{m,1}(Q)$.

If now $u \in W^{m,1}(\Omega)$, (7), (5) and (53) give

$$\|u - S_r u\|_{1,\Omega} = \|u\|_{m,1,\Omega} O\{r^{-m/n}(\log r)^n\}. \tag{54}$$

(x) $s = 1$, $1 < p < 2$. Let ε be such that $0 < \varepsilon < 1 - 1/p$ and set $p_\varepsilon = 1/(1-\varepsilon)$. For $u \in W^{m,p_\varepsilon}(\Omega)$, (54) gives

$$\|u - S_r u\|_{1,\Omega} = \|u\|_{m,p_\varepsilon,\Omega} O\{r^{-m/n}(\log r)^n\}. \tag{55}$$

Hence interpolating between $(p_\varepsilon, 1)$ and $(2, 1)$ according to Lemma 1 and using (25) and (55) we have

$$\|u - S_r u\|_{1,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-m/n}(\log r)^{nt}\}, \tag{56}$$

where

$$u \in W^{m,p}(\Omega) \quad \text{and} \quad \frac{1}{p} = \frac{t}{p_\varepsilon} + \frac{1-t}{2} = \frac{1+t}{2} - \varepsilon t.$$

Finally, choosing

$$\varepsilon \leqslant (2nt \log\log r)^{-1},$$

(56) becomes

$$\|u - S_r u)\|_{1,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-m/n}(\log r)^{n(2/p-1)}\} \quad (u \in W^{m,p}(\Omega)). \tag{57}$$

(xi) $1 < p = s < 2$. Let ε be such that $0 < \varepsilon < m/2n$ and set

$$\frac{1}{p_1} = \frac{1}{2} + \varepsilon, \quad \frac{1}{s_1} = \frac{1}{2} - \varepsilon.$$

The straight line through the points $(1/p_1, 1/s_1)$ and $(1/p, 1/p)$ intersects the line $y = 1$ in $(1/p_2, 1)$. Define t so that

$$\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2} = \frac{t}{s_1} + 1 - t.$$

Since $(1/p_1, 1/s_1)$ and $(1/p_2, 1)$ belong to the regions in (iii) and (x) respectively, from (27), (57) and Lemma 1 we have

$$(58) \quad \|u - S_r u\|_{p,\Omega} \leqslant c(p_1, s_1)\|u\|_{m,p,\Omega} r^{-t\frac{m}{n} + \frac{t}{p_1} - \frac{t}{s_1} - (1-t)\frac{m}{n}} (\log r)^{n\left(\frac{2}{p_2} - 1\right)(1-t)}$$
$$\leqslant c(p_1, s_1)\|u\|_{m,p,\Omega} r^{-\frac{m}{n} + 2t\varepsilon} (\log r)^{n\left(\frac{2}{p} - 1\right) - n\delta(\varepsilon)}$$

for every $u \in W^{m,p}(\Omega)$, where

$$\delta(\varepsilon) = \frac{2\varepsilon}{1+2\varepsilon}\left(1 + \frac{2t}{1-t}\right)\left(\frac{2}{p} - 1\right).$$

Now as $\varepsilon \to 0$ the point $(1/p_1, 1/s_1)$ tends to $(1/2, 1/2)$ and not to a point of the open segment with end-points $(m/n, 0)$ and $(1/2, 1/2)$; thus (cf. (iii)) $c(p_1, s_1)$ remains bounded as $\varepsilon \to 0$. Hence, since $t < 1$ and $\delta(\varepsilon) > 0$, choosing $\varepsilon \leqslant (2\log r)^{-1}$, (64) yields

$$(59) \quad \|u - S_r u\|_{p,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-m/n}(\log r)^{n(2/p-1)}\} \qquad (u \in W^{m,p}(\Omega)).$$

(xii) $1 < s < p < 2$. By (59), if $u \in W^{m,p}(\Omega)$,

$$\|u - S_r u\|_{s,\Omega} \leqslant \|u - S_r u\|_{p,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-m/n}(\log r)^{n(2/p-1)}\}.$$

(xiii) $(1/p, 1/s)$ *in the open triangle with vertices* $(1/2, 1/2)$, $(1, 1)$ *and* $(1, 1-m/n)$. Interpolate (Lemma 1) between a point as in (xi) and one just inside the region in (iii) to obtain, for $u \in W^{m,p}(\Omega)$,

$$\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{s}} (\log r)^{n\left[\frac{n}{ms} + \left(2 - \frac{n}{m}\right)\frac{1}{p} - 1\right]}\}.$$

(xiv) $(1/p, 1/s)$ *on one of the two open segments with end points* $(1/2, 1/2)$, $(1, 1-m/n)$ *and* $(1/2, 1/2)$, $(m/n, 0)$ *respectively*. Let σ_1 be the open segment with end-points $(1/2, 1/2)$, $(1, 1-m/n)$ and let σ_2 be the open segment with end points $(1/2, 1/2)$, $(m/n, 0)$.

If $(1/p, 1/s) \in \sigma_1$, select two points $(1/p_1, 1/s_1)$ and $(1/p_2, 1/s_2)$ arbitrarily close to $(1/p, 1/s)$ and belonging to the regions in (iii) and (xiii) respectively, to obtain, by interpolation

$$\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} O(r^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{s}}) \qquad (u \in W^{m,p}(\Omega)).$$

Suppose now that $(1/p, 1/s) \in \sigma_2$ and let $\varepsilon > 0$, $\eta > 0$ and $0 < t < 1$ be such that $t\varepsilon = (1-t)\eta$. Set

$$\frac{1}{p_1} = \frac{1}{p} - \varepsilon, \qquad \frac{1}{p_2} = \frac{1}{p} + \eta.$$

There exist $c_1(\delta)$ and $c_2(\eta)$, with

$$\lim_{\delta \to 0} c_1(\delta) = \infty, \qquad \lim_{\eta \to 0} c_2(\eta) = \infty,$$

such that (45) holds with p, $c(\delta)$ replaced by p_1, $c_1(\delta)$ and (27) with p, $c(p,s)$ replaced by p_2, $c_2(\eta)$. Hence if $u\in W^{m,p}(\Omega)$, interpolating between (p_1, s) and (p_2, s),

(60) $$\|u-S_r u\|_{s,\Omega}\leqslant c_1(\delta)^t c_2(\eta)^{1-t}\|u\|_{m,p,\Omega} r^{-\frac{m}{n}+\frac{1}{p}-\frac{1}{s}}(\log r)^{t\delta+\frac{n^2}{m}(1-t)\eta}$$

We can now choose ε, η and t so as to eliminate, in the right-hand side of (60), the dependence on either δ or η, but not both. Consequently,

$$\|u-S_r u\|_{s,\Omega}\leqslant c(\delta)\|u\|_{m,p,\Omega} r^{-\frac{m}{n}+\frac{1}{p}-\frac{1}{s}}(\log r)^{\delta}\qquad (u\in W^{m,p}(\Omega)),$$

where $c(\delta)\to\infty$ as $\delta\to 0$. This shows that (45) also holds for $(1/p, 1/s)\in\sigma_2$.

(xv) $p=\infty$, $2<s<\infty$ or $p=1$, $1-m/n<1/s<1$. We have left these two cases for last since Lemma 1 does not apply to them and hence a device is needed to get around this difficulty. Matters are taken care by the following well-known lemma whose proof is entirely routine.

LEMMA 2. *Let D be any open set in R^n and suppose that $1\leqslant s_1<s<s_2\leqslant\infty$. If $u\in L^{s_1}(D)\cap L^{s_2}(D)$, then $u\in L^s(D)$ and*

$$\|u\|_{s,D}\leqslant\|u\|_{s_1,D}^{\gamma}\|u\|_{s_2,D}^{1-\gamma},\qquad \textit{where}\qquad \gamma=\frac{s_1}{s}\cdot\frac{s_2-s}{s_2-s_1}.$$

Now suppose first that $p=\infty$ and $2<s<\infty$. From (25), (36) and Lemma 2 with $D=\Omega$, $s_1=2$, $s_2=\infty$ we have for $u\in W^{m,\infty}(\Omega)$,

$$\|u-S_r u\|_{s,\Omega}\leqslant\|u-S_r u\|_{2,\Omega}^{\gamma}\|u-S_r u\|_{\infty,\Omega}^{1-\gamma}=\|u\|_{m,\infty,\Omega}O\{r^{-m/n}(\log r)^{n(1-2/s)}\}.$$

Finally, let $p=1$, $1-m/n<1/s<1$ and define s_2 by $1/s_2=1-m/n$. By [7], Théorème 3.6, p. 72 there exists a positive constant c_2 such that

$$\|u\|_{s_2,\Omega}\leqslant c_2\|u\|_{m,1,\Omega}$$

for every $u\in W^{m,1}(\Omega)$, and from this it follows that

(61) $$\|u-S_r u\|_{s_2,\Omega}\leqslant c\|u\|_{m,1,\Omega},$$

since the operators S_r: $W^{m,1}(\Omega)\to L^{s_2}(\Omega)$ are uniformly bounded. Hence, again appealing to Lemma 2 (with $D=\Omega$ and $s_1=1$), we have from (54) and (61),

(62) $$\|u-S_r u\|_{s,\Omega}\leqslant\|u-S_r u\|_{1,\Omega}^{\gamma}\|u-S_r u\|_{s_2,\Omega}^{1-\gamma}=\|u\|_{m,1,\Omega}O\{r^{-\frac{m}{n}+1-\frac{1}{s}}(\log r)^{n\left(1-\frac{n}{m}+\frac{n}{ms}\right)}\}$$

for all $u\in W^{m,1}(\Omega)$.

The theorem is now completely proved.

Remark 2. In the course of the proof of Theorem 1 we have constructed, for $m<n$, a function $f=f(r,p,s,m,n)$ which estimates the norm of the Fourier remainder map $I-S_r$: $W^{m,p}(\Omega)\to L^s(\Omega)$ and whose values are given in Fig. 1 for fixed m and n. This function is defined in the region

$$S=\left\{\left(\frac{1}{p},\frac{1}{s}\right):\ 1\leqslant p\leqslant\infty,1\leqslant s\leqslant\infty,\frac{1}{p}-\frac{1}{s}<\frac{m}{n}\right\},$$

is continuous everywhere in S with the exception of the segments

$$\sigma_1=\left\{\left(\frac{1}{p},\frac{1}{s}\right)\epsilon S:\ s=\infty,\frac{n}{m}<p<\infty\right\},$$

$$\sigma_2=\left\{\left(\frac{1}{p},\frac{1}{s}\right)\epsilon\, S:\ 2<s<\infty,\ \frac{1}{p}=\left(1-\frac{2m}{n}\right)\frac{1}{s}+\frac{m}{n}\right\}$$

and exhibits in $S\setminus\sigma_1$ a remarkable symmetry with respect to the line $1/p+1/s=1$.

The lack of continuity of f on $\sigma_1\cup\sigma_2$ is characteristic of the many-dimensional case and is equivalent to the fact that $W^{m,n/m}(\Omega)\not\subset L^\infty(\Omega)$ (see Section 4 for the critical case $p=n/m$).

We shall now remove the restriction $m<n$ from Theorem 1 and show that (18) still holds. Precisely we have

THEOREM 2. *Let p, s and Ω be as in Theorem 1 and let m be a positive integer such that $1/p-1/s<m/n$. Then for every $u\epsilon W^{m,p}(\Omega)$ we have, as $r\to\infty$,*

$$\|u-S_ru\|_{s,\Omega}=\|u\|_{m,p,\Omega}\,o(r^{-h+\varepsilon})\quad \textit{for every } \varepsilon>0, \tag{63}$$

where h is given by (18).

Proof. In view of Theorem 1 we can assume $m\geqslant n$ and refer to Fig. 2.

(i) $p=1$, $s=\infty$. We introduce the functions

$$h(\xi)=\begin{cases}0 & \text{for}\quad \xi<0,\\ 1 & \text{for}\quad \xi\geqslant 0,\end{cases}\qquad H(x)=\prod_{i=1}^{n}h(x_i),$$

and for every positive integer j we let $D_j^{(1)}(\xi)$ be that anti-derivative of $D_j(\xi)$ such that (cf. (46))

$$B_j^{(1)}(\xi)=D_j^{(1)}(\xi)-\pi h(\xi),$$

where $B_j^{(1)}(\xi)$ satisfies (49). It follows then from (49) that

$$|D_j^{(1)}(\xi)|\leqslant|B_j^{(1)}(\xi)|+\pi h(\xi)=O(1)\qquad(|\xi|\leqslant 2\pi). \tag{64}$$

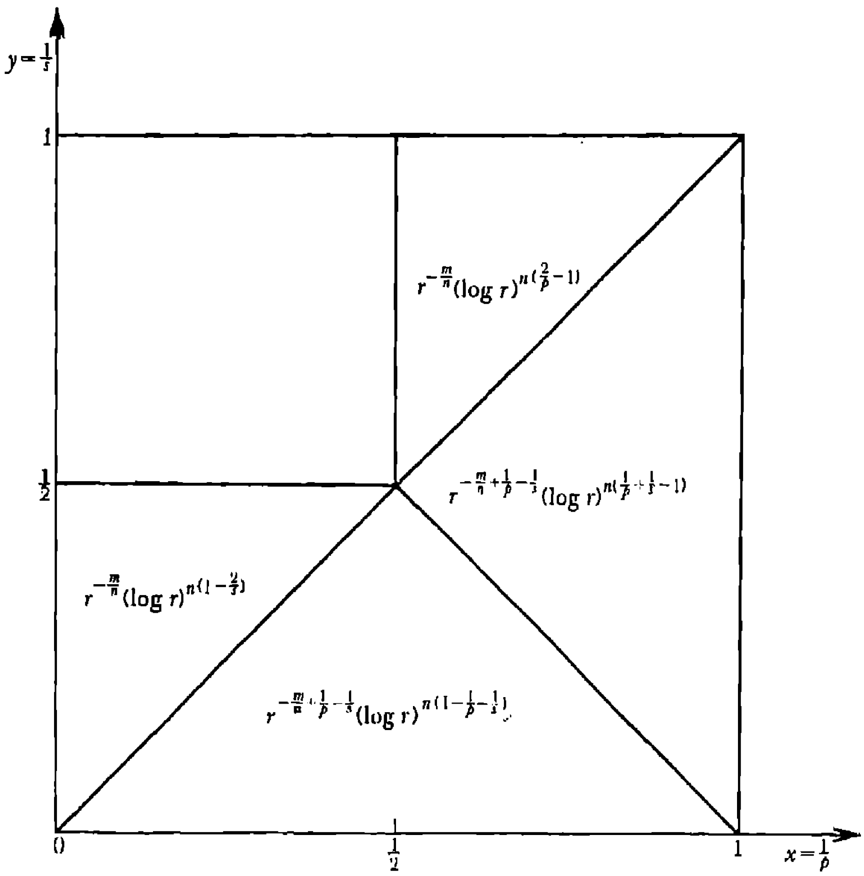

Fig. 2. Values of the function $f(r, p, s, m, n)$ estimating the norm of the remainder map of the Fourier series for $m \geqslant n$

Let now $u \in C_0^\infty(Q)$; we have

$$\begin{aligned}
&(T_{k(r)}u)(x) - u(x) \\
&= \pi^{-n} \int_Q dt \frac{\partial^n u(t)}{\partial t_1 \ldots \partial t_n} \prod_{i=1}^n D_{k_i(r)}^{(1)}(x_i - t_i) - \int_Q dt \frac{\partial^n u(t)}{\partial t_1 \ldots \partial t_n} H(x-t) \\
&= \sum_{j=1}^n \pi^{-n+j-1} \int_Q dt \frac{\partial^n u(t)}{\partial t_1 \ldots \partial t_n} \prod_{l=1}^{j-1} h(x_l - t_l) \prod_{i=j+1}^n D_{k_i(r)}^{(1)}(x_i - t_i) B_{k_j(r)}^{(1)}(x_j - t_j) \\
&= \sum_{j=1}^n \pi^{-n+j-1} \int_Q dt \frac{\partial^m u(t)}{\partial t_1 \ldots \partial t_{j-1} \partial t_j^{m-n+1} \partial t_{j+1} \ldots \partial t_n} \prod_{l=1}^{j-1} h(x_l - t_l) \times \\
&\qquad \times \prod_{i=j+1}^n D_{k_i(r)}^{(1)}(x_i - t_i) B_{k_j(r)}^{(m-n+1)}(x_j - t_j).
\end{aligned}$$

Hence by (49), (64) and (6),

$$\|T_{k(r)}u - u\|_{\infty,Q} = \|u\|_{m,1,Q}\, O(r^{-m/n+1})$$

for $u \in W_0^{m,1}(Q)$, since $C_0^\infty(Q)$ is dense in $W_0^{m,1}(Q)$.

Finally, if $u \in W^{m,1}(\Omega)$, (5) and (9) give

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,1,\Omega}\, O(r^{-m/n+1}). \tag{65}$$

(ii) $1 < p \leqslant 2$, $1/p + 1/s = 1$. Since the case $p = s = 2$ is the same as in Theorem 1 (i), by Lemma 1 we have from (24) and (65)

$$\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega}\, O(r^{-\frac{m}{n}+\frac{1}{p}-\frac{1}{s}}) \tag{66}$$

for all $u \in W^{m,p}(\Omega)$.

(iii) $s = \infty$, $1 < p \leqslant \infty$. The case $p = s = \infty$ is as in Theorem 1(iv). Now let ε be given, with $0 < \varepsilon < \min(1/2, 1/n)$. Since $W^{m,\frac{1}{1-\varepsilon}}(\Omega) \subset W^{m,1}(\Omega)$, (65) yields

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,\frac{1}{1-\varepsilon},\Omega}\, O(r^{-\frac{m}{n}+1}) \quad (u \in W^{m,\frac{1}{1-\varepsilon}}(\Omega)), \tag{67}$$

where the constant in the "O" symbol is independent of ε. Also, as in Theorem 1(v), we have (37) with $1/p = \varepsilon$, i.e.

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,\frac{1}{\varepsilon},\Omega}\, O\{r^{-\frac{m}{n}+\varepsilon}(\log r)^n\} \quad (u \in W^{m,\frac{1}{\varepsilon}}(\Omega)), \tag{68}$$

where again the constant in the "O" symbol is independent of ε. Hence interpolating between $\left(\frac{1}{1-\varepsilon}, \infty\right)$ and $\left(\frac{1}{\varepsilon}, \infty\right)$ according to Lemma 1, we obtain from (67) and (68),

$$\begin{aligned} &\|u - S_r u\|_{\infty,\Omega} \\ &= \|u\|_{m,p,\Omega}\, O\{(r^{-\frac{m}{n}+\frac{1}{p}+\varepsilon\left[\frac{1}{1-2\varepsilon}\left(\frac{2}{p}-1\right)+1-t\right]}(\log r)^{n\left(1-\frac{1}{p}\right)-\frac{\varepsilon n}{1-2\varepsilon}\left(\frac{2}{p}-1\right)}\}, \end{aligned} \tag{69}$$

for all $u \in W^{m,p}(\Omega)$, where t is such that

$$\frac{1}{p} = (1-\varepsilon)t + (1-t)\varepsilon.$$

Therefore, choosing ε so that

$$\varepsilon\left[\frac{1}{1-2\varepsilon}\left(\frac{2}{p}-1\right)+1-t\right] \leqslant (\log r)^{-1},$$

it follows from (69) that

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,p,\Omega}\, O\{r^{-\frac{m}{n}+\frac{1}{p}}(\log r)^{n\left(1-\frac{1}{p}\right)}\}$$

for all $u \in W^{m,p}(\Omega)$.

(iv) $1 < p < \infty$, $0 < \dfrac{1}{s} < \min\left(\dfrac{1}{p}, 1 - \dfrac{1}{p}\right)$. Interpolate (Lemma 1) between a point as in (ii) and one as in (iii) to obtain, for every $u \in W^{m,p}(\Omega)$,

$$\|u - S_r u\|_{\infty,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{s}} (\log r)^{n\left(1 - \frac{1}{p} + \frac{1}{s}\right)}\}.$$

(v) $p = 1$, $1 \leqslant s < \infty$. The case $p = s = 1$ being as in Theorem 1 (ix), by Lemma 2 we have from (54) and (65),

$$\|u - S_r u\|_{s,\Omega} = \|u\|_{m,1,\Omega} O\{r^{-\frac{m}{n} + 1 - \frac{1}{s}} (\log r)^{\frac{n}{s}}\}$$

for every $u \in W^{m,p}(\Omega)$.

(vi) $1 < p < 2$, $1 - 1/p < 1/s \leqslant 1/p$. The case $p = s$ can be treated exactly as in Theorem 1 (xi) to yield again (59). We can then apply Lemma 1 and interpolate between a point with $p = s$ and a point as in (ii) to obtain

$$\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} O\{r^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{s}} (\log r)^{n\left(\frac{1}{p} + \frac{1}{s} - 1\right)}\}$$

for all $u \in W^{m,p}(\Omega)$.

The proof of the theorem is completed by the observation that the remaining cases can be dealt with exactly as in Theorem 1.

From Theorem 2 by induction we obtain·

COROLLARY 1. *Let Ω be a bounded open set in R^n with minimally smooth boundary. Let $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$ and let l, m be integers such that $0 \leqslant l < m$ and $\dfrac{1}{p} - \dfrac{1}{s} < \dfrac{m-l}{n}$. Then for every $u \in W^{m,p}(\Omega)$ we have, as $r \to \infty$,*

(70) $$\|u - S_r u\|_{l,s,\Omega} = \|u\|_{m,p,\Omega} O(r^{-h+\varepsilon}) \quad \textit{for every } \varepsilon > 0,$$

where

$$h = \frac{m-l}{n} - \max\left(\frac{1}{p} - \frac{1}{s}, 0\right).$$

It is almost unnecessary to note that also in this more general case finer estimates than (70) hold, these being of the same type as the estimates obtained in the proofs of Theorems 1 and 2, but with $(m-l)/n$ instead of m/n.

3. The case $n = 1$

In this case the estimates of the previous section can be sharpened as follows:

THEOREM 3. *Let Ω be a bounded open subset of R^1 with minimally smooth boundary, let m be a positive integer and let $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$. Then for every $u \in W^{m,p}(\Omega)$ we have, as $r \to \infty$,*

(a) $\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} o(r^{-m+\varepsilon})$ *for every* $\varepsilon > 0$, *if* $s \leqslant p$;

(b) $\|u - S_r u\|_{s,\Omega} = \|u\|_{m,p,\Omega} O(r^{-m+\frac{1}{p}-\frac{1}{s}})$ *if* $s > p$.

Proof. Assume $\bar{\Omega}$ contained in the open interval $J = (-\pi, \pi)$.

(b) With j equal to the integer part of $(r-1)/2$ and $u \in W_0^{m,p}(J)$ we have (cf. 46))

$$u(x) - (T_j u)(x) = -\pi^{-1} \int_{-\pi}^{\pi} dt\, u(t) B_j(x-t). \tag{71}$$

Let q be defined by $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{s}$. Since $q > 1$, (49) gives for the anti-derivative of order m, $B_j^{(m)}$, of B_j,

$$\|B_j^{(m)}\|_{q,J} \leqslant c_1 (q-1)^{-1/q} j^{1-m-1/q} \quad \text{as} \quad j \to \infty.$$

Hence, integrating by parts the right-hand side of (71) m times and applying Young's inequality (cf. [13], Vol. 1, Theorem (1.15), p. 37),

$$\|u - T_j u\|_{s,J} \leqslant \pi^{-1} \left\| \frac{d^m u}{dx^m} \right\|_{p,J} \|B_j^{(m)}\|_{q,J} \leqslant c_2 \left(\frac{1}{p} - \frac{1}{s} \right)^{\frac{1}{p}-\frac{1}{s}-1} r^{-m+\frac{1}{p}-\frac{1}{s}} \|u\|_{m,p,J},$$

from which it follows that

$$\|u - S_r u\|_{s,\Omega} \leqslant c \left(\frac{1}{p} - \frac{1}{s} \right)^{\frac{1}{p}-\frac{1}{s}-1} r^{-m+\frac{1}{p}-\frac{1}{s}} \|u\|_{m,p,\Omega} \tag{72}$$

for all $u \in W^{m,p}(\Omega)$, whence (b).

(a) The proof proceeds as in Theorem 2, yielding the estimates obtained there, with $n = 1$.

COROLLARY 2. *Let p, s and Ω be as in Theorem 3 and let m, l be integers such that $0 \leqslant l < m$. Then for every $u \in W^{m,p}(\Omega)$ we have, as $r \to \infty$,*

(a) $\|u - S_r u\|_{l,s,\Omega} = \|u\|_{m,p,\Omega} o(r^{-m+l+\varepsilon})$ *for every* $\varepsilon > 0$, *if* $s \leqslant p$;

(b) $\|u - S_r u\|_{l,s,\Omega} = \|u\|_{m,p,\Omega} O(r^{-m+l+\frac{1}{p}-\frac{1}{s}})$ *if* $s > p$.

4. Embedding $W^{m,p}(\Omega)$ into $L^{\varphi}(\Omega)$

In this section we consider the embedding of $W^{m,p}(\Omega)$ into some Orlicz spaces $L^{\varphi}(\Omega)$. The definition of an Orlicz space is as follows.

Let $\varphi(t)$ be a non-negative convex function defined for $t \geqslant 0$ and such that

$$\varphi(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} t^{-1} \varphi(t) = \infty. \tag{73}$$

Denote by $L^{\varphi}(\Omega)$ the linear space of all measurable functions u on Ω for which there exists a $\lambda > 0$ such that

$$\int_{\Omega} \varphi\big(\lambda^{-1}|u(x)|\big)\,dx < \infty.$$

Then $L^{\varphi}(\Omega)$ becomes a Banach space when endowed with the norm

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda:\ \int_{\Omega} \varphi\big(\lambda^{-1}|u(x)|\big)\,dx \leqslant 1\Big\}. \tag{74}$$

Notice that $L^{\infty}(\Omega) \subset L^{\varphi}(\Omega) \subset L^{1}(\Omega)$. In fact, if $u \in L^{\infty}(\Omega)$, the continuity of φ implies

$$\int_{\Omega} \varphi\big(|u(x)|\big)\,dx \leqslant m(\Omega)\varphi(\|u\|_{\infty,\Omega}),$$

where $m(\Omega)$ denotes the measure of Ω. On the other hand, if $u \in L^{\varphi}(\Omega)$, then by the second condition in (73), there exists a $t_0 > 0$ such that $\varphi(t) > t$ for $t > t_0$; hence, setting

$$\Omega_0 = \Big\{x \in \Omega:\ \frac{|u(x)|}{\|u\|_{\varphi,\Omega}} \leqslant t_0\Big\},$$

we have

$$\int_{\Omega} \frac{|u(x)|}{\|u\|_{\varphi,\Omega}}\,dx = \int_{\Omega_0} \frac{|u(x)|}{\|u\|_{\varphi,\Omega}}\,dx + \int_{\Omega\setminus\Omega_0} \frac{|u(x)|}{\|u\|_{\varphi,\Omega}}\,dx$$
$$< t_0 m(\Omega_0) + \int_{\Omega\setminus\Omega_0} \varphi\left(\frac{|u(x)|}{\|u\|_{\varphi,\Omega}}\right)dx < \infty.$$

With regard to the embeddings $W^{m,p}(\Omega) \to L^{\varphi}(\Omega)$ it is now clear from Theorems 1 and 2 (see also Fig. 1) that we need only consider the case $1 \leqslant n/m \leqslant p \leqslant \infty$ for, otherwise, $W^{m,p}(\Omega)$ can only be embedded into those Orlicz spaces which are not too different, in a certain sense, from appropriate Lebesgue spaces. We also point out that the results established in Theorem 1 also hold for $\frac{1}{p} - \frac{1}{s} = \frac{m}{n}$, with the exception of $p = n/m$, $s = \infty$ due to the fact that $W^{m,n/m}(\Omega) \not\subset L^{\infty}(\Omega)$. This motivates the special attention which we shall pay to the limit case $m < n$, $p = n/m$.

Following Trudinger [11] we consider embeddings into the Orlicz spaces $L^{\varphi_\nu}(\Omega)$ associated with convex functions of the form

$$\varphi_\nu(t) = \exp(t^\nu) - 1 \qquad (t \geqslant 0,\ \nu \geqslant 1). \tag{75}$$

Theorem 4. *Let $n > 1$, $1 \leqslant m < n$, $1 \leqslant \nu \leqslant n/(n-m)$ and let Ω be a bounded open set in R^n with boundary of class C^1. Then for every $u \in W^{m,n/m}(\Omega)$*

we have, as $r\to\infty$,

(76) $$\|u-S_r u\|_{\varphi_\nu,\Omega} = \|u\|_{m,n/m,\Omega}\, O\{(\log r)^{-\frac{m}{n}+1-\frac{1}{\nu}}\}.$$

Proof. To fix the ideas we assume $m/n < 1/2$ and refer to Fig. 1, but the proof will also hold for the case $1/2 \leqslant m/n < 1$ with trivial modifications.

Let r be fixed and large. With reference to (74) we set

$$J(\lambda_r, u) = \int_\Omega \varphi_\nu\big(\lambda_r^{-1}|u(x)-S_r(u,x)|\big)\,dx.$$

Then expanding the exponential in (75) we obtain, for $u\in W^{m,n/m}(\Omega)$,

(77) $$J(\lambda_r, u) = \sum_{k=1}^{\infty}\frac{1}{k!}\lambda_r^{-\nu k}\int_\Omega |u(x)-(S_r u)(x)|^{\nu k}dx = S_1+S_2,$$

where,

$$S_1 = \sum_{1\leqslant \nu k\leqslant n/m}\frac{\lambda_r^{-\nu k}}{k!}\|u-S_r u\|_{\nu k,\Omega}^{\nu k},$$

(78) $$S_2 = \sum_{\nu k> n/m}\frac{\lambda_r^{-\nu k}}{k!}\|u-S_r u\|_{\nu k,\Omega}^{\nu k}.$$

In the course of this proof $c_1, c_2, \dots$ will denote constants possibly depending on m, n, ν and Ω only.

From (25), (40) and (41) we obtain for S_1,

(79) $$\begin{aligned} S_1 &= \sum_{1\leqslant \nu k\leqslant 2}\frac{\lambda_r^{-\nu k}}{k!}\|u-S_r u\|_{\nu k,\Omega}^{\nu k} + \sum_{2<\nu k\leqslant n/m}\frac{\lambda_r^{-\nu k}}{k!}\|u-S_r u\|_{\nu k,\Omega}^{\nu k}\\ &\leqslant \sum_{1\leqslant \nu k\leqslant 2}\frac{1}{k!}(c_1\lambda_r^{-1}r^{-m/n}\|u\|_{m,n/m,\Omega})^{\nu k}+\\ &\quad+\sum_{2<\nu k\leqslant n/m}\frac{1}{k!}(c_1\lambda_r^{-1}r^{-m/n}\|u\|_{m,n/m,\Omega})^{\nu k}(\log r)^{n(\nu k-2)}\\ &\leqslant r^{-m\nu/n}\sum_{1\leqslant \nu k\leqslant n/m}\frac{1}{k!}(c_1\lambda_r^{-1}\|u\|_{m,n/m,\Omega})^{\nu k} \end{aligned}$$

if r is large enough.

We now come to S_2. Note first that (45) gives, for $p = n/m$,

(80) $$\|u-S_r u\|_{s,\Omega}\leqslant c(\delta)\|u\|_{m,n/m,\Omega}\, r^{-\frac{1}{s}}(\log r)^{\left(\frac{n}{m}-2\right)\frac{n}{s}+\delta},$$

so that $s\to\infty$ forces $\delta\to 0$ and hence $c(\delta)\to\infty$. Thus from (78) and (80)

$$S_2\leqslant r^{-1}(\log r)^{n\left(\frac{n}{m}-2\right)}\sum_{\nu k>n/m}\frac{1}{k!}\{c(\delta)\lambda_r^{-1}(\log r)^{\delta}\|u\|_{m,n/m,\Omega}\}^{\nu k}$$

where $\delta \to 0$ and $c(\delta) \to \infty$ as $k \to \infty$. Control on the constants $c(\delta)$ for large k can be gained by using a simple device together with the following result of Edmunds and Evans [2]:

(81) $$\|u\|_{s,\Omega} \leqslant c_2 s^{1-m/n} \|u\|_{m,n/m,\Omega} \qquad (u \in W^{m,n/m}(\Omega))$$

with c_2 independent of s. This will lead to an inequality weaker than (80), though with no harm to the final result (76).

Let then s be given, with $\frac{1}{s} < \frac{1}{2} - \frac{m}{n}$, and let ε be such that $0 < \varepsilon < 1/s$. By (81) we have

$$\|u\|_{\varepsilon^{-1},\Omega} \leqslant c_3 \varepsilon^{m/n-1} \|u\|_{m,n/m,\Omega},$$

so that

(82) $$\|u - S_r u\|_{\varepsilon^{-1},\Omega} \leqslant c_4 \varepsilon^{m/n-1} \|u\|_{m,n/m,\Omega}.$$

By Lemma 2 with $s_1 = n/m$ and $s_2 = \varepsilon^{-1}$ we have from (40) and (82)

(83) $$\|u - S_r u\|_{s,\Omega} \leqslant c_5 \|u\|_{m,n/m,\Omega} \varepsilon^{(1-\gamma)\left(\frac{m}{n}-1\right)} r^{-\gamma\frac{m}{n}} (\log r)^{n\gamma\left(1-\frac{2m}{n}\right)}$$

with

$$\gamma = \frac{n}{ms} \frac{1-\varepsilon s}{1-\dfrac{\varepsilon n}{m}} = \frac{n}{m}\left(\frac{1}{s} - \varepsilon \frac{1-\dfrac{n}{ms}}{1-\dfrac{\varepsilon n}{m}}\right).$$

Since

(84) $$\frac{n}{m}\left(\frac{1}{s} - \varepsilon\right) < \gamma < \frac{n}{ms},$$

we see that $\gamma \to 0$ as $s \to \infty$. Hence, choosing for example $\varepsilon = 1/2s$ (but any $\varepsilon = \theta/s$ with $0 < \theta < 1$ would do), (83) and (84) give

(85) $$\|u - S_r u\|_{s,\Omega} \leqslant c_6 \|u\|_{m,n/m,\Omega} s^{1-\frac{m}{n}} r^{-\frac{1}{2s}} (\log r)^{\left(\frac{n}{m}-2\right)\frac{n}{s}}.$$

Finally, replacing (80) by (85) we obtain for S_2,

(86) $$S_2 \leqslant \sum_{\frac{n}{m} < \nu k \leqslant \frac{2n}{n-2m}} \frac{1}{k!} (c_7 \lambda_r^{-1} \|u\|_{m,n/m,\Omega})^{\nu k} r^{-1} (\log r)^{n\left(\frac{n}{m}-2\right)+1} + \\ + \sum_{\nu k > \frac{2n}{n-2m}} \frac{1}{k!} (c_7 \lambda_r^{-1} \|u\|_{m,n/m,\Omega})^{\nu k} (\nu k)^{\nu k\left(1-\frac{m}{n}\right)} r^{-1/2} (\log r)^{n\left(\frac{n}{m}-2\right)}.$$

If now r is large enough, it follows from (77), (79) and (86), that

$$J(\lambda_r, u) \leqslant r^{-\mu} (\log r)^{\beta n\left(\frac{n}{m}-2\right)} S$$

where

$$\mu = \min\left(\frac{1}{2}, \frac{m\nu}{n}\right), \quad \beta = \begin{cases} 1 & \text{if} \quad \mu = \frac{1}{2}, \\ 0 & \text{if} \quad \mu = m\nu/n, \end{cases}$$

and

$$S = \sum_{k=1}^{\infty} \frac{1}{k!} (\nu k)^{\nu k\left(1-\frac{m}{n}\right)} (c_8 \lambda_r^{-1} \|u\|_{m,n/m,\Omega})^{\nu k}.$$

Letting

(87) $$a = \nu\left(1-\frac{m}{n}\right), \quad x = (c_8 \nu^{1-m/n} \lambda_r^{-1} \|u\|_{m,n/m,\Omega})^{\nu},$$

we see that

$$S = S(x, a) = \sum_{k=1}^{\infty} \frac{k^{ak}}{k!} x^k$$

and hence if $a < 1$, i.e., $\nu < n/(n-m)$, (106) of Section 7, Appendix B, gives

$$S \leqslant c_0 \exp\{(2e)^{\frac{a}{1-a}} x^{\frac{1}{1-a}}\}.$$

Recalling (74), the condition $J(\lambda_r, u) \leqslant 1$ will then be satisfied if

$$c_0 \exp\{(2e)^{\frac{1}{1-a}} x^{\frac{1}{1-a}}\} \leqslant r^{\mu} (\log r)^{\beta n\left(2-\frac{n}{m}\right)},$$

i.e., by (87), if

$$\lambda_r \geqslant c_8 (2e\nu)^{1-\frac{m}{n}} \left(\mu \log r + \beta n\left(2-\frac{n}{m}\right) \log\log r - \log c_0\right)^{-\frac{m}{n}+1-\frac{1}{\nu}} \|u\|_{m,n/m,\Omega}.$$

Therefore,

(88) $$\|u - S_r u\|_{\varphi_\nu,\Omega} \leqslant c (\log r)^{-\frac{m}{n}+1-\frac{1}{\nu}} \|u\|_{m,n/m,\Omega} \quad (u \in W^{m,n/m}(\Omega)),$$

which implies (76) for $\nu < n/(n-m)$.

Finally, if $\nu = n/(n-m)$, then $a = 1$ in (87) and (107) gives

$$S \leqslant c_0 \frac{ex}{1-ex}.$$

From this it follows that

$$\|u - S_r u\|_{\varphi_\nu,\Omega} \leqslant c \|u\|_{m,n/m,\Omega} \quad (u \in W^{m,n/m}(\Omega)),$$

which is (88) for $\nu = n/(n-m)$. Thus (76) holds also for $\nu = n/(n-m)$ and the proof of the theorem is complete.

It is also of interest to consider the case $m = n \geqslant 1$, $W^{n,1}(\Omega)\to L^{\varphi_\nu}(\Omega)$, for although $W^{n,1}(\Omega) \subset L^\infty(\Omega)$ (Theorem 2 (i)), the norms of the remainder maps $I-S_r$, regarded as operators from $W^{n,1}(\Omega)\to L^\infty(\Omega)$, do not converge to 0 as $r\to\infty$ (cf. (65) for $m = n$). We find that Theorem 4 still holds in this case, as we have

THEOREM 5. *Let $n \geqslant 1$, $1 \leqslant \nu < \infty$ and let Ω be a bounded open subset of R^n with boundary of class C^1. Then for every $u\in W^{n,1}(\Omega)$ we have, as $r\to\infty$,*

$$\|u-S_r u\|_{\varphi_\nu,\Omega} = \|u\|_{n,1,\Omega} O\{(\log r)^{-1/\nu}\}. \tag{89}$$

Proof. As in (77) we write, for $u\in W^{n,1}(\Omega)$,

$$J(\lambda_r, u) = S_1+S_2 \tag{90}$$

where

$$S_1 = \lambda_r^{-\nu}\int\limits_\Omega |u(x)-(S_r u)(x)|^\nu dx; \qquad S_2 = \sum_{\nu k>1}\frac{\lambda_r^{-\nu k}}{k!}\|u-S_r u\|_{\nu k,\Omega}^{\nu k},$$

and S_1 exists only if $\nu = 1$. By Theorem 2 (v) or Theorem 3 (b), for r large enough,

$$S_1 \leqslant r^{-\nu}(\log r)^{n\nu}(c_1\lambda_r^{-1}\|u\|_{n,1,\Omega})^\nu, \tag{91}$$

$$S_2 \leqslant r^{-1}(\log r)^{n\beta}\sum_{\nu k>1}\frac{1}{k!}(c_2\lambda_r^{-1}\|u\|_{n,1,\Omega})^{\nu k}, \tag{92}$$

where $\beta = 1$ if $n>1$ and $\beta = 0$ if $n = 1$.

Then (90), (91) and (92) yield

$$\begin{aligned} J(\lambda_r, u) &\leqslant r^{-1}(\log r)^n\sum_{k=1}^{\infty}\frac{1}{k!}(c_3\lambda_r^{-1}\|u\|_{n,1,\Omega})^{\nu k} \\ &= r^{-1}(\log r)^n\big(\exp\{(c_3\lambda_r^{-1}\|u\|_{n,1,\Omega})^\nu\}-1\big), \end{aligned}$$

and hence $J(\lambda_r, u)\leqslant 1$ if

$$\lambda_r \geqslant c_3\|u\|_{n,1,\Omega}\big(\log\{r(\log r)^{-n}+1\}\big)^{-1/\nu},$$

from which (89) follows.

Finally, for $n/m < p \leqslant \infty$ the techniques used in the course of the proofs of Theorems 4 and 5 can be applied to show that the embedding $W^{m,p}(\Omega)\to L^{\varphi_\nu}(\Omega)$ behaves slightly better than the embedding $W^{m,p}(\Omega)\to L^\infty(\Omega)$. Precisely, we have

THEOREM 6. *Let $n\geqslant 1$, $m\geqslant 1$, $n/m < p \leqslant \infty$, $1\leqslant \nu < \infty$ and let Ω be a bounded open set in R^n with boundary of class C^1. Then for every $u\in W^{m,p}(\Omega)$ we have, as $r\to\infty$,*

$$\|u-S_r u\|_{\varphi_\nu,\Omega} = \|u\|_{m,p,\Omega} O(r^{-\frac{m}{n}+\frac{1}{p}}).$$

5. Embedding $W_0^{m,p}(\Omega)$ into $L^s(\Omega)$ and $L^\varphi(\Omega)$

Let Ω be any bounded open subset of R^n (with no restriction on the boundary). We can find a bounded open set $\Omega_0 \subset R^n$ such that

(i) $\bar{\Omega} \subset \Omega_0$,

(ii) $\partial\Omega_0$ is smooth,

(iii) $\operatorname{diam}(\Omega) < \operatorname{diam}(\Omega_0) < 2\operatorname{diam}(\Omega)$.

If $u \in W_0^{m,p}(\Omega)$, extending u by 0 in $\Omega_0 \setminus \Omega$ we obtain a function $\bar{u} \in W^{m,p}(\Omega_0)$ such that $\|\bar{u}\|_{m,p,\Omega_0} = \|u\|_{m,p,\Omega}$. Thus the map $u \to \bar{u}$ is an isometry of $W_0^{m,p}(\Omega)$ onto a closed subspace of $W^{m,p}(\Omega_0)$, and from the results of Sections 2, 3 and 4 we obtain

THEOREM 7. *For Ω an arbitrary bounded open set in R^n $(n \geqslant 1)$, Theorems* 1–6 *and Corollaries* 1, 2 *hold with $W^{m,p}(\Omega)$ and $W^{l,s}(\Omega)$ replaced by $W_0^{m,p}(\Omega)$ and $W_0^{l,s}(\Omega)$.*

6. Applications to the type of the embedding

The results of the previous sections can be applied to investigate the type of the embedding map of $W^{m,p}(\Omega)$ into $L^s(\Omega)$. The notion of type referred to here is due to Pietsch (cf. [8]) and is defined as follows.

Let X and Y be two Banach spaces and let T be a bounded linear map of X into Y. Denote by $\mathscr{L}(X, Y)$ the space of all bounded linear maps of X into Y endowed with the operator norm, and by $\mathscr{F}_r(X, Y)$, with r a non-negative integer, the subspace of $\mathscr{L}(X, Y)$ consisting of all maps whose range is at most r-dimensional. The number

$$a_r(T) = \inf\{\|T - F\|:\ F \in \mathscr{F}_r\}$$

is called the *r-th approximation number* of T. Clearly, we always have

$$\|T\| = a_0(T) \geqslant a_1(T) \geqslant \ldots \geqslant 0.$$

Let q be a positive real number; then the map T is said to be *of type l^q* if

$$\sum_{r=0}^{\infty} a_r(T)^q < \infty.$$

The subspace of $\mathscr{L}(X, Y)$ of all maps of type l^q is denoted by $l^q(X, Y)$.

By similarity, we are led to say that T is *of type c_0*, and write $T \in c_0(X, Y)$, if

$$\lim_{r\to\infty} a_r(T) = 0.$$

Evidently, a map of type l^q is also of type c_0, and a map of type c_0 is compact but not conversely, in virtue of Enflo's celebrated result [5] on the existence of a Banach space without the approximation property. We also recall that every map of type l^1 is nuclear, and that if X and Y are Hilbert spaces, then the maps of type l^1 and l^2 coincide with the nuclear and Hilbert–Schmidt mappings respectively.

A somewhat finer classification of the compact maps is possible if, following the lines of Triebel [20], we write $l^{p,\infty}(X, Y)$ for the family of all linear maps T from X to Y such that

$$\sup_r r^{1/p} a_r(T) < \infty.$$

Evidently $l^{p,\infty}(X, Y) \subset l^q(X, Y)$ for every $q > p$.

Coming back to the Sobolev spaces, the results obtained enable us to make more precise the "measure of compactness" of the embedding map in terms of the notion of type introduced above.

In the next theorem Ω is assumed to be, as usual, a bounded open subset of R^n with minimally smooth boundary. From Corollaries 1 and 2 we have:

THEOREM 8. *Let* $1 \leqslant p \leqslant \infty$, $1 \leqslant s \leqslant \infty$, $q > 0$ *and let* l, m, n *be integers such that* $n \geqslant 1$, $0 \leqslant l < m$ *and* $\frac{1}{p} - \frac{1}{s} < \frac{m-l}{n}$. *Then the embedding map* $W^{m,p}(\Omega) \to W^{l,s}(\Omega)$ *is in*

$$\bigcap_{0<\varepsilon<h} l^{1/(h-\varepsilon),\infty} \subset \bigcap_{t>1/h} l^t,$$

where

$$h = \frac{m-l}{n} - \left(\frac{1}{p} - \frac{1}{s}, 0\right).$$

This result is not best possible, for because of the work of Birman and Solomjak [14], [15] mentioned in the Introduction it can be seen that the embedding map of Theorem 8 is in fact in $l^{1/h,\infty}$, which is contained in $\bigcap_{0<\varepsilon<h} l^{1/(h-\varepsilon),\infty}$. From the point of view of the type of the embedding map our work naturally gives the same result as that of Birman and Solomjak.

With regard to embeddings into Orlicz spaces, Theorems 4–6 give, under the assumption that Ω is a bounded open subset of R^n with boundary of class C^1:

THEOREM 9. *Let* $n \geqslant 1$, $1 \leqslant m < n$ *and* $1 \leqslant \nu < n/(n-m)$. *Then the embedding map* $W^{m,n/m}(\Omega) \to L^{\varphi_\nu}(\Omega)$ *is of type* c_0.

THEOREM 10. *Let* $n \geqslant 1$, $m \geqslant 1$, $n/m < p \leqslant \infty$, $\nu \geqslant 1$ *and* $q > 0$. *Then the embedding map* $W^{m,p}(\Omega) \to L^{\varphi_\nu}(\Omega)$ *is of type* l^q *if*

$$\frac{m}{n} > \frac{1}{q} + \frac{1}{p}.$$

Finally, for the spaces $W_0^{m,p}(\Omega)$ we have

THEOREM 11. *For Ω an arbitrary bounded open set in R^n $(n\geqslant 1)$, Theorems 8–10 hold with $W^{m,p}(\Omega)$ and $W^{l,s}(\Omega)$ replaced by $W_0^{m,p}(\Omega)$ and $W_0^{l,s}(\Omega)$.*

7. Unfortunate technicalities

Appendix A. Estimate for the anti-derivative of order m of the kernel

$$A_n(t) = \frac{\sin nt}{\sin t} - 2\pi\,\delta(2t) \quad (n\geqslant 1,\ |t|\leqslant \pi). \tag{93}$$

LEMMA 3. *Define, for $0\leqslant t<\infty$ and m a non-negative integer,*

$$F_0(t) = \frac{\sin t}{t}, \quad F_1(t) = -\int_t^\infty F_0(s)\,ds, \quad F_{m+1}(t) = -\int_t^\infty F_m(s)\,ds.$$

Then

$$|F_m(t)|\leqslant c_m/(1+t). \tag{94}$$

Proof. Repeated integration by parts yields

$$\int_\infty^t \frac{e^{is}}{s^k}\,ds = Q_{k,l}(t)+R_{k,l}(t) \quad (t\geqslant 1,\ k=1,2,\ldots;\ l\geqslant k), \tag{95}$$

where

$$Q_{k,l}(t) = \frac{e^{it}}{i}\left\{t^{-k}+\frac{k}{i}\,t^{-k-1}+\frac{k(k+1)}{i^2}\,t^{-k-2}+\ldots+\frac{(l-1)!}{(k-1)!}\,\frac{1}{i^{l-k}}\,t^{-l}\right\}$$

and

$$|R_{k,l}(t)| = \frac{l!}{(k-1)!}\left|\frac{1}{i^{l-k+2}}\left\{e^{it}t^{-l-1}+(l+1)\int_\infty^t e^{is}s^{-l-2}\,ds\right\}\right|\leqslant \frac{l!}{(k-1)!}\,2t^{-l-1}.$$

(a) Proof of (94) for $t\geqslant 1$. With $l=m$ in (95), we have

$$\begin{aligned} F_1(t) &= \operatorname{Im}\{Q_{1,m}(t)+R_{1,m}(t)\} \\ &= \operatorname{Im}\frac{e^{it}}{i}\,\{t^{-1}+c_{1,2}t^{-2}+\ldots+c_{1,m}t^{-m}\}+O(t^{-m-1}) \end{aligned}$$

where $c_{1,j}$ $(2\leqslant j\leqslant m)$ and the constant implied by the O-symbol depend only on m. Similarly in what follows. Hence, by (95) again, but with $l=m-1$ for the $t^{-1},\ldots,t^{-m+1}$ terms in $F_1(t)$, and $l=m$ for the t^{-m} term:

$$F_2(t) = \mathrm{Im}\frac{1}{i}\{[Q_{1,m-1}(t)+R_{1,m-1}(t)]+c_{1,2}[Q_{2,m-1}(t)+R_{2,m-1}(t)]+$$
$$+\ldots+c_{1,m-1}[Q_{m-1,m-1}(t)+R_{m-1,m-1}(t)]\}+O(t^{-m})$$
$$= \mathrm{Im}\frac{e^{it}}{i^2}\{t^{-1}+c_{2,2}t^{-2}+\quad +c_{2,m-1}t^{-m+1}\}+O(t^{-m}).$$

Similarly,

$$F_n(t) = \mathrm{Im}\frac{e^{it}}{i^n}\{t^{-1}+c_{n,2}t^{-2}+\quad +c_{n,m-n+1}t^{-m+n-1}\}+O(t^{-m+n-2}),$$

for $t \geqslant 1$ and $1 \leqslant n \leqslant m$.

(b) Proof of (94) for $0 \leqslant t < 1$. Since $\left|\frac{\sin s}{s}\right| \leqslant 1$, we have

$$|F_1(t)| = \left|-\int_t^1 \frac{\sin s}{s}\,ds + F_1(1)\right| \leqslant 1+|F_1(1)|.$$

Similarly

$$|F_2(t)| = \left|-\int_t^1 F_1(s)\,ds + F_2(1)\right| \leqslant 1+|F_1(1)|+|F_2(1)|$$

and so on.

LEMMA 4. *With $F_n(t)$ as in Lemma 3, we have*

$$F_{2n+1}(0) = 0 \qquad (n = 1, 2, \ldots).$$

Proof. We use the Laplace transform

$$\hat{f}(p) = p\int_0^\infty e^{-pt}f(t)\,dt \qquad (\mathrm{Re}\,p > \text{some constant})$$

and abbreviate this to $f(t) \xrightarrow{\mathscr{L}} \hat{f}(p)$. We also note that

$$(96)\quad \cot^{-1}p = \frac{\pi}{2} - p + \frac{p^3}{3} - \quad +(-1)^n\frac{p^{2n-1}}{2n-1}+\ldots \quad \text{for} \quad |p|<1.$$

An elementary calculation yields

$$\hat{F}_1(p) = \cot^{-1}p - \frac{\pi}{2} \qquad (\mathrm{Re}\,p > 0).$$

Then

$$\int_0^t F_1(s)\,ds \xrightarrow{\mathscr{L}} \frac{1}{p}\left(\cot^{-1}p - \frac{\pi}{2}\right) \to -1$$

as $p \to 0$, by (96), giving $\int_0^\infty F_1(s)\,ds = -1$.

Hence

$$F_2(t) = -\left\{\int_0^\infty F_1(s)\,ds - \int_0^t F_1(s)\,ds\right\}$$

$$\xrightarrow{\mathscr{L}} 1 + \frac{1}{p}\left(\cot^{-1}p - \frac{\pi}{2}\right) = \frac{1}{p}\left(\cot^{-1}p - \frac{\pi}{2} + p\right).$$

Then

$$\int_0^t F_2(s)\,ds \xrightarrow{\mathscr{L}} \frac{1}{p^2}\left(\cot^{-1}p - \frac{\pi}{2} + p\right) \to 0$$

as $p \to 0$, by (96), which implies $\int_0^\infty F_2(s)\,ds = 0$. Since

$$F_3(0) = -\int_0^\infty F_2(s)\,ds,$$

this proves the result for $n = 1$. We now proceed by induction and assume that, for some fixed $n \geqslant 1$,

$$F_{2n}(t) \xrightarrow{\mathscr{L}} \frac{1}{p^{2n-1}}\left\{\cot^{-1}p - \frac{\pi}{2} + p - \quad + (-1)^{n-1}\frac{p^{2n-1}}{2n-1}\right\}. \tag{97}$$

Then

$$\int_0^t F_{2n}(s)\,ds \xrightarrow{\mathscr{L}} \frac{1}{p^{2n}}\left\{\cot^{-1}p - \frac{\pi}{2} + p - \quad + (-1)^{n-1}\frac{p^{2n-1}}{2n-1}\right\} \to 0$$

as $p \to 0$, by (96).

It follows that

$$\int_0^\infty F_{2n}(s)\,ds = 0 \tag{98}$$

and hence

$$F_{2n+1}(t) = -\left\{\int_0^\infty F_{2n}(s)\,ds - \int_0^t F_{2n}(s)\,ds\right\}$$

$$\xrightarrow{\mathscr{L}} 0 + \frac{1}{p^{2n}}\left\{\cot^{-1}p - \frac{\pi}{2} + p - \quad + (-1)^{n-1}\frac{p^{2n-1}}{2n-1}\right\},$$

$$\int_0^t F_{2n+1}(s)\,ds \xrightarrow{\mathscr{L}} \frac{1}{p^{2n+1}}\left\{\cot^{-1}p - \frac{\pi}{2} + p - \dots + (-1)^{n-1}\frac{p^{2n-1}}{2n-1}\right\}$$

$$\to \frac{(-1)^{n+1}}{2n+1}$$

as $p \to 0$, by (96). This implies

$$\int_0^\infty F_{2n+1}(s)\,ds = \frac{(-1)^{n+1}}{2n+1},$$

and so

$$F_{2n+2}(t) = -\left\{\int_0^\infty F_{2n+1}(s)\,ds - \int_0^t F_{2n+1}(s)\,ds\right\}$$

$$\xrightarrow{\mathcal{L}} \frac{1}{p^{2n+1}}\left\{\cot^{-1}p - \frac{\pi}{2} + p - \quad + (-1)^n \frac{p^{2n+1}}{2n+1}\right\}.$$

This confirms (97), and it follows from (98) that

$$F_{2n+1}(0) = -\int_0^\infty F_{2n}(s)\,ds = 0.$$

LEMMA 5. *Define for all $t \in R$ and $n = 0, 1, \ldots$*

$$G_0(t) = \frac{\sin t}{t} - \pi\delta(t), \qquad G_{n+1}(t) = \int_{-\infty}^t G_n(s)\,ds.$$

Then

$$|G_n(t)| \leqslant c_n/(1+|t|) \qquad (t \in R,\ n = 1, 2, \ldots),$$

with c_n depending only on n.

Proof. For $t < 0$,

$$G_1(t) = \int_{-\infty}^t \frac{\sin s}{s}\,ds = -F_1(-t).$$

For $t \geqslant 0$,

$$G_1(t) = \int_{-\infty}^\infty \frac{\sin s}{s}\,ds - \int_t^\infty \frac{\sin s}{s}\,ds - \pi = F_1(t).$$

Hence, for $t < 0$,

$$G_2(t) = -\int_{-\infty}^t F_1(-s)\,ds = F_2(-t).$$

Also, G_1 is an odd function and integrable on R so that $\int_{-\infty}^\infty G_1(s)\,ds = 0$ and for $t \geqslant 0$,

$$G_2(t) = \int_{-\infty}^\infty G_1(s)\,ds - \int_t^\infty G_1(s)\,ds = F_2(t).$$

Proceeding by induction, assume that, for some fixed $n \geqslant 1$,

$$(99) \qquad G_{2n}(t) = F_{2n}(|t|) \quad \text{for all } t \in R.$$

Then for $t < 0$,

$$G_{2n+1}(t) = \int_{-\infty}^{t} F_{2n}(-s)\,ds = -F_{2n+1}(-t)$$

and (since $\int_{-\infty}^{0} F_{2n}(-s)\,ds = 0$ by Lemma 4) for $t \geqslant 0$,

$$G_{2n+1}(t) = \int_0^t F_{2n}(s)\,ds = \int_0^{\infty} F_{2n}(s)\,ds - \int_t^{\infty} F_{2n}(s)\,ds = F_{2n+1}(t).$$

Next, for $t < 0$,

$$G_{2n+2}(t) = -\int_{-\infty}^{t} F_{2n+1}(-s)\,ds = F_{2n+2}(-t),$$

and since G_{2n+1} is odd and integrable on R we have $\int_{-\infty}^{\infty} G_{2n+1} = 0$. Hence for $t \geqslant 0$,

$$G_{2n+2}(t) = \int_{-\infty}^{\infty} G_{2n+1}(s)\,ds - \int_t^{\infty} G_{2n+1}(s)\,ds = F_{2n+2}(t).$$

Thus we have confirmed (99) and also shown that

$$G_{2n+1}(t) = F_{2n+1}(t) \ (t \geqslant 0), \qquad G_{2n+1}(t) = -F_{2n+1}(-t) \ (t < 0).$$

The estimate for $|G_k(t)|$ now follows from Lemma 3.

PROPOSITION 1. *The kernel A_n, defined by (93), has an anti-derivative of order m, $A_n^{(m)}$, such that*

$$(100) \qquad |A_n^{(m)}(t)| \leqslant c_m n^{-m+1} \left\{ \frac{1}{1+n|t|} + \frac{1}{1+n(\pi - t)} + \frac{1}{1+n(\pi + t)} \right\}$$
$$(|t| \leqslant \pi;\ m = 1, 2, \ldots),$$

where c_m depends only on m.

Proof. We write

$$A_n^{(m)} = \sum_{j=1}^{4} A_{n,j}^{(m)},$$

where

$$A_{n,1}^{(m)}(t) = \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} \sin ns \left\{ \frac{1}{\sin s} - \frac{1}{s} - \frac{1}{\pi - s} + \frac{1}{\pi + s} \right\} ds + P_1(t),$$

$$A_{n,2}^{(m)}(t) = \int_a^t \frac{(t-s)^{m-1}}{(m-1)!}\frac{\sin ns}{\pi - s}\,ds + P_2(t),$$

$$A_{n,3}^{(m)}(t) = -\int_a^t \frac{(t-s)^{m-1}}{(m-1)!}\frac{\sin ns}{\pi + s} + P_3(t),$$

$$A_{n,4}^{(m)}(t) = \int_a^t \frac{(t-s)^{m-1}}{(m-1)!}\left\{\frac{\sin ns}{s} - 2\pi\,\delta(2s)\right\}ds + P_4(t),$$

and where $a \in [-\pi, \pi]$ and each $P_j(t)$ $(j = 1, 2, 3, 4)$ is a polynomial of degree $m-1$ to be chosen.

Setting

$$g(t, s) = \frac{(t-s)^{m-1}}{(m-1)!}\left\{\frac{1}{\sin s} - \frac{1}{s} - \frac{1}{\pi - s} + \frac{1}{\pi + s}\right\},$$

we have

$$A_{n,1}^{(m)}(t) = \operatorname{Im}\left\{\left[g(t, s)\frac{e^{ins}}{in} - D_s g(t, s)\frac{e^{ins}}{(in^2)} + \quad + \right.\right.$$

$$\left.\left. + (-1)^{m-1} D_s^{m-1} g(t, s)\frac{e^{ins}}{(in)^m}\right]_{s=a}^{s=t} + (-1)^m \int_a^t D_s^m g(t, s)\frac{e^{ins}}{(in)^m}\,ds\right\} + P_1(t).$$

Consider the expression in square brackets. At $s = t$ all terms vanish except that with $D_s^{m-1} g$, which is $O(n^{-m})$ uniformly; the contribution of $s = a$ is cancelled by choosing P_1 suitably. The integral is $O(n^{-m})$ uniformly. Thus

$$|A_{n,1}^{(m)}(t)| \leqslant c_{m,1} n^{-m}. \tag{101}$$

For $A_{n,2}^{(m)}$, putting $n(\pi - s) = \xi \geqslant 0$, we obtain

$$A_{n,2}^{(m)}(t) = n^{-m+1} \int_{n(\pi-a)}^{n(\pi-t)} \frac{\{\xi - n(\pi - t)\}^{m-1}}{(m-1)!}\cdot(-1)^{n+1}\frac{\sin\xi}{\xi}\,d\xi + P_2(t).$$

Integrating by parts,

$$A_{n,2}^{(m)}(t) = n^{-m+1}(-1)^{n+1}\left[\frac{\{\xi - n(\pi - t)\}^{m-1}}{(m-1)!}F_1(\xi) - \right.$$

$$\left. - \frac{\{\xi - n(\pi - t)\}^{m-2}}{(m-2)!}F_2(\xi) + \ldots + (-1)^{m-1}F_m(\xi)\right]_{\xi=n(\pi-a)}^{\xi=n(\pi-t)} + P_2(t)$$

$$= (-1)^{m+n} n^{-m+1} F_m\big(n(\pi - t)\big)$$

upon suitable choice of the polynomial P_2 to cancel the terms with $\xi = n(\pi - a)$. Whence, by Lemma 3,

$$|A_{n,2}^{(m)}(t)| \leqslant \frac{c_m n^{-m+1}}{1+n(\pi-t)}. \tag{102}$$

Similarly

$$|A_{n,3}^{(m)}(t)| \leqslant \frac{c_m n^{-m+1}}{1+n(\pi+t)}. \tag{103}$$

For $A_{n,4}^{(m)}$, we set $s = \sigma/n$ and note that, for any constant $c > 0$ and $s \in R$, $c\delta(cs) = \delta(s)$. Accordingly,

$$A_{n,4}^{(m)}(t) = n^{-m+1} \int_{na}^{nt} \frac{(nt-\sigma)^{m-1}}{(m-1)!} \left\{\frac{\sin\sigma}{\sigma} - \pi\delta(\sigma)\right\} d\sigma + P_4(t),$$

and, integrating by parts,

$$A_{n,4}^{(m)}(t) = n^{-m+1}(-1)^{m-1}\left[\frac{(\sigma-nt)^{m-1}}{(m-1)!} G_1(\sigma) - \frac{(\sigma-nt)^{m-2}}{(m-2)!} G_2(\sigma) + \ldots + (-1)^{m-1} G_m(\sigma)\right]_{\sigma=na}^{\sigma=nt} + P_4(t) = n^{-m+1} G_m(nt)$$

upon suitable choice of P_4 to cancel the terms with $\sigma = na$. Applying Lemma 5, we obtain

$$|A_{n,4}^{(m)}(t)| \leqslant \frac{c_m n^{-m+1}}{1+n|t|} \tag{104}$$

and, by (101), (102), (103) and (104), (100) follows.

Appendix B. Estimate for the sum of the series

$$S(x, a) = \sum_{k=1}^{\infty} \frac{k^{ak}}{k!} x^k \quad (0 \leqslant a \leqslant 1,\ x > 0). \tag{105}$$

PROPOSITION 2. (a) *If* $0 \leqslant a < 1$ *the series* $S(x, a)$, *defined by* (105), *converges for every* $x > 0$ *and there exists an absolute constant* c_0 *such that*

$$S(x, a) \leqslant c_0 \exp\{(2e)^{\frac{a}{1-a}} x^{\frac{1}{1-a}}\}. \tag{106}$$

(b) *If* $a = 1$, *then* $S(x, a)$ *converges for every* x *with* $0 < x < 1/e$ *and there exists an absolute constant* c_0 *such that*

$$S(x, 1) \leqslant c_0 \frac{ex}{1-ex}. \tag{107}$$

Proof. (a) (106) is trivial for $a = 0$; hence let $0 < a < 1$. Stirling's formula

$$k! \approx \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k}\{1+O(k^{-1})\} \qquad (k \to +\infty)$$

implies that there exist positive constants c_1 and c_2 such that

$$c_1 \leqslant \frac{k!}{k^k(k+1)^{1/2}e^{-k}} \leqslant c_2 \qquad (k \geqslant 0).$$

Hence in (105),

$$x^k k^{ak} = \left(\frac{x}{a^a}\right)^k (ak)^{ak} \leqslant \left(\frac{x}{a^a}\right)^k \frac{1}{c_1} \frac{\Gamma(ak+1)}{(ak+1)^{1/2}e^{-ak}} = \frac{1}{c_1}\gamma^k \frac{\Gamma(ak+1)}{(ak+1)^{1/2}},$$

where

$$\gamma = \frac{xe^a}{a^a}. \tag{108}$$

(Note that $a^a = e^{a\log a} \to 1$ as $a \to 0$.) Accordingly,

$$\begin{aligned} S(x, a) &\leqslant \sum_{k=1}^{\infty} \frac{1}{c_1} \frac{\gamma^k}{(ak+1)^{1/2}} \frac{\Gamma(ak+1)}{k!} \\ &= \frac{1}{c_1}\sum_{k=1}^{\infty} \frac{1}{(ak+1)^{1/2}} \frac{\gamma^k}{k!} \int_0^\infty e^{-t}t^{ak}\,dt \\ &< \frac{1}{c_1}\int_0^\infty e^{-t}\sum_{k=1}^{\infty}\frac{\gamma^k t^{ak}}{k!}\,dt = \frac{1}{c_1}\left\{\int_0^\infty e^{-t+\gamma t^a}\,dt - 1\right\}. \end{aligned} \tag{109}$$

Define $f(t) = t - \gamma t^a$ so that $f'(t) = 1 - \gamma a t^{a-1}$. Define t_0 by $f'(t_0) = \frac{1}{2}$; then

$$t_0 = (2\gamma a)^{1/(1-a)} \tag{110}$$

and also $\frac{1}{2} \leqslant f'(t) < 1$ for $t \geqslant t_0$. Thus

$$\int_{t_0}^\infty e^{-t+\gamma t^a}\,dt \leqslant 2\int_{t_0}^\infty e^{-t+\gamma t^a}(1-\gamma a t^{a-1})\,dt = 2e^{-t_0+\gamma t_0^a}.$$

Also

$$\int_0^{t_0} e^{-t+\gamma t^a}\,dt \leqslant e^{\gamma t_0^a}.$$

Hence

$$S(x, a) < \frac{1}{c_1} e^{\gamma t_0^a}(1+2e^{-t_0}),$$

which, for γ and t_0 given by (108) and (110), yields (106) with $c_0 = 2/c_1$.

(b) If $\alpha = 1$, then the last integral in (109) converges for $\gamma < 1$, giving

$$S(x, 1) < \frac{1}{c_1} \frac{\gamma}{1-\gamma},$$

which proves the second assertion of the proposition, since now $\gamma = cx$ by (108).

REFERENCES

[1] P. J. Aranda and E. Cattaneo, *Classe $\mathscr{L}_p$ de l'injection de $H_0^s(\Omega)$ dans $L_2(\Omega)$*, C. R. Acad. Sci. Paris 274 (1974), pp. 1292–1295.

[2] D. E. Edmunds and W. D. Evans, *Orlicz and Sobolev spaces on unbounded domains*, Proc. Roy. Soc. A 342 (1975), pp. 373–400.

[3] D. E. Edmunds and V. B. Moscatelli, *Approximation de Fourier et type de l'injection des espaces de Sobolev sur des ouverts bornés*, C. R. Acad. Sci. Paris 280 (1975), pp. 889–892.

[4] D. E. Edmunds, and V. B. Moscatelli, *Approximation numbers and the asymptotic distribution of eigenvalues of elliptic operators*, Ann. Math. Pura Appl. (to appear).

[5] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. (1974), pp. 309–317.

[6] K. Maurin, *Methods of Hilbert spaces*, 2nd ed., Warszawa 1972.

[7] J. Nécas, *Les méthodes directes en théorie des équations elliptiques*, Paris–Prague 1967.

[8] A. Pietsch, *Nuclear locally convex spaces*, Berlin–Heidelberg–New York 1972.

[9] L. Schwartz, *Théorie des distributions I*, Paris 1950.

[10] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.

[11] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. and Mech. 17 (1967), pp. 473–483.

[12] K. Yosida, *Functional Analysis*, Berlin 1966.

[13] A. Zygmund, *Trigonometric series*, I and II, Cambridge 1959.

[14] M. Š. Birman and M. Z. Solomjak, *Approximation of the functions of the classes W_p^α by piecewise polynomial functions*, Soviet Math. Dokl. 7 (1966), pp. 1573–1576.

[15] — — *Piecewise polynomial approximation of functions of class W_p^α*, Mat. Sb. 73 (1967), pp. 331–355.

[16] R. S. Ismagilov, *n-dimensional diameter of compacta in a Hilbert space*, Functional. Anal. i Priložen. 2 (1968), pp. 32–39.

[17] — Uspekhi Mat. Nauk 29 (1974), pp. 161–178.

[18] A. E. Kolli, *$n^{\text{ième}}$ épaisseur dans les espaces de Sobolev*, C. R. Acad. Sci. Paris 272 (1971), pp. 537–539.

[19] — *$n^{\text{ième}}$ épaisseur dans les espaces de Sobolev*, J. Approx. Theory 10 (1974), pp. 268–294.

[20] H. Triebel, *Über die Verteilung der Approximationszahlen kompakter Operatoren in Sobolev-Besov-Räumen*, Inventiones Math. 4 (1967), pp. 275–293.

[21] — *Interpolationseigenschaften von Entropie und Durchmesseridealen kompakter Operatoren*, Studia Math. 34 (1970), pp. 89–107.

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