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AN IMPROVED INDUCTIVE DEFINITION OF TWO RESTRICTED CLASSES OF TRIANGULATIONS OF THE PLANE

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The improved inductive definitions of (i) the class of all triangulations of the plane with all vertices of even degree, and (ii) the class of all triangulations of the plane without vertices of degree 3 are given. The dual inductive definitions are local and expanding and determine (i) the class of all 3-connected bipartite planar cubic graphs, and (ii) the class of all 3-connected planar cubic graphs without triangles.

The *inductive class* $\mathfrak{I} = Cn(\mathfrak{B}; \mathfrak{R})$ is defined by giving the class $\mathfrak{B}$ of initial objects – the *basis* of $\mathfrak{I}$, and the class $\mathfrak{R}$ of *generating rules.* Any such rule applied to an appropriate sequence of objects, already in $\mathfrak{I}$, produces an object of $\mathfrak{I}$. The inductive class $\mathfrak{I}$ consists exactly of the objects which can be constructed from the basis by a finite number of applications of generating rules.

The inductive definition is *local* iff basic graphs, and for each rule the part of the graph on the left side of the rule, are connected, and is *expanding* iff the application of each rule increases the size (number of vertices, number of edges, etc.) of the graph.

The inductive definitions of the class of all triangulations of the plane without vertices of degree 3, and of the class of all triangulations of the plane with all vertices of even degree were given already in [2]. In the present paper improved inductive definitions are presented which give us in their dual form local and expanding inductive definitions of the class of all 3-connected bipartite planar cubic graphs and of the class of all 3-connected planar cubic graphs without triangles.

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Inductive definitions of the class of all triangulations with all vertices of degree at least 5 and of the corresponding class of all 3-connected planar cubic graphs without triangles and quadrangles are given in [1].

Inductive definition of the class of all 3-connected bipartite planar cubic graphs is interesting in relation to Barnette's hamiltonicity conjecture: Every 3-connected bipartite planar cubic graph is Hamiltonian. By the use of a similar (but nonlocal) inductive definition Holton, Manvel and McKay [4] proved that Barnette's conjecture holds for $n < 65$.

In the inductive definitions in this paper the base graphs and rules should be understood as embedded in the plane (sphere). The small triangles attached to the vertices in the description of the rule denote any number (zero or more) of edges; and the halfedges indicate that there must be an edge.

A triangulation is called *even* if it has all vertices of even degree.

THEOREM 1. *The inductive class* $Cn(O; P, Q)$ (see Fig. 1) *is equal to the class* T_2 *of all even triangulations of the plane.*

Proof. The basic graph O in Fig. 1 is an even triangulation and the rule P or Q produces from a graph belonging to T_2 a new graph also belonging to T_2. Therefore, by inductive generalization, it follows that $Cn(O; P, Q) \subseteq T_2$.

To prove that also $T_2 \subseteq Cn(O; P, Q)$ we must show that every even triangulation G, $G \neq O$, can be reduced by the inverse rules P^- and Q^- to a smaller triangulation of the same type.

From Euler's polyhedral formula it follows that in each $G \in T_2$ there exist at least 6 vertices of degree 4. Let x be a vertex of degree 4 and let neighbours of x in cyclic order be v, w, v, z. There are two cases to be considered.

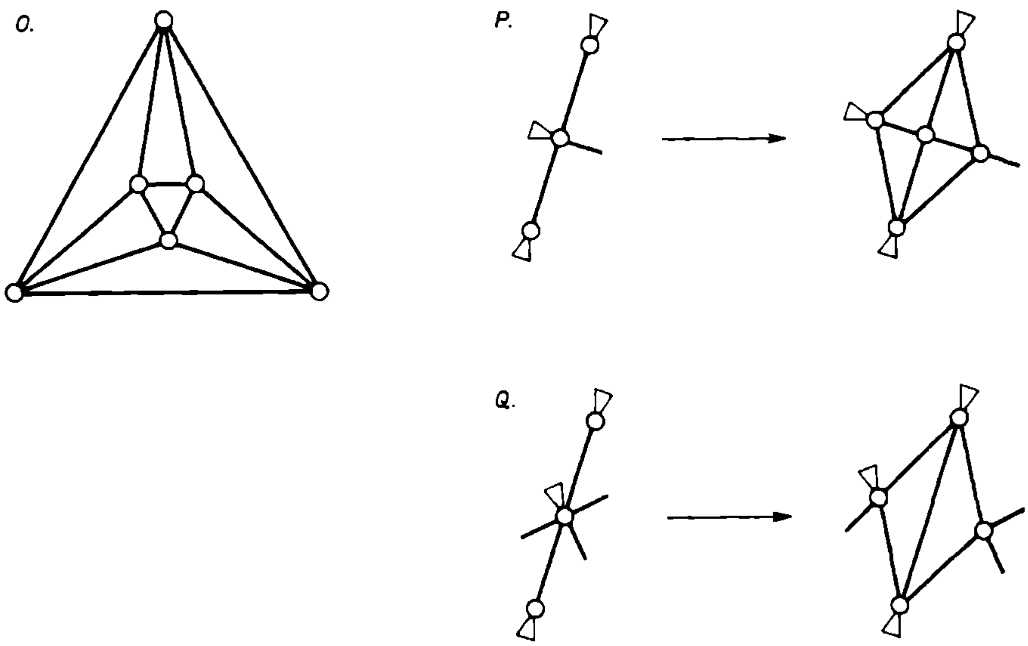

Fig. 1

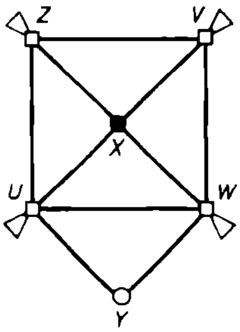

Fig. 2a

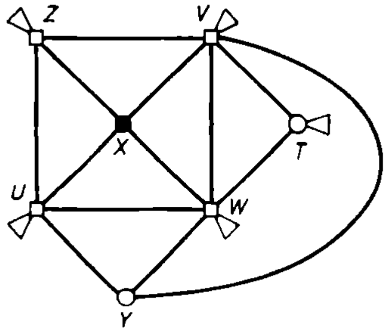

Fig. 2b

Case A. All vertices u, v, w and z adjacent to x (see Fig. 2) are of degree at least 6. We denote in figures the vertex of degree 4 by a black circle and the vertex of degree at least 6 by a square. Refer to Figs. 2a, 2b for definition of y and t.

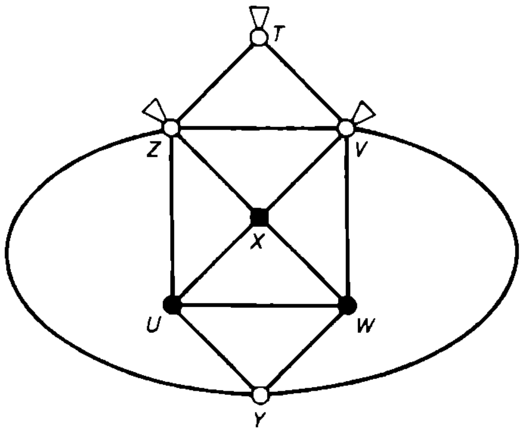

Fig. 3

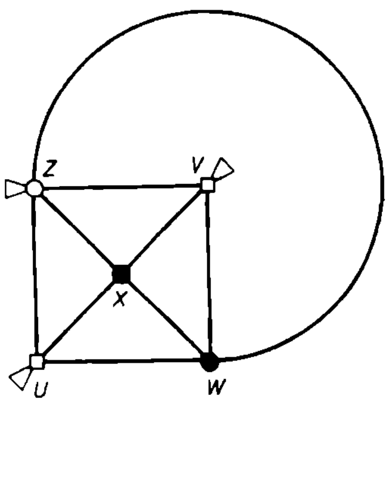

Fig. 4

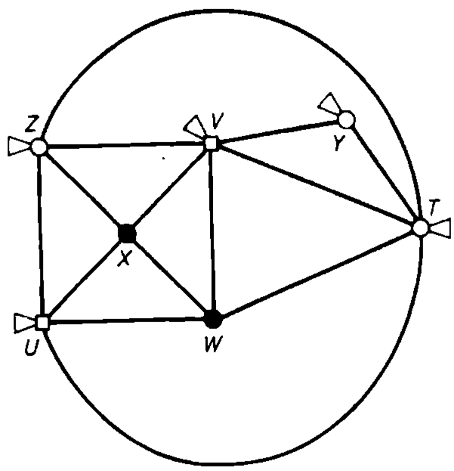

Fig. 5

If $YV \notin E$ and $YZ \notin E$ we can apply the inverse rule Q^- to the quadrangle $XUYW$ without producing multiple edges. Otherwise, because of the symmetry, suppose that $YV \in E$ (see Fig. 2b). Then $TZ \notin E$ and $TU \notin E$ and we can apply the rule Q^- to the quadrangle $XWTV$.

Case B. At least one vertex adjacent to X is of degree 4. Suppose that $\deg(W) = 4$. There are two possibilities.

Case B1. One of the common neighbours U and V of the vertices X and W is also of degree 4 (see Fig. 3). Let this be the vertex U. Because $G \neq O$ the vertices Z and V are of degree at least 6 and we can apply the rule Q^- to the quadrangle $XVTZ$.

Case B2. Both common neighbours U and V of the vertices X and W are of degree at least 6. Again we have to consider several possibilities.

Case B2.1. The vertices X, U and V are the only common neighbours of the vertices W and Z and each path between W and Z not passing through U, X or V has at least two internal vertices. We can apply the rule P^- to the quadrangle $ZUWV$.

Case B2.2. WZ is an edge of G (see Fig. 4). This is impossible because it contradicts the 3-connectedness of G—the set $\{Z, U\}$ is a cut-set in G.

Case B2.3. Beside the vertices X, U and V there is another common neighbour T of the vertices W and Z (see Fig. 5). We can apply the rule Q^- to the quadrangle $VWTY$.

This completes the proof of Theorem 1.

Even triangulations represented in Fig. 6 demonstrate that the rules P and Q are independent. The inverse rule Q^- cannot be applied to the left triangulation without producing multiple edges; and P^- cannot be applied to the right triangulation because there is no pair of adjacent vertices both of degree 4.

THEOREM 2. *The inductive class* $Cn(O; R, S)$ (see Figs. 1 and 7) *is equal to the class* $T(>3)$ *of all triangulations of the plane without vertices of degree* 3.

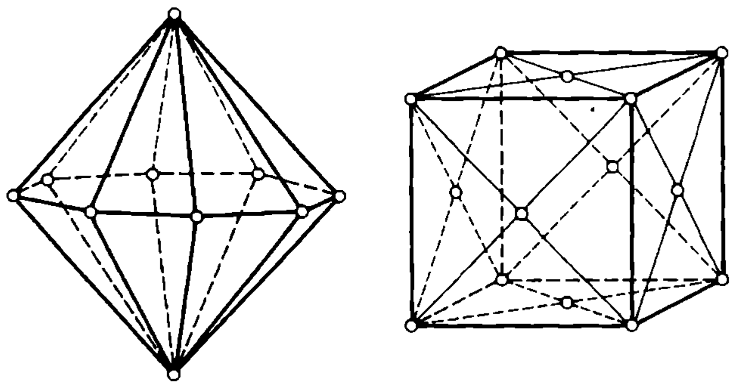

Fig. 6

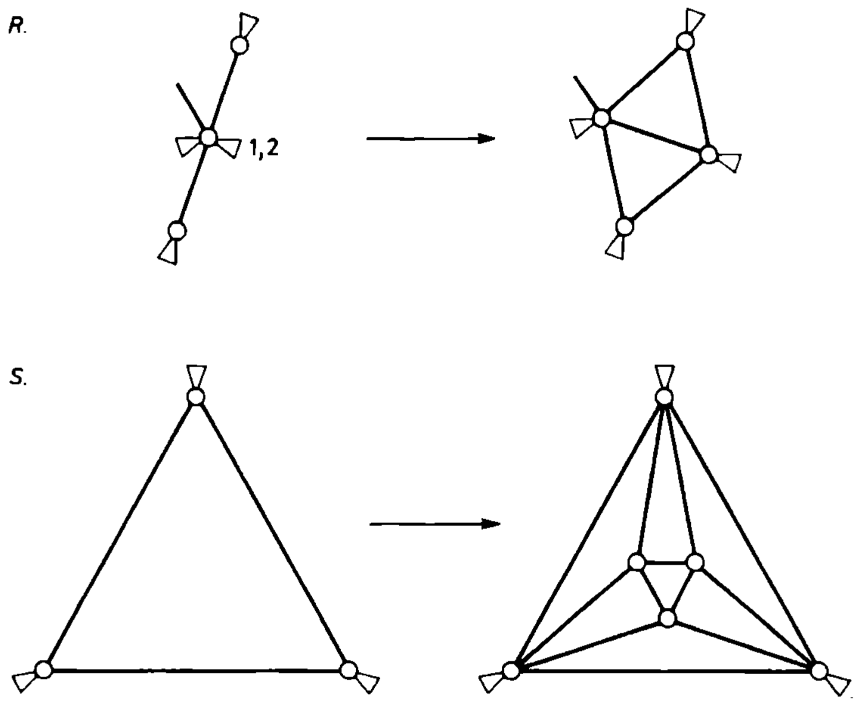

Fig. 7

The condition 1, 2 *in the description of the rule R means that on that side there should be* 1 *or* 2 *edges.*

Proof. The inclusion $Cn(O; R, S) \subseteq T(>3)$ follows by inductive generalization. To prove that also $T(>3) \subseteq Cn(O; R, S)$ we must show that every triangulation G, $G \neq O$, without vertices of degree 3 can be reduced by the inverse rules R^- and S^- to a smaller triangulation of the same type.

From Euler's formula it follows that in every triangulation there exists a vertex of degree less than 6. Let x be one of the vertices of the least degree in G. Then either $\deg(x) = 4$ or $\deg(x) = 5$. Let us analyse both cases separately.

Case A. $\deg(x) = 4$. The neighbourhood of the vertex x is represented in Fig. 8a. There are two possibilities for the degrees of the neighbours of x.

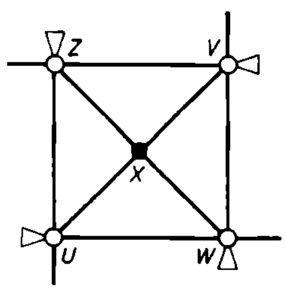

Fig. 8a

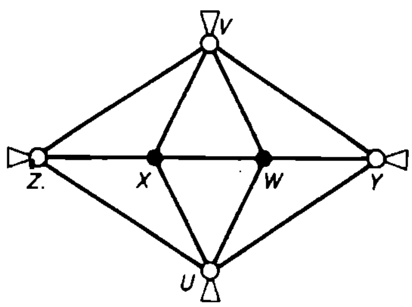

Fig. 8b

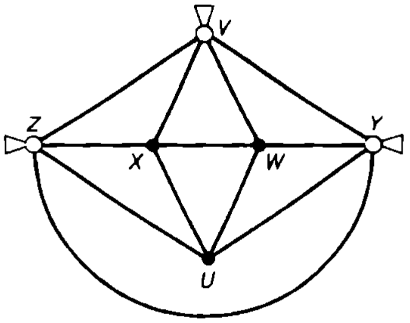

Fig. 9a

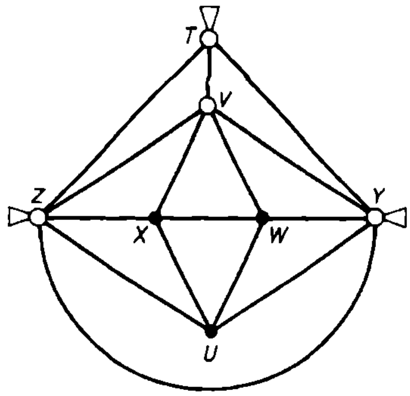

Fig. 9b

Case A1. All four neighbours are of degree at least 5. In this case at least one of the pairs uv and zw is not an edge of G. Suppose, because of symmetry, that this is the pair uv. Then we can apply the inverse rule R^- to the quadrangle $zxwv$ without producing multiple edges. The reduced graph is again a triangulation without vertices of degree 3.

Case A2. At least one neighbour, say w, is of degree 4 (see Fig. 8b). If the vertices u and v are both of degree at least 5 we can apply the rule R^- to the quadrangle $xuwv$. Otherwise, one of the vertices u and v, say u, is of degree 4 (see Fig. 9a). If the vertices v, z and y are all of degree at least 6 we can apply the rule S^-. No one among these vertices can be of degree 4 because this would imply that G is only 2-connected. Therefore, in the remaining case at least one vertex, say v, is of degree 5 (see Fig. 9b). We can apply the rule R^- to the quadrangle $zvyt$.

Case B. $\deg(x) = 5$. The neighbourhood of the vertex x is represented in Fig. 10.

Let us first show that for at least one among the vertices u, w, v, y and z all the edges of the induced subgraph are represented in the figure. Suppose the

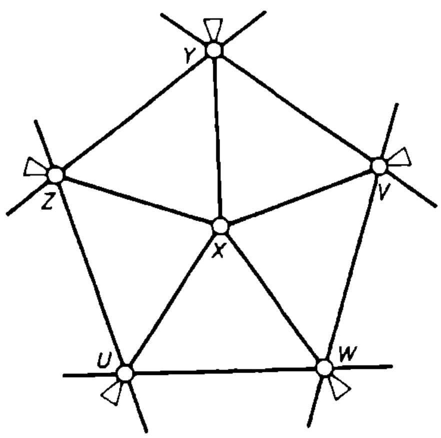

Fig. 10

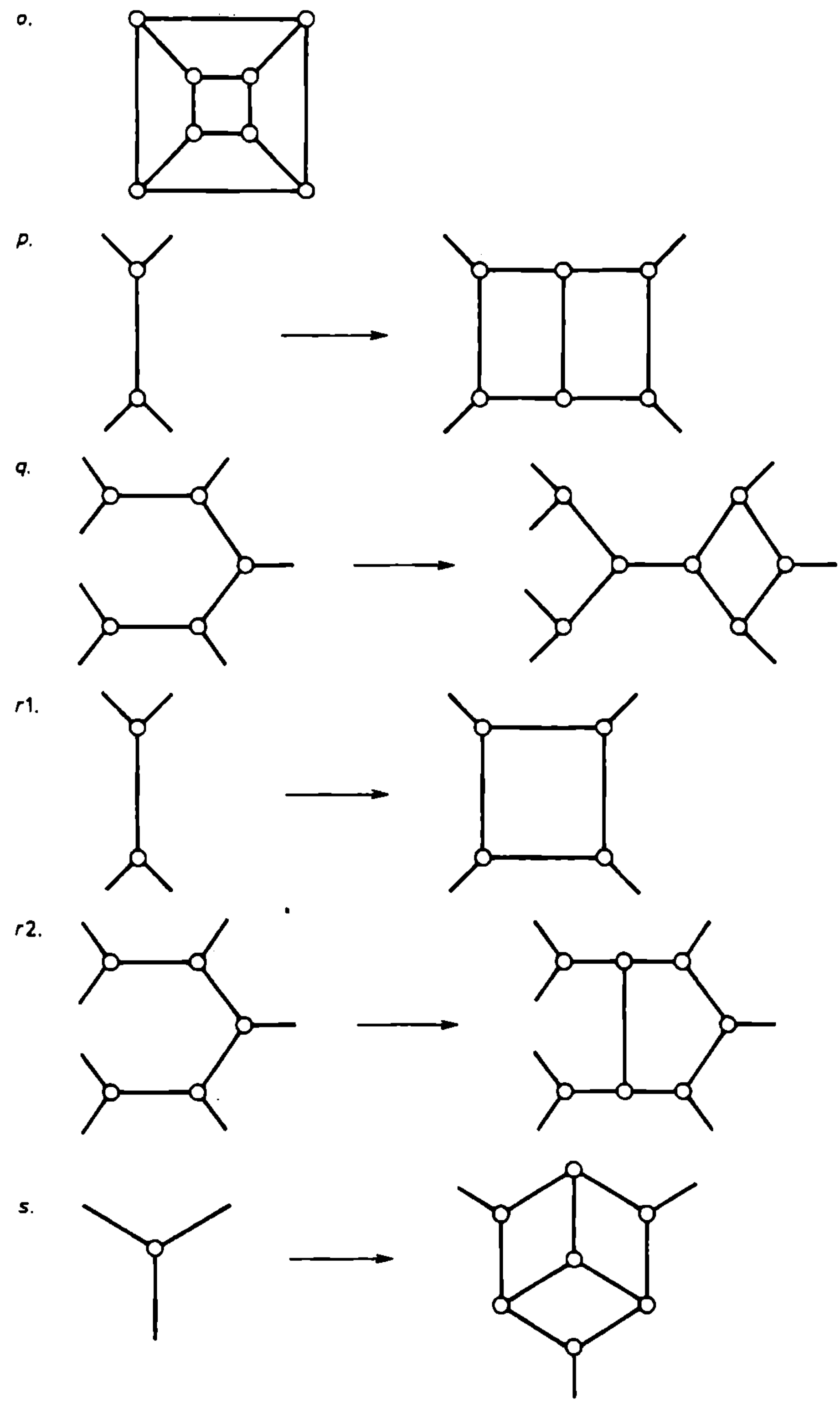

Fig. 11

contrary. Then the pair YU or the pair YW is an edge of G. Let us assume that YU is an edge. But then neither ZW nor ZV can be an edge of G — a contradiction.

So, because of symmetry, we can assume that YU and YW are not edges of G and therefore we can apply the rule R^- to the quadrangle $ZXVY$.

This completes the proof of Theorem 2.

In the dual form we can express Theorems 1 and 2 as follows (see Fig. 11).

THEOREM 1′. *The inductive class* $Cn(o;\, p,\, q)$ *is equal to the class* CBP3 *of all* 3*-connected bipartite planar cubic graphs.*

THEOREM 2′. *The inductive class* $Cn(o;\, r1, r2, s)$ *is equal to the class* CP3(>3) *of all* 3*-connected planar cubic graphs without triangles.*

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