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GLOBAL STRONG SOLUTIONS OF VLASOV'S EQUATION – NECESSARY AND SUFFICIENT CONDITIONS FOR THEIR EXISTENCE

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§ 0. Introduction

Vlasov's equation in three dimensions is the continuous equivalent of the n-body problem of celestial mechanics. The question if the initial value problem always has a global solution (for "nice" initial data) is still open.

There are certain quantities, which are connected with a solution, and global existence is equivalent to a priori estimates for them. The investigation of these matters is the subject of the present paper.

The most interesting new result is the fact that a local solution that breaks down after a finite time cannot have a support that remains in a bounded subset of position space (cf. (2.3,vii)). Hitherto only the analogous result for velocity space was known. ((2.3,vi) is a sharp version of this fact.) At least in the case of a gravitational force ($\gamma = -1$) it is easy to conceive that the system will always be kept together by mutual attraction.

§ 1. Definitions and basic facts

We use the same notation as in [7] (which is quite standard anyway). Therefore we only repeat some definitions and hope that the rest is evident.

$L_p(\boldsymbol{R}^M, \boldsymbol{R}^L)$ ($C_b^k(\boldsymbol{R}^M, \boldsymbol{R}^L)$, $\mathrm{Lip}(\boldsymbol{R}^M, \boldsymbol{R}^L)$) denotes the space of functions $f\colon \boldsymbol{R}^M \to \boldsymbol{R}^L$ such that $|f|^p$ is Lebesgue integrable (f and its derivatives of order $\leqslant k$ are bounded, f is Lipschitz continuous). If $L = 1$, we omit the second argument $\boldsymbol{R}^L$. $C_c^k(\boldsymbol{R}^M, \boldsymbol{R}^L)$ consists of the functions in $C_b^k(\boldsymbol{R}^M, \boldsymbol{R}^L)$ that have a compact support. $\mathrm{lip}(f)$ is the smallest Lipschitz constant for $f \in \mathrm{Lip}(\boldsymbol{R}^M, \boldsymbol{R}^L)$. Everything is calculated with respect to the Euclidean norm on $\boldsymbol{R}^M$ and $\boldsymbol{R}^L$. $\int f(z)\,dz$ always means an integral over the full space. If

$p \in [1, \infty[$, then p' denotes the adjoint index from Hölder's inequality with $p'^{-1} + p^{-1} = 1$.

(1.1) DEFINITION of the initial value problem. If $x \in \boldsymbol{R}^6$, we write $x = (x_s, x_v)$ with $x_s, x_v \in \boldsymbol{R}^3$ (space and velocity components). Assume $\gamma \in \{-1, +1\}$, $\varepsilon \geqslant 0$ and let for $z \in \boldsymbol{R}^3$

$$u^\varepsilon(z) := \gamma(z^2+\varepsilon)^{-1/2},$$

$$e^\varepsilon(z) := -\operatorname{grad} u^\varepsilon(z) = \gamma(z^2+\varepsilon)^{-3/2} z.$$

Assume that $\varphi \in C_c^1(\boldsymbol{R}^6)$, $\varphi \geqslant 0$ and $I \subset [0, \infty[$ is an interval with $0 \in I$. We say that $\Phi^\varepsilon\colon I \times \boldsymbol{R}^6 \to \boldsymbol{R}$ is a *(strong) solution of the (initial value) problem* P^ε on I if

(i) $\Phi^\varepsilon(0, \cdot) = \varphi$.

(ii) $\big(y \mapsto e^\varepsilon(x_s - y_s)\Phi^\varepsilon(t, y)\big) \in L_1(\boldsymbol{R}^3)$ for all $t \in I$, $x_s \in \boldsymbol{R}^3$.

(iii) If $E^\varepsilon(t, x_s) := \int e^\varepsilon(x_s - y_s)\Phi^\varepsilon(t, y)\,dy$, then E^ε is continuous on $I \times \boldsymbol{R}^3$ and $E^\varepsilon(t, \cdot)$ is Lipschitz continuous on $\boldsymbol{R}^3$, uniformly for all t in compact subsets of I, i.e. there exists a nondecreasing function $g^\varepsilon\colon I \to [0, \infty[$ such that

$$|E^\varepsilon(t, x_s) - E^\varepsilon(t, y_s)| \leqslant g^\varepsilon(t)\,|x_s - y_s|$$

for all $t \in I$, $x_s, y_s \in \boldsymbol{R}^3$.

(iv) Φ^ε is an integral of the "characteristic" system of ordinary differential equations

$$\dot{X}_s = X_v, \qquad \dot{X}_v = E^\varepsilon(t, X_s), \tag{1}$$

i.e. every solution $X\colon I \to \boldsymbol{R}^6$ of (1) satisfies

$$\Phi^\varepsilon(t, X(t)) = \Phi^\varepsilon(0, X(0)) = \varphi(X(0))$$

for all $t \in I$. We call Φ^ε a *global* solution if $I = [0, \infty[$, otherwise a *local* solution.

We denote by C "world constants", by K constants that may depend on φ. These constants will never depend on ε, dependence on ε is always explicitly denoted. (That K does not depend on ε may have been proved just at that instant or it may be an assumption at that particular point of the argument.) The same letter C or K may denote different constants, even on two sides of an inequality.

By abuse of notation we write $\Phi^\varepsilon(t)$ for the function $\Phi^\varepsilon(t, \cdot)\colon \boldsymbol{R}^6 \to \boldsymbol{R}$, $E^\varepsilon(t)$ for $E^\varepsilon(t, \cdot)$, etc.

Remarks. (i) What we are really interested in is the initial value problem P^0. The "mollified" problems P^ε, $\varepsilon > 0$, are mainly a convenient instrument for finding solutions of P^0.

(ii) A strong solution of P^ε is differentiable on $I \times \boldsymbol{R}^6$ and satisfies

"Vlasov's equation"

$$\frac{\partial}{\partial t}\Phi^\varepsilon + x_v \frac{\partial}{\partial x_s}\Phi^\varepsilon + E^\varepsilon(t, x_s)\frac{\partial}{\partial x_v}\Phi^\varepsilon = 0$$

(cf. [7]). We will not make use of this fact.

(iii) The assumption $\varphi \in C_c^1(\boldsymbol{R}^6)$ could easily be weakened along the lines of [7]. We never use the compact support of φ in an essential way. We only use the fact that φ vanishes sufficiently fast at infinity where "sufficiently" means different things in different theorems. Making minimal assumptions, we could easily create 144 theorems out of Theorem (2.3).

(iv) Our definition of "strong" solutions is equivalent to that of "classical" solutions in [7], cf. [9, remark after (1.2)]. In the present paper "solution" always means "strong solution".

(v) We only consider the three-dimensional case. In dimensions one and two there always exist global solutions, cf. [7], in dimensions greater than three there exist counterexamples, cf. [8].

(1.2) THEOREM. *If $\varepsilon > 0$, there exists a unique global solution Φ^ε of P^ε. If $\varepsilon = 0$, there exists a unique local solution Φ^0 on a maximal interval $[0, T[$ with $T \in]0, \infty]$.*

Proof: [7].

Remarks. (i) If a solution Φ^0 of P^0 exists on I, then we always have $\Phi^0 = \lim \Phi^\varepsilon$ as $\varepsilon \to 0$. Cf. [7] for details.

(ii) It is known that under additional assumptions on φ, P^0 will have a global solution, e.g. if φ has "rotational" symmetry, cf. [8], or if φ is "small", cf. [2].

(1.3) DEFINITION and THEOREM (properties of strong solutions). *Assume that Φ^ε is a solution of P^ε on I. For $x \in \boldsymbol{R}^6$, $t, \tau \in I$ let $X^\varepsilon(t, \tau, x)$ denote the solution of*

(2) $$\dot{X}^\varepsilon = (X_v^\varepsilon, E^\varepsilon(t, X_s^\varepsilon))$$

with initial condition

(3) $$X^\varepsilon(\tau, \tau, x) = x.$$

Then

(i) *X^ε is continuous on $I \times I \times \boldsymbol{R}^6$.*

(ii) *$X^\varepsilon(t, \tau)$ is a (Lebesgue) measure-preserving homeomorphism from $\boldsymbol{R}^6$ onto $\boldsymbol{R}^6$ for all $t, \tau \in I$.*

(iii) *$X^\varepsilon(t, \tau) \circ X^\varepsilon(\tau, \theta) = X^\varepsilon(t, \theta)$ and thus $(X^\varepsilon(t, \tau))^{-1} = X^\varepsilon(\tau, t)$ for all $t, \tau, \theta \in I$.*

(iv) *$\Phi^\varepsilon(t) = \varphi(X^\varepsilon(0, t))$ for all $t \in I$.*

Proof: [7].

As usual, existence of solutions is closely connected with a priori estimates. In the case of P^0 it is convenient to formulate them in terms of the solutions of P^ε, $\varepsilon > 0$. Before we state these estimates we have to define the quantities we are interested in.

(1.4) DEFINITION. Assume that Φ^ε is a solution of P^ε on I. For $t \in I$, $x_s \in \boldsymbol{R}^3$ let

(i) $\varrho^\varepsilon(t, x_s) := \int \Phi^\varepsilon(t, x)\,dx_v$

(the density in position space $\boldsymbol{R}^3$),

(ii) $U^\varepsilon(t, x_s) := \int u^\varepsilon(x_s - y_s)\,\varrho^\varepsilon(t, y_s)\,dy_s$

(the potential),

(iii) $j^\varepsilon(t, x_s) := \int x_v\,\Phi^\varepsilon(t, x)\,dx_v$

(the current),

(iv) $f_V^\varepsilon(t) := \sup\{\big||X_v^\varepsilon(\tau, 0, x)| - |x_v|\big|;\ x \in \boldsymbol{R}^6,\ 0 \leqslant \tau \leqslant t\}$

(a function that measures how much the size of the velocities has changed).

Remark. Instead of f_V^ε slightly different functions were used in [6] and [7]:

$$f_I^\varepsilon(t) := \sup\{\big||X_v^\varepsilon(t, 0, x)| - |x_v|\big|;\ x \in \boldsymbol{R}^6\} \quad \text{in [6]},$$

$$f_v^\varepsilon(t) := \sup\{|X_v(\tau, 0, x) - x_v|;\ x \in \boldsymbol{R}^6,\ 0 \leqslant \tau \leqslant t\} \quad \text{in [7]}.$$

Basically they serve the same purpose. By definition

$$f_I^\varepsilon(t) \leqslant f_V^\varepsilon(t) \leqslant f_v^\varepsilon(t).$$

As a matter of fact, f_V^ε is just f_I^ε "made monotone", which makes it somewhat easier to handle, e.g. we do not have to worry about measurability. We preferred f_V^ε to f_v^ε in view of equation (7) below and its consequences. Its only disadvantage is an aesthetic one: It is not Galilei invariant.

(1.5) PROPOSITION. *Assume that Φ^ε is a solution of P^ε on I. Then*

(i) $\Phi^\varepsilon(t) \in C_b^1(\boldsymbol{R}^6) \cap L_1(\boldsymbol{R}^6)$ *for all* $t \in I$.

(ii) $\varrho^\varepsilon(t) \in C_b^1(\boldsymbol{R}^3) \cap L_1(\boldsymbol{R}^3)$ *for all* $t \in I$ *and*

$$\|\varrho^\varepsilon(t)\|_\infty \leqslant K\left(1 + f_V^\varepsilon(t)\right)^3.$$

(iii) $j^\varepsilon(t) \in C_b^0(\boldsymbol{R}^3, \boldsymbol{R}^3) \cap L_1(\boldsymbol{R}^3, \boldsymbol{R}^3)$ *for all* $t \in I$ *and*

$$\|j^\varepsilon(t)\|_\infty \leqslant K\left(1 + f_V(t)\right)^4.$$

(iv) $U^\varepsilon \in C_b^1(I \times \boldsymbol{R}^3)$ *and for all* $t \in I$, $x_s \in \boldsymbol{R}^3$

$$E^\varepsilon(t, x_s) = -\operatorname{grad}_{x_s} U^\varepsilon(t, x_s),$$

$$\dot{U}^\varepsilon(t, x_s) := \frac{\partial}{\partial t} U^\varepsilon(t, x_s) = -\int e^\varepsilon(x_s - y_s)\,j^\varepsilon(t, y_s)\,dy_s, \tag{4}$$

$$\|U^\varepsilon(t)\|_\infty \leqslant C\,\|\varrho^\varepsilon(t)\|_p^\lambda \|\varrho^\varepsilon(t)\|_q^{1-\lambda}$$

for all $1 \leqslant q < 3/2 < p \leqslant \infty$ *with* $\lambda = (1/q-2/3)/(1/q-1/p)$;

$$\|E^\varepsilon(t)\|_\infty \leqslant C\,\|\varrho^\varepsilon(t)\|_p^\lambda \|\varrho^\varepsilon(t)\|_q^{1-\lambda}$$

for all $1 \leqslant q < 3 < p \leqslant \infty$ *with* $\lambda = (1/q-1/3)/(1/q-1/p)$;

$$\|\dot U^\varepsilon(t)\|_\infty \leqslant C\,\|j^\varepsilon(t)\|_p^\lambda \|j^\varepsilon(t)\|_q^{1-\lambda}$$

for all $1 \leqslant q < 3 < p \leqslant \infty$ *with* $\lambda = (1/q-1/3)/(1/q-1/p)$.

Furthermore, $U^\varepsilon(t) \in C_b^2(\boldsymbol{R}^3)$ *and, if* $\varepsilon = 0$,

$$\Delta_{x_s} U^0(t, x_s) = -4\pi\gamma\varrho^0(t, x_s). \tag{5}$$

Proof. All this can be found in [6] and [7].

Remark. (5) is Poisson's equation. Therefore some authors call P^0 the initial value problem for the Vlasov–Poisson system.

(1.6) LEMMA. *Assume* $0 \leqslant \alpha \leqslant \beta$, $\psi \in L_\infty(\boldsymbol{R}^6)$, $\psi \geqslant 0$, $J := \int |x_v|^\alpha \psi(x)\,dx < \infty$. *Let*

$$\psi_\beta(x_s) := \int |x_v|^\beta \psi(x)\,dx_v, \qquad x_s \in \boldsymbol{R}^3.$$

Then $\psi_\beta \in L_r(\boldsymbol{R}^3)$ *with* $r = (3+\alpha)/(3+\beta)$ *and*

$$\|\psi_\beta\|_r \leqslant C\,\|\psi\|_\infty^{1/r'} J^{1/r}.$$

Proof. Analogous to [8, (5.5)].

Remark. The lemma shows that the r-norms of $\varrho^\varepsilon(t)$ and $j^\varepsilon(t)$ can be estimated by large velocities $|x_v|$. This is a typical result.

(1.7) LEMMA (Sobolev's inequality). *Let* $r, q \in]1, \infty[$, $\lambda \in [0, M[$, $r^{-1} - q^{-1} + \lambda/M = 1$, $f_\lambda(z) = |z|^{-\lambda}$ *for* $z \in \boldsymbol{R}^M$, $\sigma \in L_r(\boldsymbol{R}^M)$. *Then we have* $\sigma * f_\lambda \in L_q(\boldsymbol{R}^M)$ *and*

$$\|\sigma * f_\lambda\|_q \leqslant C\,\|\sigma\|_r.$$

($*$ denotes convolution.)

Proof: [12, p. 31, Example 3].

(1.8) LEMMA. *Assume that* $\sigma \in L_\infty(\boldsymbol{R}^3)$, $\sigma \geqslant 0$, $\alpha \in]3, 6]$ *and*

$$J := \int |z|^\alpha \sigma(z)\,dz < \infty.$$

Let $\beta \in]3, \alpha[$ *such that* $2\beta < 3+\alpha$ *and let* $\gamma := (3+\alpha-\beta)/\beta > 1$. *Then we have for all* $w \in \boldsymbol{R}^3$

$$\int \frac{\sigma(z)}{|w-z|^2}\,dz \leqslant C\,|w|^{-\gamma}\,\|\sigma\|_\infty^{1-1/\beta} J^{1/\beta}.$$

Proof. We note that $\beta'\alpha/\beta < 3$ and $\beta'(2+\alpha/\beta) > 3$. Thus

$$\int \frac{\sigma(z)}{|w-z|^2}\,dz = \int \frac{1}{|w-z|^2\,|z|^{\alpha/\beta}}\,|z|^{\alpha/\beta}\,\sigma(z)\,dz$$

$$\leqslant \left(\int \frac{dz}{|w-z|^{2\beta'}\,|z|^{\beta'\alpha/\beta}}\right)^{1/\beta'} \left(\int |z|^{\alpha}\big(\sigma(z)\big)^{\beta}\,dz\right)^{1/\beta}$$

$$= C\,|w|^{-\gamma}\left(\int |z|^{\alpha}\big(\sigma(z)\big)^{\beta}\,dz\right)^{1/\beta} \leqslant C\,|w|^{-\gamma}\,\|\sigma\|_{\infty}^{1-1/\beta}\,J^{1/\beta}.$$

§ 2. Global existence and a priori estimates

We start with a collection of estimates that do not depend on ε.

(2.1) THEOREM. *Assume that Φ^{ε} is a solution of P^{ε} on I. Then we have for all $t \in I$*

(i) $\|\Phi^{\varepsilon}(t)\|_p = \|\varphi\|_p$ *if* $1 \leqslant p \leqslant \infty$.

(ii) $\|\varrho^{\varepsilon}(t)\|_1 = \|\varrho^{\varepsilon}(0)\|_1 = \|\varphi\|_1$.

(iii) $\int |x_v|^{\alpha}\,\Phi^{\varepsilon}(t,x)\,dx \leqslant K$ *if* $0 \leqslant \alpha \leqslant 2$.

(iv) $\int |x_s|^{\alpha}\,\Phi^{\varepsilon}(t,x)\,dx \leqslant K(1+t)^{\alpha}$ *if* $0 \leqslant \alpha \leqslant 2$.

(v) $\|\varrho^{\varepsilon}(t)\|_p \leqslant K$ *if* $1 \leqslant p \leqslant 5/3$.

(vi) $\|j^{\varepsilon}(t)\|_p \leqslant K$ *if* $1 \leqslant p \leqslant 5/4$.

(vii) $\|E^{\varepsilon}(t)\|_p \leqslant K$ *if* $3/2 < p < 15/4$.

(viii) $\|U^{\varepsilon}(t)\|_p \leqslant K$ *if* $3 < p \leqslant \infty$.

(ix) $\|\dot{U}^{\varepsilon}(t)\|_p \leqslant K$ *if* $3/2 < p < 15/7$.

Proof. (i) is obvious since $\Phi^{\varepsilon}(t) = \varphi \circ X^{\varepsilon}(0,t)$ and $X^{\varepsilon}(0,t)$ is measure-preserving.

(ii) follows from (i).

(iii) is proved for $\alpha = 2$ in [8, (5.8)] and for $\alpha = 0$ in (ii). For $0 < \alpha < 2$ it follows from Hölder's inequality.

(iv) for $\alpha = 0$ follows from (ii). We will show it for $\alpha = 2$, for general α it then follows from Hölder's inequality. We use this opportunity to present an important method of proof with more detail. In order to compute the derivative of $\int |x_s|^2\,\Phi^{\varepsilon}(t,x)\,dx$ we use Theorem (1.3,ii–iv) and the differential equation (2):

$$\frac{d}{dt}\int |x_s|^2\,\Phi^{\varepsilon}(t,x)\,dx = \frac{d}{dt}\int |x_s|^2\,\varphi\big(X^{\varepsilon}(0,t,x)\big)\,dx$$

$$= \frac{d}{dt}\int |X_s^{\varepsilon}(t,0,x)|^2\,\varphi(x)\,dx = \int 2X_s^{\varepsilon}(t,0,x)\,X_v^{\varepsilon}(t,0,x)\,\varphi(x)\,dx$$

$$= \int 2x_s\,x_v\,\Phi^{\varepsilon}(t,x)\,dx.$$

We will take this kind of computation for granted from now on. A rigorous justification of the differentiation under the integral sign can be found in [8]. Thus we have

$$\frac{d}{dt}\left(\int |x_s|^2 \Phi^\varepsilon(t, x)\, dx\right)^{1/2} = \int x_s\, x_v\, \Phi^\varepsilon(t, x)\, dx / \left(\int |x_s|^2 \Phi^\varepsilon(t, x)\, dx\right)^{1/2}$$

$$\leqslant \left(\int |x_v|^2 \Phi^\varepsilon(t, x)\, dx\right)^{1/2} \leqslant K$$

by Cauchy–Schwarz and (iii) and thus

$$\int |x_s|^2 \Phi^\varepsilon(t, x)\, dx \leqslant K(1+t)^2.$$

(v) and (vi) follow from (ii), (iii) and Lemma (1.6).

(vii), (viii) for $p < \infty$ and (ix) follow from (v), equation (4) and Sobolev's inequality. (viii) for $p = \infty$ follows from (v) and (1.5,iv).

We have mentioned in § 1 why we need estimates for high velocities. The obvious estimate is the following:

$$\big||X_v^\varepsilon(t, 0, x)| - |x_v|\big| \leqslant |X_v^\varepsilon(t, 0, x) - x_v|$$

$$= \left|\int_0^t E^\varepsilon(\tau, X_s(\tau, 0, x))\, d\tau\right| \leqslant \int_0^t \|E^\varepsilon(\tau)\|_\infty\, d\tau.$$

Less obvious but very helpful is the next result:

(2.2) LEMMA. *Assume that $g \in L_1(\mathbf{R})$, $g \geqslant 0$, $I \subset \mathbf{R}$ is an interval and that $f\colon I \to \mathbf{R}$ is twice continuously differentiable with*

$$|f''(t)| \leqslant g(f(t))$$

for almost all $t \in I$. Then we have for all $t_1, t_2 \in I$

$$|f'(t_1) - f'(t_2)| \leqslant 2^{3/2}\|g\|_1^{1/2}.$$

Proof: [8, (6.5)].

We are now ready for our main theorem, which gives necessary and sufficient conditions for the existence of a solution of P^0 on an interval I. We formulate everything for compact intervals I. If I_1 is not compact, a solution exists on I_1 if and only if it exists on every compact subinterval.

(2.3) THEOREM. *Assume that I is the interval $[0, T]$ with some $T > 0$. Then the following statements are equivalent*:

(o) *There exists a solution of P^0 on I.*

(i) $\sup f_V^\varepsilon(t) < \infty$.

(ii) $\sup \|E^\varepsilon(t)\|_\infty < \infty$.

(iii) $\sup \|E^\varepsilon(t)\|_6 < \infty$.

(iv) $\sup \|\varrho^\varepsilon(t)\|_\infty < \infty$.

(v) $\sup \|\varrho^\varepsilon(t)\|_2 < \infty$.

(vi) $\sup \int |x_v|^3 \Phi^\varepsilon(t, x)\,dx < \infty$.

(vii) $\sup \int |x_s|^\alpha \Phi^\varepsilon(t, x)\,dx < \infty$ *for some* $\alpha > 3$.

(viii) $\sup \|j^\varepsilon(t)\|_\infty < \infty$.

(ix) $\sup \|j^\varepsilon(t)\|_{3/2} < \infty$.

(x) $\sup \|\dot{U}^\varepsilon(t)\|_\infty < \infty$.

(xi) $\sup \operatorname{lip} E^\varepsilon(t) < \infty$.

(xii) $\sup \operatorname{lip} \varrho^\varepsilon(t) < \infty$.

Each supremum is to be taken over all $t \in I$, $\varepsilon > 0$.

Remark. It is possible to reformulate the theorem in a way that makes no use of the solutions Φ^ε with $\varepsilon > 0$: Assume that Φ^0 is a solution of P^0 on $[0, T[$. Then a solution exists on $[0, T]$ (and, as we can then solve the initial value problem at the time T, on some larger interval) if and only if one (and then each) of the quantities above with $\varepsilon = 0$ remains bounded on $[0, T[$. The proof would use exactly the same estimates, but we prefer to make use of the functions Φ^ε, which are defined for all $t > 0$. Moreover, the functions Φ^ε may be helpful for the numerical solution of P^0.

Proof of (2.3). (o)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv) is proved in [7].

(ii)$\Rightarrow$(i):

$$f_v^\varepsilon(t) = \sup\{\,\big||X_v^\varepsilon(\tau, 0, x)| - |x_v|\big|;\ x \in \mathbf{R}^6,\ 0 \leqslant \tau \leqslant t\}$$
$$\leqslant \int_0^t \|E^\varepsilon(\tau)\|_\infty\, d\tau.$$

(i)$\Rightarrow$(o) is proved in [6].

(ii)$\Rightarrow$(iii): $\|E^\varepsilon(t)\|_6 \leqslant \|E^\varepsilon(t)\|_\infty^{2/3} \|E^\varepsilon(t)\|_2^{1/3}$. In combination with (2.1,vii) this proves the result.

(iii)$\Rightarrow$(vi):

$$\frac{d}{dt}\int |x_v|^3 \Phi^\varepsilon(t, x)\,dx = \int 3\,|x_v|\,x_v\, E^\varepsilon(t, x_s)\,\Phi^\varepsilon(t, x)\,dx$$
$$\leqslant 3\int |E^\varepsilon(t, x_s)| \Big(\int |x_v|^2 \Phi^\varepsilon(t, x)\,dx_v\Big)dx_s$$
$$\leqslant 3\,\|E^\varepsilon(t)\|_6 \Big(\int\Big(\int |x_v|^2 \Phi^\varepsilon(t, x)\,dx_v\Big)^{6/5} dx_s\Big)^{5/6}$$
$$\leqslant K\Big(\int |x_v|^3 \Phi^\varepsilon(t, x)\,dx\Big)^{5/6}$$

by Lemma (1.6). The result now follows from Gronwall's lemma (cf. [7, (2.7)]).

(vi)$\Rightarrow$(v) follows from Lemma (1.6).

(v)$\Rightarrow$(i): From Proposition (1.5,iv) we know that

$$\|E^\varepsilon(t)\|_\infty \leqslant C\,\|\varrho^\varepsilon(t)\|_2^{2/3}\,\|\varrho^\varepsilon(t)\|_\infty^{1/3} \leqslant K\big(1 + f_v^\varepsilon(t)\big).$$

Thus we get

$$f_V^\varepsilon(t) \leqslant \int_0^t \|E^\varepsilon(\tau)\|_\infty d\tau \leqslant K \int_0^t \left(1+f_V^\varepsilon(\tau)\right) d\tau$$

and with the help of Gronwall's lemma we find a bound for f_V^ε.

(ii) $\Rightarrow$ (vii): We first note that $\sup \int |x_v|^\alpha \Phi^\varepsilon(t, x)\,dx < \infty$. This follows from

$$\frac{d}{dt}\int |x_v|^\alpha \Phi^\varepsilon(t, x)\,dx = \int \alpha\,|x_v|^{\alpha-2} x_v E^\varepsilon(t, x_s)\,\Phi^\varepsilon(t, x)\,dx$$

$$\leqslant \alpha\,\|E^\varepsilon(t)\|_\infty \int |x_v|^{\alpha-1} \Phi^\varepsilon(t, x)\,dx \leqslant K\left(\int |x_v|^\alpha \Phi^\varepsilon(t, x)\,dx\right)^{1-1/\alpha}$$

and Gronwall's lemma. And thus

$$\frac{d}{dt}\int |x_s|^\alpha \Phi^\varepsilon(t, x)\,dx = \int \alpha\,|x_s|^{\alpha-2} x_s x_v \Phi^\varepsilon(t, x)\,dx \leqslant \int \alpha\,|x_s|^{\alpha-1} |x_v|\,\Phi^\varepsilon(t, x)\,dx$$

$$\leqslant K\left(\int |x_s|^\alpha \Phi^\varepsilon(t, x)\,dx\right)^{1-1/\alpha}$$

and again by Gronwall's lemma the result follows.

(vii) $\Rightarrow$ (i): We have two estimates for $|E^\varepsilon(t, x_s)|$.

First estimate (which makes no use of (vii)):

$$\|E^\varepsilon(t)\|_\infty \leqslant \|\varrho^\varepsilon(t)\|_{5/3}^{5/9}\,\|\varrho^\varepsilon(t)\|^{4/9} \leqslant K\left(1+f_V^\varepsilon(t)\right)^{4/3}$$

as follows from (1.5, ii, iv) and (2.1, v).

Second estimate: W.l.o.g. we assume $\alpha \leqslant 6$. (If $\alpha > 6$, then (vii) is also satisfied for $\tilde{\alpha} = 6$.) Take $\beta \in]3, \alpha[$ such that

$$7\beta < 15 + 2\alpha. \tag{6}$$

This implies $2\beta < 3+\alpha$ and therefore we get from Lemma (1.8)

$$|E^\varepsilon(t, x_s)| \leqslant K\,|x_s|^{-\gamma}\,\|\varrho^\varepsilon(t)\|_\infty^{1-1/\beta} \leqslant K\,|x_s|^{-\gamma}\left(1+f_V^\varepsilon(t)\right)^{3-3/\beta}$$

with $\gamma = (3+\alpha-\beta)/\beta$.

Now take any fixed $\varepsilon > 0$ and let $F := f_V^\varepsilon(T) = \sup\{f_V^\varepsilon(t);\ 0 \leqslant t \leqslant T\}$. Then our estimates imply

$$|E^\varepsilon(t, x_s)| \leqslant g(|x_s|)$$

with $g(r) := K \min\{(1+F)^{4/3},\ |r|^{-\gamma}(1+F)^{3-3/\beta}\}$, $r \in \mathbf{R}$.

For fixed $x \in \mathbf{R}^6$ and fixed $i \in \{1, 2, 3\}$ we let $f(t) := X_i^\varepsilon(t, 0, x)$. Then we get

$$|f''(t)| = \left|E_i^\varepsilon\left(t, X_s^\varepsilon(t, 0, x)\right)\right| \leqslant g\left(|X_s^\varepsilon(t, 0, x)|\right) \leqslant g\left(f(t)\right).$$

Lemma (2.2) now implies

$$|f'(t) - f'(0)| = |X_{v_i}^\varepsilon(t, 0, x) - x_{v_i}| \leqslant C\,\|g\|_1^{1/2}$$

and therefore

$$F \leqslant C\,\|g\|_1^{1/2} = K(1+F)^k$$

with $k := (3+4\alpha+\beta)/(18+6\alpha-6\beta) < 1$ (because of (6)). This shows that there exists a bound for F that does not depend on ε.

(i) $\Rightarrow$ (viii) follows from (1.5, iii).

(viii) $\Rightarrow$ (ix) follows from (2.1, vi) and Hölder's inequality.

(ix) $\Rightarrow$ (i): Using (1.5, iii, iv) and (2.1, vi, viii) we get for all $t \in I$

$$\begin{aligned}\|\dot U^\varepsilon(t)\|_\infty &\leqslant C\,\|j^\varepsilon(t)\|_{3/2}^{1/2}\,\|j^\varepsilon(t)\|_\infty^{1/2}\\ &\leqslant K\,\|j^\varepsilon(t)\|_\infty^{1/2} \leqslant K\big(1+f_V^\varepsilon(t)\big)^2\end{aligned}$$

and

$$\|U^\varepsilon(t)\|_\infty \leqslant K.$$

Furthermore, it is easily shown by differentiation that for all $t \in I$, $x \in \boldsymbol{R}^6$

$$\begin{aligned}(7)\quad & |X_v^\varepsilon(t,0,x)|^2 - |x_v|^2\\ &= 2\Big(U^\varepsilon(0,x_s) - U^\varepsilon\big(t, X_s^\varepsilon(t,0,x)\big) + \int_0^t \dot U^\varepsilon\big(\tau, X_s^\varepsilon(\tau,0,x)\big)\,d\tau\Big).\end{aligned}$$

This shows that

$$\begin{aligned}f_V^\varepsilon(t)^2 &= \sup\big\{\big||X_v^\varepsilon(\tau,0,x)| - |x_v|\big|^2;\ x \in \boldsymbol{R}^6,\ 0 \leqslant \tau \leqslant t\big\}\\ &\leqslant \sup\big\{\big||X_v^\varepsilon(\tau,0,x)|^2 - |x_v|^2\big|;\ x \in \boldsymbol{R}^6,\ 0 \leqslant \tau \leqslant t\big\}\\ &\leqslant K\Big(1 + \int_0^t \big(1+f_V^\varepsilon(\tau)\big)^2\,d\tau\Big).\end{aligned}$$

With Gronwall's lemma we now get a bound for $f_V^\varepsilon(t)$ that does not depend on ε.

(viii) $\Rightarrow$ (x) follows from (1.5, iv) and (2.1, vi).

(x) $\Rightarrow$ (i) can be shown by an argument similar to, but easier than, the implication (ix) $\Rightarrow$ (i): The right-hand side of (7) is bounded uniformly in ε.

(o) $\Rightarrow$ (xi) is shown in [7].

(xi) $\Rightarrow$ (ii) follows from [9, (2.2)] and (2.1, ii).

(ii) $\Rightarrow$ (xii) is shown in [7].

(xii) $\Rightarrow$ (iv): It is easy to show that every $\sigma \in \mathrm{Lip}(\boldsymbol{R}^M) \cap L_1(\boldsymbol{R}_M)$ is bounded. In fact $\|\sigma\|_\infty \leqslant C\,\|\sigma\|_1^{1/(M+1)}(\mathrm{lip}\,\sigma)^{M/(M+1)}$. Therefore (2.1, ii) and (xii) yield (iv).

Concluding remark. Let us assume that there exists a function $\varphi \in C_c^1(\boldsymbol{R}^6)$ such that the initial value problem P^0 has no global solution. Then we have a local solution Φ^0 on an interval $[0, T[$ that cannot be continued to a (strong) solution on $[0, T]$. What can we say about this elusive object Φ^0? Its support is unbounded in velocity space (cf. (2.3, vi)) and position space (cf.

(2.3, vii)), but there are not too many high positions and velocities (cf. (2.1, iii, iv)). It cannot have rotational symmetry (cf. [8]). It cannot be too small (cf. [2]). It can be continued to a weak solution on [0, T] (cf. [9], note that the unique strong solution on [0, T[and the not necessarily unique weak solution constructed in [9] coincide). The potential U^ε remains bounded (cf. (2.1, viii)), but its derivatives $-E^\varepsilon$ and $\dot{U}^\varepsilon$ do not (cf. (2.3, ii, x)).

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