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ON THE NEW APPROACH TO THE CONSTRUCTIONS OF THE INDEX TRANSFORMS

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1. Introduction. As is known, the classical one-dimensional integral transforms on the half-axis $\mathbb{R}_+$ are of the form

$$g(x) = \int\limits_{\mathbb{R}_+} H(x,y) f(y)\, dy, \tag{1.1}$$

where $H(x,y)$ is some given function (the kernel of the transform), $f(y)$ is an original in a certain space of functions and $g(x)$ is the image of the function $f(y)$. All classical integral transforms may be divided into two classes: the Mellin convolution type transforms (or the Fourier type transforms) [13]

$$g(x) = \int\limits_{\mathbb{R}_+} k(xy) f(y)\, dy, \tag{1.2}$$

with the kernel $H(x,y) = k(xy)$, which is the function of the variable $z = xy$, and transforms whose kernel, generally speaking, is essentially a function of two variables. We shall call the last class of integral transforms the index transforms, since in some known examples of such transforms the kernel $H(x,y)$ is a special function [1] and the variable x is its index (a parameter). We note the most important Mellin transform [8]

$$\mathfrak{M}\{f(y);x\} = \int\limits_{\mathbb{R}_+} y^{x-1} f(y)\, dy, \tag{1.3}$$

the Kontorovich–Lebedev transform [3]

$$g(x) = \int\limits_{0}^{\infty} K_{ix}(y) f(y)\, dy \tag{1.4}$$

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with the Macdonald function $K_{ix}(y)$ [2] of imaginary index; the Mehler–Fock transform ([2], [9])

$$g(x) = \int_0^\infty P_{-1/2+ix}(\cosh(y)) f(y)\, dy \tag{1.5}$$

with the spherical Legendre function of the first kind $P_{-1/2+ix}(\cosh(y))$ [1], and the most general transform pair with the Meijer G-function as kernel [8], which contains the formulae (1.4), (1.5); it was first given by J. Wimp in [7] and then simplified by the author in [8] as follows:

$$g(x) = \int_0^\infty G^{m,n+2}_{p+2,q}\left(y \,\middle|\, \begin{matrix} 1-\nu+ix, 1-\nu-ix, (\alpha_p) \\ (\beta_q) \end{matrix}\right) f(y)\, dy, \tag{1.6}$$

$$\begin{aligned} f(x) &\\ = \frac{1}{\pi^2} \int_0^\infty G^{q-m,p-n+2}_{p+2,q}&\left(x \,\middle|\, \begin{matrix} \nu+i\tau, \nu-i\tau, -(\alpha_p^{n+1}), -(\alpha_n) \\ -(\beta_q^{m+1}), -(\beta_m) \end{matrix}\right) \tau \sinh(2\pi\tau) g(\tau)\, d\tau, \end{aligned} \tag{1.7}$$

where $m, n, p, q \in \mathbb{N}$, $0 \le n \le p$, $0 \le m \le q$, ν, $(\alpha_p) = (\alpha_1, \dots, \alpha_p)$, $(\beta_q) = (\beta_1, \dots, \beta_q)$, $-(\alpha_p^{n+1}) = (-\alpha_{n+1}, \dots, -\alpha_p)$, $-(\beta_q^{m+1}) = (-\beta_{m+1}, \dots, -\beta_q)$ are the parameters of G-functions.

Hence, from the formulas (1.6)–(1.7) in accordance with the table of particular cases for G-functions [11], a set of index transforms was obtained, including some well-known transforms such as the Olevskiĭ transform with the Gauss hypergeometric function as kernel [10], the modified Mehler–Fock transform [8] with the modified spherical Legendre function $P^k_{-1/2+i\tau}(x)$ [1], the Lebedev transform with the function $K^2_{i\tau}(x)$ [4] and others ([14], [18], [19]).

The main aim of this paper is to investigate the mapping properties of index transforms, both known and new ones, in the Hilbert space $L_2(\omega(x); \mathbb{R}_+)$ with weight $\omega(x)$ by means of their factorization into a composition of general Fourier and Watson transforms [15]. Moreover, we shall introduce a general index transform and establish the Plancherel theorems by the universal composition method.

2. General Fourier and Watson transforms.

In accordance with [3] the transform

$$[Kf](x) = g(x) = \underset{N\to 0}{\text{l.i.m.}} \int_0^N k(xy) f(y)\, dy \tag{2.1}$$

is called a *general Fourier transform*, where $f \in L_2(\mathbb{R}_+)$; k is called a *Fourier kernel* and the integral (2.1) converges in the L_2-sense. The operator (2.1) is bounded in $L_2(\mathbb{R}_+)$ and its inversion in general case has the form

$$[\widehat{K}g](x) = f(x) = \mathop{\mathrm{l.i.m.}}_{N\to\infty} \int_0^N \widehat{k}(xy)g(y)\,dy, \tag{2.2}$$

where $\widehat{k}(x)$ is the conjugate Fourier kernel and we have the following equality for the Mellin transforms $k^*(s) = \mathfrak{M}\{k(y); s\}$ and $\widehat{k}^*(s) = \mathfrak{M}\{\widehat{k}(y); s\}$:

$$k^*(s)\widehat{k}^*(1-s) = 1, \qquad \mathrm{Re}(s) = 1/2. \tag{2.3}$$

For the K-transform and $\widehat{K}$-transform we have the Parseval type relation

$$\int_0^\infty [Kf](u)[\widehat{K}f](u)\,du = \int_0^\infty f^2(y)\,dy \tag{2.4}$$

with

$$\int_0^\infty |[Kf](u)|^2\,du < c\int_0^\infty |f(y)|^2\,dy, \tag{2.5}$$

$$\int_0^\infty |[\widehat{K}f](u)|^2\,du < \widehat{c}\int_0^\infty |f(y)|^2\,dy, \qquad c, \widehat{c} = \mathrm{const}, \tag{2.6}$$

where the integrals are understood as improper.

In particular, if $k(x) = \widehat{k}(x)$ and $|k^*(s)| = 1$, then the operator (2.1) is unitary in $L_2(\mathbb{R}_+)$, i.e.

$$\int_0^\infty |[Kf](u)|^2\,du = \int_0^\infty |f(y)|^2\,dy \tag{2.7}$$

and the inversion formula has the symmetric form

$$[Kg](x) = f(x) = \mathop{\mathrm{l.i.m.}}_{N\to\infty} \int_0^N k(xy)g(y)\,dy. \tag{2.8}$$

Many examples of such kernels can be found in [3]. Note the pairs of symmetric cos- and sin-Fourier transforms

$$[F_\mathrm{c}f](x) = g(x) = \sqrt{2/\pi}\,\mathop{\mathrm{l.i.m.}}_{N\to\infty} \int_0^N \cos(xy)f(y)\,dy, \tag{2.9}$$

$$[F_\mathrm{s}f](x) = g(x) = \sqrt{2/\pi}\,\mathop{\mathrm{l.i.m.}}_{N\to\infty} \int_0^N \sin(xy)f(y)\,dy \tag{2.10}$$

and the Hankel transform

$$[J_\nu f](x) = g(x) = \mathop{\mathrm{l.i.m.}}_{N\to\infty} \int_0^N \sqrt{xy}J_\nu(xy)f(y)\,dy, \qquad \nu > -1, \tag{2.11}$$

with the Bessel function as kernel.

Now we can introduce the transform (2.1) in terms of Watson kernels. A function $k_1(x)$ is called a *Watson kernel* if

$$k_1(x) = \int_0^x k(y)\,dy = \frac{x}{2\pi i}\operatorname*{l.i.m.}_{N\to\infty}\int_{1/2-iN}^{1/2+iN}\frac{k^*(s)}{1-s}x^{-s}\,ds, \tag{2.12}$$

where $k^*(s)$ is defined by (2.3). For example, the functions $\sqrt{2/\pi}\sin(x)$, $\sqrt{2/\pi}(1-\cos(x))$, $(1/\pi)\ln\left|\frac{1+x}{1-x}\right|$, $(1/\pi)\ln|1-x^2|$ are Watson kernels. If $k_1(x)$ is a differentiable function on the half-axis $(0,\infty)$ then it is not difficult to see that $k_1'(x) = k(x)$, where $k(x)$ is a Fourier kernel. A function $\widehat{k}_1(x)$ is called a *conjugate Watson kernel* if

$$\widehat{k}_1(x) = \frac{x}{2\pi i}\operatorname*{l.i.m.}_{N\to\infty}\int_{1/2-iN}^{1/2+iN}\frac{\widehat{k}^*(s)}{1-s}x^{-s}\,ds \tag{2.13}$$

and the equality (2.3) holds. Since the functions $k^*(s)$, $\widehat{k}^*(s)$ are bounded on the line $\mathrm{Re}(s) = 1/2$, the integrals (2.12)–(2.13) exist in the L_2-sense and by the theory of the Mellin transform, $k_1(x)/x$, $\widehat{k}_1(x)/x$ belong to $L_2(\mathbb{R}_+)$.

Hence by the Mellin–Parseval equality

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\varphi^*(s)\psi^*(1-s)x^{-s}\,ds = \int_0^\infty \varphi(xy)\psi(y)\,dy, \tag{2.14}$$

where φ, $\psi \in L_2(\mathbb{R}_+)$ and $\varphi^*(s)$, $\psi^*(s)$ are their Mellin transforms in $L_2(\mathbb{R}_+)$, we can evaluate the next improper integral (so-called Watson's condition) for all $x, y > 0$:

$$\int_0^\infty \frac{k_1(xu)\widehat{k}_1(yu)}{u^2}du = \min(x,y), \tag{2.15}$$

which provides the boundedness in $L_2(\mathbb{R}_+)$ of the corresponding pair of Watson transforms

$$[Kf](x) = g(x) = \frac{d}{dx}\int_0^\infty k_1(xy)f(y)\frac{dy}{y}, \tag{2.16}$$

$$[\widehat{K}g](x) = f(x) = \frac{d}{dx}\int_0^\infty \widehat{k}_1(xy)g(y)\frac{dy}{y}. \tag{2.17}$$

Moreover, as is shown in [3] the Watson condition (2.15) is necessary for the existence of the pair of dual transforms (2.16), (2.17).

3. The Plancherel type L_2-theorems for the Kontorovich–Lebedev and Mehler–Fock transforms. In this section we shall demonstrate the Plancherel theorems for the Kontorovich–Lebedev and Mehler–Fock index transforms by the composition method. Earlier such theorems were considered in [5], [6].

We shall base on the following integral representations for the Macdonald function $K_\nu(x)$ [1]:

$$K_\nu(x)\cos(\nu\pi/2)=\int_0^\infty \cos(x\sinh(u))\cosh(\nu u)\,du,\quad x>0,\ -1<\mathrm{Re}(\nu)<1, \tag{3.1}$$

$$K_\nu(x)\sin(\nu\pi/2)=\int_0^\infty \sin(x\sinh(u))\sinh(\nu u)\,du,\quad x>0,\ -1<\mathrm{Re}(\nu)<1. \tag{3.2}$$

Let $f(x)$ be a C^∞-function with compact support on $(0,\infty)$ and consider the following composition of sin-Fourier transforms (2.10):

$$\begin{aligned} g(\tau)&=[F_{\mathrm{s}}[F_{\mathrm{s}}f(y)/\sqrt{y}](\tau)(\sinh(t))(\tau)\\ &=\frac{2}{\pi}\,\underset{N\to\infty}{\mathrm{l.i.m.}}\int_0^N \sin(\tau u)\,du\,\underset{M\to\infty}{\mathrm{l.i.m.}}\int_0^M \sin(y\sinh(u))f(y)\frac{dy}{\sqrt{y}}. \end{aligned} \tag{3.3}$$

The inner integral is absolutely and uniformly convergent with respect to $u\in(0,N)$, therefore we can pass to the limit and after changing the order of integration in (3.3) and using the representation (3.2) we obtain

$$\begin{aligned} g(\tau)&=\frac{2}{\pi}\,\underset{N\to\infty}{\mathrm{l.i.m.}}\int_0^N \sin(\tau u)\,du\int_0^\infty \sin(y\sinh(u))f(y)\frac{dy}{\sqrt{y}}\\ &=\frac{2}{\pi}\lim_{N\to\infty}\int_0^\infty f(y)\frac{dy}{\sqrt{y}}\int_0^N \sin(\tau u)\sin(y\sinh(u))\,du\\ &=\frac{2}{\pi}\sinh(\pi\tau/2)\int_0^\infty K_{i\tau}(y)f(y)\frac{dy}{\sqrt{y}}. \end{aligned} \tag{3.4}$$

The last equality follows from the Lebesgue theorem and the estimate (use integration by parts)

$$\Big|\int_0^N \sin(\tau u)\sin(y\sinh(u))\,du\Big|\le\frac{1}{y\cosh(N)}+\frac{1}{y}\int_0^N\frac{\tau\cosh(u)+\sinh(u)}{\cosh^2(u)}\,du.$$

Since $[F_{\mathrm{s}}f(y)/\sqrt{y}](\sinh(x))\in L_2(\mathbb{R}_+)$ if $[F_{\mathrm{s}}f(y)/\sqrt{y}](x)\in L_2(\mathbb{R}_+)$, from (3.3) it follows that $g(\tau)\in L_2(\mathbb{R}_+)$.

Further, consider the factorization

$$\begin{aligned} h(\tau)&=\tau[F_{\mathrm{c}}[F_{\mathrm{c}}f(y)/\sqrt{y}](\sinh(t))](\tau)\\ &=\frac{1}{\pi}\,\underset{N\to\infty}{\mathrm{l.i.m.}}\int_0^N \tau\cos(\tau u)\,du\int_0^\infty \cos(y\sinh(u))f(y)\frac{dy}{\sqrt{y}}. \end{aligned} \tag{3.5}$$

After integration by parts we have

$$h(\tau)=-\frac{2}{\pi}\,\underset{N\to\infty}{\mathrm{l.i.m.}}\int_0^N \sin(\tau u)\,du\frac{d}{du}\int_0^\infty \cos(y\sinh(u))f(y)\frac{dy}{\sqrt{y}}. \tag{3.6}$$

Moreover, it is not difficult to show that terms outside the integral vanish as $N \to \infty$ in (3.6). On the other hand, applying the representation (3.1) we obtain analogously

$$h(\tau) = \frac{2}{\pi}\tau \cosh(\pi\tau/2) \int_0^\infty K_{i\tau}(y) f(y) \frac{dy}{\sqrt{y}}. \tag{3.7}$$

On the other hand, since

$$\begin{aligned} \frac{d}{dt}[F_{\mathrm{c}} f(y)/\sqrt{y}](\sinh(t)) &= \sqrt{2/\pi} \frac{d}{dt} \int_0^\infty \cos(y \sinh(t)) f(y) \frac{dy}{\sqrt{y}} \\ &= \sqrt{2/\pi} \frac{\cosh(t)}{\sinh^2(t)} \int_0^\infty \sin(y \sinh(t)) \frac{d}{dy}[f(y)/\sqrt{y}]\, dy \\ &\quad - \sqrt{2/\pi} \frac{1}{\sinh(t)} \frac{d}{dt} \int_0^\infty \sin(y \sinh(t)) \frac{d}{dy}[f(y)/\sqrt{y}]\, dy \\ &= \sqrt{2/\pi} \coth(t) \int_0^\infty \frac{\sin(y \sinh(t))}{y \sinh(t)} y \frac{d}{dy}[f(y)/\sqrt{y}]\, dy \\ &\quad - \sqrt{2/\pi} \coth(t) \frac{d}{d \sinh(t)} \int_0^\infty \frac{\sin(y \sinh(t))}{y} y \frac{d}{dy}[f(y)/\sqrt{y}]\, dy, \end{aligned}$$

from the Mellin–Parseval equality (2.14) and the estimate

$$\left| \int_0^\infty \frac{\sin(y \sinh(t))}{y \sinh(t)} y \frac{d}{dy}[f(y)/\sqrt{y}]\, dy \right| \le M \frac{1}{\sqrt{\sinh(t)}}, \qquad M = \text{const}.$$

for $y \frac{d}{dy}[f(y)/\sqrt{y}] \in L_2(\mathbb{R}_+)$ if $f \in C_0^\infty$ with compact support on $(0, \infty)$ and for the Watson kernel $\sqrt{2/\pi} \sin(x)$, we conclude that $\frac{d}{dx}[F_{\mathrm{c}} f(y)/\sqrt{y}](\sinh(x)) \in L_2(a, \infty)$, $a > 0$. But

$$\frac{d}{dt}[F_{\mathrm{c}} f(y)/\sqrt{y}](\sinh(t)) = -\sqrt{2/\pi} \cosh(t) \int_0^\infty \sin(y \sinh(t)) f(y) \sqrt{y}\, dy$$

and thus $\frac{d}{dx}[F_{\mathrm{c}} f(y)/\sqrt{y}](\sinh(t)) \in L_2(0, a)$. Therefore $\frac{d}{dx}[F_{\mathrm{c}} f(y)/\sqrt{y}](\sinh(t)) \in L_2(\mathbb{R}_+)$. From (3.6) it follows that $h \in L_2(\mathbb{R}_+)$.

Hence applying the Parseval relation for the sin-Fourier transform we derive

$$\begin{aligned} \int_0^\infty h(\tau)\overline{g(\tau)}\, d\tau &= -\int_0^\infty \frac{d}{du}[F_{\mathrm{c}} f(y)/\sqrt{y}](\sinh(u)) [\overline{F_{\mathrm{s}} f(y)/\sqrt{y}}(\sinh(u))\, du \\ &= -\int_0^\infty \frac{d}{dt}[F_{\mathrm{c}} f(y)/\sqrt{y}](t) [\overline{F_{\mathrm{s}} f(y)/\sqrt{y}}](t)\, dt \\ &= \int_0^\infty [F_{\mathrm{c}} f(y)/\sqrt{y}](t) \frac{d}{dt}[F_{\mathrm{s}} \overline{f(y)}/\sqrt{y}](t)\, dt \end{aligned} \tag{3.8}$$

$$= \int_0^\infty [F_c f(y)/\sqrt{y}](t)[F_c \sqrt{y}\bar{f}(y)](t)\,dt.$$

Continuing the chain of equalities (3.8) with application of the Parseval equality for the cos-Fourier transform we finally obtain the Parseval relation for the Kontorovich–Lebedev transform (3.4) in the space of C_0^∞-functions with compact support on $(0,\infty)$ as follows:

$$\int_0^\infty h(\tau)\overline{g(\tau)}\,d\tau = \int_0^\infty |f(y)|^2\,dy, \tag{3.9}$$

or since $h(\tau) = \tau\coth(\pi\tau/2)g(\tau)$,

$$\int_0^\infty \tau\coth(\pi\tau/2)|g(\tau)|^2\,d\tau = \int_0^\infty |f(y)|^2\,dy. \tag{3.10}$$

Now let $f(x)$ be an arbitrary function from $L_2(\mathbb{R}_+)$. The set of C_0^∞-functions with compact support on $(0,\infty)$ is dense in $L_2(\mathbb{R}_+)$, i.e. $f(x) = \text{l.i.m.}_{n\to\infty} f_n(x)$, where $f_n \in C_0^\infty(0,\infty)$. Therefore the equalities (3.9)–(3.10) hold for all functions from $L_2(\mathbb{R}_+)$ and we conclude that $g(x) \in L_2(x\coth(\pi x/2);\mathbb{R}_+)$. Moreover, putting $f_N(x) = f(x)$, $1/N \le x \le N$, $f_N(x) = 0$, $x \notin [1/N, N]$, we obtain

$$g(\tau) = [K_{i\tau}f] = \frac{2}{\pi}\sinh(\pi\tau/2)\,\underset{N\to\infty}{\text{l.i.m.}}\int_{1/N}^N K_{i\tau}(y)f(y)\frac{dy}{\sqrt{y}}. \tag{3.11}$$

The Parseval equality (3.10) for two different functions $f(y)$, $\widehat{f}(y)$ and their Kontorovich–Lebedev transforms (3.11) $g(\tau)$, $\widehat{g}(\tau)$ can be written as follows:

$$\int_0^\infty \tau\coth(\pi\tau/2)g(\tau)\overline{\widehat{g}(\tau)}\,d\tau = \int_0^\infty f(y)\overline{\widehat{f}(y)}\,dy. \tag{3.12}$$

Supposing $\widehat{f}(y) = 1$, $0 < y \le x$ and $\widehat{f}(y) = 0$, $y > x$ we shall have

$$\int_0^\infty f(y)\,dy = \frac{2}{\pi}\int_0^\infty \tau\cosh(\pi\tau/2)g(\tau)\int_0^x K_{i\tau}(y)\frac{dy}{\sqrt{y}}d\tau$$

or

$$f(x) = \frac{2}{\pi}\,\underset{N\to\infty}{\text{l.i.m.}}\int_0^N t\cosh(\pi\tau/2)\frac{K_{i\tau}(x)}{\sqrt{x}}g(\tau)\,d\tau. \tag{3.13}$$

THEOREM 3.1. *The Kontorovich–Lebedev transform* (3.11) *maps the space* $L_2(\mathbb{R}_+)$ *onto* $L_2(x\coth(\pi x/2);\mathbb{R}_+)$ *and the Parseval equality* (3.10) *holds*, *together with the inversion formula* (3.13).

In order to establish the Plancherel type theorem for the Mehler–Fock transform (1.5) we need the following integral representation (see [11], Vol. 2, formula 2.16.21.1):

$$\tanh(\pi\tau/2)P_{-1/2+i\tau/2}(2x^2+1) = \frac{2}{\pi}\sinh(\pi\tau/2)\int_0^\infty J_0(xu)K_{i\tau}(u)\,du. \tag{3.14}$$

So in accordance with the definition of the Hankel transform (2.11), for any $f \in L_2(\mathbb{R}_+)$ we consider the composition

$$\begin{aligned} g(\tau) &= [K_{i\tau}[J_0 f]] \\ &= \frac{2}{\pi}\sinh(\pi\tau/2)\,\underset{N\to\infty}{\text{l.i.m.}}\int_0^N K_{i\tau}(y)\frac{1}{\sqrt{y}}\,\underset{M\to\infty}{\text{l.i.m.}}\int_0^M \sqrt{yu}J_0(yu)f(u)\,du\,dy. \end{aligned} \tag{3.15}$$

Since the integral (3.14) is absolutely and uniformly convergent we have

$$g(\tau) = \tanh(\pi\tau/2)\,\underset{M\to\infty}{\text{l.i.m.}}\int_0^M P_{-1/2+i\tau/2}(2u^2+1)\sqrt{u}f(u)\,du. \tag{3.16}$$

Thus applying the Parseval equality (3.10) and the unitarity of the Hankel transform (2.11) we have the Parseval equality for the Mehler–Fock transform (3.16):

$$\int_0^\infty \tau\coth(\pi\tau/2)|g(\tau)|^2\,d\tau = \int_0^\infty |[J_0 f]|^2(y)\,dy = \int_0^\infty |f(y)|^2\,dy.$$

THEOREM 3.2. *The Mehler–Fock index transform* (3.16) *maps the space* $L_2(\mathbb{R}_+)$ *onto* $L_2(x\coth(\pi x/2);\mathbb{R}_+)$ *and the dual inversion formula*

$$f(x) = \sqrt{x}\,\underset{N\to\infty}{\text{l.i.m.}}\int_0^N \tau P_{-1/2+i\tau/2}(2x^2+1)g(\tau)\,d\tau \tag{3.17}$$

holds.

4. Plancherel type L_2-theorems for the Lebedev–Skalskaya and some other index transforms. In [7] the following Lebedev–Skalskaya transform pairs with the Macdonald functions [8] were introduced:

$$\begin{aligned} [\operatorname{Re} f](\tau) &= \frac{2}{\pi}\cosh(\pi\tau/2)\int_0^\infty \operatorname{Re} K_{1/2+i\tau}(u)f(u)\,du, \\ f(x) &= \frac{2}{\pi}\int_0^\infty \operatorname{Re} K_{1/2+i\tau}(x)\frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)}[\operatorname{Re} f](\tau)\,d\tau, \end{aligned} \tag{4.1}$$

$$\begin{aligned} [\operatorname{Im} f](\tau) &= \frac{2}{\pi}\cosh(\pi\tau/2)\int_0^\infty \operatorname{Im} K_{1/2+i\tau}(u)f(u)\,du, \\ f(x) &= \frac{2}{\pi}\int_0^\infty \operatorname{Im} K_{1/2+i\tau}(x)\frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)}[\operatorname{Im} f](\tau)\,d\tau, \end{aligned} \tag{4.2}$$

where $\operatorname{Re} K_{1/2+i\tau}(x)$, $\operatorname{Im} K_{1/2+i\tau}(x)$ are defined as follows:

$$\begin{Bmatrix}\operatorname{Re}\\ \operatorname{Im}\end{Bmatrix} K_{1/2+i\tau}(x) = \frac{K_{1/2+i\tau}(x) \pm K_{1/2-i\tau}(x)}{2\left\{\begin{smallmatrix}1\\ i\end{smallmatrix}\right\}}. \tag{4.3}$$

For the transform pairs (4.1)–(4.2) it is not difficult to get the corresponding integral representations from (3.1)–(3.2). Setting $\nu = 1/2 + i\tau$, we have

$$\begin{aligned} &\cosh(\pi\tau/2) \operatorname{Re} K_{1/2+i\tau}(x) \qquad (4.4)\\ &= \frac{1}{\sqrt{2}} \int_0^\infty [\cos(x\sinh(u))\cosh(u/2) + \sin(x\sinh(u))\sinh(u/2)]\cos(\tau u)\, du, \quad x > 0,\end{aligned}$$

$$\begin{aligned} &\sinh(\pi\tau/2) \operatorname{Re} K_{1/2+i\tau}(x) \qquad (4.5)\\ &= \frac{1}{\sqrt{2}} \int_0^\infty [\sin(x\sinh(u))\cosh(u/2) - \cos(x\sinh(u))\sinh(u/2)]\sin(\tau u)\, du, \quad x > 0,\end{aligned}$$

$$\begin{aligned} &\cosh(\pi\tau/2) \operatorname{Im} K_{1/2+i\tau}(x) \qquad (4.6)\\ &= -\frac{1}{\sqrt{2}} \int_0^\infty [\cos(x\sinh(u))\sinh(u/2) + \sin(x\sinh(u))\cosh(u/2)]\sin(\tau u)\, du, \quad x > 0,\end{aligned}$$

$$\begin{aligned} &\sinh(\pi\tau/2) \operatorname{Im} K_{1/2+i\tau}(x) \qquad (4.7)\\ &= \frac{1}{\sqrt{2}} \int_0^\infty [\sin(x\sinh(u))\sinh(u/2) - \cos(x\sinh(u))\cosh(u/2)]\cos(\tau u)\, du, \quad x > 0.\end{aligned}$$

In accordance with formulae (4.4)–(4.5) for the Re-transform (4.1) we consider the following compositions of cos- and sin-Fourier transforms for C_0^∞-functions with compact support on $(0, \infty)$:

$$\begin{aligned} [\operatorname{Re} f](\tau) &= \frac{1}{\sqrt{2}}[[F_{\mathrm{c}} \cosh(u/2)[F_{\mathrm{c}} f](\sinh(u))](\tau) \qquad (4.8)\\ &\quad + [F_{\mathrm{c}} \sinh(u/2)[F_{\mathrm{s}} f](\sinh(u))](\tau)],\\ \tanh(\pi\tau/2)[\operatorname{Re} f](\tau) &= \frac{1}{\sqrt{2}}[[F_{\mathrm{s}} \cosh(u/2)[F_{\mathrm{s}} f](\sinh(u))](\tau) \qquad (4.9)\\ &\quad - [F_{\mathrm{s}} \sinh(u/2)[F_{\mathrm{c}} f](\sinh(u))](\tau)].\end{aligned}$$

Applying the Parseval relation for cos- and sin-Fourier transforms to equalities (4.8), (4.9) and adding them we derive

$$\begin{aligned} &\int_0^\infty \frac{\cosh(\pi\tau)}{\cosh^2(\pi\tau/2)} |[\operatorname{Re} f](\tau)|^2\, d\tau \qquad (4.10)\\ &\quad = \frac{1}{2} \int_0^\infty \cosh(u)[|[F_{\mathrm{c}} f](\sinh(u))|^2 + |[F_{\mathrm{s}} f](\sinh(u))|^2]\, du = \int_0^\infty |f(y)|^2\, dy.\end{aligned}$$

For the Im-transform (4.2) acting analogously from representations (4.6)–(4.7) we shall have the corresponding compositions and the Parseval relation in the form

$$\begin{aligned}[\operatorname{Im} f](\tau) = &-\frac{1}{\sqrt{2}}[[F_{\mathrm{s}} \sinh(u/2)[F_{\mathrm{c}} f](\sinh(u))](\tau) \\ &+ [F_{\mathrm{s}} \cosh(u/2)[F_{\mathrm{s}} f](\sinh(u))](\tau)], \end{aligned} \tag{4.11}$$

$$\begin{aligned}\tanh(\pi\tau/2)[\operatorname{Im} f](\tau) = &\frac{1}{\sqrt{2}}[[F_{\mathrm{c}} \sinh(u/2)[F_{\mathrm{s}} f](\sinh(u))](\tau) \\ &- [F_{\mathrm{c}} \cosh(u/2)[F_{\mathrm{c}} f](\sinh(u))](\tau)], \end{aligned} \tag{4.12}$$

$$\int_0^\infty \frac{\cosh(\pi\tau)}{\cosh^2(\pi\tau/2)} |[\operatorname{Im} f](\tau)|^2 \, d\tau = \int_0^\infty |f(y)|^2 \, dy. \tag{4.13}$$

The Parseval equalities (4.10), (4.13) can be extended to any function $f(x)$ from $L_2(\mathbb{R}_+)$ as in Section 3.

In the similar manner we can establish the dual formulae for Lebedev–Skalskaya transform pairs. From the Parseval relations (4.10), (4.13) we obtain almost everywhere for $x > 0$ the corresponding dual formulae

$$f(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^\infty \frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)} [\operatorname{Re} f](\tau) \int_0^x \operatorname{Re} K_{1/2+i\tau}(y) \, dy \, d\tau, \tag{4.14}$$

$$f(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^\infty \frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)} [\operatorname{Im} f](\tau) \int_0^x \operatorname{Im} K_{1/2+i\tau}(y) \, dy \, d\tau \tag{4.15}$$

and the final

THEOREM 4.1. *The Lebedev–Skalskaya transform pairs* (4.1)–(4.2), *where the integrals are understood as square-summable*, *map the space* $L_2(\mathbb{R}_+)$ *onto* $L_2(\cosh(\pi\tau)/\cosh^2(\pi\tau/2); \mathbb{R}_+)$. *Moreover*, *the Perseval relations* (4.10), (4.13) *are true and almost everywhere the dual formulas* (4.14)–(4.15) *hold or their* L_2-*equivalents*, *namely*

$$f(x) = \frac{2}{\pi} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)} \operatorname{Re} K_{1/2+i\tau}(x)[\operatorname{Re} f](\tau) \, d\tau, \tag{4.16}$$

$$f(x) = \frac{2}{\pi} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)} \operatorname{Im} K_{1/2+i\tau}(x)[\operatorname{Im} f](\tau) \, d\tau. \tag{4.17}$$

We note that this Plancherel type theorem is new. Only some sufficient conditions for the existence and inverting of these transform pairs were known (see, for instance [7], [12]).

Consider now the following compositions of the Fourier and Hankel transforms (2.10)–(2.11):

$$g(\tau) = [F_{\mathrm{s}} \sqrt{\sinh(u)}[J_\nu f(t)](\sinh(u))](\tau), \qquad \nu > -1/2, \tag{4.18}$$

$$\widehat{g}(\tau) = \left[F_{\mathrm{s}} \frac{\cosh(u)}{\sqrt{\sinh(u)}}[J_\nu f(t)](\sinh(u))\right](\tau), \qquad \nu > -1/2. \tag{4.19}$$

First, using the integral representations 2.12.26.1–2 from [11], Vol. 2, we can obtain the following integrals with Bessel functions:

$$\sqrt{x}\int_0^\infty J_\nu(x\sinh(u))\sinh(u)\sin(\tau u)\,du = \frac{\sqrt{x}}{2}\,\mathrm{Im}[I_{(\nu-1-i\tau)/2}(x/2) \times K_{(\nu+1+i\tau)/2}(x/2) + I_{(\nu+1+i\tau)/2}(x/2)K_{(\nu-1-i\tau)/2}(x/2)], \quad \nu > -1, \tag{4.20}$$

$$\sqrt{x}\int_0^\infty J_\nu(x\sinh(u))\cosh(u)\sin(\tau u)\,du = \frac{\sqrt{x}}{2}\,\mathrm{Im}[I_{(\nu-1-i\tau)/2}(x/2) \times K_{(\nu+1+i\tau)/2}(x/2) - I_{(\nu+1+i\tau)/2}(x/2)K_{(\nu-1-i\tau)/2}(x/2)], \quad \nu > -1. \tag{4.21}$$

THEOREM 4.2. *Let the Hankel transform* (2.11) *be such that* $[J_\nu f](x) \in L_2(1/x;(0,1))$ *for* $\nu > -1/2$. *Then the index transform*

$$g(\tau) = \frac{1}{\sqrt{2\pi}} \times \operatorname*{l.i.m.}_{N\to\infty}\int_0^N \mathrm{Im}\Big[\sum_{m=1}^{2} I_{(\nu+(-1)^m(1+i\tau))/2}(y/2)K_{(\nu-(-1)^m(1+i\tau))/2}(y/2)\Big]\sqrt{y}f(y)\,dy \tag{4.22}$$

satisfies the following estimate for $L_2(\mathbb{R}_+)$ *norms*:

$$c_1\|f\| < \|g\| < c_2\|f\|, \qquad c_i = \text{const}, \qquad i = 1,2.$$

The inversion formula

$$f(x) = \frac{\sqrt{x}}{\sqrt{2\pi}} \times \operatorname*{l.i.m.}_{N\to\infty}\int_0^N \mathrm{Im}\Big[\sum_{m=1}^{2}(-1)^m I_{(\nu-(-1)^m(1+i\tau))/2}(x/2)K_{(\nu+(-1)^m(1+i\tau))/2}(x/2)\Big]g(\tau)\,d\tau \tag{4.23}$$

and the Parseval equality

$$\int_0^\infty g(\tau)\overline{\widehat{g}(\tau)}\,d\tau = \int_0^\infty |f(y)|^2\,dy \tag{4.24}$$

also hold.

Proof. Indeed, according to the representations (4.20)–(4.21) in the space of smooth functions with compact support we can get the transform (4.22) from the composition (4.18). Applying the unitarity of transforms (2.10)–(2.11), for every $f \in L_2(\mathbb{R}_+)$ which satisfies the assumption of the theorem we show that $g(\tau)$, $\widehat{g}(\tau)$ belong to $L_2(\mathbb{R}_+)$ (using the properties of the Fourier and Hankel transforms) and we find that the Parseval equality (4.24), the estimate of the norms and the dual formulas (4.22)–(4.23) hold. The additional condition on the Hankel transform is required because we need that the inner function in the composition (4.19) is in $L_2(\mathbb{R}_+)$ for all $f \in L_2(\mathbb{R}_+)$.

A particular case of this new index transform is interesting. If $\nu = 0$, then the pair of the index transforms (4.22)–(4.23) reduces to

$$g(\tau) = \frac{1}{\sqrt{2\pi}} \operatorname*{l.i.m.}_{\nu\to\infty} \int_0^N \operatorname{Im}[(I_{-(1+i\tau)/2}(y/2) + I_{(1+i\tau)/2}(y/2))K_{(1+i\tau)/2}(y/2)]\sqrt{y}f(y)\,dy, \tag{4.25}$$

$$f(x) = \frac{\sqrt{2x}}{\pi\sqrt{\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \cosh(\pi\tau/2)\operatorname{Im} K^2_{(1+i\tau)/2}(x/2)g(\tau)\,d\tau. \tag{4.26}$$

In the case $\nu = 1$ we immediately get the index transform pair

$$g(\tau) = \frac{\sqrt{2}}{\pi\sqrt{\pi}}\sinh(\pi\tau/2) \operatorname*{l.i.m.}_{N\to\infty} \int_0^N K_{i\tau/2}(y/2)\operatorname{Re}[K_{1+i\tau/2}(x/2)]\sqrt{y}f(y)\,dy, \tag{4.27}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \tau K_{i\tau/2}(x/2)\operatorname{Re}[I_{i\tau/2}(x/2)]g(\tau)\,d\tau, \tag{4.28}$$

which are easily found to be Lebedev's transforms from [4].

5. Some general compositions. We can use this technique to consider general compositions and to obtain some general index transforms with Watson kernels. It is known [8], [20] that particular cases of index transforms and L_2-theorems for them can also be obtained from the pair (1.6)–(1.7). Now we shall investigate more general compositions with arbitrary functions $f, \widehat{f} \in L_2(\mathbb{R}_+)$, namely

$$g(x) = [K(\mu'(t))^{\alpha}[Hf](\mu(t))](x), \tag{5.1}$$

$$\widehat{g}(x) = [\widehat{K}(\mu'(t))^{1-\alpha}[\widehat{H}\widehat{f}](\mu(t))](x), \quad 0 < \alpha \le 1/2, \tag{5.2}$$

where the functions $k(x)$, $h(x)$ are arbitrary Fourier kernels and $[Kf](x)$, $[Hf](x)$ and their conjugate ones are the corresponding general Fourier transforms defined by (2.1)–(2.2). Let also $\mu(x)$ be an increasing positive differentiable function on $\mathbb{R}_+$, with $\mu(0) = 0$, $\mu'(0) \neq 0$. We suppose that $(\mu'(t)/\mu(t)) = O(1)$, $t \to \infty$ and for any function $f \in L_2(\mathbb{R}_+)$, $[\widehat{H}f](x)$ belongs to the weighted L_2-space $L_2(x^{1-2\alpha};\mathbb{R}_+)$, $0 < \alpha \le 1/2$. Then it is not difficult to conclude that for any functions f, $\widehat{f} \in L_2(\mathbb{R}_+)$ we have $(\mu'(x))^{\alpha}[Hf](\mu(x))$, $(\mu'(x))^{1-\alpha}[\widehat{H}\widehat{f}](\mu(x)) \in L_2(\mathbb{R}_+)$ and we have the next Parseval equality

$$\int_0^\infty g(\tau)\overline{\widehat{g}(\tau)}\,d\tau = \int_0^\infty f(y)\overline{\widehat{f}(y)}\,dy \tag{5.3}$$

together with the inequalities

$$\int_0^\infty |g(u)|^2\,du < c_1 \int_0^\infty |f(y)|^2\,dy, \tag{5.4}$$

$$\int\limits_0^\infty |\widehat{g}(u)|^2\,du < \widehat{c}_1 \qquad c_1, \widehat{c}_1 = \text{const.} \tag{5.5}$$

Using the properties of Fourier kernels (see Section 2) and the representations (2.16)–(2.17), integrating by parts, we can treat the composition (5.1) as follows:

$$g(x) = \frac{d}{dx}\int\limits_0^\infty K_{kh}^{\mu,\alpha}(x,y)f(y)\,dy, \tag{5.6}$$

where

$$K_{kh}^{\mu,\alpha}(x,y) = \frac{\partial}{\partial y}\int\limits_0^\infty \frac{k_1(xt)}{t} h_1(\mu(t)y)\frac{(\mu'(t))^\alpha}{\mu(t)}\,dt, \tag{5.7}$$

and setting $\mu(t) = v$ and using the properties of Watson kernels it is not difficult to show that for any $x>0$, $K_{kh}^{\mu,\alpha}(x,y) \in L_2(\mathbb{R}_+)$. So we understand the integral (5.6) as an improper one. If we suppose that for any $y > 0$, $\widehat{h}_1(yt)/t^{1/2+\alpha} \in L_2(\mathbb{R}_+)$, then taking $\widehat{f}(y) = 1$, $0 < y \le x$, $\widehat{f}(y) = 0$, $y > x$, from the composition (5.2) and the equality (5.3) we get the dual formula

$$f(x) = \frac{d}{dx}\int\limits_0^\infty \widehat{K}_{\hat{k}\hat{h}}^{\mu,1-\alpha}(u,x)g(u)\,du, \tag{5.8}$$

where

$$\widehat{K}_{\hat{k}\hat{h}}^{\mu,1-\alpha}(u,x) = \frac{\partial}{\partial u}\int\limits_0^\infty \frac{\widehat{k}_1(ut)}{t}\widehat{h}_1(\mu(t)x)\frac{(\mu'(t))^{1-\alpha}}{\mu(t)}\,dt. \tag{5.9}$$

THEOREM 5.1. *Let the function* $\mu(x)$ *satisfy the conditions mentioned above and* $(\mu'(x)/\mu(x)) = O(1)$, $x \to \infty$. *Let the kernel* $\widehat{h}_1(x)$ *be such that for any* $y > 0$, $\widehat{h}_1(yt)/t^{1/2+\alpha} \in L_2(\mathbb{R}_+)$, $0 < \alpha \le 1/2$. *Then for any function* $f \in L_2(\mathbb{R}_+)$ *its index transform* (5.6) *belongs to* $L_2(\mathbb{R}_+)$ *with the corresponding estimate* (5.4). *Moreover, we have the Parseval equality* (5.3), *where* $\widehat{f} \in L_2(\mathbb{R}_+)$ *and* $\widehat{g}(x)$ *is defined by* (5.2) *and it belongs to* $L_2(\mathbb{R}_+)$ *if* $[\widehat{H}\widehat{f}](x) \in L_2(\mathbb{R}_+; x^{1-2\alpha})$, $0 < \alpha \le 1/2$. *Finally, almost everywhere on* $\mathbb{R}_+$ *the dual inverse formula* (5.8) *holds.*

We note that according to the Parseval equality (2.4) from the definitions (5.7), (5.9) of the kernels $K_{kh}^{\mu,\alpha}(x,y)$, $\widehat{K}_{\hat{k}\hat{h}}^{\mu,1-\alpha}(u,x)$ and the Watson condition (2.15) we obtain the Watson condition for the general index transform, namely

$$\int\limits_0^\infty \widehat{K}_{kh}^{\mu,\alpha}(u,x)\widehat{K}_{\hat{k}\hat{h}}^{\mu,1-\alpha}(u,y)\,du = \min(x,y). \tag{5.10}$$

More exact and simple Plancherel type theorems are obtained for $\alpha = 1/2$, because some conditions of Theorem 5.1 are immediate for such kernels. Moreover, we have new examples of index transforms. Indeed, if $\mu(t) = e^t - 1$ and (a) $h(y) = \widehat{h}(y) = \sqrt{2/\pi}\sin(y)$ or (b) $h(y) = \widehat{h}(y) = \sqrt{2/\pi}\cos(y)$, then the general pair (5.6), (5.8) reduces to the integral transforms with the Lommel functions $\mathcal{S}_{\mu,\nu}(z)$ [1], namely in the case (a),

$$\begin{aligned} g(x) &= \frac{2}{\pi} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} [y^{-ix}\mathcal{S}_{ix,1/2}(y)] f(y)\, dy, \\ f(x) &= \frac{2}{\pi} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} [x^{-iy}\mathcal{S}_{iy,1/2}(x)] g(y)\, dy, \end{aligned} \tag{5.11}$$

and in the case (b),

$$\begin{aligned} g(x) &= \frac{8}{\pi} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} [(1/2-ix)(y/4)^{ix}\mathcal{S}_{-1+ix,1/2}(y)] y^{-2} f(y)\, dy, \\ f(x) &= \frac{8}{\pi} x^{-2} \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} [(1/2-iy)(x/4)^{iy}\mathcal{S}_{-1+iy,1/2}(x)] g(y)\, dy. \end{aligned} \tag{5.12}$$

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