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VARIATIONAL APPROXIMATION METHODS FOR EIGENVALUES CONVERGENCE THEOREMS

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0. Introduction

Consider an eigenvalue problem of the form

$$Bu = \lambda Au, \tag{0.1}$$

where A, B are linear transformations of a (complex) vector space V into a vector space W. Assume that there exists a sesquilinear functional (or pairing) σ on $V \times W$ which separates the points of W and that A and B are symmetric with respect to this pairing:

$$\sigma(u, Av) = \overline{\sigma(v, Au)}, \quad \sigma(u, Bv) = \overline{\sigma(v, Bu)}.$$

A and B then define real quadratic forms $\mathfrak{A}(u) = \sigma(u, Au)$, $\mathfrak{B}(u) = \sigma(u, Bu)$ on V. If $\mathfrak{A}$ is positive definite and $\mathfrak{B}$ is semi-bounded above with respect to $\mathfrak{A}$, then we may, in many applications, replace the problem of finding the eigenvalues in (0.1) by that of finding the stationary values of the Rayleigh quotient $\mathfrak{B}(u)/\mathfrak{A}(u)$. This variational formulation of the eigenvalue problem allows one to apply variational techniques to find approximations to the eigenvalues.

In applications the linear transformations A and B are often differential operators, V and W are spaces of smooth functions, and σ is the L^2 scalar product. Also, A and B are symmetric (formally self-adjoint), A is elliptic, and the order of B is less than that of A.

Our aim in what follows is to give a brief introduction to variational approximation methods for eigenvalues, including a detailed discussion of convergence of the approximations. Since these methods are detailed quite thoroughly in the standard reference texts [1], [8], [13], [14], we include here neither proofs nor extensive references. Exceptions to this policy are the convergence theorems of Sections 4 and 5. These theorems are not found in the standard texts cited, and we include complete proofs for them.

A precise definition for the variational eigenvalues of $\mathfrak{B}$ with respect to $\mathfrak{A}$ on V is given in Section 1, and the Monotony Theorems are stated. These theorems are basic to the general variational method of eigenvalue approximation described in Section 2. In Section 3 a brief description of the well-known Rayleigh–Ritz approximation method is given. Section 4 contains a description of the Weinstein method; a convergence theorem for this method is proved. Section 5 is devoted to the Aronszajn method. Using a construction due to Weinberger, we are able to deduce convergence theorems for Aronszajn's method from the convergence theorem of Section 4. Section 6 is entirely taken up by a counterexample using Aronszajn's method. The counterexample shows that certain natural conditions which imply convergence of the approximations in many special circumstances are not sufficient to insure convergence in all cases.

Section 7 contains a few concluding remarks.

The reader should be warned that terminology and notations in this field are not completely standardized. We use notations and definitions similar to those of [1] and [13].

1. Variational eigenvalue problems and the Monotony Theorems

We shall consider, for the remainder of this talk, eigenvalue problems for pairs $\mathfrak{A}, \mathfrak{B}$ of real quadratic forms on vector spaces V which satisfy three hypotheses:

HYPOTHESIS 1.1. $\mathfrak{A}$ is positive definite on V; i.e. $\mathfrak{A}(u) > 0$ for all $u \in V$, $u \neq 0$.

HYPOTHESIS 1.2. $\mathfrak{B}$ is semi-bounded above with respect to $\mathfrak{A}$; i.e., there exists a real constant c such that $\mathfrak{B}(u) \leqslant c\mathfrak{A}(u)$ for all $u \in V$.

The real quadratic forms $\mathfrak{A}$ and $\mathfrak{B}$ uniquely determine associated symmetric sesquilinear forms, also denoted by $\mathfrak{A}$ and $\mathfrak{B}$ respectively. By Hypothesis 1.1 $\mathfrak{A}$ determines a scalar product on V. Let H be the Hilbert space completion of V with respect to $\mathfrak{A}$; then $\mathfrak{B}$ determines a sesquilinear form in H with domain V. If $\mathfrak{B}$ is closable, then its closure (also denoted by $\mathfrak{B}$) is a closed form with domain $\mathfrak{D}(\mathfrak{B}) \supset V$, and there exists an associated self-adjoint operator T in H such that

$$\mathfrak{B}(u, v) = \mathfrak{A}(Tu, v) \qquad \forall\, u \in \mathfrak{D}(T),\ v \in \mathfrak{D}(\mathfrak{B}). \tag{1.1}$$

(See [9], p. 322, Theorem 2.1.) We assume

HYPOTHESIS 1.3. The symmetric sesquilinear form $\mathfrak{B}$ with domain V is closable in H, and $\mathfrak{D}(T) \supset V$.

It follows from (1.1) that

$$\mathfrak{B}(u, v) = \mathfrak{A}(Tu, v) \quad \forall\, u, v \in V. \tag{1.2}$$

Given $\mathfrak{A}, \mathfrak{B}$, and V satisfying Hypotheses 1.1, 1.2, and 1.3 we define the (upper) *variational eigenvalues* $\{\lambda_j\}$ of $\mathfrak{B}$ with respect to $\mathfrak{A}$ on V as follows:

$$\lambda_1 = \sup\left\{\frac{\mathfrak{B}(u)}{\mathfrak{A}(u)}:\ u \in V,\ u \neq 0\right\},$$

and, for $j = 1, 2, \ldots$

$$\lambda_{j+1} = \inf \lambda_{j+1}(V_j),$$

where the infimum is taken over all subspaces $V_j \neq (0)$ of V with $\dim V_j \leqslant j$, and where

$$\lambda_{j+1}(V_j) = \sup\left\{\frac{\mathfrak{B}(u)}{\mathfrak{A}(u)}:\ u \in V,\ u \neq 0,\ u \perp V_j\right\}.$$

The variational eigenvalues form a decreasing sequence. Let $\sigma_e(T)$ be the essential spectrum of T and $l^* = \sup \sigma_e(T)$. Then $c \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots$ $\ldots \geqslant l^*$. Let $J_0 = \sup\{j:\ \lambda_j > l^*\}$. Then $\{\lambda_j:\ j = 1, \ldots, J_0\}$ is precisely the upper point spectrum of T; i.e., the set of isolated eigenvalues of T of finite multiplicity which are greater than l^*. (And each such eigenvalue of T will be listed as many times as its multiplicity.)

Corresponding to the sequence $\{\lambda_j:\ j = 1, \ldots, J_0\}$ is an orthonormal sequence $\{u_j:\ j = 1, \ldots, J_0\}$ of eigenvectors: $T_j u_j = \lambda_j u_j$, $j = 1, \ldots, J_0$. It follows from Hypothesis 1.3 and (1.1) that $\mathfrak{B}(u_j, v) = \lambda_j \mathfrak{A}(u_j, v)$ for all $j = 1, \ldots, J_0$ and all $v \in V$. Thus, in particular, if $\mathfrak{A}$ and $\mathfrak{B}$ originate from an eigenvalue problem of the form (0.1) as described in the introduction, then a variational eigenvalue $\lambda_j > l^*$ is an eigenvalue of (0.1) provided that $u_j \in V$.

Variational techniques for eigenvalue approximations are based on the so-called Monotony Theorems. These theorems, easily proved using the variational definition for eigenvalues given earlier, show how, in special situations, eigenvalues corresponding to different variational problems may be compared. Specifically, let $\{\lambda_j^{(1)}\}$ be the eigenvalues of $\mathfrak{B}_1$ with respect to $\mathfrak{A}_1$ on V_1, and $\{\lambda_j^{(2)}\}$ be the eigenvalues of $\mathfrak{B}_2$ with respect to $\mathfrak{A}_2$ on V_2. (See [13].)

MONOTONY THEOREM 1.1. *Let $V_1 \supset V_2$, and assume $\mathfrak{A}_1 = \mathfrak{A}_2$ and $\mathfrak{B}_1 = \mathfrak{B}_2$ on V_2. Then $\lambda_j^{(1)} \geqslant \lambda_j^{(2)}$ $\forall\, j = 1, 2, \ldots$*

Monotony Theorem 1.2. *Let* $V_1 = V_2$, *and assume* $\mathfrak{B}_1 \geqslant \mathfrak{B}_2$ *and* $\mathfrak{A}_1 \leqslant \mathfrak{A}_2$ *on* V_2. *Then* $\lambda_j^{(1)} \geqslant \lambda_j^{(2)}$ $\forall j = 1, 2, \ldots$ *such that either* $\lambda_j^{(1)} > 0$ *or* $\lambda_j^{(2)} > 0$.

Monotony Theorem 1.2′. *Let* $V_1 = V_2$, *and assume* $\mathfrak{B}_1 \geqslant \mathfrak{B}_2$ *and* $\mathfrak{A}_1 = \mathfrak{A}_2$ *on* V_2. *Then* $\lambda_j^{(1)} \geqslant \lambda_j^{(2)}$ $\forall j = 1, 2, \ldots$

2. General outline of variational approximation techniques

Variational approximation techniques are based on the Monotony Theorems given in the preceding section and are used to approximate the eigenvalues of a variational eigenvalue problem when the problem is not explicitly solvable. In the following sections we shall discuss specific variational methods. In this section we give a general, necessarily vague, outline of the technique, which is common to all of them.

Suppose one wishes, e.g., to find upper bounds for the first J eigenvalues $\lambda_1, \ldots, \lambda_J$ of $\mathfrak{B}$ with respect to $\mathfrak{A}$ on V. The first step is to replace this *given problem* by another problem, the base problem, defined by forms $\mathfrak{A}_0, \mathfrak{B}_0$ on a vector space V_0, with eigenvalues $\{\lambda_j^{(0)}\}$. We assume, of course, that $\mathfrak{A}_0$, $\mathfrak{B}_0$, and V_0 satisfy Hypotheses 1.1, 1.2, and 1.3. In addition, the base problem should satisfy two conditions:

Condition 2.1. One of the Monotony Theorems applies and implies that $\lambda_j^{(0)} \geqslant \lambda_j$, $j = 1, \ldots, J$.

Condition 2.2. The base problem is sufficiently tractable; i.e., the eigenvalues $\lambda_1^{(0)}, \ldots, \lambda_J^{(0)}$ are computable, and such other spectral properties of the base problem as are needed can be obtained.

The eigenvalue $\lambda_j^{(0)}$ is usually not a very good approximation to λ_j, $j = 1, \ldots, J$. Therefore one introduces *intermediate problems* using forms $\mathfrak{A}_n, \mathfrak{B}_n$ on a vector space V_n and with eigenvalues $\{\lambda_j^{(n)}\}$. These problems are so chosen that, in addition to Hypotheses 1.1, 1.2, and 1.3, the following conditions hold:

Condition 2.3. The Monotony Theorems imply that $\lambda_j^{(0)} \geqslant \lambda_j^{(1)} \geqslant \ldots \ldots \geqslant \lambda_j^{(n)} \geqslant \ldots \geqslant \lambda_j$, $j = 1, \ldots, J$.

Condition 2.4. Each intermediate problem is, in an appropriate sense, a finite dimensional perturbation of the base problem [2].

The last condition implies, for each $n = 1, 2, \ldots$, the existence of a perturbation determinant $M_n(\zeta)$ which is meromorphic for those $\zeta \in \boldsymbol{C}$ with $\operatorname{Re}\zeta > l_0^* = \sup \sigma_e(T_0)$. ($T_0$ is the self-adjoint operator associated with the base problem.) Moreover, $M_n(\zeta)$ can be explicitly computed if enough is known about T_0 (see Condition 2.2). The eigenvalues $\{\lambda_j^{(n)}\}$

are related to the eigenvalues $\{\lambda_j^{(0)}\}$ by Aronszajn's Rule:

$$m_n(\lambda) = o(\lambda) + m_0(\lambda), \tag{2.1}$$

where $m_n(\lambda)$ is the multiplicity of λ as an eigenvalue of $\mathfrak{B}_n$ with respect to $\mathfrak{A}_n$ on V_n, $m_0(\lambda)$ is the multiplicity of λ as an eigenvalue of $\mathfrak{B}_0$ with respect to $\mathfrak{A}_0$ on V_0, and $o(\lambda)$ is the order of λ as a zero or pole of $M_n(\zeta)$. Since $m_0(\lambda)$ is known and $o(\lambda)$ can be computed, Aronszajn's Rule enables one (at least theoretically) to find $\lambda_1^{(n)}, \ldots, \lambda_J^{(n)}$.

Remark 2.1. Other methods can also be used to find the eigenvalues of the intermediate problems from those of the base problem. See, e.g., [4], [5], [15], and the Standard References.

$\lambda_j^{(n)}$ is in general a better approximation to λ_j than is $\lambda_j^{(0)}$, $j = 1, \ldots, J$. In fact, for each $j = 1, \ldots, J$, $\{\lambda_j^{(n)}\}$ is a non-increasing sequence. If it can be shown that $\lim_n \lambda_j^{(n)} = \lambda_j$, then by choosing n sufficiently large one can obtain as good an upper bound for λ_j as one wishes.

Analogous techniques can be used to give lower bounds $\hat{\lambda}_j^{(m)}$ for λ_j, $j = 1, \ldots, J$. For any choice of m, n one therefore has $\hat{\lambda}_j^{(m)} \leqslant \lambda_j \leqslant \lambda_j^{(n)}$, $j = 1, \ldots, J$, and the difference $\lambda_j^{(n)} - \hat{\lambda}_j^{(m)}$ gives an *a posteriori* error bound.

3. Lower bounds. The Rayleigh–Ritz method

One of the best known and most used variational approximation methods is the Rayleigh–Ritz method. This method is used to obtain lower bounds and is based on Monotony Theorem 1.1. To find approximations to eigenvalues $\lambda_1, \ldots, \lambda_J$ of $\mathfrak{B}$ with respect to $\mathfrak{A}$ on V (the given problem) one chooses a linearly independent sequence $\{p_k\}$ in V and defines V_m to be the subspace of V spanned by $p_1, \ldots, p_m$. The mth intermediate problem is simply the restriction of the given problem to V_m instead of V. By Theorem 1.1 the eigenvalues of this problem satisfy $\hat{\lambda}_j^{(m)} \leqslant \lambda_j$, $j = 1, \ldots, J$. If one takes as base problem the restriction of the given problem to the trivial subspace $V_0 = (0)$, then Conditions 2.1–2.4 are satisfied. However, the intermediate problems are all finite dimensional problems and so can be solved (at least in theory) without the use of a base problem or of perturbation theory.

It is well known that if the sequence $\{p_k\}$ is complete in the Hilbert space completion H of V with respect to $\mathfrak{A}^{1/2}$, then $\hat{\lambda}_j^{(m)} \to \lambda_j$, $j = 1, \ldots$ Moreover, for special classes of problems arising from elliptic differential operators, the elements $\{p_k\}$ can be chosen using splines or finite element methods. For such special choices *a priori* error estimates are even available [3], [7].

4. Upper bounds. The Weinstein method

In the Weinstein method the given problem involving forms $\mathfrak{A}, \mathfrak{B}$ on the vector space V is replaced by a base problem involving forms $\mathfrak{A}_0, \mathfrak{B}_0$ on V_0, where $V_0 \supset V$ and $\mathfrak{A} = \mathfrak{A}_0$, $\mathfrak{B} = \mathfrak{B}_0$ on V. Thus by Monotony Theorem 1.1 the eigenvalues $\{\lambda_j^{(0)}\}$ of the base problem give upper bounds for the eigenvalues $\{\lambda_j\}$ of the given problem. Also, one may consider the Hilbert space completion H of V with respect to $\mathfrak{A}^{1/2}$ to be a closed subspace of the completion H_0 of V_0 with respect to $\mathfrak{A}_0^{1/2}$. Let T, T_0 be the self-adjoint operators associated with the forms $\mathfrak{B}, \mathfrak{B}_0$ on H, H_0 respectively. One assumes that $T = QT_0Q$ on H, where Q is the orthogonal projection of H_0 onto H.

Choose a linearly independent sequence $\{p_j\}$ in the orthogonal complement $H^{\perp}$ of H in H_0; let H_n be the orthogonal complement of span $\{p_1, \ldots, p_n\}$ in H_0; Q_n be the orthogonal projection of H_0 onto H_n; and T_n be the restriction of $Q_nT_0Q_n$ to H_n. It is important to note that $\mathfrak{D}(T) = \mathfrak{D}(T_n) \cap H$.

The nth intermediate problem is defined by taking $\mathfrak{A}_n, \mathfrak{B}_n$ to be the restrictions of $\mathfrak{A}_0, \mathfrak{B}_0$ respectively to $V_n = \mathfrak{D}(T_n) \subset H_n$. Monotony Theorem 1.1 shows that the corresponding variational eigenvalues satisfy

$$\lambda_j \leqslant \ldots \leqslant \lambda_j^{(n+1)} \leqslant \lambda_j^{(n)} \leqslant \ldots \leqslant \lambda_j^{(1)} \leqslant \lambda_j^{(0)}, \quad j = 1, 2, \ldots$$

All four conditions of Section 2 will hold, provided the base problem is nice enough (see Condition 2.2). The nth perturbation determinant is the celebrated Weinstein determinant

$$W_n(\lambda) = \det\{\mathfrak{A}_0((T_0-\lambda)^{-1}p_i, p_j)\} \tag{4.1}$$

(i.e., $W_n(\lambda)$ is the determinant of the $n \times n$ matrix whose general term is

$$\mathfrak{A}_0((T_0-\lambda)^{-1}p_i, p_j), \quad i, j = 1, \ldots, n).$$

There remains the question of convergence of the approximations.

LEMMA 4.1. *Let $\{x_n\}$ be a sequence in $\mathfrak{D}(T_0)$ such that $\mathfrak{A}_0(x_n) = 1$, $x_n \rightharpoonup 0$, and $\mathfrak{A}_0(T_0x_n, x_n) \to \lambda'$. Then $\lambda' \leqslant l_0^* = \sup \sigma_e(T_0)$.*

Proof. Assume that $\lambda' > l_0^*$. Choose $\hat{\lambda}$ such that $\lambda' > \hat{\lambda} > l_0^*$ and such that there are no eigenvalues of T_0 between $\hat{\lambda}$ and λ'. Let $\lambda_1^{(0)} \geqslant \lambda_2^{(0)} \geqslant \ldots \ldots \geqslant \lambda_M^{(0)}$ be those eigenvalues of T_0 which are $\geqslant \lambda'$. (There may be none, in which case the necessary modifications in what follows will be clear.) Then

$$T_0 = \int_{-\infty}^{\infty} \lambda dE_0(\lambda) = \int_{-\infty}^{\hat{\lambda}} \lambda dE_0(\lambda) + \sum_{j=1}^{M} \lambda_j^{(0)} \mathfrak{A}_0(\cdot, u_j)u_j, \tag{4.2}$$

where $E_0(\lambda)$ is the resolution of the identity corresponding to T_0, and $u_1, \ldots, u_M$ is a set of orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1^{(0)}, \ldots, \lambda_M^{(0)}$ of T_0. Define $\|x\|^2 = \mathfrak{A}_0(x)$, $x \in H_0$. Then

$$(4.3) \qquad 1 = \|x_n\|^2 = \int_{-\infty}^{\infty} d\|E_0(\lambda)x_n\|^2 = \int_{-\infty}^{\hat{\lambda}} d\|E_0(\lambda)x_n\|^2 + \sum_{j=1}^{M} |\mathfrak{A}_0(x_n, u_j)|^2.$$

Since $x_n \rightharpoonup 0$, therefore $\lim_n \sum_{j=1}^{M} |\mathfrak{A}_0(x_n, u_j)|^2 = 0$, and (4.3) implies that

$$(4.4) \qquad 1 = \lim_n \int_{-\infty}^{\hat{\lambda}} d\|E_0(\lambda)x_n\|^2.$$

From (4.2) follows

$$\mathfrak{A}_0(T_0x_n, x_n) = \int_{-\infty}^{\hat{\lambda}} \lambda d\|E_0(\lambda)x_n\|^2 + \sum_{j=1}^{M} \lambda_j^{(0)} |\mathfrak{A}_0(x_n, u_j)|^2$$

$$\leqslant \hat{\lambda} \int_{-\infty}^{\hat{\lambda}} d\|E_0(\lambda)x_n\|^2 + \sum_{j=1}^{M} \lambda_j^{(0)} |\mathfrak{A}_0(x_n, u_j)|^2,$$

which, together with (4.4) implies

$$\lambda' = \lim_n \mathfrak{A}_0(T_0x_n, x_n) \leqslant \hat{\lambda},$$

a contradiction. The lemma is proved.

LEMMA 4.2. *Assume the sequence $\{p_j\}$ is complete in $H^\perp$. If $\lambda_J > l_0^*$, then $\lim_n \lambda_J^{(n)} = \lambda'$, where λ' is an eigenvalue of T and $\lambda' \geqslant \lambda_J$. Moreover, let u_n be a normalized eigenvector of T_n corresponding to the eigenvalue $\lambda_J^{(n)}$, $n = 1, 2, \ldots$ Then there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \to u$, where u is an eigenvector of T corresponding to λ'.*

Proof. $\{\lambda_J^{(n)}\}$ is a non-increasing sequence such that $\lambda_J^{(n)} \geqslant \lambda_J$ for all $n = 0, 1, 2, \ldots$ Hence $\lambda' = \lim_n \lambda_J^{(n)}$ exists, and $\lambda' \geqslant \lambda_J$. By hypothesis

$$\|u_n\| = 1, \quad u_n \in H_n, \quad T_n u_n = \lambda_J^{(n)} u_n, \quad n = 1, 2, \ldots$$

Since $\{u_n\}$ is a bounded sequence in H_0, there exists $u \in H_0$ and a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup u$. We shall show that u is an eigenvector of T corresponding to the eigenvalue λ'.

First, since $Q_n u_n = u_n$,

$$\mathfrak{A}_0(T_0 u_{n_j}, u_{n_j}) = \mathfrak{A}_0(T_{n_j} u_{n_j}, u_{n_j}) = \lambda_J^{n_j} \to \lambda' \geqslant \lambda_J > \lambda_0^*.$$

By Lemma 4.1 therefore $u \neq 0$. Second, for each $k = 1, 2, \ldots,$

$$\mathfrak{A}_0(u_{n_j}, p_k) = 0 \quad \text{for} \quad n_j > k.$$

Thus

$$\mathfrak{A}_0(u, p_k) = \lim_{n_j} \mathfrak{A}_0(u_{n_j}, p_k) = 0, \quad k = 1, 2, \ldots;$$

i.e., $u \in H$. Finally, for every $v \in \mathfrak{D}(T) = \mathfrak{D}(T_{n_j}) \cap H \subset H$,

$$\begin{aligned} \mathfrak{A}(\lambda' u, v) &= \lambda' \mathfrak{A}_0(u, v) = \lim_{n_j} \lambda_J^{(n_j)} \mathfrak{A}_0(u_{n_j}, v) \\ &= \lim_{n_j} \mathfrak{A}_0(T_{n_j} u_{n_j}, v) = \lim_{n_j} \mathfrak{A}_0(u_{n_j}, T_{n_j} v) \\ &= \lim_{n_j} \mathfrak{A}_0(u_{n_j}, T_0 v) = \mathfrak{A}_0(u, T_0 v) = \mathfrak{A}(u, Tv). \end{aligned}$$

Thus $u \in \mathfrak{D}(T^*) = \mathfrak{D}(T)$ and $Tu = \lambda' u$. The lemma is proved.

THEOREM 4.1. *Assume $\{p_k\}$ is complete in $H^\perp$, and let $\lambda_J > l_0^*$. Then $\lim_n \lambda_j^{(n)} = \lambda_j$, $j = 1, \ldots, J$. Moreover, for each $n = 0, 1, \ldots$ let $u_1^{(n)}, \ldots, u_J^{(n)}$ be an orthonormal set of eigenvectors of T_n corresponding to the eigenvalues $\lambda_1^{(n)}, \ldots, \lambda_J^{(n)}$. Then there is a linearly independent set $u_1, \ldots, u_J$ of eigenvectors of T corresponding to $\lambda_1, \ldots, \lambda_J$ and a subsequence $\{n_k\}$ of the non-negative integers such that $u_j^{(n_k)} \rightharpoonup u_j$, $j = 1, \ldots, J$.*

Proof. We proceed by induction on $j = 1, \ldots, J$:

For $j = 1$ the theorem follows from Lemma 4.2. Assume the theorem true for $j = 1, \ldots, k-1$, where $k \leqslant J$. By Lemma 4.2 $\lambda' = \lim_n \lambda_k^{(n)}$ is an eigenvalue of T, and clearly $\lambda_{k-1} \geqslant \lambda' \geqslant \lambda_k$. Let $u_1^{(n)}, \ldots, u_J^{(n)}$ be as in the statement of the theorem. By the induction hypothesis and Lemma 4.2 there are subsequences $u_j^{(n_m)} \rightharpoonup u_j$, $j = 1, \ldots, k-1$ and $u_k^{(n_m)} \rightharpoonup u$, where $u_1, \ldots, u_{k-1}$ are linearly independent eigenvectors of T corresponding to eigenvalues $\lambda_1, \ldots, \lambda_{k-1}$ respectively, and u is an eigenvector of T corresponding to λ'.

There are two possibilities:

1. If $\lambda_{k-1} > \lambda'$, then $\lambda' = \lambda_k \neq \lambda_{k-1}$. Then $u_k = u$ is orthogonal to $u_1, \ldots, u_{k-1}$ and therefore $u_1, \ldots, u_k$ are linearly independent eigenvectors corresponding to $\lambda_1, \ldots, \lambda_k$. Thus the theorem holds for $j = k$ as required.

2. If $\lambda' = \lambda_{k-1}$, then there is a smallest integer l, $1 \leqslant l \leqslant k-1$, such that $\lambda_l = \lambda_{k-1}$. Thus $\lambda_{l-1} > \lambda_l = \lambda_{l+1} = \ldots = \lambda_{k-1} = \lambda'$, and $u_l, u_{l+1}, \ldots, u_{k-1}, u$ are all eigenvectors corresponding to the multiple eigenvalue λ_l. We need only show that these $k-l+1$ vectors are linearly independent. For then λ_l has multiplicity $\geqslant k-l+1$, so that $\lambda_k = \lambda_{k-1} = \lambda'$. We then may take $u_k = u$ and the theorem will be proved.

By way of contradiction assume that $u_l, u_{l+1}, \ldots, u_{k-1}, u$ are linearly dependent. Then there are constants $a_l, a_{l+1}, \ldots, a_k$ such that $\sum_{j=l}^{k} |a_j|^2 = 1$ and $\sum_{j=l}^{k-1} a_j u_j + a_k u = 0$. For $n = 1, 2, \ldots$ let $x_n = \sum_{j=l}^{k} a_j u_j^{(n)}$. Then $x_n \in \mathfrak{D}(T_n) \subset \mathfrak{D}(T_0)$, $\|x_n\|^2 = \sum_{j=l}^{k} |a_j|^2 = 1$, $x_n \rightharpoonup \sum_{j=l}^{k-1} a_j u_j + a_k u = 0$, while

$$\mathfrak{A}_0(T_0 x_n, x_n) = \mathfrak{A}_0(T_n x_n, x_n) = \sum_{j=l}^{k} \lambda_j^{(n)} |a_j|^2 \to \lambda_{k-1} \sum_{j=l}^{k} |a_j|^2 = \lambda_{k-1} \geqslant \lambda_J > l_0^*,$$

contradicting Lemma 4.1. The theorem is proved.

Remark 4.1. Lemma 4.1 applies to any self-adjoint operator T_0 which is semi-bounded above. For bounded operators it implies that the essential numerical range of T_0 lies to the left of l_0^*. In [11] the notion of essential numerical range is used to study convergence for approximation methods based on orthogonal projections converging strongly to the identity.

5. Upper bounds. Aronszajn's method

Aronszajn's method is based on Monotony Theorem 1.2 (or 1.2′). The given problem, defined by forms $\mathfrak{A}, \mathfrak{B}$ on the (infinite dimensional) space V, is replaced by a base problem using forms $\mathfrak{A}_0, \mathfrak{B}_0$ on V, where $\mathfrak{A} \geqslant \mathfrak{A}_0$ and $\mathfrak{B} \leqslant \mathfrak{B}_0$. Thus

$$\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{A}', \quad \mathfrak{B} = \mathfrak{B}_0 - \mathfrak{B}',$$

where $\mathfrak{A}', \mathfrak{B}'$ are positive forms on V.

As usual H_0 will denote the Hilbert space completion of V with respect to $\mathfrak{A}_0^{1/2}$. For convenience we assume that $\mathfrak{A}', \mathfrak{B}'$ have positive lower bounds on V (see Remark 5.2 however) and that the associated self-adjoint operators M, N respectively have domains in H_0 which include V. Thus

$$\mathfrak{A}'(u, v) = \mathfrak{A}_0(Mu, v), \quad \mathfrak{B}'(u, v) = \mathfrak{A}_0(Nu, v) \quad \text{for} \quad u, v \in V.$$

We extend $\mathfrak{A}', \mathfrak{B}'$ to closed forms by setting

$$\mathfrak{A}'(u, v) = \mathfrak{A}_0(M^{1/2}u, M^{1/2}v), \quad \mathfrak{B}'(u, v) = \mathfrak{A}_0(N^{1/2}u, N^{1/2}v)$$

with domains $\mathfrak{D}(M^{1/2})$, $\mathfrak{D}(N^{1/2})$ respectively. (Note that $\mathfrak{D}(M^{1/2})$, $\mathfrak{D}(N^{1/2})$ are themselves Hilbert spaces with scalar products induced by $\mathfrak{A}', \mathfrak{B}'$ respectively.)

We further assume that, for sufficiently large k,

CONDITION 5.1. The operator $T_0 - kM - N$ is self-adjoint in H_0.

Note then that $T_0 - kI - kM - N$ is the self-adjoint operator corresponding to the form $\mathfrak{B} - k\mathfrak{A}$ in H_0.

To form the intermediate problems for Aronszajn's method we choose two linearly independent sequences $\{\zeta_a\}$ and $\{\psi_\beta\}$ in V, and combine them into a single sequence $\{p_k\}$ having $\{\zeta_a\}$ and $\{\psi_\beta\}$ as disjoint subsequences. Then for each $n = 1, 2, \ldots$ the finite sequence $\{p_1, \ldots, p_n\}$ is formed using sequences $\{\zeta_1, \ldots, \zeta_a\}$ and $\{\psi_1, \ldots, \psi_b\}$, where $a+b = n$. Let P_a be the orthogonal projection of the Hilbert space $\mathfrak{D}(M^{1/2})$ onto $\operatorname{span}\{\zeta_1, \ldots, \zeta_a\}$ and Q_b be the orthogonal projection of $\mathfrak{D}(N^{1/2})$ onto $\operatorname{span}\{\psi_1, \ldots, \psi_b\}$. The nth intermediate problem is defined by the forms

$$\mathfrak{A}_a(u) = \mathfrak{A}_0(u) + \mathfrak{A}'(P_a u), \qquad \mathfrak{B}_b(u) = \mathfrak{B}_0(u) - \mathfrak{B}'(Q_b u)$$

on V. By Monotony Theorem 1.2

$$\lambda_j \leqslant \ldots \leqslant \lambda_j^{(n+1)} \leqslant \lambda_j^{(n)} \leqslant \ldots \leqslant \lambda_j^{(1)} \leqslant \lambda_j^{(0)}$$

for all j such that $\lambda_j > 0$. The conditions of Section 2 will hold provided Condition 2.2 does. The associated perturbation determinant (the Aronszajn determinant) is

$$(5.1) \quad M_n(\zeta)$$

$$= \det\begin{bmatrix} \mathfrak{A}_0\left((T_0-\lambda)^{-1}M\zeta_i - \frac{1}{\lambda}\zeta_i,\ M\zeta_j\right) & \mathfrak{A}_0\left((T_0-\lambda)^{-1}M\zeta_i, N\psi_\beta\right) \\ \mathfrak{A}_0\left((T_0-\lambda)^{-1}N\psi_a, M\zeta_j\right) & \mathfrak{A}_0\left((T_0-\lambda)^{-1}N\psi_a - \psi_a, N\psi_\beta\right) \end{bmatrix}$$

(where $i, j = 1, \ldots, a$ and $a, \beta = 1, \ldots, b$).

In order to investigate convergence for Aronszajn's method we shall use a slightly modified form of a construction due to Weinberger [13], Chapter 4.5, which allows one to deduce convergence for Aronszajn's method from that for Weinstein's method. Accordingly, choose $k > 2c_0$, where c_0 is the upper bound of $\mathfrak{B}_0$ with respect to $\mathfrak{A}_0$ on V (see Hypothesis 1.2). Then the self-adjoint operator $L = k - T_0$ is positive definite and continuously invertible, and the form

$$\hat{\mathfrak{A}}(u) = \mathfrak{A}_0(L^{1/2}u), \qquad u \in \mathfrak{D}(L^{1/2})$$

is closed. Note that $\mathfrak{D}(L^{1/2})$ is itself a Hilbert space with scalar product induced by $\hat{\mathfrak{A}}$.

Define the Hilbert space

$$\mathscr{H}_0 = \mathfrak{D}(L^{1/2}) \times \mathfrak{D}(M^{1/2}) \times \mathfrak{D}(N^{1/2}),$$

with norm determined by the positive definite quadratic form

$$\mathscr{C}_0([u, v, w]) = \hat{\mathfrak{A}}(u) + k\mathfrak{A}'(v) + \mathfrak{B}'(w), \qquad [u, v, w] \in \mathscr{H}_0.$$

On $\mathscr{H}_0$ define the real quadratic form

$$\mathscr{D}_0([u, v, w]) = k\mathfrak{A}_0(u) - \hat{\mathfrak{A}}(u) - \mathfrak{B}'(w).$$

It is not difficult to see that $\mathscr{D}_0$ is bounded with respect to $\mathscr{C}_0$ on $\mathscr{H}_0$ and the associated (bounded) self-adjoint operator S_0 is defined by

$$S_0([u, v, w]) = [(kL^{-1} - I)u, 0, -w]. \tag{5.2}$$

The spectrum of S_0 is

$$\sigma(S_0) = \{-1\} \cup \{0\} \cup \{\mu = \lambda/(k-\lambda) \colon \lambda \in \sigma(T_0)\},$$

and for $\mu \notin \sigma(S_0)$

$$\begin{aligned}(S_0 - \mu I)^{-1}([u, v, w]) \\ = \left[\frac{1}{1+\mu} L\left(T_0 - \frac{\mu k}{\mu+1}\right)^{-1} u, \ -\frac{1}{\mu} v, \ -\frac{1}{1+\mu} w\right].\end{aligned} \tag{5.3}$$

Note in particular that $S_0([0, v, 0]) = 0$ for all $v \in \mathfrak{D}(M^{1/2})$, so $0 \in \sigma_e(S_0)$ and $\sup \sigma_e(S_0) = \max\{0, l_0^*/(k - l_0^*)\}$. The variational eigenvalues of $\mathscr{D}_0$ with respect to $\mathscr{C}_0$ on $\mathscr{H}_0$ are precisely those $\mu_i^{(0)} = \lambda_i^{(0)}/(k - \lambda_i^{(0)})$ with $\lambda_i^{(0)} \geqslant \max\{0, l_0^*\}$.

Define

$$\mathscr{H} = \{[u, u, u] \in \mathscr{H}_0 \colon u \in \mathfrak{D}(L^{1/2}) \cap \mathfrak{D}(M^{1/2}) \cap \mathfrak{D}(N^{1/2})\}.$$

Then $\mathscr{H}$ is a closed subspace of $\mathscr{H}_0$, and $\mathscr{H}^\perp$ is the closure in $\mathscr{H}_0$ of the space $\hat{\mathscr{H}} = \{[-L^{-1}(M\zeta + N\psi), k^{-1}\zeta, \psi] \colon \zeta \in \mathfrak{D}(M), \ \psi \in \mathfrak{D}(N)\}$. Using Condition 5.1, one sees that $\mathscr{V} = \{[u, u, u] \colon u \in \mathfrak{D}(L) \cap \mathfrak{D}(M) \cap \mathfrak{D}(N)\}$ is a dense subspace of $\mathscr{H}$. Therefore the variational eigenvalues $\{\mu_i\}$ of $\mathscr{D}_0$ with respect to $\mathscr{C}_0$ on $\mathscr{H}$ are the same as those of $\mathscr{D}_0$ with respect to $\mathscr{C}_0$ on $\mathscr{V}$. But for $[u, u, u] \in \mathscr{V}$ (or $u \in \mathfrak{D}(L) \cap \mathfrak{D}(M) \cap \mathfrak{D}(N)$)

$$\frac{\mathscr{D}_0([u, u, u])}{\mathscr{C}_0([u, u, u])} = \frac{\mathfrak{B}(u)}{k\mathfrak{A}(u) - \mathfrak{B}(u)},$$

so $\mu_i = \lambda_i/(k - \lambda_i)$, $i = 1, 2, \ldots$

We consider the eigenvalue problem for $\mathscr{D}_0$ with respect to $\mathscr{C}_0$ on $\mathscr{V}$ (or $\mathscr{H}$) as a given problem, and that for $\mathscr{D}_0$ with respect to $\mathscr{C}_0$ on $\mathscr{H}_0$ as a corresponding base problem. Let $\lambda_J > \max\{0, l_0^*\}$. Then $\mu_J > \sup \sigma_e(S_0)$ and so, by Monotony Theorem 1.1,

$$\mu_j^{(0)} = \frac{\lambda_j^{(0)}}{k - \lambda_j^{(0)}} \geqslant \mu_j = \frac{\lambda_j}{k - \lambda_k}, \quad j = 1, \ldots, J.$$

Note also that $\mu_j^{(0)}$ is known, since $\lambda_j^{(0)}$ is.

We form intermediate problems using Weinstein's method and the sequence $\{\mathscr{P}_k\}$ in $\mathscr{H}^\perp$ defined by

$$\mathscr{P}_k = \begin{cases} [-L^{-1}M\zeta_a, k^{-1}\zeta_a, 0] & \text{if} \quad p_k = \zeta_a, \\ [-L^{-1}N\psi_\beta, 0, \psi_\beta] & \text{if} \quad p_k = \psi_\beta. \end{cases} \tag{5.4}$$

(Recall the sequence $\{p_k\}$ in V is formed using $\{\zeta_a\}$ and $\{\psi_\beta\}$.) The eigenvalues $\{\mu_j^{(n)}\}$ of the nth intermediate problem satisfy

$$\mu_j \leqslant \ldots \leqslant \mu_j^{(n+1)} \leqslant \mu_j^{(n)} \leqslant \ldots \leqslant \mu_j^{(1)} \leqslant \mu_j^{(0)}, \quad j = 1, \ldots, J,$$

and (see equation (4.1)) can be found from the Weinstein determinant

$$\mathscr{W}_n(\mu) = \det\{\mathscr{C}_0((S_0-\mu)^{-1}\mathscr{P}_i, \mathscr{P}_j)\}_{i,j=1,\ldots,n}. \tag{5.5}$$

Moreover, by Theorem 4.1:

THEOREM 5.1. *Let $\lambda_J > \max\{0, l_0^*\}$. If $\{\mathscr{P}_k\}$ is complete in $\mathscr{H}^\perp$, then.*

$$\lim_n \mu_j^{(n)} = \mu_j, \quad j = 1, \ldots, J.$$

We use $\mathscr{W}_n(\mu)$ to investigate the intermediate eigenvalues $\{\mu_j^{(n)}\}$ From (5.1), (5.3), (5.4), and (5.5) we obtain

$$\mathscr{W}_n(\mu) = c_n M_n(\lambda),$$

where $\mu = \lambda/(k-\lambda)$, $a+b = n$, and c_n is a non-zero constant. (The ambiguity in sign results from a rearrangement of rows and columns in the matrix defining $\mathscr{W}_n(\mu)$.) It follows that $\mu_j^{(n)} = \lambda_j^{(n)}/(k-\lambda_j^{(n)})$, $j = 1, \ldots, J$. In particular, $\lim_n \mu_j^{(n)} = \mu_j$ if and only if $\lim_n \lambda_j^{(n)} = \lambda_j$, $j = 1, \ldots, J$.

THEOREM 5.2. *Let $\lambda_J > \max\{0, l_0^*\}$. If $\{\mathscr{P}_k\}$ is complete in $\mathscr{H}^\perp$, then $\lim_n \lambda_j^{(n)} = \lambda_j$, $j = 1, \ldots, J$.*

COROLLARY 5.1. *Let $\lambda_J > \max\{0, l_0^*\}$. If $\{M(\zeta_a)\}$ and $\{N(\psi_\beta)\}$ are complete in H_0, then $\lim_n \lambda_j^{(n)} = \lambda_j$, $j = 1, \ldots, J$.*

Proof. Assume that $\{M(\zeta_a)\}$ and $\{N(\psi_\beta)\}$ are complete in H_0. If $[u, v, w] \in \mathscr{H}_0$ is orthogonal to every $\mathscr{P}_k$, $k = 1, 2, \ldots$, then for every a

$$\begin{aligned} 0 &= \mathscr{C}_0([u, v, w], [-L^{-1}M\zeta_a, k^{-1}\zeta_a, 0]) \\ &= \hat{\mathfrak{A}}(u, -L^{-1}M\zeta_a) + \mathfrak{A}'(v, \zeta_a) \\ &= -\mathfrak{A}_0(u, M\zeta_a) + \mathfrak{A}_0(v, M\zeta_a) = \mathfrak{A}_0(v-u, M\zeta_a), \end{aligned} \tag{5.6}$$

which implies that $v = u$. Similarly,

$$\begin{aligned} 0 &= \mathscr{C}_0([u, v, w], [-L^{-1}N\psi_\beta, 0, \psi_\beta]) \\ &= -\mathfrak{A}_0(u, N\psi_\beta) + \mathfrak{B}'(w, \psi_\beta) = \mathfrak{A}_0(w-u, N\psi_\beta) \end{aligned} \tag{5.7}$$

for every β implies $w = u$. Hence $[u, v, w] = [u, u, u]$ with $u \in \mathfrak{D}(L^{1/2}) \cap \cap \mathfrak{D}(M^{1/2}) \cap \mathfrak{D}(N^{1/2})$, i.e., $[u, v, w] \in \mathscr{H}$. It follows that $\{\mathscr{P}_k\}$ is complete in $\mathscr{H}^{\perp}$, and the corollary follows from Theorem 5.2.

Remark 5.1. The hypothesis of Corollary 5.1 that $\{M(\zeta_\alpha)\}$ be complete in H_0 may be replaced by the condition:

(a) $\quad \mathfrak{D}(L^{1/2}) \subset \mathfrak{D}(M^{1/2})$ and $\{\zeta_\alpha\}$ is complete in $\mathfrak{D}(M^{1/2})$.

For, under condition (a), equation (5.6) will still imply that $v = u$. Similarly, the hypothesis that $\{N(\psi_\alpha)\}$ be complete in H_0 be replaced by:

(b) $\quad \mathfrak{D}(L^{1/2}) \subset \mathfrak{D}(N^{1/2})$ and $\{\psi_\beta\}$ is complete in $\mathfrak{D}(N^{1/2})$.

The condition $\mathfrak{D}(L^{1/2}) \subset \mathfrak{D}(N^{1/2})$ will hold automatically if $\mathfrak{D}(N^{1/2})$ contains the domain of the form closure of $\mathfrak{B}$ in H_0. Simpler still, it will hold if N is bounded in H_0. In such cases (b) can be replaced by:

(b′) $\quad \{\psi_\beta\}$ is complete in $\mathfrak{D}(N^{1/2})$.

Similar remarks apply to the condition (a). In particular, when $\mathfrak{A}$ and $\mathfrak{A}_0$ are equivalent, then $\mathfrak{A}'$ will be bounded and (a) can be replaced by:

(a′) $\quad \{\zeta_\alpha\}$ is complete in $\mathfrak{D}(M^{1/2})$.

(See [1].) In general, however, conditions (a′) and (b′) are not sufficient to insure the conclusions of Corollary 5.1. This fact will be demonstrated by the example of Section 6.

Remark 5.2. The conclusions of this section can be modified to include the case that $\mathfrak{A}'$, $\mathfrak{B}'$ are only positive semi-definite (see [13], Section 4.5). The necessary modifications are especially easy in case $\mathfrak{A}' \equiv 0$ or $\mathfrak{B}' \equiv 0$. When $\mathfrak{A}' \equiv 0$ the forms $\mathfrak{B}_0$ and $\mathfrak{B}$ are defined in the same Hilbert space $H_0 = H$, with associated self-adjoint operators T_0 and $T_0 - N$ respectively. This case is the one considered in [14], Chapter 5.

In this case, also, we may replace the condition $\lambda_J > \max\{l_0^*, 0\}$ in Corollary 5.1 by the simpler condition $\lambda_J > l_0^*$. To see this we need only choose $d > 0$ such that $\lambda_J + d > 0$, and replace the forms $\mathfrak{B}_0, \mathfrak{B}$ by $\mathfrak{B}_0 + d\mathfrak{A}_0$, $\mathfrak{B} + d\mathfrak{A}_0$ respectively. (Note that this replacement does not affect $\mathfrak{A}_0$ or $\mathfrak{B}'$.)

Finally, in the case $\mathfrak{A}' \equiv 0$, compare Corollary 5.1 and condition (b) of Remark 5.1 with the results of [1], [6], [10], [12].

6. A counterexample for Aronszajn's method

In this section we construct an example of Aronszajn's method with $\mathfrak{A}' = 0$, $\lambda_1 > l_0^*$, and $\{\psi_\beta\}$ complete in $\mathfrak{D}(N^{1/2})$, but with $\lim_n \lambda_1^{(n)} \neq \lambda_1$.

Let N be an unbounded, self-adjoint positive definite operator in a Hilbert space H_0, and let N be bounded below by a positive constant d; i.e., $(Nx, x) \geqslant d\|x\|^2 \ \forall\, x \in H_0$. Let $\{\psi_\beta\}$ be a sequence in $\mathfrak{D}(N)$ such that span$\{\psi_\beta\}$ is a core of $N^{1/2}$ but not of N. Then ([9], problem III. 5.19) there exists $x_0 \in H_0$, $\|x_0\| = 1$, such that $(x_0, N\psi_\beta) = 0 \ \forall\, \beta = 1, 2, \ldots$

Choose $y \in \vartheta(N)$ such that $\|y\| = 1$ and $(y, x_0) \neq 0$. Then choose a strictly decreasing sequence $\{\lambda_j^{(0)}\}$ of positive numbers converging to zero, with $\lambda_1^{(0)} > (Nx_0, x_0)|(y, x_0)|^{-2}$. Let T_0 be a bounded self-adjoint operator in H_0 whose (upper) eigenvalues are precisely the $\lambda_j^{(0)}$'s and with $T_0 x_0 = \lambda_1^{(0)} x_0$.

Define quadratic forms associated to T_0, N on $V = \mathfrak{D}(N)$ by the formulas

$$\mathfrak{B}_0(x) = (T_0 x, x), \qquad \mathfrak{B}'(x) = (Nx, x)$$

respectively, and let $\mathfrak{B} = \mathfrak{B}_0 - \mathfrak{B}'$. The self-adjoint operator T associated to $\mathfrak{B}$ is $T = T_0 - N$, and $\mathfrak{D}(T) = \mathfrak{D}(N) = V$. The operators associated to $\mathfrak{B}_0$ and $\mathfrak{B}'$ are T_0 and N respectively.

Let $\{\lambda_j\}$ be the sequence of (upper) eigenvalues of T, and note that $\mathfrak{B}(x) \leqslant \mathfrak{B}_0(x) - d \leqslant \lambda_1^{(0)} - d < \lambda_1^{(0)} \ \forall\, x \in V$. Hence

$$\lambda_1 < \lambda_1^{(0)}. \tag{6.1}$$

But also,

$$\lambda_1 \geqslant (Ty, y) \geqslant \lambda_1^{(0)} |(y, x_0)|^2 - (Ny, y) > 0 = \sup \sigma_e(T_0). \tag{6.2}$$

Use the eigenvalue problem for $\mathfrak{B}$ with respect to $\mathfrak{A}_0(\cdot) = \|\cdot\|^2$ on V as a given problem and that for $\mathfrak{B}_0$ with respect to $\mathfrak{A}_0$ on V as a base problem. Use the sequence $\{\psi_\beta\}$ to form intermediate problems following Aronszajn's method. We modify the Weinberger construction of Section 5 to take account of the fact that $\mathfrak{A}' = 0$, and find:

$$\mathscr{H}_0 = H_0 \times \mathfrak{D}(N^{1/2}), \qquad \mathscr{C}_0([u, w]) = \hat{\mathfrak{A}}(u) + \mathfrak{B}'(w),$$

$$\mathscr{D}_0([u, w]) = k\mathfrak{A}_0(u) - \hat{\mathfrak{A}}(u) - \mathfrak{B}'(w), \qquad \mathscr{H} = \{[u, u] \colon u \in \mathfrak{D}(N^{1/2})\},$$

and

$$\hat{\mathscr{H}} = \{[-L^{-1}N\psi, \psi] \colon \psi \in \mathfrak{D}(N)\}.$$

The nth Weinstein intermediate problem in this case is formed using the restrictions of $\mathscr{D}_0$ and $\mathscr{C}_0$ to the orthogonal complement $\mathscr{H}_n$ of span $\{\mathscr{P}_1, \ldots, \mathscr{P}_n\}$, where $\mathscr{P}_\beta = [-L^{-1}N\psi_\beta, \psi_\beta]$, $\beta = 1, \ldots, n$. Note that

$$\mathscr{C}([x_0, 0], [-L^{-1}N\psi_\nu, \psi_\beta]) = -(x_0, N\psi_\beta) = 0 \qquad \forall\, \beta = 1, 2, \ldots,$$

so that $[x_0, 0] \in \mathscr{H}_n \ \forall\, n = 1, 2, \ldots$

On the other hand,

$$\frac{\mathscr{D}_0([x_0, 0])}{\mathscr{C}_0([x_0, 0])} = \frac{\lambda_1^{(0)}}{k - \lambda_1^{(0)}} = \mu_1^{(0)},$$

so $[x_0, 0]$ is an eigenvector of $\mathscr{D}_0$ with respect to $\mathscr{C}_0$ on $\mathscr{H}_n$. In fact, $\mu_1^{(n)} = \mu_1^{(0)}$ and $\lambda_1^{(n)} = \lambda_1^{(0)}$ $\forall n = 1, 2, \ldots$

We therefore have, in view of (6.1) and (6.2), that $\lim_n \lambda_1^{(n)} = \lambda_1^{(0)} > \lambda_1$. Thus the conclusion of Corollary 5.1 fails even though $\lambda_1 > \sup \sigma_e(T_0)$ and $\{\psi_\beta\}$ is complete in $\mathfrak{D}(N^{1/2})$ (and in H_0).

Remark 6.1. Similar counterexamples can be constructed when $\mathfrak{B}' = 0$. See, e.g., [6].

7. Concluding remarks

In this introduction to variational approximation methods we have considered only three explicit methods. There are, besides these three most basic methods, numerous extensions, modifications, and refinements. These other methods are covered in the standard texts [8], [13], [14], which also include detailed proofs, extensive references, and numerous examples and applications. The convergence results proved in Sections 4 and 5 can be applied to several of these modified methods (e.g., the Bazley distinguished choice, the Bazley–Fox method).

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