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MORE ABOUT THE CLOSURE OF *NP*

GERD WECHSUNG

Department of Mathematics of the Friedrich Schiller University Jena, G.D.R.

Recently the Boolean closure of NP has been studied in terms of nondeterministic TM's with modified acceptance [9]. Here we introduce a refinement of the classes considered there. This refinement is motivated by the fact that apparently the classes known so far are not sufficient for a complexity classification of many natural problems.

1. Introduction

Recently, the Boolean closure of NP (denoted by $BC(NP)$ has been studied from several points of views. Since F. Hausdorff ([4]) the Boolean closure of a class of sets closed unter union and intersection is known to be the union of classes which we call the *Hausdorff hierarchy* generated by K. In the case of NP this hierarchy is defined by $D_1 = NP$, $C_1 = \mathrm{co}NP$

$$D_{n+1} = C_n \vee NP =_{\mathrm{df}} \{X \cup Y\colon\ X \in C_n \wedge Y \in NP\},$$
$$C_{n+1} = D_n \wedge \mathrm{co}NP =_{\mathrm{df}} \{X \cap Y\colon\ X \in D_n \wedge Y \in \mathrm{co}NP\}.$$

The inclusion relationships between these classes which are known so far follow from

$$C_n \subseteq C_{n+1} \cap D_{n+1} \quad \text{and} \quad C_n = \mathrm{co}D_n \quad (n = 1, 2, \ldots).$$

In [9] and [10] new acceptance types for nondeterministic Turing machines (NTM's) have been introduced in such a way that the polynomial time complexity classes with respect to these new notions are exactly the classes of the Hausdorff hierarchy. Characterizations of these classes in terms of restricted bounded truth-table reducibility are given. In [5] the same hierarchy is studied from the point of view of bounded truth-table reducibility and complete sets, and in [8] natural complete problems for the classes of the Hausdorff hierarchy are exhibited. In [6] the case of C_2 has been considered.

In this paper we give a very brief review of [9] including some new results and then we mainly consider a quantitative analogon of the acceptance types introduced in [9]. This leads to a refinement of the Boolean closure of NP. In particular, we shall study the class $1NP$ of those languages which are acceptable by polynomial time NTM's by just one accepting path. This class has been introduced in [1], and the following inclusions are known for it:

$$\mathrm{co}NP \subseteq 1NP \subseteq NP \wedge \mathrm{co}NP.$$

Many problems concerning this refined hierarchy are still open. A more detailed version will be presented in a forthcoming joint paper with Thomas Gundermann [3]. I would like to emphasize that the cooperation with him promoted also this paper a good deal.

2. Qualitative acceptance notions

In this section we briefly review the main notions and results from [9].

We consider NTM's attached with polynomial clocks. We assume that every NTM has its distinguished set S of final states and that it reaches on every path some state from S.

DEFINITION. 1. $(\mathfrak{A}, S)$ is an acceptance type if $\mathfrak{A}$ is a strict subset of the power set of S not containing the empty set.

2. $\mathrm{Leaf}_M(x) = \{s\colon s \in S \wedge s$ is reached by M on input x on at least one path$\}$.

3. M accepts x in the sense $(\mathfrak{A}, S) \leftrightarrow df\ \mathrm{Leaf}_M(x) \in \mathfrak{A}$, provided M has set S of final states.

4. $L_{(\mathfrak{A},S)}(M) = \{x\colon x$ accepted by M in the sense of $(\mathfrak{A}, S)\}$.

5. $(\mathfrak{A}, S)P = \{L_{(\mathfrak{A},S)}(M)\colon M$ is a polynomial time $NTM\}$ is the polynomial time complexity class corresponding to the acceptance type $(\mathfrak{A}, S)$. NP and coNP are special cases.

The new acceptance types can be considered to be formalizations of weak forms of parallelity. Namely, the condition that the final states reached on a computation should form a set belonging to $\mathfrak{A}$ requires in a sense that certain paths have to cooperate in a prescribed way. This is not the case for the usual nondeterministic acceptance, but it is already present in the case of coNP. The degree of parallelity reachable in this way is not too high because the $(\mathfrak{A}, S)P$ are subclasses of Δ_2^P (see [9]).

More precisely, we have shown that every $(\mathfrak{A}, S)P$ is some class C_i or D_i of the Hausdorff-hierarchy generated by NP and vice versa. Because of $BC(NP) = \bigcup_{i=1}^{\infty} C_i$ the classes $(\mathfrak{A}, S)P$ are exactly the Hausdorff classes within

$BC(NP)$. Furthermore, $BC(NP)$ is shown to be equal to

$$R^P_{btt}(NP) = \{X: \bigvee_{Y \in NP} (X \leqslant^P_{btt} Y)\}.$$

Recall that X is polynomially time bounded truth table reducible to Y ($X \leqslant^P_{btt} Y$) if and only if there exists a natural number m and a function f computable within polynomial time such that

$$x \in A \leftrightarrow \alpha(C_B(x_1), \ldots, C_B(x_n)) = \text{true},$$

where α is a Boolean function of arity $n \leqslant m$, α, $x_1, \ldots, x_n$ are determined by

$$f(x) = \langle x_1, \ldots, x_n, \text{code}(\alpha) \rangle$$

and C_B is the characteristic function of B. (Here $\langle \ldots \rangle$ denotes a 1-1-mapping of N^* onto N and code is some fixed encoding of the Boolean functions by natural numbers.) But

$$R^P_{btt}(NP) \subseteq R^P_T(NP) = \{X: \bigvee_{Y \in NP} X \leqslant^P_T Y\} = \Delta^P_2.$$

3. Relativization results

It is an open question whether the Hausdorff hierarchy is finite or infinite. The following proposition is elementary:

PROPOSITION 1. *For every $k \geqslant 1$ the following statements are mutually equivalent*

(1) $$C_k = C_{k+1},$$

(2) $$\bigwedge_{n \geqslant k} C_k = C_n,$$

(3) $$C_k = D_k.$$

If $P = NP$ then it collapses to P. So proving that it is a strict hierarchy (at least up to a certain level) amounts to proving $P \subset NP$.(¹)

In such cases one usually looks for relativizations. In [2] we prove the following results

THEOREM 2. *There exists a recursive oracle A such that*

$$C^A_0 \subset C^A_1 \subset C^A_2 \subset \ldots \subset C^A_n \ldots \blacksquare$$

THEOREM 3. *For every $k \geqslant 0$ there exists a recursive oracle A such that*

$$C^A_0 \subset \ldots \subset C^A_k = C^A_{k+1} = \ldots \blacksquare$$

(¹) $\subset$ denotes strict inclusion.

4. Quantitative acceptance notions

The classes of the Hausdorff hierarchy are too coarse for complexity classification of many interesting problems. Consider, for instance,

$$1SAT =_{\mathrm{df}} \{H\colon H \text{ is a Boolean formula having exactly one satisfying assignment}\}$$

or

$$\{2, 5\}SAT =_{\mathrm{df}} \{H\colon H \text{ is a Boolean formula having either 2 or 5 satisfying assignments}\},$$

or

$$(1 \vee 1)SAT =_{\mathrm{df}} \{(H_1, H_2)\colon H_1, H_2 \text{ are Boolean formulas such that at least one of them has exactly one satisfying assignment}\}.$$

$1SAT$ has been considered in [1]. Since $1SAT$ is the intersection of SAT and $SAT(\leqslant 1) =_{\mathrm{df}} \{H\colon H$ is a Boolean formula having no more than one satisfying assignment$\}$ and since

$$SAT(\leqslant 1) \equiv^P_m \overline{SAT} = \{H\colon H \text{ is a Boolean formula having no satisfying assignment}\},$$

it follows that $1SAT \in NP \wedge \mathrm{co}NP = C_2$. But it is very unlikely that $1SAT$ is $\leqslant^P_m$-complete in C_2. By a close inspection of Cook's proof that SAT is $\leqslant^P_m$-complete in NP one gets that $1SAT$ is $\leqslant^P_m$-complete in $1NP$, where $1NP$ is the class of all those sets which can be accepted by polynomial time NTM's, M by exactly one successful path.

$\{2, 5\}$ SAT and $(1 \vee 1)SAT$ can be found to be complete in similar classes.

A general framework for dealing with such classes is provided by the following acceptance notions. Again we consider clocked NTM's reaching on every path some final state.

Definition. 1. $\mathfrak{A}$ is called an *acceptance type* if there is a natural number k such that $\mathfrak{A} \subseteq N^k$, where N is the set of natural numbers.

2. Let M be an NTM having the set $\{s_0, \ldots, s_k\}$ of final states. $\mathrm{Leaf}^i_M(x)$ = number of paths of M reaching a final configuration with state s_i. $\mathrm{Leaf}_M(x) = (\mathrm{Leaf}^1_M(x), \ldots, \mathrm{Leaf}^k_M(x))$ is called the *leaf number vector of M on x*. Note that we do not ask how often s_0 is reached.

3. M $\mathfrak{A}$-accepts $x \underset{\mathrm{df}}{\leftrightarrow} \mathrm{Leaf}_M(x) \in \mathfrak{A}$, provided M has set $\{s_0, \ldots, s_k\}$ of final states.

4. $L_{\mathfrak{A}}(M) = \{x\colon M\ \mathfrak{A}\text{-accepts } x\}$.

5. $\mathfrak{A}NP = \{L_{\mathfrak{A}}(M)\colon M \text{ is an } NTM \text{ working in polynomial time}\}$.

We call $\mathfrak{A}$ a *quantitative acceptance type* in contrast to $(\mathfrak{A}, S)$ because the number of accepting paths is here counted. Note however, that formally speaking every qualitative acceptance type is also a quantitative one.

In this paper we restrict ourselves to studying only those classes defined by quantitative acceptance notions which are closely connected with $1NP$.

5. The set ring generated by $1NP$

In terms of the new acceptance notions, $1NP$ is defined to be $\{1\}NP$. This means, x is accepted by an NTM M in the sense of $1NP$ if and only if M works in polynomial time, has two distinguished final states s_0, s_1, and on input x the state s_1 is reached on exactly one path. $1NP$ is not to be confused with the class U of languages accepted by unambiguous polynomial time Turing machines (for relativizations including U see [7]).

Unambiguous Turing machines are such NTM's having either no or exactly one accepting path. Every polynomial time NTM accepts some set in the sense of $1NP$ (inputs causing more than one accepting paths are rejected), but as soon as it has for some input more than one accepting path it is not unambiguous.

PROPOSITION 4. *$1NP$ is closed under intersection, i.e. $1NP \wedge 1NP = 1NP$.*

Proof. Let $X, Y \in 1NP$ and let M_1, M_2 be NTM's accepting X and Y, resp., in the sense of $1NP$. We construct a new machine M working on input x first like M_1 and if and when state s_1 is reached it works like M_2 on input x. This combined machine works in polynomial time, and on input x it reaches state s_1 on exactly one path if and only if both M_1 and M_2 on input x reach state s_1 on exactly one path. This means, $X \cap Y \in 1NP$. ∎

A similar technique for proving $1NP \vee 1NP = 1NP$ is not known. We have, on the contrary, reasons to believe that $1NP$ is not closed under union (see Theorem 5).

Let $H(1NP)$ be the closure of $1NP$ under $\cup$ and $\cap$. According to [4] $H(1NP) = \bigcup_{\substack{k \in N \\ i_1, \ldots, i_k}} A_{i_1, \ldots, i_k}$, where

$$A_i =_{\mathrm{df}} \underbrace{1NP \vee \ldots \vee 1NP}_{i \text{ times}} \quad \text{and} \quad A_{i_1, \ldots, i_k} = A_{i_1} \wedge \ldots \wedge A_{i_k}.$$

The only known set inclusions are the trivial ones given by $A_{i_1 \ldots i_k} \subset A_{j_1 \ldots j_t}$ if either $k = t \wedge i_1 \leqslant j_1 \wedge \ldots \wedge i_k \leqslant j_k$ or

$$k < t \wedge \bigvee_{1 \leqslant n_1, \ldots, n_k \leqslant t} ((\mathrm{card}\,\{n_1, \ldots, n_k\}) = k \wedge i_1 \leqslant j_{n_1} \wedge \ldots \wedge i_k \leqslant j_{n_k}).$$

An appropriate picture of this hierarchy could best be drawn in an infinite dimensional space. Figure 1 shows the small fragment of those classes having no more than two indices.

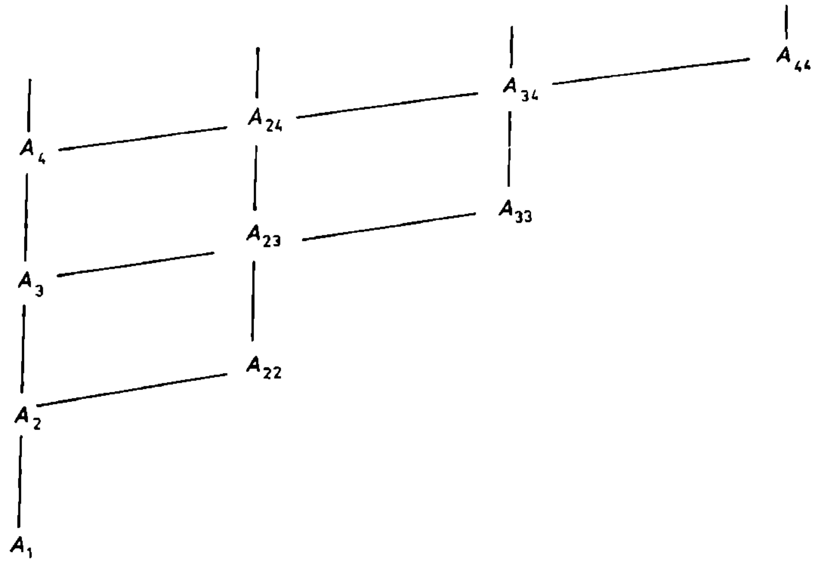

Fig. 1

In [3] we prove several relativization results on $H(1NP)$, for instance

THEOREM 5. *There exists a recursive oracle A such that*

$$A_1^A \subset A_2^A \subset \ldots \subset A_k^A \subset \ldots \ \blacksquare$$

Another interesting question is how the classes A_k are compared with the classes of the Hausdorff hierarchy generated by NP.

We know that $1NP \subseteq C_2$ and hence $A_k \subseteq C_{2k}$. The next theorem shows that according to a suitable oracle we cannot do better.

THEOREM 6. *There exists a recursive oracle A such that for every* $k \geqslant 1$

$$C_{2k-1}^A \subset A_k^A. \ \blacksquare$$

The following proposition is evident

PROPOSITION 7. *Every class* $A_{i_1 \ldots i_k}$ *is closed under* $\leqslant_m^P$*-reducibility.*

It is very easy to construct problems which are $\leqslant_m^P$-complete in $A_{i_1 \ldots i_k}$. It is sufficient to illustrate this by some examples.

$(1 \vee 1)SAT$ is complete in A_2,

$\{(H_1, H_2, G_1, G_2, G_3, k)$: $H_1, \ldots, G_3$ are graphs and $k \in N$ and (at least one of the graphs H_1, H_2 has a uniquely determined clique of size k) and (at least one of the graphs G_1, G_2, G_3 has a uniquely determined clique of size k)$\}$ is $\leqslant_m^P$-complete in A_{23}.

6. Finite acceptance types

In this section we consider the sets ANP for finite sets A. We call these acceptance types finite acceptance types.

PROPOSITION 8. *For every singleton* A,

$$ANP = 1NP.$$

Proof. Let $A = \{i\}$.

1. $\supseteq$. This inclusion is trivial because every path leading to state s_1 can be split up into paths leading to state s_1.

2. $\subseteq$. Let M be a machine accepting $X \in ANP$ by reaching exactly i times the state s_1 for exactly the inputs $x \in X$. We assume for every input x a linear order of the paths of M. Now we construct a new machine $M' = \underbrace{M \cdot \ldots \cdot M}_{i \text{ times}}$ working as follows on input x. First it works like M on x. On those α_1 where s_1 is reached, M is again applied on x, and the computation path α_2 of the second application is compared with the computation path α_1. If $\alpha_1 > \alpha_2$ then apply again M on x and so forth. M' reaches s_1 on such paths which correspond to i paths $\alpha_1, \alpha_2, \ldots, \alpha_i$ satisfying the condition $\alpha_1 > \alpha_2 > \ldots > \alpha_i$ such that M on each of these paths reaches s_1. On all remaining paths M' reaches s_0. It is evident that M' reaches s_1 exactly once if and only if M reaches s_1 exactly i times. ■

The remaining classes with finite acceptance type are related to the classes A_n:

PROPOSITION 9. *If* card $A = n$ *then* $ANP \subseteq A_n$.

Proof. Let $A = \{a_1, \ldots, a_n\}$. If $X \in ANP$ then there exists an NTM M such that $X = L_A(M)$. But, evidently, $L_A(M) = L_{\{a_1\}}(M) \cup \ldots \cup L_{\{a_n\}}(M)$, and because of Proposition 8 this union belongs to A_n.

It is unlikely that $ANP = A_n$ for card $A = n$. This is suggested by

THEOREM 10. *For every* $n \geqslant 2$ *and for every* $A \subseteq N \setminus \{0\}$ *with* card $A = n$ *there exists a recursive oracle* B *such that*

$$ANP^B \subset A_n^B. \ ■$$

For sets A having sufficiently many gaps one gets the following result.

THEOREM 11. *If* $a = \{a_1, \ldots, a_n\}$ *satisfies the condition that there exist natural numbers* $b_1, \ldots, b_{n-1} \notin A$ *such that* $a_1 < b_1 < \ldots < b_{n-1} < a_n$, *then there exists a recursive oracle* B *such that*

$$C_{2n-1}^B \subset ANP^B. \ ■$$

By dovetailing we construct a recursive oracle B such that

$$C_3^B \subset \{1, 3\}NP^B \subset (1NP \vee 1NP)^B \subset C_4^B \subset \ldots \subset C_5^B \subset \{1, 3, 5\}NP^B$$
$$\cdot \subset (1NP \vee 1NP \vee 1NP)^B \subset \ldots$$

This shows that for suitable relativizations the sets ANP as well as the A_n are cofinal with the sequence of C_n in $BC(NP)$. We close with a few results concerning the classes ANP for $\text{card}\, A = 2$.

THEOREM 12. *For every i and every j, k such that $k \geqslant j+2$ there exists a recursive oracle B such that*

$$\{i, i+1\}\, NP^B \subset \{j, k\}\, NP^B.$$

PROPOSITION 13. *For every $i, j, k > 0$*

$$\{i, j\}\, NP \subseteq \{i+k, j+k\}\, NP.$$

Proof. Every machine M accepting a set in the sense of $\{i, j\}\, NP$ becomes a machine accepting in the sense of $\{i+k, i+j\}\, NP$ if k paths leading to s_1 are added to M.

PROPOSITION 14. *For $i < j < k$,*

$$\{i, j\}\, NP \subseteq \{i, k\}\, NP.$$

Proof. Let M be a machine accepting X in the sense of $\{i, j\}\, NP$. Define $M' = \underbrace{M \cdot \ldots \cdot M}_{j\text{ times}}$. The final states of M' are redefined as follows:

$s_1 s_1 \ldots s_1 = s_1$ if every copy of M reaches s_1 on the same path,

$s_1 s_1 \ldots s_1 = s_2$ if there are paths $\alpha_1 < \alpha_2 < \ldots < \alpha_j$ such that the νth copy of M reaches s_1 on path α_ν,

$x_1 \ldots x_j = s_0$ in all other cases.

Now M' is changed into M'' by the modification that state s_2 splits into $k-j$ states s_1. Now, if M reaches s_1 i' times where $i' < j$, then M' reaches s_1 i' times and s_2 is not reached. If M reaches s_1 j times, then M' reaches j times s_1 and once s_2, and hence M'' reaches s_1 exactly $j+k-j = k$ times. If M reaches s_1 more than j times then M'' reaches s_2 more than k times. ∎

It seems to be difficult to prove inclusions of the form $ANP \subseteq BNP$, where B contains numbers less then those in A. The best result of this kind we know of so far is

PROPOSITION 15. *For every $k \geqslant 2$ and $l > k$,*

$$\{k, l\}\, NP \subseteq \left\{1, \binom{l}{k}\right\} NP.$$

Proof. Let M accept a set in the sense of $\{k, l\}\, NP$. Define $M' = \underbrace{M \cdot \ldots \cdot M}_{k}$ as in the proof of Proposition 8.

It is clear: if M reaches s_1 exactly k times then M' reaches s_1 exactly once. If M reaches s_1 exactly l times, then M' reaches s_1 exactly $\binom{l}{k}$ times. If

M reaches s_1 j times where $j \notin \{k, l\}$, then M' reaches s_1 either not at all (for $j < k$) or $\binom{j}{k}$ times which is different from 1 and $\binom{l}{k}$.

The results of the last proposition are summarized in Figure 2. Note that according to Theorem 10 for suitable relativizations this whole picture is below $1NP \vee 1NP$.

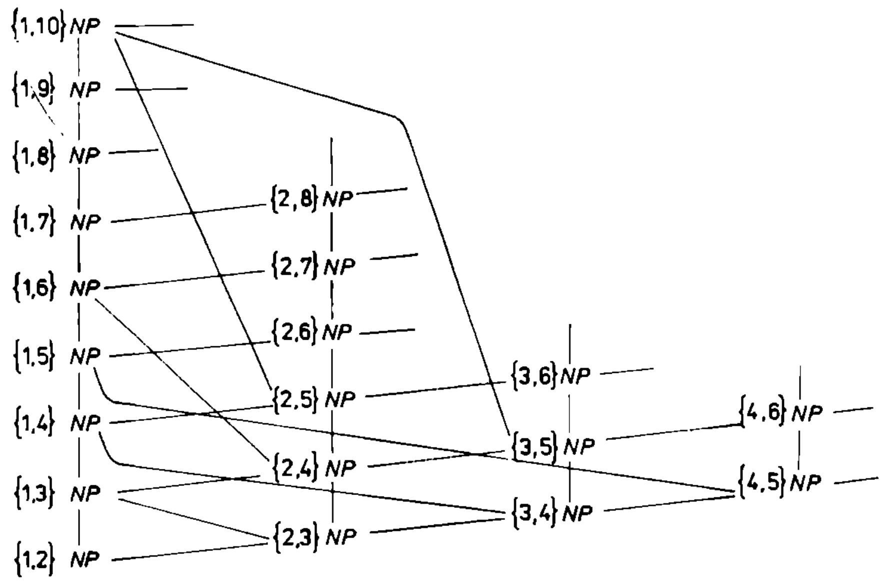

Fig. 2

Conclusion

This paper which is a slightly extended version of the authors lectures given at the Banach semester does not intend to give a complete and exhaustive representation of recent results.

Instead we tried to illustrate by some selected sample results the type of problems which are studied in connection with the new acceptance types. The reader is referred to the forthcoming papers [2] and [3] which deal with this topic in greater detail.

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