POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCXXXIV

LECH MALIGRANDA

Indices and interpolation

WARSZAWA 1985
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-05818-8 ISSN 0012-3862

WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

0. Introduction

The purpose of this paper is to give a systematic exposition of several aspects of indices of functions (indices of submultiplicative and measurable functions), various indices of Orlicz functions and indices of rearrangement invariant spaces defined by Boyd and Zippin, and to provide various settings from interpolation theory in which indices play an important role. The class of Orlicz or symmetric spaces has been extensively studied in the literature in connection with interpolation properties (see e.g., Bennett [2], Boyd [4], [6], Calderón [9], Dmitriev and Krein [67], Dmitriev and Semenov [12], Fehér [14], [69], Gustavsson and Peetre [16], Krein, Petunin and Semenov [22], Krein and Semenov [23], Lindenstrauss and Tzafriri [28], Lorentz and Shimogaki [30], [32], Maligranda [34], [35], [36], [70], Pavlov [47], [71], Semenov [49], Sharpley [51], [52], [53], Shimogaki [54], [56], Simonenko [58], Torchinsky [60], Zippin [64], Zygmund [65]). The above authors have investigated the relationship between the interpolation property of Orlicz or symmetric spaces and the indices of these spaces.

Indices have also other numerous applications in analysis, namely:

to the study of Orlicz spaces (separability, reflexivity, local boundedness and local convexity if F is not convex, subspaces of Orlicz sequence and function spaces, compact operators between Orlicz spaces – see, e.g. [19], [25], [26], [27], [28], [45]).

to the study of symmetric spaces (subspaces and isomorphisms, [18], [28]; orthogonal series and Schauder basis, [22], pp. 233–249),

to the theory of integral operators on symmetric spaces (the Hardy-Littlewood maximal function, [22], pp. 191–193; convolution operator, [22], pp. 199–202; Hilbert transform, see [4], [22], pp. 203–211; conjugate operator, [22], pp. 231–233; averaging operators P_θ, Q_θ, [6], [22] or more generally P_φ, Q_φ – [35], [36]),

to the best approximation, see [1], [68],

to the theory of regularly varying functions and Frullani integrals, [39], [46], [50], [66],

to Pólya peaks, [13], [57],

to L. C. Young's series and the Riemann–Stieltjes integral, [24],

to packing problem in Orlicz spaces, [10],

to the Schur property of generalized Orlicz sequence spaces, [63].

There are various applications in probability theory, see e.g. [15], [37]. We note that indices can be found on investigating entire and meromorphic functions, convergence of Dirichlet series and Lyapunov's stability of solutions of differential equations.

The paper consists of six sections. The table of contents gives a general idea of the topics considered.

In Section 1 we give the definition of indices α, β and $\bar{\alpha}$, $\bar{\beta}$ of finite measurable submultiplicative or non-decreasing submultiplicative functions, and many examples of submultiplicative functions. We show that a finite concave submultiplicative function v such that $v(1) = 1$ has the form $v(s) = s^p$ for some p, $0 \leqslant p \leqslant 1$.

In Section 2 we define indices p_i, q_i and $\bar{p}_i$, $\bar{q}_i$ for $i = a, 0, \infty$ of a finite positive measurable function defined on $(0, \infty)$. We compute the above indices, for example for functions $\bar{\psi}(t) = t^a \psi(t^b)^c$ where $a, b, c \in R$, $\bar{\psi}(t) = \psi_1(t)\psi(\psi_2(t)/\psi_1(t))$ and the indices of the inverse function. We prove that the indices p_i, q_i are invariant with respect to some equivalence relation. This proof is due to Matuszewska and Orlicz [41]. Finally we show that p_∞, q_∞ and the Drasin-Shea indices μ_*, ϱ_* coincide.

In Section 3 we recall the Yamamuro, Matuszewska-Orlicz, Simonenko, and Lindenstrauss-Tzafriri indices of Orlicz functions and investigate the relations between them.

In Section 4 we define symmetric spaces in the sense of Semenov with the Fatou property of the norm i.e. the so-called rearrangement invariant spaces (r.i. spaces in short) and their indices. It turns out that the Boyd and Zippin indices are useful in interpolating operators, but they are not suitable for the characterization of separability and reflexivity of r.i. spaces.

Section 5 is concerned with interpolation theorems for operators of a weak type on r.i. spaces. Theorem 5.6 contains the result of D. W. Boyd [6] for Lebesgue spaces and fills a gap left by the papers of E. M. Semenov [49] and M. Zippin [64]. A proof of the sufficiency part of Theorem 5.6 and some of its generalizations are also to be found in [22] and [23]. Our main results are the necessity in Theorem 5.6, which is a confirmation of the conjecture of Zippin [64], and counterexamples showing that the necessary condition of theorem 5.6 cannot be replaced by the sufficient condition and vice versa (Examples 14 and 15).

Applying Theorems 5.5 and 5.6 we give a generalization of a certain result of Pavlov and Dmitriev and Semenov (Theorems 5.12 and 5.15, respectively).

In Section 6, the Boyd indices of Lorentz–Orlicz spaces $\Lambda(X, L_F)$ are given in the case where $F(u) = u^p$ or $\varphi_X(t) = t^{1/p}$ The general case remains open. We also give some remarks and problems concerning a generalization of the strong type. The indices and the interpolation of compact operators are presented in the final part.

1. Submultiplicative functions and indices

We say that the function $v: R_+ \to [0, \infty]$, where $R_+ = (0, \infty)$, is *submultiplicative* if $v \not\equiv 0$, $v \not\equiv \infty$ and

$$(1.1) \qquad v(s_1 \cdot s_2) \leqslant v(s_1) v(s_2) \quad \text{for all} \quad s_1, s_2 \in R_+$$

(where $0 \cdot \infty = \infty \cdot 0 = \infty$ by convention).

Since $v(1) = v(1 \cdot 1) \leqslant v(1) v(1)$, we have $v(1) \geqslant 1$ or $v(1) = 0$. If $v(s)$ is finite for each $s \in R_+$, then $v(s) > 0$ for all $s \in R_+$. Namely, if $v(s_0) = 0$ for some $s_0 \in R_+$, then we obtain a contradiction $1 \leqslant v(1) \leqslant v(s_0) v(1/s_0) = 0$ (the case $v(1) = 0$ gives $v(s) \leqslant v(1) v(s) = 0$ for all $s > 0$, i.e. $v \equiv 0$).

LEMMA 1.1. *If v is a non-decreasing submultiplicative function, then only two cases are possible*:

$$v(s) < \infty \textit{ for any } s > 1 \quad \textit{or} \quad v(s) = \infty \textit{ for any } s > 1.$$

Proof. If $v(s_0) < \infty$ for some $s_0 > 1$, then for all $s > 1$ we have $s \leqslant s_0^n$ for some $n \in N$, and from the monotonicity and submultiplicativity of v we get $v(s) \leqslant v(s_0^n) \leqslant v(s_0)^n < \infty$. If $v(s_0) = \infty$ for some $s_0 > 1$ then for all $s > 1$ we have $s_0 \leqslant s^n$ and $\infty = v(s_0) \leqslant v(s^n) \leqslant v(s)^n$.

THEOREM 1.2. (see Hille and Phillips [17], pp. 241, 244 and 250, the additive version) *Let v be a finite measurable and submultiplicative function. Then*

(i) *v is bounded in any interval $[a, b] \subset R_+$*;

(ii) *there exist numbers (indices) α and β such that $-\infty < \alpha \leqslant \beta < +\infty$ and*

(1.2) *for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$s^\alpha \leqslant v(s) \leqslant s^{\alpha - \varepsilon} \textit{ for } 0 < s < \delta \quad \textit{and}$$

$$s^\beta \leqslant v(s) \leqslant s^{\beta + \varepsilon} \textit{ for } s > 1/\delta;$$

(iii) *we have*

$$(1.3) \qquad \begin{aligned} \alpha = \alpha(v) &= \sup_{0<s<1} \frac{\ln v(s)}{\ln s} = \lim_{s \to 0^+} \frac{\ln v(s)}{\ln s}, \\ \beta = \beta(v) &= \inf_{s>1} \frac{\ln v(s)}{\ln s} = \lim_{s \to \infty} \frac{\ln v(s)}{\ln s}; \end{aligned}$$

(iv) *if $\bar{\alpha} = \bar{\alpha}(v) = \inf\limits_{0<s<1} \dfrac{\ln v(s)}{\ln s}$ and $\bar{\beta} = \bar{\beta}(v) = \sup\limits_{s>1} \dfrac{\ln v(s)}{\ln s}$ are finite, then* $\bar{\alpha} = \lim\limits_{s \to 1^-} \dfrac{\ln v(s)}{\ln s}$ *and* $\bar{\beta} = \lim\limits_{s \to 1^+} \dfrac{\ln v(s)}{\ln s}$.

Evidently, $\bar{\alpha} \leqslant \alpha \leqslant \beta \leqslant \bar{\beta}$ and for any $\varepsilon > 0$ there exists a $C \geqslant 1$ such that

$$\max(s^{\alpha}, s^{\beta}) \leqslant v(s) \leqslant C \max(s^{\alpha-\varepsilon}, s^{\beta+\varepsilon}) \tag{1.4}$$

for all $s \in R_+$. To obtain inequality in (1.4) it is enough to take for C the supremum of the function $v(s)/\max(s^{\alpha-\varepsilon}, s^{\beta+\varepsilon})$ on the interval $[\delta, 1/\delta]$.

Remark 1.3. (a) The assumptions of finiteness and measurability of v in Theorem 1.2 are necessary since for the submultiplicative measurable function $v_0(s) = 1$ for s rational, $= \infty$ for s irrational or for the multiplicative function $v_1(s) = \exp h(\ln s)$, where h is a non-measurable Hamel function, the limits do not exist. Moreover, v_0 and v_1 are unbounded on any interval.

(b) If v is a non-decreasing and submultiplicative function (not necessarily finite) such that $v(1) = 1$, then (1.3) holds and $0 \leqslant \alpha(v) \leqslant \beta(v) \leqslant \infty$ (cf. Matuszewska and Orlicz [42], pp. 12–13). As regards the meaning of formulae (1.3) we shall keep the convention $\ln 0 = -\infty$, $\ln \infty = \infty$, and the same conventions are adopted in analogous situations.

For any function v: $R_+ \to R_+$ we define the *order* ϱ and the *lower order* μ of v by

$$\varrho = \varrho(v) = \limsup_{s\to\infty} \frac{\ln v(s)}{\ln s}, \qquad \mu = \mu(v) = \liminf_{s\to\infty} \frac{\ln v(s)}{\ln s}.$$

If v is submultiplicative or v is a log-convex function, i.e., $v(s) = w(\ln s)$ where w: $R \to R_+$ is a convex function, then $\varrho = \mu = \beta$.

The definition of the order and the lower order is quite similar to the well-known definition of the order and the lower order of an entire function. Namely, if f is an entire function then $\varrho(f) = \varrho(v)$ and $\mu(f) = \mu(v)$, where $v(s) = \ln M_f(s)$ with $M_f(s) = \max_{|z| \leqslant s} |f(z)|$. If $w(s) = v(\ln s)$ then $\lambda(v) = \varrho(w)$ $= \limsup_{t\to\infty} \frac{\ln v(t)}{t}$ denotes Lyapunov's characteristic index of v used in the stability theory of differential equations.

The next theorem will be devoted to the construction of submultiplicative functions from given functions of a simple type.

PROPOSITION 1.4. (a) *If v and v_1, v_2 are three submultiplicative functions and if $a \geqslant 1$, $b \geqslant 1$, then av, $av_1 + bv_2$, $\max(av_1, bv_2)$, $av_1 v_2$ and $\bar{v}$ where $\bar{v}(s) = v(1/s)$ are also submultiplicative.*

(b) *If v_1 is a non-decreasing submultiplicative function and v_2 is a finite submultiplicative function, then the composite function $v_1 \circ v_2$ is submultiplicative.*

(c) *If u: $R \to R$ is a subadditive function, i.e., $u(s_1 + s_2) \leqslant u(s_1) + u(s_2)$ for all $s_1, s_2 \in R$, then $v_3(s) = \exp u(\ln s)$ is a finite and submultiplicative function.*

(d) *If u is a subadditive and non-negative function, and if $a \geqslant 1$, then $v_4(s) = a+u(\ln s)$ is a finite and submultiplicative function.*

Proof. The proof of submultiplicativity of functions is immediate from the definitions. Moreover, the following relations hold:

(a) $\alpha(av) = \alpha(v), \quad \beta(av) = \beta(v)$,

$$\alpha(\max(av_1, bv_2)) \leqslant \max(\alpha(v_1), \alpha(v_2)) \leqslant \max(\beta(v_1), \beta(v_2)) \leqslant \beta(\max(av_1, bv_2)),$$

$$\alpha(av_1 v_2) = \alpha(v_1)+\alpha(v_2), \quad \beta(av_1 v_2) = \beta(v_1)+\beta(v_2),$$

$$\alpha(\bar{v})+\beta(v) = \alpha(v)+\beta(\bar{v}) = 0;$$

(b) if additionally v_2 is a measurable function such that $v_2(0^+) = 0$ and $v_2(+\infty) = +\infty$, then

$$\alpha(v_1 \circ v_2) = \alpha(v_1)\alpha(v_2), \quad \beta(v_1 \circ v_2) = \beta(v_1)\beta(v_2);$$

(c) $\alpha(v_3) = \lim_{s\to-\infty} \frac{u(s)}{s}$, $\beta(v_3) = \lim_{s\to+\infty} \frac{u(s)}{s}$.

Analogous relations hold for indices $\bar{\alpha}$ and $\bar{\beta}$.

Remark 1.5. (a) If v is a submultiplicative function such that $v(s) = v(1/s)$ for all $s > 0$, then $v(s) \geqslant 1$ for all $s > 0$. From this $\alpha(v)+\beta(v) = 0$ and $\alpha(v) \leqslant 0 \leqslant \beta(v)$.

(b) Since $u_1(s) = [s]+a(a \geqslant 1)$, $u_2(s) = -[s]$, $u_3(s) = s-[s]$, $u_4(s) = |\sin s|$ and $u_5(s) = 3+\sin s$ are subadditive functions, where $[s]$ denotes the integer part of s, it follows by Proposition 1.4 that we have some examples of submultiplicative functions.

We give other examples of submultiplicative functions and we compute the indices of these functions.

Example 1. For $-\infty < p_1 \leqslant p_2 \leqslant +\infty$ let

$$v(s) = \begin{cases} s^{p_1}, & 0 < s \leqslant 1, \\ s^{p_2}, & s > 1, \end{cases} \quad \text{with the convention} \quad s^{\infty} = \begin{cases} 0, & 0 < s < 1, \\ 1, & s = 1, \\ \infty, & s > 1. \end{cases}$$

Then v is a submultiplicative function and $\bar{\alpha}(v) = \alpha(v) = p_1 \leqslant p_2 = \beta(v) = \bar{\beta}(v)$. Moreover, if $-\infty < p_1 \leqslant p_2 \leqslant 0$ or $1 \leqslant p_1 \leqslant p_2 < \infty$ then v is a convex function.

Example 2. Let A be a subset of R_+. Define

$$v(s; A) = \begin{cases} a, & s \in A, \\ b, & s \notin A. \end{cases}$$

If $1 \leqslant a \leqslant b^2$ and $A \cdot A = \{s \cdot t \colon s \in A, t \in A\} \subset A$ or if $1 \leqslant a \leqslant b^2 \leqslant a^4$, then v is a submultiplicative function. Moreover, $\alpha(v) = \beta(v) = 0$ and if $a \neq b$ then

$\bar{\alpha}(v) = -\infty$, $\bar{\beta}(v) = +\infty$. In particular, if A is the set of rational numbers and $a \neq b$, then $v(\ ;A)$ is a two-valued measurable submultiplicative function which is discontinuous for all $s > 0$.

EXAMPLE 3. For any fixed $a \geqslant 1$ let $v_1(s) = a+[s]$, $v_2(s) = a+s$, $v_3(s) = a+|\ln s|$ and $v_4(s) = a+|\sin \ln s|$. Then v_i are submultiplicative functions such that

$$v_1(R_+) = \{a, a+1, a+2, \ldots\}, \quad v_2(R_+) = (a, \infty), \quad v_3(R_+) = [a, \infty), \quad v_4(R_+) = [a, a+1]$$

and

$$\alpha(v_i) = \beta(v_j) = 0 \quad (i = 1, 2, 3, 4,\ j = 3, 4), \quad \beta(v_1) = \beta(v_2) = 1.$$

EXAMPLE 4. Let $p \geqslant 1$. Define

$$v(s) = s^p(1+|\ln s|);$$

v is an increasing continuous submultiplicative function such that $v(1) = 1$, $v(R_+) = R_+$ and $\bar{\alpha}(v) = p-1 < p = \alpha(v) = \beta(v) = p < p+1 = \bar{\beta}(v)$. The function v is a regularly varying function at infinity in the sense of Karamata with the index p, i.e. $\lim\limits_{t\to\infty} \dfrac{v(ts)}{v(t)} = s^p$ for each $s > 0$ (for further information on regularly varying functions — see [50]).

Moreover, if $p \geqslant (3+\sqrt{5})/2$ then v is a convex function (in 1931, Birnbaum and Orlicz [3] studied a class of N'-functions and gave (p. 30) this example as a submultiplicative N'-function. Krasnoselski and Ruticki [21] defined an N-function as a convex N'-function and asserted (pp. 15, 27, 33) that v for $p > 1$ is an N-function. This is false since v is no convex on (0, 1) for $p < (3+\sqrt{5})/2$. A similar misunderstanding can be found in [27], p. 146 and [28], p. 210).

EXAMPLE 5. Let $p \geqslant 1$. Define

$$v(s) = s^p(1+|\sin \ln s|).$$

v is a continuous increasing submultiplicative function such that $v(e^{2\pi n}s)/v(e^{2\pi n}) = v(s)$ for any $n = 0, \pm 1, \pm 2,$ and any $s > 0$. Hence, $\limsup\limits_{t\to\infty} v(ts)/v(t) = v(s)$ and v is not a regularly varying function. Moreover,

$$\bar{\alpha}(v) = p-1 < p = \alpha(v) = \beta(v) = p < p+1 = \bar{\beta}(v).$$

THEOREM 1.6. *Let v be a finite submultiplicative function such that $v(1) = 1$.*

(a) *If $v'(1)$ exists, then $v(s) = s^p$ for some $p \in R$.*

(b) *If v is a concave function, then $v(s) = s^p$ for some $p \in [0, 1]$.*

Proof. (a) We show that v must be differentiable at each point s of R_+ and that $v'(s) = v'(1)v(s)/s$.

Using (1.1), we see that, for small t,

$$v(s+t)-v(s) \leqslant v(s)\left[v\left(1+\frac{t}{s}\right)-1\right],$$

$$v(s)\left[\frac{1}{v(s/(s+t))}-1\right] \leqslant v(s+t)-v(s).$$

Thus for small, positive t,

$$\frac{v(s)}{s+t}\cdot\frac{v\left(1-\frac{t}{s+t}\right)-1}{-\frac{t}{s+t}}\cdot\frac{1}{v\left(\frac{s}{s+t}\right)} \leqslant \frac{v(s+t)-v(s)}{t} \leqslant \frac{v(s)}{s}\cdot\frac{v\left(1+\frac{t}{s}\right)-1}{\frac{t}{s}}$$

and the reverse inequalities hold for small, negative t. Hence $v'(s)$ exists and equals $v'(1)\frac{v(s)}{s}$ for each s in R_+. We put $p = v'(1)$. It follows that

$$\frac{d}{ds}[s^{-p}v(s)] = s^{-p}\left[v'(s)-p\frac{v(s)}{s}\right] = 0$$

for every s in R_+, and so $v(s) = Cs^p$ for some constant C. Since $v(1) = 1$, we have $C = v(1) = 1$, and the theorem is proved.

(b) Since v is a concave function, it follows that v is absolutely continuous on any $[a, b] \subset R_+$ and v has, at every point, a finite right and left derivative v'_+, v'_- such that $v'_+(s) \leqslant v'_-(s)$ (cf. [21] — for a convex function).

We prove that $v'(1)$ exists.

It is sufficient to prove that $v'_-(1) \leqslant v'_+(1)$. Since $[v(1+s)-v(1)]/s \to v'_+(1)$ and $\left[v(1)-v\left(1-\frac{s}{1+s}\right)\right]/s \to v'_-(1)$ as $s \to 0^+$, it follows that $v(1+s) = 1+v'_+(1)s+o(s)$ and $v\left(\frac{1}{1+s}\right) = 1-v'_-(1)s+o(s)$ as $s \to 0^+$ From the assumptions on v and the above we have

$$1 = v(1) \leqslant v(1+s)v(1/(1+s)) = 1+[v'_+(1)-v'_-(1)]s+o(s).$$

Hence, $v'_-(1) \leqslant v'_+(1)$, i.e., $v'(1)$ exists. The assertion (b) follows from (a).

We observe, that the examples $v(s) = \max(s^{1/3}, s^{1/2})$ and $v(s) = 1+s$ show that the assumptions of concavity of v on R_+ and $v(1) = 1$, and the assumption about the differentiability of v at the point $s_0 = 1$ are important.

2. Indices of measurable functions

For any measurable function $\psi\colon R_+ \to R_+$ we define submultiplicative functions M, M_0, M_∞ by

$$M(s) = M^l(s, \psi) = \sup_{t\in I, ts\in I} \frac{\psi(ts)}{\psi(t)} = \sup_{t\in(0,\min(1,1/s)l)} \frac{\psi(ts)}{\psi(t)},$$

where $I = (0, l)$, $0 < l \leqslant \infty$,

(2.1)

$$M_0(s) = M_0(s, \psi) = \limsup_{t\to 0^+} \frac{\psi(ts)}{\psi(t)},$$

$$M_\infty(s) = M_\infty(s, \psi) = \limsup_{t\to\infty} \frac{\psi(ts)}{\psi(t)}.$$

If M, M_0 and M_∞ are measurable finite or non-decreasing functions, then by Theorem 1.2 and Remark 1.3. (b) it is possible to define indices of the function ψ:

(2.2)
$$p(\psi) = p^l(\psi) = \alpha(M^l(\cdot, \psi)), \quad q(\psi) = q^l(\psi) = \beta(M^l(\cdot, \psi)),$$
$$p_i(\psi) = \alpha(M_i(\cdot, \psi)), \quad q_i(\psi) = \beta(M_i(\cdot, \psi)), \quad i = 0, \infty.$$

The indices $\bar{p}$, $\bar{q}$, $\bar{p}_i$, $\bar{q}_i$ we define as above, but replacing α, β by $\bar{\alpha}$, $\bar{\beta}$ respectively.

In the sequel the symbols ψ, ψ_1, ψ_2 denote measurable positive functions on R_+ for which the M (M_0 or M_∞, respectively) is either a measurable and finite or a non-decreasing function.

Remark 2.1. (a) The measurability of ψ does not imply the measurability of M, M_0 and M_∞ (cf. Rubel [48]).

(b) If ψ is a submultiplicative function and $\psi(1) = 1$, then

$$M^\infty(s, \psi) = \psi(s).$$

It is easy to notice that if, for some $a \geqslant 0$, $t^{-a}\psi(t)$ is a non-decreasing function then $p(\psi) \geqslant a$, but if $t^{-a}\psi(t)$ is a non-increasing function then $q(\psi) \leqslant a$. Moreover, as

$$M_0(s, \psi) \leqslant M^l(s, \psi) \leqslant M^\infty(s, \psi) \quad \text{and} \quad M_\infty(s, \psi) \leqslant M^\infty(s, \psi),$$

we have

(2.3)
$$p^\infty(\psi) \leqslant p^l(\psi) \leqslant p_0(\psi) \leqslant q_0(\psi) \leqslant q^l(\psi) \leqslant q^\infty(\psi),$$
$$p^\infty(\psi) \leqslant p_\infty(\psi) \leqslant q_\infty(\psi) \leqslant q^\infty(\psi).$$

THEOREM 2.2 (Boyd [8]). *If $l < \infty$, then $p^l(\psi) = p_0(\psi)$.*

Proof. It is sufficient to prove the inequality

$$M_0(s, \psi) \geqslant s^{p^l(\psi)} \quad \text{for} \quad 0 < s < 1.$$

Define $A(t, s) = \psi(ts)/\psi(t)$. Let m be a positive integer, and let s be fixed and not greater than 1. Then there is a $t_m < l$ such that

$$M^l(s^m, \psi) \leqslant A(t_m, s^m)/s.$$

Using (1.4), we have

$$A(t_m, s^m) \geqslant s^{mp^l(\psi)+1}$$

If n_m is such that

$$A(t_{2m} s^{n_m}, s) = \max\{A(t_{2m} s^n, s)\colon m \leqslant n \leqslant 2m-1\},$$

one has

$$A(t_{2m} s^{n_m}, s)^m \geqslant \prod_{n=m}^{2m-1} A(t_{2m} s^n, s) = \frac{A(t_{2m}, s^{2m})}{A(t_{2m}, s^m)}$$
$$\geqslant \frac{s^{2mp^l(\psi)+1}}{Cs^{mp^l(\psi)-\varepsilon(s)}} = C^{-1} s^{mp^l(\psi)+1+\varepsilon(s)},$$

and thus

$$A(t_{2m} s^{n_m}, s) \geqslant C^{-1/n} s^{p^l(\psi)+1/m+\varepsilon(s)/m}.$$

But

$$t_{2m} s^{n_m} \leqslant ls^{n_m} \leqslant ls^m \to 0 \quad \text{as} \quad m \to \infty.$$

Hence

$$M_0(s, \psi) \geqslant \limsup_{m\to\infty} A(t_{2m} s^{n_m}, s) \geqslant s^{p^l(\psi)}.$$

Remark 2.3. (a) If ψ is defined on $I = (0, l)$, then we can consider only the indices p^l, q^l and p_0, q_0.

(b) We shall denote $p^\infty(\psi)$ and $q^\infty(\psi)$ also by $p_a(\psi)$ and $q_a(\psi)$, respectively.

Example 6. Let $\psi(t) = t^a \psi_1(t^b)^c$, where $a, b, c \in R^1$ Then

$$M^\infty(s, \psi) = s^a M^\infty(s^{b\operatorname{sgn} c}, \psi_1)^{|c|},$$

$$M_0(s, \psi) = \begin{cases} s^a M_0(s^{b\operatorname{sgn} c}, \psi_1)^{|c|} & \text{if} \quad b \geqslant 0, \\ s^a M_\infty(s^{b\operatorname{sgn} c}, \psi_1)^{|c|} & \text{if} \quad b < 0, \end{cases}$$

$$M_\infty(s, \psi) = \begin{cases} s^a M_\infty(s^{b\operatorname{sgn} c}, \psi_1)^{|c|} & \text{if} \quad b \geqslant 0, \\ s^a M_0(s^{b\operatorname{sgn} c}, \psi_1)^{|c|} & \text{if} \quad b < 0. \end{cases}$$

Hence

$$p^{\infty}(\psi) = \begin{cases} a+bc\,p^{\infty}(\psi_1) & \text{if} \quad bc \geqslant 0, \\ a+bc\,q^{\infty}(\psi_1) & \text{if} \quad bc < 0, \end{cases}$$

$$q^{\infty}(\psi) = \begin{cases} a+bc\,q^{\infty}(\psi_1) & \text{if} \quad bc \geqslant 0, \\ a+bc\,p^{\infty}(\psi_1) & \text{if} \quad bc < 0, \end{cases}$$

and in each of the four cases: $b \geqslant 0$ and $c \geqslant 0$, $b \geqslant 0$ and $c < 0$, $b < 0$ and $c \geqslant 0$, $b < 0$ and $c < 0$, we have respectively

$$p_0(\psi) = \begin{cases} a+bc\,p_0(\psi_1), \\ a+bc\,q_0(\psi_1), \\ a+bc\,q_{\infty}(\psi_1), \\ a+bc\,p_{\infty}(\psi_1), \end{cases} \qquad q_0(\psi) = \begin{cases} a+bc\,q_0(\psi_1), \\ a+bc\,p_0(\psi_1), \\ a+bc\,p_{\infty}(\psi_1), \\ a+bc\,q_{\infty}(\psi_1), \end{cases}$$

$$p_{\infty}(\psi) = \begin{cases} a+bc\,p_{\infty}(\psi_1), \\ a+bc\,q_{\infty}(\psi_1), \\ a+bc\,q_0(\psi_1), \\ a+bc\,p_0(\psi_1), \end{cases} \qquad q_{\infty}(\psi) = \begin{cases} a+bc\,q_{\infty}(\psi_1), \\ a+bc\,p_{\infty}(\psi_1), \\ a+bc\,p_0(\psi_1), \\ a+bc\,q_0(\psi_1). \end{cases}$$

EXAMPLE 7. Let $\psi(t) = t^a \psi_1(t^b)^c$, where $a, b, c \in R^1$ and $\psi_1(t) = \log(1+t)$. Since

$$p^{\infty}(\psi_1) = p_{\infty}(\psi_1) = q_{\infty}(\psi_1) = 0,$$

$$q^{\infty}(\psi_1) = p_0(\psi_1) = q_0(\psi_1) = 1,$$

we have

$$p^{\infty}(\psi) = \min(a, a+bc), \; q^{\infty}(\psi) = \max(a, a+bc),$$

$$p_0(\psi) = q_0(\psi) = \begin{cases} a+bc & \text{if} \quad b \geqslant 0, \\ a & \text{if} \quad b < 0, \end{cases}$$

$$p_{\infty}(\psi) = q_{\infty}(\psi) = \begin{cases} a & \text{if} \quad b \geqslant 0, \\ a+bc & \text{if} \quad b < 0. \end{cases}$$

EXAMPLE 8. Let $\psi_n(t) = e^t - \sum_{k=0}^{n-1} \frac{t^k}{k!}$, $n = 1, 2, \ldots$ Then

$$M^{\infty}(s, \psi_n) = \begin{cases} s^n & \text{if} \quad 0 < s \leqslant 1, \\ \infty & \text{if} \quad s > 1, \end{cases}$$

$$M_0(s, \psi_n) = s^n, \; M_{\infty}(s, \psi_n) = \begin{cases} 0 & \text{if} \quad 0 < s < 1, \\ \infty & \text{if} \quad s > 1, \end{cases}$$

and so

$$p^{\infty}(\psi_n) = n, \quad q^{\infty}(\psi_n) = \infty,$$
$$p_0(\psi_n) = q_0(\psi_n) = n, \quad p_{\infty}(\psi_n) = q_{\infty}(\psi_n) = \infty.$$

Positive functions $\psi_1, \psi_2\colon R_+ \to R_+$ are called *equivalent* and this equivalence is denoted by $\psi_1(t) \approx \psi_2(t)$ if there exist positive numbers C_1 and C_2 such that

$$C_1\psi_1(t) \leqslant \psi_2(t) \leqslant C_2\psi_1(t) \quad \forall t \in R_+. \tag{2.4}$$

Let $\tilde{\psi}$ denote the smallest concave majorant of the function ψ, e.g.,

$$\tilde{\psi}(t) = \inf\{\bar{\psi}\colon \bar{\psi} \text{ is concave and } \psi(t) \leqslant \bar{\psi}(t)\ \forall t \in R_+\}.$$

THEOREM 2.4. (a) *If a positive non-decreasing function ψ is such that $\psi(t)/t$ is a non-increasing function, then*

$$\tfrac{1}{2}\tilde{\psi}(t) \leqslant \psi(t) \leqslant \tilde{\psi}(t) \quad \forall t \in R_+.$$

(b) *If $0 < p_a(\psi) \leqslant q_a(\psi) < 1$ then $\psi(t) \approx \tilde{\psi}(t)$.*

(c) *If $0 < p_a(\psi) \leqslant q_a(\psi) < \infty$ then*

$$\int_0^t \frac{\psi(s)}{s}\,ds \approx \psi(t).$$

(d) *If $-\infty < p_a(\psi) \leqslant q_a(\psi) < 0$ then*

$$\int_t^{\infty} \frac{\psi(s)}{s}\,ds \approx \psi(t).$$

Proof of Theorem 2.4 see [22] pp. 68–81. Theorem 2.4 (c), (d) is due to Bari and Stečkin [1].

The symbol $\psi_1 \overset{a}{\sim} \psi_2$ ($\psi_1 \overset{0}{\sim} \psi_2$ or $\psi_1 \overset{\infty}{\sim} \psi_2$) will mean that there exist positive constants C_1, C_2, C_3, C_4 (t_0 or t_∞) such that

$$C_1\psi_1(C_2t) \leqslant \psi_2(t) \leqslant C_3\psi_1(C_4t)$$

for all $t > 0$ ($0 < t \leqslant t_0$ or $t \geqslant t_\infty$, respectively).

THEOREM 2.5 (Matuszewska and Orlicz [41]). *If $\psi_1 \overset{i}{\sim} \psi_2$ then $p_i(\psi_1) = p_i(\psi_2)$ and $q_i(\psi_1) = q_i(\psi_2)$ for $i = a, 0, \infty$.*

Proof. For example we shall prove for $i = \infty$ that if $\psi_1 \overset{\infty}{\sim} \psi_2$ then for $t \geqslant \max(t_\infty, 1/s)$

$$\frac{C_1\psi_1((C_2/C_4)sC_4t)}{C_3\psi_1(C_4t)} \leqslant \frac{\psi_2(st)}{\psi_2(t)} \leqslant \frac{C_3\psi_1((C_4/C_2)sC_2t)}{C_1\psi_1(C_2t)},$$

i.e.,

$$\frac{C_1}{C_3} M_\infty\left(\frac{C_2}{C_4} s, \psi_1\right) \leqslant M_\infty(s, \psi_2) \leqslant \frac{C_3}{C_1} M_\infty\left(\frac{C_4}{C_2} s, \psi_1\right),$$

and so

$$p_\infty(\psi_1) = p_\infty(\psi_2) \quad \text{and} \quad q_\infty(\psi_1) = q_\infty(\psi_2).$$

Let us take note of some differences between the properties of the indices p_i, q_i and those of $\bar{p}_i$, $\bar{q}_i$. Namely $\bar{p}_i$, $\bar{q}_i$ indices are not invariants with respect to $\overset{i}{\sim}$ equivalence.

The converse of Theorem 2.5 is not true in general.

EXAMPLE 9. Let

$$\psi_1(t) = \begin{cases} t^p, & 0 < t \leqslant 1, \\ t^p \ln(1+t)/\ln 2, & t > 1, \end{cases}$$

$$\psi_2(t) = \begin{cases} t^p \ln\dfrac{e}{t}, & 0 < t \leqslant 1, \\ t^p \ln^2(1+t)/\ln^2 2, & t > 1. \end{cases}$$

Then ψ_1 and ψ_2 do not satisfy $\psi_1 \overset{0}{\sim} \psi_2$, $\psi_1 \overset{\infty}{\sim} \psi_2$, and all indices are equal to p.

From Theorem 2.5 it follows that if ψ_1 is a non-decreasing function and $\psi(t) = \int_0^t \psi_1(s)\,ds$ then $p^\infty(\psi) = 1 + p^\infty(\psi_1)$, $q^\infty(\psi) = 1 + q^\infty(\psi_1)$. For other indices it is sufficient that ψ_1 is a non-decreasing function near zero or infinity.

THEOREM 2.6. *Let ψ be an increasing continuous function. We denote by ψ^{-1} the inverse function of ψ.*

(a) *If $\psi(R_+) = R_+$ then $p^\infty(\psi^{-1}) = q^\infty(\psi)^{-1}$.*

(b) *If $\psi(0^+) = \lim_{t\to 0^+} \psi(t) = 0$ then $p_0(\psi^{-1}) = q_0(\psi)^{-1}$*

(c) *If $\psi(\infty) = \lim_{t\to\infty} \psi(t) = \infty$ then $p_\infty(\psi^{-1}) = q_\infty(\psi)^{-1}$*

As regards the meaning of the above formulas we shall keep the conventions $1/0 = \infty$, $1/\infty = 0$, and the same conventions are tacitly adopted in analogous situations.

Proof. (for another proof in the case $i = 0$ see Matuszewska [38]). For example we shall prove (a). We prove the other cases in the same manner.

Let $q^\infty(\psi) < \infty$. Then by (1.2) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{\psi(ts)}{\psi(t)} \leqslant M^\infty(s, \psi) \leqslant s^{q^\infty(\psi)+\varepsilon}$$

for all $t \in R_+$ and $s > 1/\delta$, and hence

$$ts \leqslant \psi^{-1}\left(s^{q^\infty(\psi)+\varepsilon}\psi(t)\right).$$

Taking $t = \psi^{-1}(u)$, we can write

$$\psi^{-1}(u)s \leqslant \psi^{-1}(s^{q^\infty(\psi)+\varepsilon}u),$$

i.e.,

$$\frac{\psi^{-1}(s_1 v)}{\psi^{-1}(v)} \leqslant s_1^{1/(q^\infty(\psi)+\varepsilon)}, \quad \text{where} \quad s_1 = s^{-1}$$

Thus (the assumption $\psi(R_+) = R_+$ implies that $u \in R_+$)

$$M^\infty(s_1, \psi^{-1}) \leqslant s_1^{(q^\infty(\psi)+\varepsilon)^{-1}}$$

and hence

$$p^\infty(\psi^{-1}) \geqslant (q^\infty(\psi)+\varepsilon)^{-1}.$$

If $q^\infty(\psi) = 0$ then the arbitrariness of ε implies $p^\infty(\psi^{-1}) = 0$; however, if $q^\infty(\psi) > 0$ then $p^\infty(\psi^{-1}) \geqslant q^\infty(\psi)^{-1}$. If $p^\infty(\psi^{-1}) = (q^\infty(\psi)-\lambda)^{-1}$ for some $\lambda > 0$ then we proceed in the same way as above and we get

$$M^\infty(s, \psi) \leqslant s^{[(q^\infty(\psi)-\lambda)^{-1}-\varepsilon]^{-1}} \qquad \forall s > 1/\delta,$$

or

$$q^\infty(\psi) \leqslant [(q^\infty(\psi)-\lambda)^{-1}-\varepsilon]^{-1}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain a contradiction

$$q^\infty(\psi) \leqslant q^\infty(\psi)-\lambda.$$

Therefore $p^\infty(\psi^{-1}) = q^\infty(\psi)^{-1}$.

Let $q^\infty(\psi) = \infty$; then $M^\infty(s, \psi) = \infty$ for any $s > 1$. Hence there exist $t_n > 0$ such that

$$\frac{\psi(t_n s)}{\psi(t_n)} \geqslant s^n \qquad \forall s > 1,\ n = 1, 2,$$

Taking $\psi(t_n) = u_n$, we can write

$$\frac{\psi^{-1}(s_1 u_n)}{\psi^{-1}(u_n)} \geqslant s_1^{1/n}, \quad \text{where} \quad s_1 = s^{-1},$$

i.e.,

$$M^\infty(s_1, \psi^{-1}) \geqslant s_1^{1/n}.$$

Hence

$$p^\infty(\psi^{-1}) \leqslant 1/n.$$

Since n is arbitrary, we have $p^\infty(\psi^{-1}) = 0$.

PROPOSITION 2.7. *Let* $\bar{\psi}(t) = \psi_1(t)\psi(\psi_2(t)/\psi_1(t))$, *where* ψ *is a non-decreasing function. We denote by* p, q *the indices* p_a, q_a *or* p_0, q_0 *or* p_∞, q_∞, *respectively.*

(a) *If* $p(\psi_2/\psi_1) > 0$, *then* $p(\bar{\psi}) \geq p(\psi_1) + p(\psi)\,p(\psi_2/\psi_1)$.
If $q(\psi_2/\psi_1) > 0$, *then* $q(\bar{\psi}) \leq q(\psi_1) + q(\psi)\,q(\psi_2/\psi_1)$.

(b) *Let* $u^{-1}\psi(u)$ *be a non-increasing function on* R_+.
If $p(\psi) > 0$ *and* $p(\psi_1) < p(\psi_2)$ *then* $p(\bar{\psi}) \geq [1-p(\psi)]\,p(\psi_1) + p(\psi)\,p(\psi_2)$.
If $q(\psi) > 0$ *and* $q(\psi_1) < q(\psi_2)$ *then* $q(\bar{\psi}) \leq [1-q(\psi)]\,q(\psi_1) + q(\psi)\,q(\psi_2)$.

Proof. (a) It is not hard to verify that

$$M(s, \bar{\psi}) \leq M(s, \psi_1)\,M(M(s, \psi_2/\psi_1), \psi) \tag{2.5}$$

and analogously for M_0 and M_∞. From (2.6) and (1.4) we have the theorem.

(b) The proof, for example for p_a and q_a, follows from the inequalities (1.4) and the inequality

$$M(s, \bar{\psi}) \leq M(s, \psi_1)\,M\big(M(s, \psi_2)/M(s, \psi_1), \psi\big). \tag{2.6}$$

The theory of Pólya peaks (Drasin and Shea [13]; see also Silverman [57]) may be interpreted in terms of indices. We consider a positive function f on R_+ such that

$$t_1^{-a} f(t_1) \leq C t_2^{-a} f(t_2) \quad (0 < t_1 \leq t_2) \tag{2.7}$$

holds for some $a > -\infty$ and $C > 0$.

Define the indices (see Drasin and Shea [13])

$$\begin{aligned} \mu_* &= \mu_*(f) = \inf\{p\colon \liminf_{t,s\to\infty} s^{-p} f(st)/f(t) = 0\}, \\ \varrho_* &= \varrho_*(f) = \sup\{p\colon \limsup_{t,s\to\infty} s^{-p} f(st)/f(t) = \infty\}. \end{aligned} \tag{2.8}$$

It is directly obvious from (2.8) that $a \leq \mu_* \leq \liminf_{t\to\infty} \ln f(t)/\ln t \leq \limsup_{t\to\infty} \ln f(t)/\ln t \leq \varrho_* \leq \infty$, where a is the constant in (2.7).

DEFINITION. Sequences $r_n, t_n \to \infty$ such that

$$f(sr_n)/f(r_n) \leq s^p(1+\varepsilon_n), \quad \text{where} \quad a_n^{-1} \leq s \leq a_n,$$
$$f(st_n)/f(t_n) \geq s^p(1-\varepsilon_n), \quad \text{where} \quad a_n^{-1} \leq s \leq a_n$$

hold for some $a_n \to \infty$, $\varepsilon_n \to 0$ are called *Pólya peaks of order* p, of the first and second kinds respectively, for f. Drasin and Shea proved that f has Pólya peaks of the first and the second kind of order $p(<\infty)$ if and only if $\mu_* \leq p \leq \varrho_*$.

THEOREM 2.8. *We have*

$$\mu_*(\psi) = p_\infty(\psi) \quad \text{and} \quad \varrho_*(\psi) = q_\infty(\psi).$$

Proof. If $q_\infty(\psi) < \infty$ then from (1.4) and from local uniformity, for every $b > q_\infty(\psi)$, there exist positive constants C, t_0 such that $\psi(st)/\psi(t) \leqslant Cs^b$, for $s \geqslant 1$ and $t \geqslant t_0$. This gives $\limsup_{t,s\to\infty} s^{-b}\psi(st)/\psi(t) \leqslant C < \infty$, i.e., $\varrho_*(\psi) \leqslant b$. Hence $\varrho_*(\psi) \leqslant q_\infty(\psi)$.

On the other hand, if $\varrho_*(\psi) < \infty$ then for every $b > \varrho_*(\psi)$ we have $\limsup_{t,s\to\infty} s^{-b}\psi(st)/\psi(t) < \infty$, whence $M_\infty(s, \psi) \leqslant Cs^b$ for $s \geqslant s_0$, and so $q_\infty(\psi) \leqslant b$. Thus, $q_\infty(\psi) \leqslant \varrho_*(\psi)$. Equivalence, $q_\infty(\psi) = \infty \Leftrightarrow \varrho_*(\psi) = \infty$, is evident. The proof that $\mu_*(\psi) = p_\infty(\psi)$ is similar.

3. Indices of Orlicz spaces

Let F be a convex *Orlicz function* on $[0, \infty)$, i.e., a finite continuous convex increasing function satisfying $F(0) = 0$ and $F(u) \to \infty$ as $u \to \infty$. The *Orlicz space* L_F is the space of all (equivalence classes of) measurable functions f on a possibly infinite interval $I = (0, l)$, so that

$$\int_0^l F(|f(t)|/r)\,dt < \infty$$

for some $r = r(f) > 0$. The *Luxemburg norm* in L_F is defined by

$$\|f\|_{L_F} = \inf\{r > 0: \int_0^l F(|f(t)|/r)\,dt \leqslant 1\}.$$

By $L_{(F)}$ we denote the Orlicz space with *Orlicz norm*

$$\|f\|_{L_{(F)}} = \sup\{\int_0^l |f(t)g(t)|\,dt: g \in L_{F^*} \quad \text{and } \|g\|_{L_{F^*}} \leqslant 1\},$$

where F^* is a complementary function to F.

If for elements we take sequences and write series instead of integrals, then we have *Orlicz sequence spaces* l_F and $l_{(F)}$. There is a close connection between L_F and $L_{(F)}$: they consist precisely of the same elements and

$$\|f\|_{L_F} \leqslant \|f\|_{L_{(F)}} \leqslant 2\|f\|_{L_F} \tag{3.1}$$

(for l_F and $l_{(F)}$ analogously).

The definitions of lower and upper indices of an Orlicz function (Orlicz space) are connected with the investigation of Orlicz space over infinite, finite or discrete measure spaces.

Yamamuro in 1953 considered indices of modular spaces and, in a particular case, indices p_F^a, q_F^a of an Orlicz function F. He also used these indices to study the Schur property of Musielak–Orlicz sequence spaces $l_{(F_n)}$

(see [63]). For an Orlicz function F define

$$p_F^a = \sup\{p > 0: \quad u^{-p}F(u) \text{ is non-decreasing for all } u > 0\},$$

$$q_F^a = \inf\{p > 0: \quad u^{-p}F(u) \text{ is non-increasing for all } u > 0\},$$

$$p_F^\infty = \sup\{p > 0: \quad u^{-p}F(u) \text{ is non-decreasing for large } u\},$$

$$q_F^\infty = \inf\{p > 0: \quad u^{-p}F(u) \text{ is non-increasing for large } u\},$$

$$p_F^0 = \sup\{p > 0: \quad u^{-p}F(u) \text{ is non-decreasing near zero}\},$$

$$q_F^0 = \inf\{p > 0: \quad u^{-p}F(u) \text{ is non-increasing near zero}\}.$$

Simonenko in 1964 defined explicitly indices a_F^a, b_F^a and a_F^∞, b_F^∞ and used them for interpolation and extrapolation in Orlicz spaces. Lindberg [25] considered p_F^0 and q_F^0 indices. Recall that if F is a convex Orlicz function then $F'(u)$ is a non-decreasing function defined for each $u \geqslant 0$ except a countable set of points in which we take $F'(u)$ as the derivative from the right. The above indices are defined by

$$a_F^a = \inf_{u>0}\frac{uF'(u)}{F(u)}, \qquad b_F^a = \sup_{u>0}\frac{uF'(u)}{F(u)},$$

$$a_F^\infty = \liminf_{u\to\infty}\frac{uF'(u)}{F(u)}, \qquad b_F^\infty = \limsup_{u\to\infty}\frac{uF'(u)}{F(u)},$$

$$a_F^0 = \liminf_{u\to 0^+}\frac{uF'(u)}{F(u)}, \qquad b_F^0 = \limsup_{u\to 0^+}\frac{uF'(u)}{F(u)}.$$

In 1960, Matuszewska and Orlicz studied indices s_F^a, σ_F^a and s_F^∞, σ_F^∞, and in 1965 the s_F^0, σ_F^0 indices of an Orlicz function F. The Matuszewska–Orlicz indices are topologically invariant and play a part in the theory of Orlicz spaces. They are also useful in various problems of analysis and probability theory. The definition of the Matuszewska–Orlicz indices is as follows:

$$s_F^a = \lim_{\lambda\to\infty}\left[\ln\left\{\inf_{u>0}\frac{F(\lambda u)}{F(u)}\right\}\Big/\ln\lambda\right] = p_a(F),$$

$$\sigma_F^a = \lim_{\lambda\to\infty}\left[\ln\left\{\sup_{u>0}\frac{F(\lambda u)}{F(u)}\right\}\Big/\ln\lambda\right] = q_a(F),$$

$$s_F^\infty = \lim_{\lambda\to\infty}\left[\ln\left\{\liminf_{u\to\infty}\frac{F(\lambda u)}{F(u)}\right\}\Big/\ln\lambda\right] = p_\infty(F),$$

$$\sigma_F^\infty = \lim_{\lambda\to\infty}\left[\ln\left\{\limsup_{u\to\infty}\frac{F(\lambda u)}{F(u)}\right\}\Big/\ln\lambda\right] = q_\infty(F),$$

$$s_F^0 = \lim_{\lambda\to\infty}\left[\ln\left\{\liminf_{u\to 0^+}\frac{F(\lambda u)}{F(u)}\right\}\Big/\ln\lambda\right] = p_0(F),$$

$$\sigma_F^0 = \lim_{\lambda\to\infty}\left[\ln\left\{\limsup_{u\to 0^+}\frac{F(\lambda u)}{F(u)}\right\}\Big/\ln\lambda\right] = q_0(F).$$

Lindenstrauss and Tzafriri proved in 1972 that the set of p's such that $l_p (1 \leqslant p < \infty)$ is isomorphic to a subspace of a separable l_F space coincides with the interval $[\alpha_F^0, \beta_F^0]$; Kalton [19] proved this for $p > 0$. The structure of Orlicz function spaces is naturally far more complicated than that of Orlicz sequence spaces. However, Lindenstrauss and Tzafriri proved that the set of p's such that $l_p(1 \leqslant p < \infty)$ is isomorphic to a subspace of separable $L_F(0, 1)$ space with $\alpha_F^\infty \geqslant 2$ coincides with the set $\{2\} \cup [\alpha_F^\infty, \beta_F^\infty]$. We note that l_F contains an infinite-dimensional locally bounded (Banach) subspace if and only if $\beta_F^0 > 0 (\beta_F^0 \geqslant 1)$ — see [19]. Define

$$\alpha_F^a = \sup\{p:\ \inf_{\substack{u>0\\ \lambda\geqslant 1}} \lambda^{-p}F(\lambda u)/F(u) > 0\},$$

$$\beta_F^a = \inf\{p:\ \sup_{\substack{u>0\\ \lambda\geqslant 1}} \lambda^{-p}F(\lambda u)/F(u) < \infty\},$$

$$\alpha_F^\infty = \sup\{p:\ \inf_{u,\lambda\geqslant 1} \lambda^{-p}F(\lambda u)/F(u) > 0\},$$

$$\beta_F^\infty = \inf\{p:\ \sup_{u,\lambda\geqslant 1} \lambda^{-p}F(\lambda u)/F(u) < \infty\},$$

$$\alpha_F^0 = \sup\{p:\ \inf_{\substack{0<u\leqslant 1\\ \lambda\geqslant 1}} \lambda^{-p}F(\lambda u)/F(u) > 0\},$$

$$\beta_F^0 = \inf\{p:\ \sup_{\substack{0<u\leqslant 1\\ \lambda\geqslant 1}} \lambda^{-p}F(\lambda u)/F(u) < \infty\}.$$

We shall be interested in the relations between those indices and the conditions on the finiteness of upper indices. Before proving the relations between the indices of Orlicz functions we give a definition and an auxiliary lemma.

We shall say that an Orlicz function F satisfies the Δ_2^a (Δ_2^∞ or Δ_2^0)-condition if there exists a constant $C > 1$ such that

$$F(2u) \leqslant CF(u) \tag{3.2}$$

for all $u > 0$ (large or near zero, respectively) and we shall write $F \in \Delta_2^a (F \in \Delta_2^\infty$ or $F \in \Delta_2^0$, respectively).

LEMMA 3.1. *If* $a_1 \leqslant uF'(u)/F(u) \leqslant a_2$ *for all* $u > 0$ ($u \geqslant u_\infty$ *or* $0 < u \leqslant u_0$), *then*

$$\min(s^{a_1}, s^{a_2}) \leqslant \frac{F(su)}{F(u)} \leqslant \max(s^{a_1}, s^{a_2}) \tag{3.3}$$

for all $u, s > 0$ $(u \geqslant u_\infty,\ su \geqslant u_\infty$ *or* $0 < u \leqslant u_0,\ 0 < su \leqslant u_0$, *respectively*).

Proof. Let $G(s) = sF'(s)/F(s)$. Then $F(s) = C\exp\int_0^s (G(t)/t)\,dt$. Let $u \geqslant u_\infty$ and $su \geqslant u_\infty$. We have

$$\frac{F(su)}{F(u)} = \exp\int_u^{su} \frac{G(t)}{t}\,dt.$$

Since $a_1 \leqslant G(t) \leqslant a_2$, for $t \geqslant u_\infty$, we have for $s > 1$

$$\ln s^{a_1} = \int_u^{su} \frac{a_1}{t}\,dt \leqslant \int_u^{su} \frac{G(t)}{t}\,dt \leqslant \int_u^{su} \frac{a_2}{t}\,dt = \ln s^{a_2}.$$

Hence

$$s^{a_1} \leqslant \frac{F(su)}{F(u)} \leqslant s^{a_2}.$$

We prove the other cases in the same manner.

THEOREM 3.2. (a) *We have*

$$[\alpha_F^i, \beta_F^i] = [s_F^i, \sigma_F^i] \subset [a_F^i, b_F^i] = [p_F^i, q_F^i] \subset [1, \infty],$$

for $i = a,\ \infty,\ 0$.

(b) $F \notin \Delta_2^i$ *iff* $\beta_F^i = \sigma_F^i = b_F^i = q_F^i = \infty$, *for* $i = a,\ \infty,\ 0$.

Proof. (a) For example we shall prove the theorem for $i = a$. First of all we prove $[\alpha_F^a, \beta_F^a] = [s_F^a, \sigma_F^a]$.

Since $C_1\lambda^{\alpha_F^a - \varepsilon} \leqslant F(\lambda u)/F(u) \leqslant C_2\lambda^{\beta_F^a + \varepsilon}\ \forall u > 0,\ \lambda \geqslant 1,\ \varepsilon > 0$, we have

$$\alpha_F^a \leqslant s_F^a \quad \text{and} \quad \sigma_F^a \leqslant \beta_F^a.$$

We assume that $s_F^a = \alpha_F^a + \delta$ and $\sigma_F^a = \beta_F^a - \delta < \infty$ for some $\delta > 0$. Then

$$-\delta/2 < \ln\inf_{u>0} \frac{F(\lambda u)}{F(u)}\Big/\ln\lambda - \alpha_F^a - \delta$$

and

$$\ln\sup_{u>0} \frac{F(\lambda u)}{F(u)}\Big/\ln\lambda - \beta_F^a + \delta < \delta/2$$

for large λ, i.e., $\alpha_F^a \geqslant \alpha_F^a + \delta/2$ and $\beta_F^a \leqslant \beta_F^a - \delta/2$, contradictions. Hence

$$\alpha_F^a = s_F^a \quad \text{and} \quad \beta_F^a = \sigma_F^a.$$

The first inclusion follows from Lemma 3.1; however, the second equality follows from the fact that

$$[u^{-p} F(u)]' = u^{-p-1} F(u)\left[\frac{uF'(u)}{F(u)} - p\right]. \tag{3.4}$$

(b) From (a) it is sufficient to prove that $F \in \Delta_2^i \Rightarrow q_F^i < \infty$ and $\beta_F^i < \infty \Rightarrow F \in \Delta_2^i$, for $i = a, \infty, 0$. For example, we shall prove this for $i = 0$. If $F(2u) \leqslant CF(u)$ for some constant $C > 1$ and for $0 < u \leqslant u_0$, then

$$uF'(u) \leqslant \int_u^{2u} F'(t)\,dt = F(2u) - F(u) \leqslant (C-1)F(u)$$

and so

$$[u^{1-C} F(u)]' = F(u)u^{-C}\left[\frac{uF'(u)}{F(u)} - C + 1\right] \leqslant 0,$$

i.e.,

$$q_F^0 \leqslant C - 1.$$

If $\beta_F^0 < \infty$, then

$$F(\lambda u)/F(u) \leqslant C\lambda^b \quad \text{for} \quad 0 < u \leqslant 1,\ \lambda \geqslant 1.$$

Hence, for $\lambda \geqslant 2$,

$$F(2u) \leqslant F(\lambda u) \leqslant C\lambda^b F(u)$$

for all $0 < u \leqslant 1$, i.e., $F \in \Delta_2^0$.

We shall give an example of an Orlicz function such that the inclusions in Theorem 3.2 (a) cannot be replaced by equalities.

Example 10. Let $F(0) = 0$ and $F(u) = u^p(1 + c\sin(p\ln u))$ for $u > 0$, where $0 < c < 1/2$ and $p \geqslant \left(1 - \frac{\sqrt{2}c}{\sqrt{1-2c^2}}\right)^{-1}$ It can easily be checked that this function is an Orlicz function on $[0, \infty)$. Since $F(u) \approx u^p$ and

$$\frac{uF'(u)}{F(u)} = p + cp\frac{\cos(p\ln u)}{1 + c\sin(p\ln u)},$$

we have

$$s_F^a = \sigma_F^a = p$$

and

$$a_F^a = p_F^a = p\left(1 - \frac{c}{\sqrt{1-c^2}}\right), \quad b_F^a = q_F^a = p\left(1 + \frac{c}{\sqrt{1-c^2}}\right).$$

We denote

$$A_F^i = \sup\{a_G^i\colon\ G \overset{i}{\sim} F\},$$
$$B_F^i = \inf\{b_G^i\colon\ G \overset{i}{\sim} F\},$$
$$P_F^i = \sup\{p_G^i\colon\ G \overset{i}{\sim} F\},$$
$$Q_F^i = \inf\{q_G^i\colon\ G \overset{i}{\sim} F\},$$

for $i = a,\ \infty,\ 0$.

THEOREM 3.3. *We have*

$$[P_F^i, Q_F^i] = [A_F^i, B_F^i] = [\alpha_F^i, \beta_F^i],$$

for $i = a, \infty, 0$.

Proof (The idea of the proof is taken from Bari and Stečkin [1] and Mazur and Orlicz [43] p. 107; Lindenstrauss and Tzafiri [26] proved this theorem in the case $i = 0$). By Theorem 2.5 and Theorem 3.2 we have

$$P_F^i \leqslant \alpha_F^i \quad \text{and} \quad \beta_F^i \leqslant Q_F^i,$$

for $i = a, \infty, 0$.

We prove the reverse inequalities:

($i = a$). Take $p < \alpha_F^a$ and put

$$g(x) = \begin{cases} 0 & \text{for } x = 0, \\ x^{-p}F(x) & \text{for } x > 0. \end{cases}$$

The function g is continuous on $[0, \infty)$ and it follows from the definition of α_F^a that $x^{-p}F(x) \leqslant Cy^{-p}F(y)$ for all $0 < x \leqslant y$. The function

$$P(t) = \sup_{0 \leqslant x \leqslant t} g(x)$$

is continuous on $[0, \infty)$ and

$$t^{-p}F(t) \leqslant P(t) \leqslant C^{-1}t^{-p}F(t) \qquad \forall t > 0.$$

Hence $G(u) = \int_0^u P(t)t^{p-1}\,dt$ is an Orlicz function and

$$F(u/2) \leqslant G(u) \leqslant C^{-1}F(u).$$

Moreover,

$$\frac{uG'(u)}{G(u)} \geqslant p, \quad \text{i.e.,} \quad p_F^a \geqslant p.$$

This proves that $\alpha_F^a = P_F^a$, and $\beta_F^a = Q_F^a$ is proved similarly.

($i = 0$). We prove this case in the same manner.

($i = \infty$). It is sufficient to take

$$g(x) = \begin{cases} F(1)x & \text{for} \quad 0 \leqslant x < 1, \\ x^{-p}F(x) & \text{for} \quad x \geqslant 1 \end{cases}$$

and

$$P(t) = \sup_{1 \leqslant x \leqslant t} g(x) \quad \text{on } [1, \infty).$$

Remark 3.4. The indices can also be defined for a non-convex Orlicz function.

4. Indices of rearrangement invariant spaces

Let $L^0(I, m)$ be the space of all equivalence classes of real valued Lebesgue measurable functions on $I = (0, l)$, $0 < l \leqslant \infty$. If $f \in L^0(I, m)$: then the distribution function of f is defined by

$$d_f(y) = m\{t \in I\colon |f(t)| > y\}, \qquad y > 0.$$

The non-increasing rearrangement of f onto $(0, \infty)$ is given by

$$f^*(t) = d_{d_f}(t) = m\{y > 0\colon d_f(y) > t\} = \sup\{y > 0\colon d_f(y) > t\}$$
$$= \inf\{y > 0\colon d_f(y) \leqslant t\}.$$

A Banach subspace X of $L^0(I, m)$ is called a *Banach function space* on I if

(4.1) $|g| \leqslant |f|$ a.e., $g \in L^0(I, m), f \in X \Rightarrow g \in X$ and $\|g\|_X \leqslant \|f\|_X$;

(4.2) $\{f_n\}_{n=1}^{\infty} \subset X$ such that $0 \leqslant f_n \uparrow f$ a.e., then either $\|f_n\|_X \uparrow \|f\|_X$ or $\|f_n\|_X \uparrow \infty$.

Property (4.2) is called the *Fatou property*.

The Banach function space on I is called a *rearrangement invariant* (*r.i.*) *space* or a symmetric space in the terminology of Semenov with the Fatou property of the norm if

(4.3) for any $g \in L^0(I, m)$ equimeasurable to $f \in X$, that is $d_g = d_f$, one has $g \in X$ and $\|g\|_X = \|f\|_X$.

Since $d_f = d_{f^*}$, we have for an r.i. space

$$\|f\|_X = \|f^*\|_X.$$

Examples of r.i. spaces include the Lebesgue L_p-spaces, the Orlicz L_F-spaces and the Lorentz spaces Λ, M and L_{pq}. Also if X, Y are r.i. spaces, so are $X \cap Y$, $X+Y$.

We shall write $\langle f, g\rangle$ for the inner product $\int_I f(t)g(t)\,dt$.

The associated space X' of the Banach function space X is defined by

$$X' = \{g \in L^0(I, m)\colon \operatorname{supp} g \subset \operatorname{supp} X,\ \|g\|_{X'} = \sup_{\|f\|_X \leqslant 1} \langle |f|, |g|\rangle < \infty\}.$$

If X is an r.i. space then $\|g\|_{X'} = \sup_{\|f\|_X \leqslant 1} \langle f^*, g^*\rangle$ and it follows that X' is also an r.i. space under the norm $\|\cdot\|_{X'}$. According to a result due to Lorentz and Luxemburg, X is isometrically isomorphic to X'', and in particular the norm has the representation

$$\|f\|_X = \sup\{\langle f^*, g^*\rangle\colon \|g\|_{X'} \leqslant 1\}, \tag{4.4}$$

whence follows the important Hölder inequality

$$\langle |f|, |g|\rangle \leqslant \langle f^*, g^*\rangle \leqslant \|f\|_X \|g\|_{X'}. \tag{4.5}$$

The smallest and the largest of the r.i. spaces are respectively $L^1 \cap L^\infty$ and $L^1 + L^\infty$ in the sense that the continuous embeddings

$$\begin{cases} L^1 \cap L^\infty \subset X \subset L^1 + L^\infty & \text{if} \quad l = \infty, \\ L^\infty \subset X \subset L^1 & \text{if} \quad l < \infty \end{cases} \tag{4.6}$$

hold for any r.i. space X (see [22]).

The *fundamental function* φ_X of an r.i. space X is given by

$$\varphi_X(t) = \|\chi_{(0,t)}\|_X, \quad t \in I. \tag{4.7}$$

The fundamental function is a positive non-decreasing function on I which is absolutely continuous on $[\varepsilon, l)$ for each $\varepsilon > 0$ and satisfies

(4.8) $\quad \varphi_X(t)\cdot\varphi_{X'}(t) = t \quad$ for all $t \in I$,

(4.9) $\quad \dfrac{d\varphi_X(t)}{dt} \leqslant \dfrac{\varphi_X(t)}{t} \quad$ a.e. on I,

(4.10) X has an equivalent r.i. norm $\|\cdot\|_0$ such that the fundamental function φ_{X_0} is concave and, moreover, $\varphi_X(t) \leqslant \varphi_{X_0}(t) \leqslant 2\varphi_X(t)$, $t \in I$.

A straightforward computation shows that $\varphi_{L_F}(t) = 1/F^{-1}(1/t)$, $\varphi_{L_{(F)}}(t) = tF^{*-1}(1/t)$, $\varphi_{L_{(F)}}$ is a concave function, and φ_{L_F} does not have to be a concave function.

The r.i. *Lorentz spaces* associated with the r.i. space X, defined by

$$\Lambda(X) = \{f \in L^0(I, m):\ \|f\|_{\Lambda(X)} = \int_0^l f^*(s)\, d\varphi_X(s)$$

$$= \varphi_X(0^+)\|f\|_{L_\infty} + \int_0^l f^*(s)\, \varphi_X'(s)\, ds < \infty\}$$

and

$$M(X) = \left\{f \in L^0(I, m):\ \|f\|_{M(X)} = \sup_{t \in I} \frac{\varphi_X(t)}{t} \int_0^t f^*(s)\, ds < \infty\right\}$$

are extremal in the sense that

$$\Lambda(X) \subset X \subset M(X) \tag{4.11}$$

with continuous embeddings. We consider also an r.i. quasi-Banach function space (Marcinkiewicz space) and an r.i. Lorentz space

$$M^*(X) = \{f \in L^0(I, m):\ \|f\|_{M^*(X)} = \sup_{t \in I} \varphi_X(t) f^*(t) < \infty\},$$

$$\Lambda(X, p) = \{f \in L^0(I, m):\ \|f\|_{\Lambda(X,p)} = \left[\int_0^l f^*(s)^p\, d\varphi_X(s)\right]^{1/p} < \infty\}.$$

For example, if $X = L_{pq}$, then $\varphi_X(t) = t^{1/p}$, so that $\Lambda(X) = L_{p1}$ and $M^*(X) = L_{p\infty}$.

If $\psi(t) = \varphi_X(t)$, $t \in I$, then $p(X) = p(\varphi_X)$ and $q(X) = q(\varphi_X)$ are *lower* and *upper Zippin indices* of an r.i. space X (see [64]).

Let $[X]$ denote the Banach space of bounded linear operators from X into itself with the operator norm and let $M(s, X) = M^I(s, \varphi_X)$.

For every $s > 0$, let E_s denote the *dilation operator* on X:

$$E_s f(t) = \begin{cases} f(ts) & \text{if} \quad ts \in I, \\ 0 & \text{elsewhere}, \end{cases} \qquad t \in I, \tag{4.12}$$

$E_s \in [X]$ and $\min(1, s) \leqslant \|E_{1/s}\|_{[X]} \leqslant \max(1, s)$, and $h(s, X) = \|E_{1/s}\|_{[X]}$ is a submultiplicative function (see [22]). The numbers

$$\alpha(X) = \alpha^I(X) = \alpha(h(\cdot, X)), \qquad \beta(X) = \beta^I(X) = \beta(h(\cdot, X)), \tag{4.13}$$

are *lower* and *upper Boyd indices* of an r.i. space X (see [6]).

Remark 4.1. (a) The above indices are not invariant under isomorphism. For example, $L_p(0, \infty) \cap L_2(0, \infty)$ is isomorphic to $L_p(0, \infty)$ if $2 < p < \infty$ (see [18], [28]) and

$$q(L_p) = \beta(L_p) = \frac{1}{p} < \frac{1}{2} = \beta(L_p \cap L_2) = q(L_p \cap L_2).$$

(b) The Zippin and the Boyd indices can be defined also for r.i. spaces on integers, i.e., for sequences r.i. spaces (see [64] p. 283, [6] pp. 1250–1251 and [28] pp. 131–132).

Since $M^l(s, X) = \sup_{t\in I, ts\in I} \frac{\|E_{1/s}\chi_{(0,t)}\|_X}{\|\chi_{(0,t)}\|_X} \leqslant h(s, X)$ for $s > 0$ and $\varphi_X(t)\varphi_{X'}(t) = t$, $h(s, X') = sh(s^{-1}, X)$, where X' is the associated space of X, and so these indices satisfy

$$0 \leqslant \alpha(X) \leqslant p(X) \leqslant q(X) \leqslant \beta(X) \leqslant 1 \tag{4.14}$$

and

$$p(X')+q(X) = p(X)+q(X') = 1, \quad \alpha(X')+\beta(X) = \alpha(X)+\beta(X') = 1. \tag{4.15}$$

For familiar r.i. spaces such as the Lebesgue spaces L_p, $1 \leqslant p \leqslant \infty$, the Orlicz spaces L_F, the Lorentz spaces L_{pq}, $\Lambda(\varphi)$, $M(\varphi)$, both $p(\cdot) = \alpha(\cdot)$ and $q(\cdot) = \beta(\cdot)$ hold. For the Orlicz spaces L_F, these equalities are derived from the inequality $h(s, L_F) \leqslant 2M^l(s, L_F)$, which is a direct consequence of Theorem 6 in [31] or theorem in [8]. D. W. Boyd also proved in [8] the same equality for the Orlicz sequence space l_F.

T. Shimogaki has given in [55], [56] an example of non-separable r.i. spaces X_a on (0, 1) such that $p^1(X_a) = q^1(X_a) = a$ and $\alpha(X_a) = 0$ for every $0 < a < 1$. However, if we let $\bar{X}_a$ be the norm closure of the simple functions with finite support in X_a, then $\bar{X}_a$ is a separable r.i. space such that $p^1(\bar{X}_a) = q^1(\bar{X}_a) = a$ and $\alpha(\bar{X}_a) = 0$ for every $0 < a < 1$.

THEOREM 4.2. *Let L_F be an Orlicz space on $(0, l)$, $l < \infty$. Then*

$$[\alpha(L_F), \beta(L_F)] = [p(L_F), q(L_F)] = \left[\frac{1}{\sigma_F^\infty}, \frac{1}{s_F^\infty}\right] = \left[\frac{1}{\beta_F^\infty}, \frac{1}{\alpha_F^\infty}\right]$$
$$\subset \left[\frac{1}{b_F^\infty}, \frac{1}{a_F^\infty}\right] = \left[\frac{1}{q_F^\infty}, \frac{1}{p_F^\infty}\right] \subset [0, 1].$$

Analogous results hold for Orlicz spaces L_F on $(0, \infty)$ and for sequence spaces l_F.

Proof. From the above equalities, Theorem 2.2, Theorem 2.6 and Theorem 3.2 we have

$$\alpha^l(L_F) = p^l(L_F) = p^l(F^{-1}(1/\cdot)) = p_0(F^{-1}(1/\cdot)) = p_\infty(F^{-1})$$
$$= \frac{1}{q_\infty(F)} = \frac{1}{\sigma_F^\infty} = \frac{1}{\beta_F^\infty} \geqslant \frac{1}{b_F^\infty} = \frac{1}{q_F^\infty} \geqslant 0;$$

moreover, since $L'_F = L_{(F^*)}$ and

$$t \leqslant F^{-1}(t)F^{*-1}(t) \leqslant 2t \qquad \forall t > 0 \text{ (see [21])},$$

we have

$$\beta^l(L_F) = q^l(L_F) = 1-p^l(L_{(F^*)}) = 1-p^l(L_{F^*}) = 1-p_\infty(F^{*-1})$$
$$= 1-[1-q_\infty(F^{-1})] = q_\infty(F^{-1}) = \frac{1}{p_\infty(F)} = \frac{1}{s_F^\infty} = \frac{1}{\alpha_F^\infty} \leqslant \frac{1}{a_F^\infty} = \frac{1}{p_F^\infty} \leqslant 1.$$

The proof of the assertion concerning L_F on $(0, \infty)$ is the same. However, we shall not give the proofs in the sequence case since, while they are essentially the same as those for function spaces, they often require a somewhat different notation.

EXAMPLE 11. For any numbers a and b satisfying $0 \leqslant a < b \leqslant 1$ there exists an r.i. space X on $(0, \infty)$ such that

$$p^\infty(X) = a, \quad q^\infty(X) = b. \tag{4.16}$$

If we take $X = M(Y)$, where $\omega_Y(t) = t^a \log(1+t^{b-a})$ then (4.16) and $p^1(X) = q^1(X) = b$, however, if $\varphi_Y(t) = \max(t^a, t^b)$ then (4.16) and $p^1(X) = q^1(X) = a$.

EXAMPLE 12. There exist reflexive r.i. spaces X for which $p(X) = 0$.

We consider the space $\Lambda(Y, p)$ over the interval I, where $1 < p < \infty$ and $p(Y) = 0$. It is known [29] that the space $\Lambda(Y, p)$ is reflexive as a Banach space if $1 < p < \infty$. On the other hand,

$$p(\Lambda(Y, p)) = \frac{1}{p} p(Y) = 0.$$

EXAMPLE 13. There exist non-reflexive and non-separable r.i. spaces X for which $0 < \alpha(X) \leqslant \beta(X) < 1$. It is sufficient to take $X = L_{p\infty}$, $1 < p < \infty$

We shall repeatedly use the following simple inequality of Hardy:

LEMMA 4.3. *If f_1, f_2 are non-decreasing absolutely continuous functions such that $f_1(0) = f_2(0) = 0$ and if $f_1(s) \leqslant f_2(s)$ for all $s \in I$, then for any non-negative non-increasing function g*

$$\int_I g(s)\, df_1(s) \leqslant \int_I g(s)\, df_2(s). \tag{4.17}$$

5. Interpolation theorems for weak type operators

In the sequel X_1, X_2, X and Y_1, Y_2, Y denote r.i. spaces on I.

A linear operator L is said to be *strong* (*M-weak* or *weak*) *type* (X, Y) if L is bounded from X into Y (from X into $M^*(Y)$ or from $\Lambda(X)$ into $M^*(Y)$, respectively).

DEFINITION 5.1. Let X be embedded in $X_1 + X_2$ (resp. in $\Lambda(X_1) + \Lambda(X_2)$).

We say that (X, Y) is a *strong* (resp. *weak*) *interpolation pair* for the segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$ if every linear operator which is strong (resp. weak) type (X_1, Y_1) and (X_2, Y_2) acts as a bounded operator from X into Y. If $X_i = Y_i$, $i = 1, 2$, and $X = Y$ then X is called, in short, a *strong* (resp. *weak*) *interpolation space* for (X_1, X_2).

We assume that the following condition holds:

$$\min\{\varphi_{X_1}(0^+), \varphi_{X_2}(0^+)\} = 0. \tag{5.1}$$

We now define a *Calderón operator* $S(\sigma)$, with $\mathscr{D}_{S(\sigma)} \subset L^0(I, m)$ of the form

$$S(\sigma)[f](t) = \int_0^l f(s)\, d \min_{i=1,2} \{\varphi_{X_i}(s)/\varphi_{Y_i}(t)\}, \qquad t \in I, \tag{5.2}$$

where

$$\mathscr{D}_{S(\sigma)} = \{f \in L^0(I, m): \int_0^l |f(s)|\, d \min_{i=1,2} \{\varphi_{X_i}(s)/\varphi_{Y_i}(t)\} < \infty \text{ a.e.}\}$$
$$\supset \Lambda(X_1) + \Lambda(X_2).$$

THEOREM 5.2 (Sharpley [51], Theorem 4.7). *Let* $X \subset \Lambda(X_1) + \Lambda(X_2)$. *$(X, Y)$ is a weak interpolation pair for σ if and only if the operator $S(\sigma)$ is bounded from X into Y.*

Let $X_i = Y_i$, $i = 1, 2$, and let $\varphi_{12}(t) := \varphi_{X_1}(t)/\varphi_{X_2}(t)$ be a non-increasing function. Then the Calderón operator has the form

$$S(\sigma)[f](t) = \varphi_{X_2}(t)^{-1} \int_0^t f(s)\, d\varphi_{X_2}(s) + \varphi_{X_1}(t)^{-1} \int_t^l f(s)\, d\varphi_{X_1}(s) \tag{5.3}$$
$$=: (P_{X_2} f)(t) + (Q_{X_1} f)(t).$$

If we apply Theorem 5.2 in order to obtain the interpolation theorem for this case, it is sufficient to investigate the P_{X_2} and Q_{X_1} operators.

LEMMA 5.3 (a) *$(P_{X_2} f^*)(t)$ and $(Q_{X_1} f^*)(t)$ are non-increasing functions of $t \in I$.*

(b) *The following inequalities hold*:

$$(P_{X_2} f)^*(t) \leqslant (P_{X_2} f^*)(t), \qquad (Q_{X_1} f)^*(t) \leqslant (Q_{X_1} f^*)(t).$$

Proof. (a) If $0 < t_1 < t_2 < l$, then

$$(P_{X_2} f^*)(t_2) - (P_{X_2} f^*)(t_1) \leqslant [\varphi_{X_2}(t_2)^{-1} - \varphi_{X_2}(t_1)^{-1}] \int_0^{t_1} f^*(t_1)\, d\varphi_{X_2}(s) +$$
$$+ \varphi_{X_2}(t_2)^{-1} \int_{t_1}^{t_2} f^*(s)\, d\varphi_{X_2}(s)$$
$$\leqslant f^*(t_1)[\varphi_{X_2}(t_1)/\varphi_{X_2}(t_2) - 1] + f^*(t_1)[1 - \varphi_{X_2}(t_1)/\varphi_{X_2}(t_2)] = 0.$$

The monotonicity of $(Q_{X_1} f^*)(t)$ is obvious.

(b) Applying the well-known inequality of Hardy, Littlewood and Pólya

$$\int_I |fg|\,dt \leqslant \int_I f^* g^*\,dt,$$

we have

$$(P_{X_2}f)(t) \leqslant (P_{X_2}f^*)(t),$$

$$(Q_{X_1}f)(t) \leqslant \varphi_{X_1}(t)^{-1} \int_0^{l-t} |f(t+u)|\,d\varphi_{X_1}(t+u)$$

$$\leqslant \varphi_{X_1}(t)^{-1} \int_0^{l-t} f^*(t+u)\,d\varphi_{X_1}(t+u) = (Q_{X_1}f^*)(t).$$

Hence, from (a) we obtain

$$(P_{X_2}f)^* \leqslant (P_{X_2}f^*)^* = P_{X_2}f^*, \quad (Q_{X_1}f)^* \leqslant (Q_{X_1}f^*)^* = Q_{X_1}f^*.$$

We note that if $\beta(X) < p(X_2)$ then $X \subset L_\infty + \Lambda(X_2)$ (see [35], Lemma 4.2) and if $q(X_1) < \alpha(X) \leqslant \beta(X) < p(X_2)$ then $X \subset \Lambda(X_1) + \Lambda(X_2)$ (see [22], p. 174).

LEMMA 5.4. (a) *The following assertions are equivalent:*

(i) $$\beta(X) < p(X_2);$$

(ii) $$C_0(X_2, X) := \int_0^1 h(s^{-1}, X)\,dM(s, X_2) < \infty \quad \textit{and} \quad p(X_2) > 0.$$

(b) *The following assertions are equivalent:*

(i′) $$\alpha(X) > q(X_1) > 0;$$

(ii′) $$C_1(X_1, X) := \int_1^\infty h(s^{-1}, X)\,dM(s, X_1) < \infty \quad \textit{and} \quad q(X_1) > 0.$$

Proof. (a) (i)$\Rightarrow$(ii). Let us take $\varepsilon > 0$ such that $\beta(X) + 2\varepsilon < p(X_2)$; then, since $M(s, X_2) \leqslant 1$ for $s \leqslant 1$ and $h(s, X) \leqslant \max(s, 1)$, we have by Lemma 4.3

$$C_0(X_2, X) \leqslant \int_0^\delta s^{-\beta(X)-\varepsilon}\,ds^{p(X_2)-\varepsilon} + 1 + \int_\delta^1 s^{-2}\,ds < \infty.$$

(ii)$\Rightarrow$(i). Since $s^{p(X_2)} \leqslant M(s, X_2)$ and $C_0(X_2, X) < \infty$, we have $s_0^{p(X_2)-1} h(s_0^{-1}, X) < s_0^{-1}$ for some $0 < s_0 < 1$, and so

$$\beta(X) = \inf_{s>1} \frac{\log h(s, X)}{\log s} < p(X_2).$$

(b) In the same manner as (a).

THEOREM 5.5 (Generalized Hardy inequalities). (a) *If X is any r.i. space such that its upper index $\beta(X) < p(X_2)$, then*

$$\|P_{X_2} f\|_X \leqslant C_0(X_2, X)\|f\|_X \tag{5.4}$$

holds for all $f \in X$.

(b) *If X is any r.i. space such that its lower index $\alpha(X) > q(X_1)$, then*

$$\|Q_{X_1} f\|_X \leqslant C_1(X_1, X)\|f\|_X \tag{5.5}$$

holds for all $f \in X$

Proof. (a) and (b) can be proved in the same manner as in [35] and [36]. For example we shall prove (b). Using Lemma 5.3, Lemma 4.3, the Fubini Theorem and the Hölder inequality, we have

$$\langle (Q_{X_1} f)^*, g^* \rangle \leqslant \langle Q_{X_1} f^*, g^* \rangle$$

$$= \int_0^l (\varphi_{X_1}(t))^{-1} \int_1^\infty f^*(ts)\chi_l(ts)\, d\varphi_{X_1}(ts)\, g^*(t)\, dt$$

$$\leqslant \int_1^\infty \Big(\int_0^l E_s f^*(t) g^*(t)\, dt\Big) dM(s, X_1) \leqslant \int_1^\infty \|E_s f^*\|_X \|g^*\|_{X'}\, dM(s, X_1)$$

$$\leqslant \int_1^\infty h(s^{-1}, X)\, dM(s, X_1)\|f^*\|_X \|g^*\|_{X'} = C_1(X_1, X)\|f\|_X \|g\|_{X'}.$$

Taking the supremum over all $g \in X'$, $\|g\|_{X'} \leqslant 1$, we obtain (5.5).

In Theorem 5.5 the indices $p(X_2)$ and $q(X_1)$ cannot be replaced by the indices $q(X_2)$ and $p(X_1)$, respectively.

EXAMPLE 14. Let $l = \infty$ and $X_2 = L_p \cap L_q$, $X = L_r$, where $1 \leqslant p < r \leqslant q \leqslant \infty$. Then $\varphi_{X_2}(t) = \max(t^{1/p}, t^{1/q})$ and

$$p(X_2) = 1/q \leqslant 1/r = \alpha(X) = \beta(X) < 1/p = q(X_2).$$

Since

$$B_2 := \sup_{s>0} \left[\int_s^\infty \frac{dt}{\varphi_{X_2}(t)^r}\right]^{1/r} \cdot \left[\int_0^s \varphi'_{X_2}(t)^{r'}\, dt\right]^{1/r'} = \infty,$$

where $1/r + 1/r' = 1$, it follows by Theorem 1 of Muckenhoupt [44] that $P_{X_2} \notin [L_r]$. Analogously, if $X_1 = L_p \cap L_q$, $X = L_r$, where $1 \leqslant p \leqslant r < q \leqslant \infty$, then

$$p(X_1) = 1/q < 1/r = \alpha(X) = \beta(X) \leqslant 1/p = q(X_1)$$

and

$$B_1 := \sup_{s>0}\left[\int_0^s \frac{dt}{\varphi_{X_1}(t)^r}\right]^{1/r}\cdot\left[\int_s^\infty \varphi'_{X_1}(t)^{r'}\,dt\right]^{1/r'} = \infty,$$

and so, by Theorem 2 of Muckenhoupt [44], $Q_{X_1}\notin[L_r]$.

THEOREM 5.6. (a) *If X is any r.i. space such that $P_{X_2}\in[X]$ and $q(X_2)>0$, $\varphi_{X_2}(0^+)=0$, then its upper index $\beta(X)$ is smaller than $q(X_2)$.*

(b) *If X is any r.i. space such that $Q_{X_1}\in[X]$ and $p(X_1)>0$, then its lower index $\alpha(X)$ is larger than $p(X_1)$.*

Proof. (a) By modifying the proof from [35] we get the theorem with the assumptions $q(X_2)>0$ and $\varphi_{X_2}(0^+)=0$ instead of the assumption $p(X_2)>0$. If $P_{X_2}\in[X]$ and $0<\lambda<\min\{1, r(P_{X_2})^{-1}\}$, where $r(P_{X_2})$ denotes the spectral radius of the operator P_{X_2} and I is the identity operator, then $(I-\lambda P_{X_2})^{-1}$ exists and $(I-\lambda P_{X_2})^{-1}\in[X]$, and

$$(I-\lambda P_{X_2})^{-1} = \sum_{n=0}^{\infty}\lambda^n P_{X_2}^n,$$

where the series converges in the operator norm and $P_{X_2}^n$ denotes the nth iterate of P_{X_2}. Since for $f\in X$

$$(P_{X_2}^n f)(t) = \varphi_{X_2}(t)^{-1}\int_0^1 \frac{1}{(n-1)!}\left(\ln\frac{\varphi_{X_2}(t)}{\varphi_{X_2}(ts)}\right)^{n-1} f(ts)\,d\varphi_{X_2}(ts),$$

we have

$$(\bar{P}f)(t) = P_{X_2}(I-\lambda P_{X_2})^{-1}f(t) = \varphi_{X_2}(t)^{\lambda-1}\int_0^1 f(ts)\,\varphi_{X_2}(ts)^{-\lambda}\,d\varphi_{X_2}(ts)$$

$$= \varphi_{X_2}(t)^{\lambda-1}(1-\lambda)^{-1}\int_0^t f(u)\,d\{\varphi_{X_2}(u)^{1-\lambda}\}.$$

For $t\in I$ and $s>1$ we have

$$(\bar{P}f^*)(t) \geqslant \varphi_{X_2}(t)^{\lambda-1}(1-\lambda)^{-1}\int_0^{t/s} f^*(u)\,d\{\varphi_{X_2}(u)^{1-\lambda}\}$$

$$\geqslant \varphi_{X_2}(t)^{\lambda-1}(1-\lambda)^{-1} f^*(t/s)\,\varphi_{X_2}(t/s)^{1-\lambda} \geqslant \frac{f^*(t/s)}{(1-\lambda)M(s, X_2)^{1-\lambda}},$$

i.e.,

$$f^*(t/s) \leqslant (1-\lambda)M(s, X_2)^{1-\lambda}(\bar{P}f^*)(t).$$

Consequently

$$\|E_{1/s}f^*\|_X \leqslant \text{const} \quad M(s, X_2)^{1-\lambda}\|f\|_X.$$

Hence $h(s, X) \leqslant \text{const}\, M(s, X_2)^{1-\lambda}$ and

$$\beta(X) \leqslant (1-\lambda)q(X_2) < q(X_2).$$

(b) Let $Q_{X_1} \in [X]$. We proceed in the same way as in (a) and we have

$$(Q^n_{X_1}f)(t) = \varphi_{X_1}(t)^{-1}\int_1^l \frac{1}{(n-1)!}\left(\ln\frac{\varphi_{X_1}(ts)}{\varphi_{X_1}(t)}\right)^{n-1} f(ts)\chi_I(ts)\,d\varphi_{X_1}(ts)$$

and

$$(\bar{Q}f)(t) = Q_{X_1}(I-\lambda Q_{X_1})^{-1}f(t) = \sum_{n=0}^{\infty}\lambda^n(Q^{n+1}_{X_1}f)(t)$$

$$= \int_1^{l/t}\left(\frac{\varphi_{X_1}(ts)}{\varphi_{X_1}(t)}\right)^{\lambda} f(ts)\frac{d\varphi_{X_1}(ts)}{\varphi_{X_1}(t)}$$

$$= (1+\lambda)^{-1}\varphi_{X_1}(t)^{-1-\lambda}\int_1^l f(ts)\chi_I(ts)\,d\{\varphi_{X_1}(ts)^{1+\lambda}\}.$$

For $t \in I$ and $0 < s < 1$ we have

$$(\bar{Q}f^*)(t) = (1+\lambda)^{-1}\varphi_{X_1}(t)^{-1-\lambda}\int_t^l f^*(u)\,d\{\varphi_{X_1}(u)^{1+\lambda}\}$$

$$\geqslant (1+\lambda)^{-1}\varphi_{X_1}(t)^{-1-\lambda}\int_t^{t/s}\chi_I(t/s)f^*(u)\,d\{\varphi_{X_1}(u)^{1+\lambda}\}$$

$$\geqslant f^*(t/s)\chi_I(t/s)\frac{\varphi_{X_1}(t/s)^{1+\lambda}-\varphi_{X_1}(t)^{1+\lambda}}{(1+\lambda)\varphi_{X_1}(t)^{1+\lambda}}$$

$$\geqslant \frac{f^*(t/s)\chi_I(t/s)}{1+\lambda}\cdot\left[\frac{1}{M(s, X_1)^{1+\lambda}}-1\right].$$

Hence

$$f^*(t/s)\chi_I(t/s) \leqslant (1+\lambda)(\bar{Q}f^*)(t)\frac{M(s, X_1)^{1+\lambda}}{1-M(s, X_1)^{1+\lambda}},$$

i.e., $h(s, X) \leqslant \text{const}\dfrac{M(s, X_1)^{1+\lambda}}{1-M(s, X_1)^{1+\lambda}}$ and

$$\alpha(X) \geqslant (1+\lambda)p(X_1) > p(X_1).$$

In Theorem 5.6 the indices $q(X_2)$ and $p(X_1)$ cannot be replaced by the indices $p(X_2)$ and $q(X_1)$, respectively.

EXAMPLE 15. Let $l = \infty$ and $X = M(Y)$, where $q(Y) < 1$.

(a) We take φ_{X_2} such that $0 < p^\infty(\varphi_{X_2}/\varphi_Y) \leqslant q^\infty(\varphi_{X_2}/\varphi_Y) < \infty$. Then

$$\begin{aligned}\|P_{X_2}f\|_{M^*(Y)} &\leqslant \sup_{t>0}\frac{\varphi_Y(t)}{\varphi_{X_2}(t)}\int_0^t f^*(s)\,d\varphi_{X_2}(s)\\ &\leqslant \sup_{t>0}\frac{\varphi_Y(t)}{\varphi_{X_2}(t)}\int_0^t \frac{\|f\|_{M^*(Y)}}{\varphi_Y(s)}\,d\varphi_{X_2}(s)\\ &\leqslant \sup_{t>0}\frac{\varphi_Y(t)}{\varphi_{X_2}(t)}\int_0^t \frac{\varphi_{X_2}(s)}{\varphi_Y(s)}\frac{ds}{s}\|f\|_{M^*(Y)}.\end{aligned}$$

Since $q(Y) < 1$, we have $\|f\|_{M(Y)} \leqslant C(Y)\|f\|_{M^*(Y)}$ (see Sharpley [51], Th. 2.2) and by Theorem 2.4 (c)

$$\int_0^t \frac{\varphi_{X_2}(s)}{\varphi_Y(s)}\frac{ds}{s} \leqslant C_1\frac{\varphi_{X_2}(t)}{\varphi_Y(t)} \qquad \forall t > 0.$$

Hence

$$P_{X_2} \in [M(Y)].$$

In particular, let $\varphi_{X_2}(t) = t^{a+bc}\log(1+t^b)$ and $\varphi_Y(t) = t^a\log(1+t^b)$, where $a, b > 0$, $0 < c \leqslant 1$ and $a+b+bc \leqslant 1$. Then

$$p_a(\varphi_{X_2}/\varphi_Y) = q_a(\varphi_{X_2}/\varphi_Y) = bc, \qquad q(Y) = a+b < 1,$$

i.e., $P_{X_2} \in [M(Y)]$, and

$$p(X_2) = a+bc \leqslant a+b = \beta(M(Y)) = q(Y) < q(X_2) = a+b+bc.$$

(b) We take φ_{X_1} such that $-\infty < p_a(\varphi_{X_1}/\varphi_Y) \leqslant q_a(\varphi_{X_1}/\varphi_Y) < 0$. Then by Theorem 2.4 (d) we have $Q_{X_1} \in [M(Y)]$.

In particular, let $\varphi_{X_1}(t) = t^a\log(1+t^b)$ and $\varphi_Y(t) = t^{a+bc}\log(1+t^b)$, where $a, b > 0$, $0 < c \leqslant 1$ and $a+b+bc < 1$. Then

$$q(Y) = a+b+bc < 1 \quad \text{and} \quad p_a(\varphi_{X_1}/\varphi_Y) = q_a(\varphi_{X_1}/\varphi_Y) = -bc,$$

i.e., $Q_{X_1} \in [M(Y)]$, and

$$p(X_1) = a < a+bc = p(Y) = \alpha(M(Y)) \leqslant a+b = q(X_1).$$

Remark. 5.7. (a) From Examples 14 and 15 it follows that there are r.i. spaces X_i, Y_i such that $0 < p(X_i) \leqslant \beta(Y_i) < q(X_i)$, $i = 1, 2$ and $P_{X_1} \notin [Y_1]$, and $P_{X_2} \in [Y_2]$. This confirms my Conjecture (see [35], Conjecture 3.8).

(b) From Example 15 it follows that there exist an r.i. space X and function φ such that

$$\left\| \varphi(t)^{-1} \int_0^t f(s) \frac{\varphi(s)}{s} ds \right\|_X \leqslant \text{const } \|f\|_X$$

and

$$0 < p^\infty(\varphi) \leqslant \beta(X) < q^\infty(\varphi) < 1.$$

This fact gives a counterexample to my problem (see [36]).

THEOREM 5.8 (Interpolation Theorem). *Let* $\varphi_{X_1}(t)/\varphi_{X_2}(t)$ *be a non-increasing function.*

(a) *If* $q(X_1) < \alpha(X) \leqslant \beta(X) < p(X_2)$, *then* X *is a weak interpolation space for* (X_1, X_2).

(b) *If* X *is a weak interpolation space for* (X_1, X_2) *and* $p(X_1) > 0$, *then*

$$p(X_1) < \alpha(X) \leqslant \beta(X) < q(X_2).$$

Proof. The theorem follows immediately from Theorems 5.2, 5.5 and 5.6.

Remark 5.9. (a) If $\varphi_{X_1}(t) = \text{const}$, then in Theorem 5.8 (a) it is sufficient to assume $\beta(X) < p(X_2)$. Moreover, Theorem 5.8 (b) has the following form: if X is a weak interpolation space for (X_1, X_2) and $q(X_2) > 0$, then $\beta(X) < q(X_2)$.

(b) To prove Theorem 5.8 (b) it is sufficient to use Theorem 5.6 (a) and the following remarks (compare [14]):

(i) P_{X_2} is of weak type (Y, Y) for any r.i. space Y such that $\varphi_Y(t)/\varphi_{X_2}(t)$ is non-increasing and $p(Y) > 0$;

(ii) $\langle P_Y f^*, g^* \rangle \approx \langle f^*, Q_Y g^* \rangle$, for Y such that $0 < p(Y) \leqslant q(Y) < 1$.

In [6] Boyd gives a necessary and sufficient condition for X to be a weak interpolation space for (L_{p_1}, L_{p_2}). This condition is

$$1/p_1 < \alpha(X) \leqslant \beta(X) < 1/p_2.$$

Zippin, applying Theorem 1 of Semenov [49], proves in [64] that if L_∞ is dense in all spaces under consideration then the condition

$$q(X_1) < p(X) \leqslant q(X) < p(X_2)$$

is sufficient for X to be a weak interpolation space for (X_1, X_2). On the other hand, Shimogaki [55] gives an example of an r.i. space X which satisfies

$$1/p_1 < p(X) \leqslant q(X) < 1/p_2,$$

but X is not a weak interpolation space for (L_{p_1}, L_{p_2}). In [23] Krein and Semenov recognize that Zippin's result must be altered and also suggest that the result holds true if X satisfies the property $h(s, X) \leqslant CM(s, X)$. Of course, this is easily seen to imply that $\alpha(X) = p(X)$ and $\beta(X) = q(X)$. In [53] Sharpley has given an example of an r.i. space which is a counterexample to Theorem 1 of Semenov [49] as well as to Zippin's Interpolation Theorem. From Theorem 5.8 (a) it follows that Zippin's Interpolation Theorem holds true if we replace Zippin's assumptions on indices by the assumptions

$$q(X_1) < \alpha(X) \leqslant \beta(X) < p(X_2).$$

In [14] Fehér, applying the theorems obtained in [35], proved theorem 5.8 with the additional assumptions

$$0 < p(X_1) = q(X_1) < 1 \quad \text{and} \quad 0 < p(X_2) = q(X_2).$$

In [17] Pavlov proved that if $l = \infty$ and the space X is a weak interpolation space for (X_1, X_2), then

$$p(X_1) \leqslant \alpha(X) \leqslant \beta(X) \leqslant q(X_2).$$

Sharp inequalities in Theorem 5.8 (b) improve Pavlov's result and give a positive answer to the modified conjecture of Zippin (see [64], Conjecture 5.4). Examples 14 and 15 show that the necessary condition of Theorem 5.8 cannot be replaced by the sufficient condition and vice versa. Krein and Semenov (see [23] or [22]) gave a generalization of Theorem 5.5 and thus of Theorem 5.8 (a) to the segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$ with the assumption of monotonicity of function $\varphi_{X_1}(t)/\varphi_{X_2}(t)$. Pavlov [47] proved these theorems without the assumption of monotonicity of the function $\varphi_{X_1}(t)/\varphi_{X_2}(t)$.

THEOREM 5.10 (Pavlov [47]). *Let $q(X_1) < \alpha(X) \leqslant \beta(X) < p(X_2)$ and $q(Y_1) < 1$ or $q(X_2) < \alpha(X) \leqslant \beta(X) < p(X_1)$ and $q(Y_2) < 1$. Then the Calderón operator acts boundedly from X into $X_{\delta,\mu}$, where $\delta(t)$ and $\mu(t)$ stand for measurable solutions of the equations*

$$\varphi_{Y_i}(\delta(t)) = \mu(t)\,\varphi_{X_i}(t), \quad i = 1, 2,$$

and

$$\|f\|_{X_{\delta,\mu}} \doteq \|f^{**}(\delta(t))\,\mu(t)\|_X, \quad f^{**}(t) = (1/t)\int_0^t f^*(s)\,ds.$$

THEOREM 5.11 (Generalized Krein–Semenov Interpolation Theorem). *Let $q(X_1) < \alpha(X) \leqslant \beta(X) < p(X_2)$ and $q(Y_1) < 1$ or $q(X_2) < \alpha(X) \leqslant \beta(X) < p(X_1)$ and $q(Y_2) < 1$, then the pair $(X, X_{\delta,\mu})$ is a weak interpolation pair for the segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$.*

Proof. The theorem follows immediately from Theorems 5.2 and 5.10. Applying Theorem 5.6 and Pavlov's estimation from [47], we have Pavlov's Theorem 3 strengthened and we also obtain some generalization of Theorem 5.8 (b).

THEOREM 5.12. *If the ratio* $\varphi_{X_1}(t)/\varphi_{X_2}(t)$ *is non-increasing,* $\varphi_{Y_i}(t)$ *are submultiplicative and* $\alpha(Y_i) < 1$, $i = 1, 2$, *and* $p(X_1) > 0$, *then a necessary condition for the Calderón operator to act boundedly from* X *into* $Y_{\delta,u}$ *is*

$$p(X_1) < \alpha(X) \leqslant \beta(X) < q(X_2).$$

Proof. We have

$$\mu(t) S(\sigma) [f^*]^{**}(\sigma(t)) \geqslant \int_0^1 \min_{i=1,2} \{\varphi_{Y_i}(u)^{-1}\} du \int_0^l f^*(s) d\left\{\min_{i=1,2} \frac{\varphi_{X_l}(s)}{\varphi_{X_i}(t)}\right\}$$

and from the assumption follows the finiteness of the first integral of the inequality on the right (see [47]). If the Calderón operator S is bounded from X into $X_{\delta,\mu}$, then, under these assumptions, S is bounded from X into itself. Hence the theorem follows from Theorem 5.6.

Remark 5.13. (a) Theorems 5.2–5.12 hold true for an r.i. space without the Fatou property. Only Theorem 5.5, where the Fatou property has really been applied, requires a different proof. Namely, applying the inequality

$$(P_{X_2} f)^*(t) \leqslant P_{X_2} f^*(t) = \varphi_{X_2}(t)^{-1} \int_0^1 f^*(ts) d\varphi_{X_2}(ts) \leqslant \int_0^1 f^*(ts) dM(s, X_2)$$
$$= \int_0^1 E_s f^*(t) dM(s, X_2)$$

and Lemma 4.7 from [22] p. 136, we have the theorem. Analogously for the operator Q_{X_1}.

(b) Theorem 5.5. holds true for any Banach function space (not necessarily an r.i. space) with the Fatou property and satisfying condition $E_s \in [X]$. Evidently, the definition of Boyd indices for these Banach function spaces is correct.

Many operators playing an important part in the theory of orthogonal series are weak-type (L_1, L_1) (see [65], [22], [12]). Therefore, the interpolation of these operators is interesting.

Let $l = 1$. We assume that $\log 1/t \in X$ and $\tilde{X} = \{f \in L^0(0, 1; m): \|f\|_{\tilde{X}} = \|f^*(t) \log 1/t\|_X < \infty\}$. For example, $\widetilde{L^p \log^\alpha L} = L^p \log^{\alpha+1} L$ for $\alpha \geqslant 0$.

LEMMA 5.14. *For any r.i. space* X *on* $(0, 1)$ *such that* $\log 1/t \in X$ *we have* $\tilde{X} \subset X$. *Moreover,* P_{L_1} *is a bounded operator from* $\tilde{X}$ *into* X.

Proof. Applying the Fubini Theorem and the Hölder inequality, we have

$$\langle (P_{L_1}f)^*, g^*\rangle \leqslant \langle P_{L_1}f^*, g^*\rangle = \int_0^1 \left(\frac{1}{t}\int_0^t f^*(s)\,ds\right) g^*(t)\,dt$$
$$= \int_0^1 \left(\int_s^1 \frac{g^*(t)}{t}\,dt\right) f^*(s)\,ds \leqslant \int_0^1 g^*(s)\left(\int_s^1 \frac{dt}{t}\right) f^*(s)\,ds$$
$$= \langle f^*(\cdot)\log 1/(\cdot), g^*\rangle \leqslant \|f^*(t)\log 1/t\|_X \|g^*\|_{X'} = \|f\|_{\tilde{X}} \|g\|_{X'}.$$

Taking the supremum over all $g \in X'$, $\|g\|_{X'} \leqslant 1$, we obtain

$$\|P_{L_1}f\|_X \leqslant \|P_{L_1}f^*\|_X \leqslant \|f\|_{\tilde{X}}.$$

Moreover,

$$\|f\|_X = \|f^*\|_X \leqslant \|P_{L_1}f^*\|_X \leqslant \|f\|_{\tilde{X}},$$

i.e., $\tilde{X} \subset X$.

THEOREM 5.15. *Let* $l = 1$. *If* $q(X_1) < \alpha(X) \leqslant \beta(X) = 1$, *then* $(\tilde{X}, X)$ *is a weak interpolation pair for the segment* $\sigma = [(X_1, X_1), (L_1, L_1)]$.

Proof. We have

$$S(\sigma)[f](t) = P_{L_1}f(t) + Q_{X_1}f(t).$$

By Theorem 5.5 (b), $Q_{X_1} \in [X]$. Applying Lemma 5.14 and Theorem 5.2, we obtain the theorem.

In the proof of Theorem 5.15 the assumption $q(X_1) < \alpha(X)$ has been used immediately. However the assumption $\beta(X) = 1$ is meaningful since if the condition $\beta(X) < 1$ holds then the result follows from Theorem 5.11.

Theorem 5.15 was obtained by Dmitriev–Semenov [12] in the particular case $X_1 = L_p$, $1 < p < \infty$.

Remark 5.16. Interpolation theorems hold true for quasilinear operators.

6. Some additional remarks and open problems

A. Indices of Lorentz–Orlicz spaces. We assume that an r.i. space X on $(0, \infty)$ has a concave fundamental function φ_X and $\varphi_X(0^+) = 0$. Let F denote the Orlicz function on $[0, \infty)$. We define the functional $I_{X,F}$ on $L^0(0, \infty; m)$ by

$$I_{X,F}(f) = \int_0^\infty F(f^*(t))\,\varphi_X'(t)\,dt \tag{6.1}$$

and the *Lorentz–Orlicz space*

$$\Lambda(X, L_F) = \{f \in L^0(0, \infty; m): I_{X,F}(f/r) < \infty \text{ for some } r > 0\}.$$

The Lorentz–Orlicz space is an r.i. space on $(0, \infty)$ with the norm

$$\|f\|_{\Lambda(X,L_F)} = \inf\{r > 0\colon\ I_{X,F}(f/r) \leqslant 1\}; \tag{6.2}$$

the triangle inequality follows from the equalities: $F(f^*) = F(|f|)^*$ and $I_{X,F}(f) = \langle F(|f|)^*, \varphi'_X\rangle = \sup\{\langle F(|f|), g\rangle\colon\ g \in L^0(0, \infty; m)$ and $d_g = d_{\varphi'_X}\}$.

We also have for all $s > 0$

$$h(s, \Lambda(X, L_F)) \geqslant M(s, \Lambda(X, L_F)) = \sup_{t>0} \frac{F^{-1}(1/\varphi_X(t))}{F^{-1}(1/\varphi_X(ts))}. \tag{6.3}$$

PROBLEM 6.1. Does there exist a positive constant $C > 0$ such that for all $s > 0$ we have

$$h(s, \Lambda(X, L_F)) \leqslant CM(s, \Lambda(X, L_F))? \tag{6.4}$$

For many cases the answer to Problem 6.1 is positive.

THEOREM 6.2. (a) *For all $s > 0$ we have*

$$h(s, \Lambda(X, L_p)) = M(s, \Lambda(X, L_p)) = M(s, X)^{1/p}$$

(b) *If X is an r.i. space such that for some constant $C > 0$*

$$M(s, X)M(1/s, X) \leqslant C \qquad \forall s > 0, \tag{6.5}$$

then (6.4) *holds.*

Proof. (a) Applying Lemma 4.3, we have

$$\|E_{1/s}f\|^p_{\Lambda(X,L_p)} = \int_0^\infty f^*(t/s)^p\, d\varphi_X(t) = \int_0^\infty f^*(u)^p\, d\varphi_X(su)$$

$$\leqslant \sup_{u>0} \frac{\varphi_X(su)}{\varphi_X(u)} \int_0^\infty f^*(u)^p\, d\varphi_X(u) = M(s, X)\|f\|^p_{\Lambda(X,L_p)}.$$

Hence from this and (6.3) we have

$$h(s, \Lambda(X, L_p)) = M(s, X)^{1/p}$$

(b) Since for all $t > 0$

$$F[F^{-1}(1/\varphi_X(t))/M(s, \Lambda(X, L_F))] \leqslant 1/\varphi_X(ts),$$

putting $u = F^{-1}(1/\varphi_X(t))$, we have

$$F(u/M(s, \Lambda(X, L_F))) \leqslant \frac{\varphi_X(t)}{\varphi_X(ts)} F(u) \leqslant M(1/s, X)F(u).$$

Suppose that $f \in \Lambda(X, L_F)$. Then

$$I_{X,F}\left(\frac{E_{1/s}f}{M(s, \Lambda(X, L_F))}\right) \leqslant M(1/s, X) \int_0^\infty F(f^*(t))\, d\varphi_X(st)$$

$$\leqslant M(1/s, X)\, M(s, X)\, I_{X,F}(f).$$

By (6.5) we have $h(s, \Lambda(X, L_F)) \leqslant CM(s, \Lambda(X, L_F))$.

If $p(X) = q(X)$ then condition (6.5) holds true and if X is such as in Example 11 then φ_X does not have to satisfy condition (6.5).

THEOREM 6.3. *The Boyd indices of the Lorentz–Orlicz spaces* $\Lambda(X, L_p)$ *and* $\Lambda(L_p, L_F)$ *are given by*

$$\alpha(\Lambda(X, L_p)) = \frac{1}{p} p(X), \qquad \beta(\Lambda(X, L_p)) = \frac{1}{p} q(X),$$

$$\alpha(\Lambda(L_p, L_F)) = \frac{1}{p} p(L_F), \qquad \beta(\Lambda(L_p, L_F)) = \frac{1}{p} q(L_F).$$

B. Marcinkiewicz interpolation theorem in Orlicz spaces. A. Zygmund, A. P. Calderón, S. Koizumi, I. B. Simonenko, W. Riordan, H. P. Heinig and A. Torchinsky (for references see [58] and [60]) extended the theorem of Marcinkiewicz concerning the interpolation of operators of M-weak type or of mixed M-weak and strong type to include Orlicz spaces as interpolation classes. The assumptions of these theorems contain for a continuous increasing function ψ on $[0, \infty)$ with $\psi(0) = 0$ the following relations:

$$(A_r^a \text{ or } A_r^0) \qquad \int_0^t s^{-r}\psi(s)\frac{ds}{s} = O(t^{-r}\psi(t)) \quad \text{for all } t > 0 \text{ or as } t \to 0,$$

$$(A_r^\infty) \qquad \int_1^t s^{-r}\psi(s)\frac{ds}{s} = O(t^{-r}\psi(t)) \quad \text{as} \quad t \to \infty$$

and

$$(B_r^a \text{ or } B_r^\infty) \qquad \int_t^\infty s^{-r}\psi(s)\frac{ds}{s} = O(t^{-r}\psi(t)) \quad \text{for all } t > 0 \text{ or as } t \to \infty,$$

$$(B_r^0) \qquad \int_t^1 s^{-r}\psi(s)\frac{ds}{s} = O(t^{-r}\psi(t)) \quad \text{as} \quad t \to 0.$$

These relations have exact connections with the indices introduced in Section 2.

THEOREM 6.4. (a) *The following assertions are equivalent:*

$$p_i(\psi) > r, \tag{6.6}$$

$$\psi \in A_r^i \tag{6.7}$$

for $i = a, \infty, 0$.

(b) *The following assertions are equivalent:*

$$q_i(\psi) < r, \tag{6.6'}$$

$$\psi \in B_r^i \tag{6.7'}$$

for $i = a, \infty, 0$.

Proof. For example we shall prove the theorem for $i = a$. Implication (6.6)$\Rightarrow$(6.7) follows from Theorem 2.4 (c). If (6.7) holds, then we have

$$\psi(t/2)/2 \leqslant \psi_1(t) = t^r \int_0^t s^{-r}\psi(s)\frac{ds}{s} \leqslant C\psi(t)$$

and

$$\frac{d}{dt}\big(t^{-r-\varepsilon}\psi_1(t)\big) = -\varepsilon t^{-\varepsilon-1}\int_0^t s^{-r}\psi(s)\frac{ds}{s} + t^{-\varepsilon-r-1}\psi(t)$$

$$\geqslant (1-\varepsilon C)\, t^{-\varepsilon-r-1}\psi(t) > 0$$

if $\varepsilon > 0$ is sufficiently small. Hence by (3.4), Lemma 3.1 and Theorem 2.5 we get

$$r < r+\varepsilon \leqslant p_a(\psi_1) = p_a(\psi).$$

Part (b) can be proved in a similar way.

This theorem implies the following formulas:

$$\begin{aligned} p_i(\psi) &= \sup\{r > 0\colon\ \psi \in A_r^i\},\\ q_i(\psi) &= \inf\ \{r > 0\colon\ \psi \in B_r^i\},\end{aligned} \qquad i = a, \infty, 0 \tag{6.8}$$

follow.

If we apply Theorem 6.4, the Torchinsky interpolation Theorems 2.3, 2.8 and 2.11 can be formulated in a simple way by using the indices of functions. These extensions of the theorem of Marcinkiewicz are not included in the generalized Krein–Semenov Interpolation Theorem.

THEOREM 6.5 (Torchinsky [60] and Simonenko [58] case (a)). *Let*

$$0 \leqslant \frac{1}{q_i} = \beta_i \leqslant \frac{1}{p_i} = \alpha_i < \infty, \qquad i = 1, 2,\ p_1 \neq p_2,\ q_1 \neq q_2$$

and

$$G^{-1}(u) = F^{-1}(u^{\varepsilon})u^{\gamma},$$

where

$$\varepsilon = \frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}, \quad \gamma = \frac{\beta_1/\alpha_1 - \beta_2/\alpha_2}{1/\alpha_1 - 1/\alpha_2}$$

and F, G denote Orlicz functions (not necessarily convex) such that

$$\min(q_1, q_2) \leqslant p_G^a \leqslant q_G^a \leqslant \max(q_1, q_2).$$

Suppose that a quasilinear operator T is simultaneously

(a) *of M-weak types* $(L_{p_i}(\mu), L_{q_i}(\nu))$, $i = 1, 2$ *and*

$$\min(q_1, q_2) < p_a(G) \leqslant q_a(G) < \max(q_1, q_2) < \infty$$

or

$$q_a(G) < \max(q_1, q_2) = \infty;$$

(b) *of M-weak type* $(L_{p_1}(\mu), L_{q_i}(\nu))$ *and of strong type* $(L_{p_2}(\mu), L_{q_2}(\nu))$, *and*

$$q_1 < p^a(G) \text{ if } q_1 < p_2 \quad \text{or} \quad q_a(G) < q_1 \text{ if } q_1 > q_2;$$

(c) *of strong types* $(L_{p_i}(\mu), L_{q_i}(\nu))$, $i = 1, 2$ *and*

$$\max(q_1, q_2) < \infty \quad \text{or} \quad q_G^a < \max(q_1, q_2) = \infty;$$

then

$$\int G(|Tf(s)|/C)\, d\nu \leqslant 1$$

whenever

$$\int F(|f(t)|)\, d\mu \leqslant 1,$$

where C is independent of f.

The reader will have no difficulty in modifying the results for spaces of finite measure and for sequence spaces.

Remark 6.6. From Theorems 6.4 (b) and 3.2 (b) it follows that in Marcinkiewicz Interpolation Theorems (4.22) and (4.34) formulated in [65] Section XII, and the Koizumi Interpolation Theorem (see [20], Theorem 4) Δ_2^i condition is unnecessary and follows from the B_r^i condition. In fact,

$$u^{-r}F(2u) = CF(2u)\int_{2u}^{4u} t^{-r-1}\,dt \leqslant C_1 \int_{u}^{\infty} t^{-r-1}F(t)\,dt \leqslant C_2 u^{-r}F(u).$$

C. Indices and strong interpolation. In [59] G. Sparr proved that if X is a strong interpolation space for (L_p, L_q), *then*

$$1/q \leqslant \alpha(X) \leqslant \beta(X) \leqslant 1/p, \quad \text{where} \quad 1 \leqslant p < q \leqslant \infty.$$

We remark that the Boyd indices of X do not characterize completely the pairs (p, q) with $1 \leqslant p < q \leqslant \infty$, so that X is a strong interpolation space for (L_p, L_q). Indeed, L_r is a strong interpolation space for (L_r, L_q) for every q, but $\alpha(L_r) = \beta(L_r) = 1/r$. Thus, from the fact that X is a strong interpolation space for (L_p, L_q) it does not follow that $1/q < \alpha(X)$ or that $\beta(X) < 1/p$. Conversely, if $1/q \leqslant \alpha(X)$ and $\beta(X) \leqslant 1/p$ and one of these inequalities is a sharp inequality, then it need not follow in general that X is a strong interpolation space for (L_p, L_q), as the following examples show.

EXAMPLE 16. Let $I = (0, \infty)$, $1 < p < \infty$, and let $\varphi_X(t) = \varphi(t) = t^{1/p} u(t)$ be a concave function such that $u(t) \geqslant 1$, $\lim_{t\to\infty} u(t) = \infty$ and $\alpha(\Lambda(X)) = \beta(\Lambda(X)) = 1/p$ (for example, $\varphi(t) = t^{1/p}\ln(e+t)$ for $p \geqslant 2$). For $s > 1$ we define $T_s f(t) = s^{-1/p} E_{s^{-1}} f(t)$. Then, for any $p < q \leqslant \infty$, $\{T_s\}$ is a family of operators that have norms $\leqslant 1$ on L_p and on L_q, but as operators on $\Lambda(X)$ they satisfy

$$\|T_s\| = s^{-1/p} M(s, \varphi) = M(s, u) \geqslant u(1)^{-1} u(s), \quad \text{and so } \lim_{s\to\infty} \|T_s\| = \infty.$$

Hence, $\Lambda(X)$ is not a strong interpolation space for (L_p, L_q) for any $p < q \leqslant \infty$.

EXAMPLE 17 (see [34], Example 1). Let $I = (0, 1)$. The Lorentz space L_{pr} is not a strong interpolation space for (L_p, L_∞) for any $p > r \geqslant 1$, but $\alpha(L_{pr}) = \beta(L_{pr}) = 1/p$.

EXAMPLE 18. Let $I = (0, \infty)$, $1 < q < \infty$, and let $\varphi_X(t) = \varphi(t) = t^{1/q} u(t)$ be a concave function such that $\lim_{t\to 0^+} u(t) = \infty$ and $\alpha(\Lambda(X)) = \beta(\Lambda(X)) = 1/q$. For $s > 1$ we define $T_s f(t) = s^{1/q} E_s f(t)$. Then $\Lambda(X)$ is not a strong interpolation space for (L_p, L_q) for any $1 \leqslant p < q$. Indeed, $\{T_s\}$ is a sequence of operators that have norms $\leqslant 1$ on L_p and on L_q, but as operators on $\Lambda(X)$ they satisfy $\|T_s\| = s^{1/q} M(s^{-1}, \varphi) = M(s^{-1}, u) \geqslant u(1)^{-1} u(s^{-1})$, and so $\lim_{s\to\infty} \|T_s\| = \infty$.

THEOREM 6.6. *If X is a strong interpolation space for (X_1, X_2), then*

$$\min\{\alpha(X_1), \alpha(X_2)\} \leqslant \alpha(X) \leqslant \beta(X) \leqslant \max\{\beta(X_1), \beta(X_2)\} \tag{6.9}$$

and

$$\min\{p(X_1), p(X_2)\} \leqslant p(X) \leqslant q(X) \leqslant \max\{q(X_1), q(X_2)\}. \tag{6.10}$$

The proof of (6.9) follows from the fact that

$$h(s, X) \leqslant C \max\{h(s, X_1), h(s, X_2)\}.$$

The proof of (6.10) results from the following theorem:

THEOREM 6.7. *If (X, Y) is a strong interpolation pair for the segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$, then there exists a positive constant C such that the* *following inequality holds*:

$$(6.11) \qquad \frac{\varphi_Y(t)}{\varphi_X(s)} \leqslant C \max\left\{\frac{\varphi_{Y_1}(t)}{\varphi_{X_1}(s)}, \frac{\varphi_{Y_2}(t)}{\varphi_{X_2}(s)}\right\} \qquad \forall t, s \in I.$$

The proof follows from the fact that the operators $T_{s,t}\colon X \to Y$ defined by

$$T_{s,t}f(u) = \left(s^{-1}\int_0^s f(v)\,dv\right)\chi_{(0,t)}(u)$$

have the norm $\varphi_Y(t)/\varphi_X(s)$. Indeed, we have

$$\|T_{s,t}f\|_Y = \left|s^{-1}\int_0^s f(v)\,dv\right| \varphi_Y(t) \leqslant s^{-1}\|f\|_X \|\chi_{(0,s)}\|_{X'} \varphi_Y(t)$$
$$= \|f\|_X \varphi_Y(t)/\varphi_X(s)$$

and

$$\|T_{s,t}\chi_{(0,s)}\|_Y = \|\chi_{(0,t)}\|_Y.$$

Example 19. The spaces $L_p(0, \infty)$ and $L_q(0, \infty)$ are not strong interpolation spaces for $(L_p(0, \infty)+L_q(0, \infty), L_p(0, \infty)\cap L_q(0, \infty))$ for any $1 \leqslant p < q \leqslant \infty$. It is sufficient to see that inequality (6.11) does not hold.

Problem 6.8. If X is a strong interpolation space for (X_1, X_2), then

$$\min\{p(X_1), p(X_2)\} \leqslant \alpha(X) \leqslant \beta(X) \leqslant \max\{q(X_1), q(X_2)\}?$$

Applying Theorem 6.6, we have a positive answer to Problem 6.8 for r.i. spaces for which Boyd and Zippin indices coincide.

D. Indices and interpolation of compact operators.

Definition 6.9. Let (X, Y) be a strong interpolation pair for the segment $\sigma = [(X_1, Y_1), (X_2, Y_2)]$. We say that (X, Y) is a *compact interpolation pair* for σ if every linear operator which is a bounded linear operator from X_1 into Y_1 and from X_2 into Y_2 and which is also a compact operator in one of the two cases has a unique extension to a compact operator from X into Y. If $X_i = Y_i$, $i = 1, 2$, and $X = Y$ then X is called, in short, a *compact interpolation space* for (X_1, X_2).

Theorem 6.10. *An r.i. space X is a compact interpolation space for (L_p, L_q), $1 \leqslant p < q \leqslant \infty$ if and only if*

$$1/q < \alpha(X) \leqslant \beta(X) < 1/p.$$

Proof. For (L_1, L_∞), see Shimogaki [54]. For (L_p, L_∞), $1 \leqslant p < \infty$, see Lorentz and Shimogaki [32]. For general case, see Sharpley [52].

THEOREM 6.11 (Sharpley [52], Th. 3). *Suppose* Y_1 *and* Y_2 *have absolutely continuous norms. If*

$$\beta(X_1) < \alpha(X) \leqslant \beta(X) < \alpha(X_2) \quad \text{and} \quad \beta(Y_1) < \alpha(Y) \leqslant \beta(Y) < \alpha(Y_2),$$

then (X, Y) *is a compact interpolation pair for* $\sigma = [(X_1, Y_1), (X_2, Y_2)]$.

It is possible to generalize and give a necessary condition, but these topics will be dealt with elsewhere.

References

[1] N. K. Bari and S. B. Stechkin, *The best approximation and differential properties of two conjugate functions*, Trudy Mosk. Mat. Obshch. 5 (1956), 483–522 (in Russian).

[2] C. Bennett, *Banach function spaces and interpolation methods I. The abstract theory*, J. Functional Analysis 17 (1974), 409–440.

[3] Z. W. Birnbaum and W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Math. 3 (1931), 1-67.

[4] D. W. Boyd, *The Hilbert transform on rearrangement invariant spaces*, Canad. J. Math. 19 (1967), 599–616.

[5] — *The spectral radius of averaging operators*, Pacific J. Math. 24 (1968), 19–28.

[6] — *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. 21 (1969), 1245–1254.

[7] — *Monotone semigroups of operators on cones*, Canad. Math. Bull. 12 (1969), 299–309.

[8] — *Indices for the Orlicz spaces*, Pacific J. Math. 38 (1971), 315–323.

[9] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), 273–299.

[10] C. E. Cleaver, *Packing spheres in Orlicz spaces*, Pacific J. Math. 65 (1976), 325–335.

[11] R. Cooper, *The converses of the Cauchy-Hölder inequality and the solutions of the inequality* $g(x+y) \leqslant g(x)+g(y)$, Proc. London Math. Soc. 26 (1927), 415–432.

[12] A. A. Dmitriev and E. M. Semenov, *On operators of weak type* (1, 1), Sibirsk. Mat. Zh. 20 (1979), 656–658 (in Russian).

[13] D. Drasin and D. F. Shea, *Pólya peaks and the oscillation of positive functions*, Proc. Amer. Math. Soc. 34 (1972), 403–411.

[14] F. Féher, *A note on weak-type interpolation and function spaces*, Bull. London Math. Soc. 12 (1980), 443–451.

[15] C. M. Goldie, *Convergence theorems for empirical Lorentz curves and their inverses*, Adv. in Appl. Probab. 9 (1977), 765–791.

[16] J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. 60 (1977), 33–59.

[17] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Providence, 1957.

[18] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric Structures in Banach Spaces*, Memoirs Amer. Math. Soc. 1979.

[19] N. J. Kalton, *Orlicz sequence spaces without local convexity*, Math. Proc. Camb. Phil. Soc. 81 (1977), 253–277.

[20] S. Koizumi, *On the singular integrals I*, Proc. Japan Acad. 34 (1958), 193–198.

[21] M. A. Krasnoselski and Ya. B. Rutícki, *Convex functions and Orlicz spaces*, P. Noordhoof, Groningen, 1961.

[22] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow 1978 (in Russian).

[23] S. G. Krein and E. M. Semenov, *Interpolation of operators of weakened type*, Funkctional. Anal. i Prilozhen 7 (1973), 89–90 (in Russian).

[24] R. Leśniewicz and W. Orlicz, *On generalized variations II*, Studia Math. 45 (1973), 71–109.

[25] K. J. Lindberg, *On subspaces of Orlicz sequence spaces*, Studia Math. 45 (1973), 119–146.

[26] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces III*, Israel J. Math. 14 (1973), 368–389.

[27] –, – *Classical Banach spaces I.* Sequence Spaces, Springer-Verlag, Berlin–Heidelberg–New York 1977.

[28] –, – *Classical Banach Spaces II. Function Spaces*, Springer-Verlag, Berlin–Heidelberg–New York 1979.

[29] G. G. Lorentz, *On the theory of spaces Λ*, Pacific J. Math. 1 (1951), 411–429.

[30] G. G. Lorentz and T. Shimogaki, *Interpolation theorems for operators in function spaces*, J. Functional Analysis 2 (1968), 31–51.

[31] –, – *Majorants for interpolation theorems*, Publ. Ramanujan Inst. 1 (1969), 115–122.

[32] –, – *Interpolation theorems for the pairs of spaces* (L^p, L^∞) *and* (L^1, L^q), Trans. Amer. Math. Soc. 159 (1971), 207–221.

[33] W. A. J. Luxemburg, *Banach function spaces*, Thesis, Delft Technical Univ. 1955.

[34] L. Maligranda, *Interpolation of Lipschitz operators for the pairs of spaces* (L^p, L^∞), *and* (l^p, C_0), $0 < p < \infty$, Functiones et Approximatio 9 (1980), 107–115.

[35] – *A generalization of the Shimogaki theorem*, Studia Math. 71 (1981), 69–83.

[36] – *Generalized Hardy inequalities in rearrangement invariant spaces*, J. Math. Pures Appl. 59 (1980), 405–415.

[37] R. A. Maller, *Relative stability, characteristic functions and stochastic compactness*, J. Austral. Math. Soc. Ser. A, 28 (1979), 499–509.

[38] W. Matuszewska, *Some further properties of φ-functions*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 9 (1961), 445–450.

[39] – *Regularly increasing functions in connection with the theory of* $L^{*\varphi}$*-spaces*, Studia Math. 21 (1962), 317–344.

[40] – *On a generalization of regularly increasing functions*, Studia Math. 24 (1964), 271–279.

[41] W. Matuszewska and W. Orlicz, *On certain properties of φ-functions*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 8 (1960), 439–443.

[42] –, – *On some classes of functions with regard to their orders of growth*, Studia Math. 26 (1965), 11–24.

[43] S. Mazur and W. Orlicz, *On some classes of linear spaces*, Studia Math. 17 (1958), 97–119.

[44] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. 44 (1972), 31–38.

[45] N. I. Nielsen, *On the Orlicz function spaces* $L_M(0, \infty)$, Israel Jour. Math. 20 (1975), 237–259.

[46] A. M. Ostrowski, *On Cauchy–Frullani integrals*, Comment. Math. Helv. 51 (1976), 57–91.

[47] E. A. Pavlov, *On the Calderón type operator*, Analysis Mathematica, 4 (1978), 117–124 (in Russian).

[48] L. A. Rubel, *A pathological Lebesgue-measurable function*, Jour. London Math. Soc. 38 (1963), 1–4.

[49] E. M. Semenov, *A new interpolation theorem*, Funkctional. Anal. i Prilozhen. 2 (1968), 158–168 (in Russian).

[50] E. Seneta, *Regularly varying functions*, Lecture Notes in Mathematics 508, Springer-Verlag, Berlin–Heidelberg–New York 1976.

[51] R. Sharpley, *Spaces* $\Lambda_\alpha(X)$ *and interpolation*, J. Functional Analysis 11 (1972), 479–513.

[52] – *Interpolation theorems for compact operators*, Indiana Univ. Math. J. 22 (1973), 965–984.

[53] – *Interpolation of n pairs and counterexamples employing indices*, J. Approximation Theory 13 (1975), 117–127.

[54] T. Shimogaki, *On the complete continuity of operators in an interpolation theorem*, Jour. Fac. Sci. Hokkaido Univ. Ser. I. 20 (1968), 109–114.

[55] — *A note on norms of compression operators on function spaces*, Proc. Japan Acad. 46 (1970), 239–242.

[56] — *Indices of function spaces* (preprint).

[57] H. Silverman, *Properties of Polya peaks*, Rocky Mountain Math. J. 1 (1971), 649–656.

[58] I. B. Simonenko, *Interpolation and extrapolation of linear operators in Orlicz spaces*, Mat. Sb. 63 (1964), 536–553 (in Russian).

[59] G. Sparr, *Interpolation of weighted L_p-spaces*, Studia Math. 62 (1978), 229–271.

[60] A. Torchinsky, *Interpolation of operations and Orlicz spaces*, Studia Math. 59 (1976), 177–207.

[61] P. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Diss. Math. 131 (1976).

[62] S. Yamamuro, *Exponents of modulared semi-ordered linear spaces*, J. Fac. Sci. Hokkaido Univ. 12 (1953), 211–253.

[63] — *Modulared sequence spaces*, Jour. Fac. Sci. Hokkaido Univ. Ser. I 13 (1954), 1–12.

[64] M. Zippin, *Interpolation of operators of weak type between rearrangement invariant spaces*, J. Functional Analysis 7 (1971), 267–284.

[65] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Vol. I, II, 1959.

References added in proof

[66] N. H. Bingham and C. M. Goldie, *Extensions of regular variation*, I: *Uniformity and quantifiers*; II: *Representations and indices*, Proc. London Math. Soc. 44 (1982), 473–496, 497–534.

[67] V. I. Dmitriev and S. G. Krein, *Interpolation of operators of weak type*, Analysis Mathematica 4 (1978), 83–99.

[68] V. K. Dzjadyk, *An introduction to the theory of uniformly approximation functions by polynomials*, Nauka, Moscow 1977 (in Russian).

[69] F. Fehér, *The Marcinkiewicz Interpolation Theorem for rearrangement-invariant function spaces and applications*, Zeit. Anal. and ihre Anwend. 2 (1983), 111–125.

[70] L. Maligranda, *On Hardy's inequality in weighted rearrangement invariant spaces and applications I, II*, Proc. Amer. Math. Soc. 88 (1983), 67–74, 75–80.

[71] E. A. Pavlov, *On Calderón's operator*, Ukr. Mat. Žurnal 33 (1981), 142–143.

[72] Y. Sagher, *Real interpolation with weights*, Indiana Univ. Math. J. 30 (1981), 113–121.

[73] J. O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. 28 (1979), 511–544.