TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

ON BERNSTEIN-GELFAND-GELFAND EQUIVALENCE AND TILTING THEORY

PIOTR DOWBOR

Institute of Mathematics, Nicholas Copernicus University
Toruń, Poland

Let Λ be the exterior algebra of k^{n+1} and let S be the symmetric algebra of the dual space $(k^{n+1})^*$. The relations between the derived categories of finitely generated **Z**-graded modules over these algebras are studied. It is shown that Λ and S are linked by means of some tilting procedure which induces a triangle equivalence $D^b(\mathrm{mod}^{\mathbf{Z}}\Lambda) \simeq D^b(\mathrm{mod}^{\mathbf{Z}}S)$.

0. Introduction

Let $\mathrm{coh}(X)$ denote the category of coherent sheaves on a projective variety X. The problem of description of the derived category $D^b(\mathrm{coh}(X))$ was intensively studied in many papers, e.g. [BGG], [Be], [K1], [K2]. The description is usually given by a triangle equivalence $D^b(\mathrm{coh}(X)) \simeq D^b(\mathrm{mod}\, A)$, where $\mathrm{mod}\, A$ is the category of left finitely generated modules over a certain finite-dimensional k-algebra A associated with X of finite global dimension (see [Be] and [K2]). Recently D. Baer in [Ba] and D. Vossieck (unpublished) explained this equivalence for projective n-space P^n using tilting theory. Interpreting A as the endomorphism ring of some sheaf on P^n satisfying tilting conditions they adopt in their proofs Happel's arguments (see [H]) who observed that the tilting construction preserves derived categories.

The description given in [BGG] is different from the above one and has the form $\underline{\mathrm{mod}}^{\mathbf{Z}}\Lambda \simeq D^b(\mathrm{coh}(P^n))$, where $\underline{\mathrm{mod}}^{\mathbf{Z}}\Lambda$ denotes the stable category of finite-dimensional **Z**-graded left Λ-modules. This triangle equivalence is defined as the composition of three triangle equivalences

$$\underline{\mathrm{mod}}^{\mathbf{Z}}\Lambda \xrightarrow{\sim} D^b(\mathrm{mod}^{\mathbf{Z}}\Lambda)/\mathscr{F} \xrightarrow{\sim} D^b(\mathrm{mod}^{\mathbf{Z}}S)/\mathscr{J} \xrightarrow{\sim} D^b(\mathrm{coh}(X)),$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\mathscr{F}$ is a thick subcategory of $D^b(\text{mod}^{\mathbf{Z}} \Lambda)$ consisting of all complexes $X^\cdot$ with all X^t projective, $\text{mod}^{\mathbf{Z}} S$ denotes the category of finitely generated $\mathbf{Z}$-graded S-modules and $\mathscr{J}$ a thick subcategory of $D^b(\text{mod}^{\mathbf{Z}} S)$ consisting of all complexes $Y^\cdot$ such that all Y^t are finite-dimensional. In contrast to the left and right triangle equivalences the middle one is rather unexpected. Its proof is done by an explicit construction of functors F: $D^b(\text{mod}^{\mathbf{Z}} \Lambda) \to D^b(\text{mod}^{\mathbf{Z}} S)$ and G: $D^b(\text{mod}^{\mathbf{Z}} S) \to D(\text{mod}^{\mathbf{Z}} \Lambda)$ which are mutually inverse and preserve the respective subcategories.

A better understanding of the nature of interrelations between $D^b(\text{mod}^{\mathbf{Z}} \Lambda)$ and $D^b(\text{mod}^{\mathbf{Z}} S)$ and an explanation of the middle triangle equivalence in the context of tilting theory are the main aims of this paper.

The main results of this paper were obtained in the autumn of 1986 during my stay at Universität-Gesamthochschule Paderborn. They were presented at the DFG Meeting "Darstellungstheorie" in January 1987 in Bad Honnef, where also Schofield announced related results obtained by J. Rickard (see [R]). Rickard's tilting complex is more general then our tilting subcategory but his proofs seem to be much more complicated than ours. The tilting procedure we need to use in the explanation requires an extra property (o) (see Section 1) which reflects the fact that we deal with directed categories and this observably simplifies the situation.

An explanation of the above triangle equivalence is also discussed in [Bu].

I would like to thank Universität-Gesamthochschule Paderborn for the hospitality during my stay there. In particular, I wish to express my deep gratitude to Helmut Lenzing and his colleagues for fruitful collaboration, which considerably influenced my work on this and related problems.

1. Preliminaries

Throughout this paper k will denote a fixed algebraically closed field. By a *k-category* we mean an additive category R whose morphism sets $R(x, y)$ are endowed with a k-vector space structure such that the composition of maps is k-bilinear. An R-module is a k-linear covariant functor from R to the category of vector spaces over k. An R-module M is *finitely generated* (*finite-dimensional*) if M is an epimorphic image of a finite direct sum of representable functors (resp. $\sum_{x \in R} \dim_k M(x)$ is finite). We denote by $\text{Mod}\, R$ the category of all R-modules, and by $\text{mod}\, R$ (resp. $\text{mod}_0 R$) the full subcategory of $\text{Mod}\, R$ formed by all finitely generated (resp. finite-dimensional) R-modules. We denote by $\text{proj}\ R$ and $\text{inj}\ R$ the full subcategory of all projective and injective modules in $\text{mod}\, R$ respectively. Following [BG], R is *locally finite* (resp. *locally bounded*) if R is a k-category satisfying the following conditions:

(a) For each $x \in R$ the endomorphism algebra $R(x, x)$ is local.
(b) Distinct objects are nonisomorphic.

(c) $\dim_k R(x, y) < \infty$ for any two objects $x, y \in R$ (resp. $\sum_{y\in R} \dim_k R(x, y) < \infty$ and $\sum_{y\in R} \dim_k R(y, x) < \infty$ for any $z \in R$).

Observe that if R is locally finite the usual k-duality yields an equivalence $\operatorname{inj} R^{\mathrm{op}} \xrightarrow{\sim} (\operatorname{proj} R)^{\mathrm{op}}$, and if R is locally bounded then $\operatorname{mod} R = \operatorname{mod}_0 R$.

For any $\mathbf{Z}$-graded k-algebra $A = \bigoplus_{n\in\mathbf{Z}} A_n$ we denote by $\operatorname{Mod}^{\mathbf{Z}} A$ (resp. $\operatorname{mod}^{\mathbf{Z}} A$, $\operatorname{mod}_0^{\mathbf{Z}} A$) the category of $\mathbf{Z}$-graded (resp. finitely generated $\mathbf{Z}$-graded, finite-dimensional $\mathbf{Z}$-graded) A-modules with homomorphisms of degree 0 and we identify it with $\operatorname{Mod} \tilde{A}$ (resp. $\operatorname{mod} \tilde{A}$, $\operatorname{mod}_0 \tilde{A}$), where $\tilde{A}$ is the *cover k-category* associated to the $\mathbf{Z}$-grading of A, with $\operatorname{ob} \tilde{A} = \mathbf{Z}$, $\tilde{A}(m, n) = A_{n-m}$ and composition given by multiplication in A. The equivalence $\operatorname{Mod} \tilde{A} \xrightarrow{\sim} \operatorname{Mod}^{\mathbf{Z}} A$ is given by the map $M \mapsto \bigoplus_{n\in\mathbf{Z}} M(n)$, where $M \in \operatorname{Mod} \tilde{A}$. Observe that if A_0 is a local ring and $\dim_k A < \infty$ (resp. $\dim_k A_m < \infty$ for each $m \in \mathbf{N}$) then the k-category $\tilde{A}$ is locally bounded (resp. locally finite).

If $\mathscr{A}$ is an additive category then a *complex* $X^{\cdot} = (X^i, d^i_{X^{\cdot}})_{i\in\mathbf{Z}}$ over $\mathscr{A}$ is a collection of objects X^i and morphisms $d^i_{X^{\cdot}}: X^i \to X^{i+1}$ such that $d^{i+1}_{X^{\cdot}} d^i_{X^{\cdot}} = 0$. A morphism $f^{\cdot} = (f^i)_{i\in\mathbf{Z}}: X^{\cdot} \to Y^{\cdot}$ of complexes is a collection of morphisms f^i: $X^i \to Y^i$ such that $f^{i+1} d^i_{X^{\cdot}} = d^i_{Y^{\cdot}} f^i$. A complex $X^{\cdot}$ is *bounded below* (resp. *above*) if $X^i = 0$ for all but finitely many $i < 0$ (resp. $i > 0$). It is *bounded* if it is bounded below and above. Denote by $C(\mathscr{A})$ the category of all complexes and their morphisms, and by $C^+(\mathscr{A})$ (resp. $C^-(\mathscr{A})$, $C^{\mathrm{b}}(\mathscr{A})$) the full subcategory of $C(\mathscr{A})$ of complexes bounded below (resp. above, bounded). The category $\mathscr{A}$ will always be identified with the full subcategory of stalk complexes in $C(\mathscr{A})$ consisting of all $X^{\cdot}$ such that $X^i = 0$ for each $i \neq 0$. The *shift functor* T: $C(\mathscr{A}) \to C(\mathscr{A})$ is defined by $(TX^{\cdot})^i = X^{i+1}$, $(d_{TX^{\cdot}})^i = -(d^{i+1}_{X^{\cdot}})$ and $(Tf^{\cdot})^i = f^{i+1}$. The *mapping cone* $C_{f^{\cdot}}$ of the morphism $f^{\cdot}$: $X^{\cdot} \to Y^{\cdot}$ is the complex $C_{f^{\cdot}} = ((TX^{\cdot})^i \oplus Y^i, d^i_{C_{f^{\cdot}}})_{i\in\mathbf{Z}}$ with the differential

$$\begin{pmatrix} -d^{i+1}_{X^{\cdot}} & 0 \\ f^{i+1} & d^i_{Y^{\cdot}} \end{pmatrix}: X^{i+1} \oplus Y^i \to X^{i+2} \oplus Y^{i+1}.$$

The mapping cone construction is functorial in the sense that for any $f^{\cdot}_1$: $X_1 \to Y_1$, $f^{\cdot}_2$: $X^{\cdot}_2 \to Y^{\cdot}_2$, $g^{\cdot}_1$: $X^{\cdot}_1 \to X^{\cdot}_2$ and $g^{\cdot}_2$: $Y^{\cdot}_1 \to Y^{\cdot}_2$ such that $g^{\cdot}_2 f^{\cdot}_1 = f^{\cdot}_2 g^{\cdot}_1$ we have the morphism $C_{(g^{\cdot}_1, g^{\cdot}_2)}$: $C_{f^{\cdot}_1} \to C_{f^{\cdot}_2}$ given by the maps $g^{i+1}_1 \oplus g^i_2$. Given $X^{\cdot} \in C^+(\mathscr{A})$ we define a complex $X'^{\cdot}$ by $X'^i = X^i$, $d^i_{X'^{\cdot}} = d^i_{X^{\cdot}}$ if $i > n$ and $X'^i = 0$ for $i \leqslant n$, where n is minimal with $X^n \neq 0$. Analogously we define $f'^{\cdot}$ for a morphism $f^{\cdot}$: $X^{\cdot} \to Y^{\cdot}$ if n is common for $X^{\cdot}$ and $Y^{\cdot}$.

If $\mathscr{A}$ is a full subcategory of an abelian category then the cohomology $H^i(X^{\cdot})$ is defined for any $i \in \mathbf{Z}$ and $X^{\cdot} \in C(\mathscr{A})$. A morphism $f^{\cdot}$: $X^{\cdot} \to Y^{\cdot}$ is a *quasi-isomorphism* if the induced cohomology morphisms $H^i(f^{\cdot})$: $H^i(X^{\cdot}) \to H^i(Y^{\cdot})$ are isomorphisms for all $i \in \mathbf{Z}$. Denote by $K(\mathscr{A})$ (resp. $K^+(\mathscr{A})$, $K^-(\mathscr{A})$, $K^{\mathrm{b}}(\mathscr{A})$) the factor categories of $C(\mathscr{A})$ (resp. $C^+(\mathscr{A})$, $C^-(\mathscr{A})$, $C^{\mathrm{b}}(\mathscr{A})$) modulo the homotopy relation, and in case $\mathscr{A}$ is abelian by $D(\mathscr{A})$ (resp.

$D^+(\mathscr{A})$, $D^-(\mathscr{A})$, $D^b(\mathscr{A})$) the respective *derived categories*, which arise from $K(\mathscr{A})$ (resp. $K^+(\mathscr{A})$, $K^-(\mathscr{A})$, $K^b(\mathscr{A})$) by localization with respect to the multiplicative system of all quasi-isomorphisms. All these categories carry the structure of *triangulated category* (see [Ha], [V]), where the translation functor is given by T, and the triangulation by the set of triangles consisting of all sextuples isomorphic to those of the form $X^{\cdot} \xrightarrow{f^{\cdot}} Y^{\cdot} \to C_{f^{\cdot}} \to TX^{\cdot}$, for any $f^{\cdot} \in C(\mathscr{A})$.

We say that a functor of triangulated categories $F\colon \mathscr{C} \to \mathscr{C}'$ is *exact* if it commutes up to isomorphism with the translation functor and sends triangles to triangles. If moreover F is an equivalence then we call F a *triangle equivalence*. The smallest triangulated subcategory of $\mathscr{C}$ containing a given subcategory $\mathscr{U}$ of $\mathscr{C}$ will always be denoted by $\langle \mathscr{U} \rangle$.

The categories $D^b(\mathscr{A})$, $D^+(\mathscr{A})$, $D^-(\mathscr{A})$ can be identified with full subcategories of $D(\mathscr{A})$ defined by an obvious condition expressed in terms of vanishing of the respective cohomologies. By the canonical functor $C(\mathscr{A}) \to K(\mathscr{A}) \to D(\mathscr{A})$, the category $\mathscr{A}$ is equivalent to the subcategory of all complexes $X^{\cdot}$ such that $H^i(X^{\cdot}) = 0$ for all $i \neq 0$. For any X, $Y \in \mathscr{A}$ we have

$$D(\mathscr{A})(X^{\cdot}, T^m Y^{\cdot}) = \begin{cases} \mathrm{Ext}^m_{\mathscr{A}}(X, Y) & \text{if } m \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the composition is given by the Yoneda product (see [Ha]). If $\mathscr{A}$ has enough injective (resp. projective) objects then the canonical functor $K^+(\operatorname{inj}\mathscr{A}) \to D^+(\mathscr{A})$ (resp. $K^-(\operatorname{proj}\mathscr{A}) \to D^-(\mathscr{A})$) is a triangle equivalence. If in addition $\operatorname{gl.dim}\mathscr{A} < \infty$ then there exists a triangle equivalence $K^b(\operatorname{inj}\mathscr{A}) \xrightarrow{\sim} D^b(\mathscr{A})$ (resp. $K^b(\operatorname{proj}\mathscr{A}) \xrightarrow{\sim} D^b(\mathscr{A})$). Now for any full subcategory $\mathscr{A}_0$ of $\mathscr{A}$ closed under extensions, we denote by $D_0(\mathscr{A})$ the full subcategory of $D(\mathscr{A})$ consisting of all $X^{\cdot}$ such that $H^i(X^{\cdot}) \in \mathscr{A}_0$ for all $i \in \mathbf{Z}$. For the basic properties of triangulated categories we refer to [Ha] and [V].

2. Tilting subcategories

Let $\mathscr{U}$ be an additive subcategory of $D(\mathscr{A})$. We are going to find a sufficient condition on $\mathscr{U}$ in order that the embedding $\mathscr{U} \to D(\mathscr{A})$ has an extension to an exact functor $K^b(\mathscr{U}) \to D(\mathscr{A})$. Note that the case $\mathscr{U} \subseteq \mathscr{A}$ is obvious.

LEMMA 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories and let $F\colon \mathscr{B} \to D(\mathscr{A})$ be an additive functor. Then there exists a functor $\check{F}\colon C^b(\mathscr{B}) \to C(\mathscr{A})$ commuting up to isomorphism with the shift functors and with the cone construction, such that $\check{F} \mid \mathscr{B} = F$.*

Proof. We extend F first to $C^{\leqslant 0}(\mathscr{B}) = \varinjlim_{i \in \mathbf{N}} C^{[-i,0]}$ and next to $C^b(\mathscr{B})$

$= \varinjlim_{i \in \mathbf{N}} C^{\leq i}$, where $C^{[-i,0]}$ consists of all $B^{\cdot}$ such that $B^t = 0$ for $t > 0$ and $t < -i$, and $C^{\leq i} = T^{-i}(C^{b,-}(\mathscr{B}))$.

Step 1°. In order to define a functor F': $C^{\leq 0}(\mathscr{B}) \to C(\mathscr{A})$ together with an isomorphism η': $F'T \xrightarrow{\sim} TF'$ we construct inductively functors F_i: $C^{[-i,0]} \to C(\mathscr{A})$ together with natural isomorphisms of functors η_i: $TF_{i-1} \xrightarrow{\sim} F_i T$, $i \in \mathbf{N}$, such that $F_{i+1} \mid C^{[-i,0]} = F_i$, $\eta_{i+1} \mid C^{[-i,0]} = \eta_i$ and the cones are preserved. If $i = 0$ then we set $F_0 = F$. Assume that for some n, F_n and η_n are constructed in such a way that all properties required are satisfied. In order to define F_{n+1} and $\eta_{n+1}^{\cdot}$ we denote by $u_{B^{\cdot}}$: $T^n(B^{-(n+1)}) \to B^{\cdot}$ for each $B^{\cdot} \in C^{[-(n+1),0]}$ the morphism in $C^{[-n,0]}$ defined by $d_{B^{\cdot}}^{-(n+1)}$. Now let $B^{\cdot} \in C^{[-(n+1),0]}$; then we set

$$F_{n+1}(B^{\cdot}) = \begin{cases} F_n(B^{\cdot}) & \text{if } B^{-(n+1)} = 0, \\ C_{F_n(u_{B^{\cdot}})} & \text{otherwise} \end{cases}$$

(because $B^{\cdot} \simeq C_{u_{B^{\cdot}}}$). For any morphism $f^{\cdot}$: $B_1^{\cdot} \to B_2^{\cdot}$ we put

$$F_{n+1}(f^{\cdot}) = e_{B_2^{\cdot}}^{-1} \circ C_{(F_n T^n(f^{-(n+1)}), F_n(f^{\cdot}))} \circ e_{B_1^{\cdot}},$$

where the map $e_{B^{\cdot}}$: $F_{n+1}(B) \to C_{F_n(u_{B^{\cdot}})}$ denotes the canonical isomorphism for each $B^{\cdot} \in C^{[-(n+1),0]}$. The construction of η_{n+1} is obvious from the inductive definition of F_n.

Step 2°. In order to define $\check{F}$ we construct functors F^i: $C^{\leq i} \to C(\mathscr{A})$ and isomorphisms of functors η^i: $TF^i \xrightarrow{\sim} F^{i-1}T$, $i \in \mathbf{N}$, preserving the cone construction and such that $F^{i+1} \mid C^{\leq i} = F^i$ and $\eta^{i+1} \mid C^{\leq i} = \eta^i$. We set $F^0 = F$. Assume that F^n and η^n are defined for some n; then we define F^{n+1} and η^{n+1} as follows. Given $B^{\cdot} \in C^{\leq n+1}$ we put

$$F^{n+1}(B^{\cdot}) = \begin{cases} F^n(B^{\cdot}) & \text{if } B^{n+1} = 0, \\ T^{-1}F^n T(B^{\cdot}) & \text{otherwise,} \end{cases}$$

$$\eta^{n+1}(B^{\cdot}) = \begin{cases} \eta^{n+1} & \text{if } B^{n+1} = 0, \\ \text{id} & \text{otherwise.} \end{cases}$$

For any map $f^{\cdot}$: $B_1^{\cdot} \to B_2^{\cdot}$ in $C^{\leq n+1}$ we set $F^{n+1}(f^{\cdot}) = T^{-1}F^n T(f^{\cdot})$. The functor $\check{F} = \varinjlim_{n \in \mathbf{N}} F^n$ has the required properties. The uniqueness follows immediately from the construction.

Remark 1. (i) Note that we can quite easily recover from the inductive construction above an explicit formula for $\check{F}$. If $B^{\cdot} \in C^b(\mathscr{B})$ then $F(B^i) = A_i^{\cdot} \in C(\mathscr{A})$ and $F(d_{B^{\cdot}}^i) = f_i^{\cdot}$: $A_i^{\cdot} \to A_{i+1}^{\cdot} \in C^b(\mathscr{A})$ for each $i \in \mathbf{N}$. Our functor F is isomorphic to a functor $\check{F}$: $C^b(\mathscr{B}) \to C(\mathscr{A})$ defined by setting $F(B^{\cdot}) = (A^n, d_{A^{\cdot}}^n)_{n \in \mathbf{Z}}$, where $A^n = \bigoplus_{i \in \mathbf{Z}} A_i^{n-i}$ and $d_{A^{\cdot}}^n$: $\bigoplus_{i \in \mathbf{Z}} A_i^{n-i} \to \bigoplus_{l \in \mathbf{Z}} A_l^{n+1-l}$

is given by a matrix $(v_{l,i})_{l,i\in\mathbf{Z}}$ with entries

$$v_{l,i} = \begin{cases} (-1)^i d_{A_i}^{n-i} & \text{if } i = l, \\ f_i^{n-i} & \text{if } i = l-1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the functor $\check{F}$ attaches to any complex $B^{\cdot} \in C^{\mathrm{b}}(\mathscr{B})$ the total complex of the bicomplex $(F(B^i), F(d^i_{B^{\cdot}}))_{i\in\mathbf{Z}}$.

(ii) If $\mathscr{A}$ has infinite direct sums then F can be extended to $C(\mathscr{B})$.

(iii) The functor F induces an exact functor $\bar{F}$: $K^{\mathrm{b}}(\mathscr{B}) \to K(\mathscr{A})$.

(iv) Let F_1, F_2: $\mathscr{B} \to C(\mathscr{A})$ be two additive functors. If α: $F_1|\mathscr{B}_0 \to F_2|\mathscr{B}_0$ is a natural transformation of functors for some full subcategory $\mathscr{B}_0$ of $\mathscr{B}$ such that $\mathrm{add}(\mathscr{B}) \simeq \mathscr{B}$ then there exists a natural transformation β: $\check{F}_1 \to \check{F}_2$ such that $\beta|\mathscr{B}_0 = \alpha$. Moreover, if $\bar{F}_1$, $\bar{F}_2$: $K^{\mathrm{b}}(\mathscr{B}) \to K(\mathscr{A})$ are exact functors induced by $\check{F}_1$ and $\check{F}_2$ then β: $\bar{F}_1 \to \bar{F}_2$ is an isomorphism of functors provided so is α: $\bar{F}_1|\mathscr{B}_0 \to \bar{F}_2|\mathscr{B}_0$. The last statement follows from the existence of a long exact sequence induced by the mapping cone.

COROLLARY 1. *Let $\mathscr{U}$ be a full additive subcategory of $D(\mathscr{A})$, where $\mathscr{A}$ is an abelian category. If there exists a full subcategory $\mathscr{B}$ of $C(\mathscr{A})$ such that the canonical functor π: $C(\mathscr{A}) \to D(\mathscr{A})$ induces an equivalence of $\mathscr{B}$ with the full subcategory $\bar{\mathscr{U}}$ of $D(\mathscr{A})$ consisting of all objects isomorphic to those in $\mathscr{U}$, then the embedding J: $\mathscr{U} \to D(\mathscr{A})$ can be extended to an exact functor $\hat{J}$: $K^{\mathrm{b}}(\mathscr{U}) \simeq K^{\mathrm{b}}(\mathscr{B}) \to D(\mathscr{A})$.*

Remark 2. Let $\mathscr{U} \subseteq D^-(\mathrm{mod}\,R) \simeq K^-(\mathrm{proj}\,R)$ and $\mathscr{B} \subseteq C^-(\mathrm{proj}\,R)$ be additive categories such that $\pi(\mathscr{B}) \subseteq \bar{\mathscr{U}}$ and the restriction $\pi|\mathscr{B}$: $\mathscr{B} \to \bar{\mathscr{U}}$ is dense, where R is a k-category. Then $\pi|\mathscr{B}$ is an equivalence iff there is no nonzero morphism in $\mathscr{B}$ homotopic to 0. An analogous fact holds for $\mathscr{U} \subseteq D^+(\mathrm{mod}\,R)$ and $\mathscr{B} \subseteq C^+(\mathrm{inj}\,R)$, where R is a locally bounded k-category.

DEFINITION. Let $\mathscr{A}_0$ be a full abelian subcategory of an abelian category $\mathscr{A}$, closed under extensions. A full subcategory $\mathscr{U}$ consisting of indecomposable objects of $D_0(\mathscr{A})$ is called an *$\mathscr{A}_0$-almost tilting subcategory* if the following conditions are satisfied:

(o) There exists a full subcategory $\mathscr{B}$ of $C(\mathscr{A})$ such that the canonical functor π: $C(\mathscr{A}) \to D(\mathscr{A})$ induces an equivalence of $\mathscr{B}$ with the full subcategory $\bar{\mathscr{U}}$ of $D(\mathscr{A})$ consisting of all objects isomorphic to those in $\mathscr{U}$.

(1) $D(\mathscr{A})(X^{\cdot}, T^n Y^{\cdot}) = 0$ for any objects $X^{\cdot}$, $Y^{\cdot} \in \mathscr{U}$ and $0 \neq n \in \mathbf{Z}$.

(2) $\langle \mathscr{U} \rangle = D_0^{\mathrm{b}}(\mathscr{A})$.

$\mathscr{U}$ is called an *$\mathscr{A}_0$-tilting subcategory* if $\mathscr{U}$ is an $\mathscr{A}_0$-almost tilting subcategory and $\mathrm{gl.dim}\,\mathrm{mod}\,\mathscr{U}^{\mathrm{op}} < \infty$. If $\mathscr{A}_0 = \mathscr{A}$ we simply call $\mathscr{U}$ an *almost tilting* (resp. a *tilting*) *subcategory*.

Remark 3. The category formed by all indecomposable direct summands of a tilting object in the sense of [Ba] is obviously a tilting subcategory.

The following simple fact observed by Beilinson in [Be] will be used.

LEMMA 2. *Let* $F: \mathscr{C} \to \mathscr{D}$ *be an exact functor of triangulated categories and let* $\mathscr{X}$ *be a set of objects in* $\mathscr{C}$. *If* F *induces an isomorphism* $\bigoplus_{n \in \mathbf{Z}} \mathscr{C}(X, T^n Y) \xrightarrow{\sim} \mathscr{D}(FX, T^n FY)$ *for any* $X, Y \in \mathscr{X}$ *then* F *induces a triangle equivalence* $\langle \mathscr{X} \rangle \xrightarrow{\sim} \langle F\mathscr{X} \rangle$.

THEOREM 1. *Let* $\mathscr{U}$ *be an* $\mathscr{A}_0$*-almost tilting subcategory of* $D_0(\mathscr{A})$, *where* $\mathscr{A}_0$ *is a full abelian subcategory of an abelian category* $\mathscr{A}$, *closed under extensions. Then the embedding* $J: \mathscr{U} \to D(\mathscr{A})$ *induces a triangle equivalence* $K^{\mathrm{b}}(\mathscr{U}) \xrightarrow{\sim} D_0^{\mathrm{b}}(\mathscr{A})$. *In case* $\mathscr{U}$ *is a tilting subcategory,* J *induces a triangle equivalence* $D^{\mathrm{b}}(\mathrm{mod}\, \mathscr{U}^{\mathrm{op}}) \xrightarrow{\sim} D^{\mathrm{b}}(\mathscr{A})$.

Proof. Let $\mathscr{U}$ be an $\mathscr{A}_0$-almost tilting subcategory. It follows from (o) that $\mathrm{add}(\mathscr{U})$ and $\mathrm{add}(\mathscr{B})$ satisfy the assumption of Corollary 1. Then one can extend the embedding $J_1: \mathrm{add}(\mathscr{U}) \to D(\mathscr{A})$ to an exact functor $\hat{J}_1: K^{\mathrm{b}}(\mathrm{add}(\mathscr{U})) \to D(\mathscr{A})$ and we conclude from Lemma 2 that $K^{\mathrm{b}}(\mathrm{proj}\, \mathscr{U}^{\mathrm{op}}) \simeq K^{\mathrm{b}}(\mathrm{add}(\mathscr{U})) \to D_0^{\mathrm{b}}(\mathscr{A})$ is a triangle equivalence. Moreover, if $\mathrm{gl.dim}\, \mathrm{mod}\, \mathscr{U}^{\mathrm{op}} < \infty$ then $D^{\mathrm{b}}(\mathrm{mod}\, \mathscr{U}^{\mathrm{op}}) \simeq K^{\mathrm{b}}(\mathrm{proj}\, \mathscr{U}^{\mathrm{op}})$ and the theorem is proved.

3. The main results

Let $\Lambda = \Lambda(k^{n+1}) = \bigoplus_{p \in \mathbf{N}} \Lambda^p$ (resp. $S = S((k^{n+1})^*) = \bigoplus_{p \in \mathbf{N}} S^p$) be the exterior (resp. symmetric) algebra of the vector space k^{n+1} with the standard basis $e_1, \ldots, e_{n+1}$ (resp. $(k^{n+1})^*$ with the standard dual basis $x_1, \ldots, x_{n+1}$) and let $\tilde{\Lambda}$ (resp. $\tilde{S}$) be its cover category. Note that $\dim_k \Lambda$ is finite. For each $m \in \mathbf{Z}$ we denote by $\Lambda_{(m)}$ the indecomposable projective $\tilde{\Lambda}$-module $\tilde{\Lambda}(m, -)$, by E_m the simple $\tilde{\Lambda}$-module corresponding to $\Lambda_{(m)}$ and by $\Lambda^*_{(m)}$ the injective hull $\Lambda(-, m)^*$ of E_m. Analogously we denote by $S_{(m)}$ the indecomposable projective $\tilde{S}$-module $\tilde{S}(m, -)$ and by E'_m the corresponding simple module.

Let $\mathscr{U}_{\tilde{\Lambda}}$ (resp. $\mathscr{U}_{\tilde{S}}$) be the full subcategory of $D(\mathrm{mod}\, \tilde{\Lambda})$ (resp. $D(\mathrm{mod}\, \tilde{S})$) consisting of all complexes $T^m E_m$ (resp. $T^m E'_m$), $m \in \mathbf{Z}$.

PROPOSITION 1. *If* $\tilde{\Lambda}$, $\tilde{S}$, $\mathscr{U}_{\tilde{\Lambda}}$, $\mathscr{U}_{\tilde{S}}$ *are as above, there exist equivalences of categories*:

(i) $\tilde{S} \xrightarrow{\sim} \mathscr{U}_{\tilde{\Lambda}}$,

(ii) $\tilde{\Lambda} \xrightarrow{\sim} \mathscr{U}_{\tilde{S}}$.

For the proof we construct some nice injective (resp. projective) resolutions for all E_p (resp. E'_p), $p \in \mathbf{Z}$, by using the Koszul complex for the sequence $x_1, \ldots, x_{n+1} \in S$

$$K\colon \quad 0 \to \Lambda^{n+1} \otimes S \to \ldots \to \Lambda^1 \otimes S \to \Lambda^0 \otimes S \to 0$$

whose differentials are determined by the maps

$$d_{r,t}\colon \Lambda^r \otimes S^t \to \Lambda^{r-1} \otimes S^{t+1}, \qquad t \in \mathbf{N},\ 1 \leqslant r \leqslant n+1,$$

given by the formula

$$d_{r,t}(e_{i_1}\wedge\ldots\wedge e_{i_r}\otimes s)=\sum_{l=1}^{r}(-1)^{l-1}e_{i_1}\wedge\ldots\wedge\hat{e}_{i_l}\wedge\ldots\wedge e_{i_r}\otimes s\cdot x_{i_l}$$

for any $s\in S^t$ and any sequence $1\leqslant i_1,\ldots,i_r\leqslant n+1$. Recall that K induces exact sequences

$$K_l\colon\quad 0\to\Lambda^{l+1}\otimes S^0\to\Lambda^{l-1}\otimes S^1\to\ldots\to\Lambda^0\otimes S^l\to 0,\qquad l\geqslant 1.$$

Now consider the operator $\bar{d}=\sum_{i=1}^{n+1}(\wedge e_i)\otimes(\cdot x_i)\in\operatorname{End}_k(\Lambda^*\otimes_k S)$, where Λ^* is the dual bimodule $\operatorname{Hom}_k({}_\Lambda\Lambda_\Lambda, k)$. It is an endomorphism of $\Lambda^*\otimes_k S$ as a left Λ-module as well as a left S-module. Moreover, $\bar{d}$ preserves gradings. Consequently, $\bar{d}$ induces morphisms

$$(\Lambda^r)^*\otimes S_{(m)}\to(\Lambda^{r-1})^*\otimes S_{(m-1)}$$

in $\operatorname{mod}\tilde{S}$ and morphisms

$$\Lambda^*_{(m)}\otimes S^t\to\Lambda^*_{(m-1)}\otimes S^{t+1}$$

in $\operatorname{mod}\tilde{\Lambda}$ for all $r,t\in\mathbf{N}$, $m\in\mathbf{Z}$. Since $\bar{d}^2=0$, for each $m\in\mathbf{Z}$ we construct from the morphisms above a complex $I^{\cdot}_m\in C^+(\operatorname{inj}\tilde{\Lambda})$, where

$$I^t_m=\begin{cases}\Lambda^*_{(m-t)}\otimes S^t & \text{if } t\geqslant 0,\\ 0 & \text{if } t<0,\end{cases}$$

and a complex $P^{\cdot}_m\in C^{\mathrm{b}}(\operatorname{proj}\tilde{S})$, where

$$P^r_m=\begin{cases}(\Lambda^{-r})^*\otimes S_{(m-r)} & \text{if } 0\geqslant r\geqslant -(n+1),\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 3. *For each* $m\in\mathbf{Z}$:

(i) $I^{\cdot}_m$ *is an injective resolution of* E_m *in* $\operatorname{mod}\tilde{\Lambda}$.

(ii) $P^{\cdot}_m$ *is a projective resolution of* E'_m *in* $\operatorname{mod}\tilde{S}$.

Proof. It is enough to show that the nontrivial component complexes $P^{\cdot}_m(d)$ over objects $d>m$ (resp. $I^{\cdot}_m(d)$ over objects $d<m$) are exact. Observe that $P^{\cdot}_m(d)=\bar{K}_{d-m}$ for $d>m$ (resp. $I^{\cdot}_m(d)=\bar{K}_{m-d}$ for $d<m$), where $\bar{K}_l$ for each $l\geqslant 1$ denotes the complex

$$0\to(\Lambda^l)^*\otimes S^0\xrightarrow{\bar{d}_{l,0}}(\Lambda^{l-1})^*\otimes S^1\xrightarrow{\bar{d}_{l-1,1}}\ldots\xrightarrow{\bar{d}_{1,l-1}}(\Lambda^0)^*\otimes S^l\to 0.$$

For each $r\in\mathbf{N}$ denote by $w_r\colon\Lambda^r\to(\Lambda^r)^*$ the vector space isomorphism determined by bijection of the bases. It is not difficult to see that for all $r,t\in\mathbf{N}$,

$$\bar{d}_{r,t}\circ(w_r\otimes\mathrm{id}_{S^t})=(w_{r-1}\otimes\mathrm{id}_{S^{t+1}})\circ d_{r,t}.$$

Thus for each $l\geqslant 1$ the maps $w_r\otimes\mathrm{id}_{S^t}$, $r+t=l$, define an isomorphism of complexes $K_l\xrightarrow{\sim}\bar{K}_l$. Consequently, all complexes $\bar{K}_l$, $l\geqslant 1$, are exact and we are done.

LEMMA 4. *For all $m, m' \in \mathbf{Z}$ and $r, t \in \mathbf{N}$ we have*

(i) $$\operatorname{Ext}^t_{\bar\Lambda}(E_m, E_{m'}) \simeq \begin{cases} S^{m'-m} & \text{if } t = m'-m, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) $$\operatorname{Ext}^r_{\bar S}(E'_m, E'_{m'}) \simeq \begin{cases} \Lambda^{m'-m} & \text{if } m'-m = r, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. (i) In order to calculate the groups $\operatorname{Ext}^t_{\bar\Lambda}(E_m, E_{m'})$ for given $m, m' \in \mathbf{Z}$ we apply the functor $\operatorname{Hom}_{\bar\Lambda}(E_m, -)$ to the injective resolution $I^{\bullet}_{m'}$ of $E_{m'}$. Since for each $t \in \mathbf{N}$ we have an isomorphism

$$\operatorname{Hom}_{\bar\Lambda}(E_m, \Lambda^*_{(m'-t)} \otimes S^t) \simeq \operatorname{Hom}_{\bar\Lambda}(E_m, \Lambda^*_{(m'-t)}) \otimes S^t \simeq \begin{cases} S^t & \text{if } m = m'-t, \\ 0 & \text{otherwise,} \end{cases}$$

(i) follows.

(ii) Analogously, we calculate the groups $\operatorname{Ext}^r_{\bar S}(E'_m, E'_{m'})$ for given $m, m' \in \mathbf{Z}$ by applying the functor $\operatorname{Hom}_{\bar S}(-, E'_{m'})$ to the projective resolution $P^{\bullet}_m$ of E'_m. Since for any $r \in \mathbf{N}$ we have an isomorphism

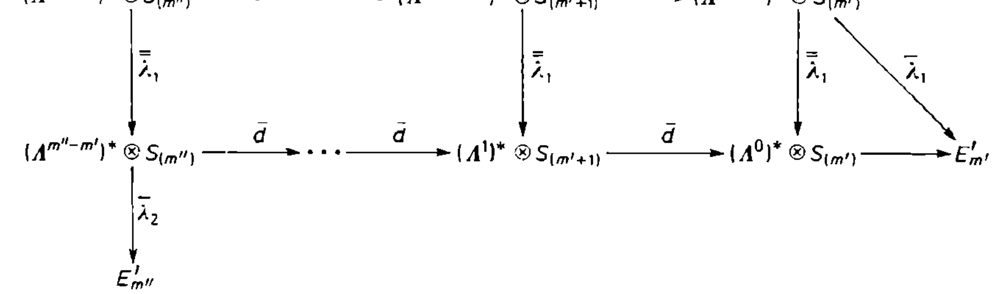
$$\operatorname{Hom}_{\bar S}((\Lambda^r)^* \otimes S_{(m+r)}, E'_{m'}) \simeq (\Lambda^r)^{**} \otimes \operatorname{Hom}_{\bar S}(S_{(m+r)}, E'_{m'}) \simeq \begin{cases} \Lambda^r & \text{if } m+r = m', \\ 0 & \text{otherwise,} \end{cases}$$

(ii) follows.

LEMMA 5. (i) *The isomorphisms $u_{m,m'}$: $\operatorname{Ext}^{m'-m}_{\bar\Lambda}(E_m, E_{m'}) \simeq S^{m'-m}$, $m \leqslant m'$, identify the Yoneda product with the multiplication in S.*

(ii) *The isomorphisms $u'_{m,m'}$: $\operatorname{Ext}^{m'-m}_{\bar S}(E'_m, E'_{m'}) \simeq \Lambda^{m'-m}$, $m \leqslant m'$, identify the Yoneda product with the multiplication in Λ.*

Proof. We only prove (ii). The proof of (i) is analogous and technically simpler. Let $m, m', m'' \in \mathbf{Z}$, $m \leqslant m' \leqslant m''$, $\lambda_1 \in \Lambda^{m'-m}$, $\lambda_2 \in \Lambda^{m''-m'}$, and let $\bar\lambda_1 = u'_{m,m'}(\lambda_1) \in \operatorname{Ext}^{m'-m}_{\bar S}(E'_m, E'_{m'})$, $\bar\lambda_2 = u'_{m',m''}(\lambda_2) \in \operatorname{Ext}^{m''-m'}_{\bar S}(E'_{m'}, E'_{m''})$ be the cocycles corresponding to λ_1 and λ_2 (in our case they are uniquely determined). Consider the following commutative diagram:



where the $\bar{\bar{\lambda}}_1$ are the canonical liftings of λ_1 along the standard projective resolutions, constructed from λ_1 by use of the homomorphism

$$\Lambda^i \to \mathrm{Hom}_k(\Lambda^j, \Lambda^{j+i}) \simeq \mathrm{Hom}_k((\Lambda^{j+i})^*, (\Lambda^j)^*) \simeq \mathrm{Hom}_{\tilde{S}}((\Lambda^{j+i})^* \otimes S_{(l)}, (\Lambda^j)^* \otimes S_{(l)}),$$

$j = m' - m$, $i \in \mathbf{N}$, which at the first step attaches to λ_1 the map $- \wedge \lambda_1$. The Yoneda product $\bar{\lambda}_1 \circ \bar{\lambda}_2 \in \mathrm{Ext}_{\tilde{S}}^{m''-m}(E'_m, E'_{m''})$ is given by the cocycle $\bar{\lambda}_2 \cdot \bar{\bar{\lambda}}_1$ which coincides with $u'_{m,m''}(\lambda_2 \wedge \lambda_1)$.

Proof of Proposition 1. (i) We define a functor φ: $\tilde{S} \to \mathscr{U}_{\tilde{\Lambda}}$. Let $m \in \mathbf{Z}$ be an object of $\tilde{S}$; then we set $\varphi(m) = T^m E_m$. Let $m, m' \in \mathbf{Z}$; then the map φ: $\tilde{S}(m, m') \to \mathscr{U}_{\tilde{\Lambda}}(\varphi(m), \varphi(m'))$ is defined as the composition

$$\tilde{S}(m, m') \simeq S^{m'-m} \simeq \mathrm{Ext}_{\tilde{\Lambda}}^{m'-m}(E_m, E_{m'}) = D(\mathrm{mod}\,\tilde{\Lambda})(T^m E_m, T^{m'} E_{m'})$$

if $m \leqslant m'$ and $\tilde{S}(m, m') = 0 = D(\mathrm{mod}\,\tilde{\Lambda})(T^m E_m, T^{m'} E_{m'})$ otherwise. Since by Lemma 5, φ preserves the composition, we see that (i) holds.

(ii) Analogously, we can extend the mapping $m \mapsto T_m E'_m$, $m \in \mathbf{Z}$, to an equivalence ψ: $\tilde{\Lambda} \xrightarrow{\sim} \mathscr{U}_{\tilde{S}}$.

PROPOSITION 2. (i) *$\mathscr{U}_{\tilde{\Lambda}}$ is a tilting subcategory in $D(\mathrm{mod}\,\tilde{\Lambda})$.*
(ii) *$\mathscr{U}_{\tilde{S}}$ is a $\mathrm{mod}_0\tilde{S}$-almost tilting subcategory in $D(\mathrm{mod}_0\tilde{S})$.*

Proof. Since by Proposition 1 and Lemma 3, $\mathrm{gl.dim}\,\mathscr{U}^{\mathrm{op}} = n+1$, we have to check properties (o), (1) and (2) of the definition.

(o) Let $\mathscr{B}_{\tilde{\Lambda}}$ denote the full subcategory of $C(\mathrm{mod}\,\tilde{\Lambda})$ formed by all complexes $T^m I^{\cdot}_m$, $m \in \mathbf{Z}$. Observe that

$$(T^m I^{\cdot}_m)^i = \begin{cases} \Lambda^*_{(-i)} \otimes S^{i+m} & \text{if } i \geqslant -m, \\ 0 & \text{otherwise,} \end{cases}$$

for any $i, m \in \mathbf{Z}$. Since $\mathrm{Hom}_{\tilde{\Lambda}}(\Lambda^*_{(m)}, \Lambda^*_{(m')}) = 0$ for $m < m'$, any homotopy connecting maps of complexes from $\mathscr{B}_{\tilde{\Lambda}}$ is zero and by Remark 2 the canonical epimorphism

$$C(\mathrm{mod}\,\tilde{\Lambda})(T^m I^{\cdot}_m, T^{m'} I^{\cdot}_{m'})$$
$$\to K(\mathrm{mod}\,\tilde{\Lambda})(T^m I^{\cdot}_m, T^{m'} I^{\cdot}_{m'}) \simeq D(\mathrm{mod}\,\tilde{\Lambda})(T^m E_m, T^{m'} E_{m'})$$

is an isomorphism for any $m, m' \in \mathbf{Z}$. Moreover, the canonical embedding of complexes $E_m \to I^{\cdot}_m$ induces an isomorphism in $D(\mathrm{mod}\,\tilde{\Lambda})$ and (o) is satisfied.

(1) Since

$$D(\mathrm{mod}\,\tilde{\Lambda})(T^m E_m, T^{m'+t} E_{m'}) = \begin{cases} \mathrm{Ext}_{\tilde{\Lambda}}^{m'+t-m}(E_m, E_{m'}) & \text{if } m'+t \geqslant m, \\ 0 & \text{otherwise,} \end{cases}$$

for all $m, m', t \in \mathbf{Z}$, by Lemma 4 we have $D(\mathrm{mod}\,\tilde{\Lambda})(T^m E_m, T^{t+m'} E_{m'}) = 0$ for $t \neq 0$ and (1) is proved.

(2) Observe that $\langle \mathscr{U}_{\tilde{\Lambda}} \rangle \subseteq D^{\mathrm{b}}(\mathrm{mod}\,\tilde{\Lambda})$ because $\mathscr{U}_{\tilde{\Lambda}} \subseteq D^{\mathrm{b}}(\mathrm{mod}\,\tilde{\Lambda})$. Therefore

for the proof of (2) it suffices to show that $\operatorname{mod}\tilde{\Lambda} \subseteq \langle \mathscr{U}_{\tilde{\Lambda}} \rangle$. Since $E_m \in \langle \mathscr{U}_{\tilde{\Lambda}} \rangle$ for each $m \in \mathbf{Z}$ the crucial inclusion follows from the remark below. Consequently, $\mathscr{U}_{\tilde{\Lambda}}$ is a tilting subcategory.

The proof of (ii) is analogous if for $\mathscr{B}_{\tilde{S}}$ we take the full subcategory of all complexes $T^m P_m^{\cdot}$, $m \in \mathbf{Z}$.

Remark 4. Let $\mathscr{C}$ be a triangulated subcategory of $D(\mathscr{A})$. If for some $A \in \mathscr{A}$ there exists a subobject A' such that A' and $A'' = A/A'$ belong to $\mathscr{C}$ then A belongs to $\mathscr{C}$. This follows from the existence of the triangle $T^{-1}A'' \xrightarrow{e} A' \xrightarrow{w} A \xrightarrow{p} A''$, where e: $0 \to A' \xrightarrow{w} A \xrightarrow{p} A'' \to 0$ is an element of $\operatorname{Ext}^1_{\mathscr{A}}(A'', A') = D(\mathscr{A})(T^{-1}A'', A')$.

Now we can construct two compositions of exact functors

$$\Phi\colon D^{\mathrm{b}}(\operatorname{mod}\tilde{S}) \simeq K^{\mathrm{b}}(\operatorname{proj}\tilde{S}) \xrightarrow{\check{F}} K^{+}(\operatorname{inj}\tilde{\Lambda}) \subseteq K^{+}(\operatorname{mod}\tilde{\Lambda}) \to D^{+}(\operatorname{mod}\tilde{\Lambda}),$$

where $\check{F}$ is an extension of the composition of functors (see Lemma 1)

$$F\colon \operatorname{proj}\tilde{S} \simeq \operatorname{add}(\tilde{S}^{\mathrm{op}}) \simeq \operatorname{add}(\tilde{S}) \simeq \operatorname{add}(\mathscr{U}_{\tilde{\Lambda}}) \simeq \operatorname{add}(\mathscr{B}_{\tilde{\Lambda}}) \subseteq C^{+}(\operatorname{inj}\tilde{\Lambda}),$$

and

$$\Psi\colon K^{\mathrm{b}}(\operatorname{inj}\tilde{\Lambda}) \xrightarrow{\check{G}} K^{\mathrm{b}}(\operatorname{proj}\tilde{S}) \subseteq K^{\mathrm{b}}(\operatorname{mod}\tilde{S}) \to D^{\mathrm{b}}(\operatorname{mod}\tilde{S}),$$

where $\check{G}$ is an extension of the composition of functors

$$G\colon \operatorname{inj}\tilde{\Lambda} \simeq \operatorname{add}(\tilde{\Lambda}^{\mathrm{op}}) \simeq \operatorname{add}(\tilde{\Lambda}) \simeq \operatorname{add}(\mathscr{U}_{\tilde{S}}) \simeq \operatorname{add}(\mathscr{B}_{\tilde{S}}) \subseteq C^{\mathrm{b}}(\operatorname{proj}\tilde{S}).$$

Now we are able to prove our main result.

THEOREM 2. *The functor Φ induces a triangle equivalence*

$$D^{\mathrm{b}}(\operatorname{mod}\tilde{S}) \xrightarrow{\sim} D^{\mathrm{b}}(\operatorname{mod}\tilde{\Lambda})$$

and the functor Ψ induces a triangle equivalence

$$K^{\mathrm{b}}(\operatorname{inj}\tilde{\Lambda}) \xrightarrow{\sim} D^{\mathrm{b}}_{0}(\operatorname{mod}\tilde{S}).$$

Moreover, regarding $K^{\mathrm{b}}(\operatorname{inj}\tilde{\Lambda})$ as a full subcategory of $D^{\mathrm{b}}(\operatorname{mod}\tilde{\Lambda})$, the restriction $\Phi_|$ of Φ to $D^{\mathrm{b}}_0(\operatorname{mod}\tilde{S})$ is a quasi-inverse of Ψ and Φ induces a triangle equivalence

$$D^{\mathrm{b}}(\operatorname{mod}\tilde{S})/D^{\mathrm{b}}_0(\operatorname{mod}\tilde{S}) \xrightarrow{\sim} D^{\mathrm{b}}(\operatorname{mod}\tilde{\Lambda})/K^{\mathrm{b}}(\operatorname{inj}\tilde{\Lambda}).$$

Proof. The fact that Φ and Ψ induce the required triangle equivalences follows from Theorem 1 and Proposition 2. In order to show that $\Phi_|$ is a quasi-inverse of Ψ one need only construct an isomorphism of functors $\Phi\Psi \simeq \operatorname{Id}_{K^{\mathrm{b}}(\operatorname{inj}\tilde{\Lambda})}$. Since our functors satisfy the assumption of Remark 1(iv) it is enough to show that the canonical projections $\check{F}\check{G}(\Lambda^*_{(m)}) = \check{F}(T^{-m}P_m^{\cdot}) \to \Lambda^*_{(m)}$, $m \in \mathbf{Z}$, in $C^{+}(\operatorname{inj}\tilde{\Lambda})$ are quasi-isomorphisms. For this, consider the bicomplex

$$
\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & (\Lambda^{n+1})^*\otimes\Lambda^*_{(m-n-2)}\otimes S^1 & \to \ldots \to & (\Lambda^1)^*\otimes\Lambda^*_{(m-n-2)}\otimes S^{n+1} & \to & (\Lambda^0)^*\otimes\Lambda^*_{(m-n-2)}\otimes S^{n+2} & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & (\Lambda^{n+1})^*\otimes\Lambda^*_{(m-n-1)}\otimes S^0 & \to \ldots \to & (\Lambda^1)^*\otimes\Lambda^*_{(m-n-1)}\otimes S^n & \to & (\Lambda^0)^*\otimes\Lambda^*_{(m-n-1}\otimes S^{n+1} & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & \vdots & & \vdots & \\
 & & & \uparrow & & \uparrow & \\
 & & 0 \to & (\Lambda^1)^*\otimes\Lambda^*_{(m-1)}\otimes S^0 & \to & (\Lambda^0)^*\otimes\Lambda^*_{(m-1)}\otimes S^1 & \to 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & \to & (\Lambda^0)^*\otimes\Lambda^*_{(m)}\otimes S^0 & \to 0 \quad (-m)\text{th row} \\
 & & & & & \uparrow & \\
 & & & & & 0 & \\
 & & & & & m\text{th column} &
\end{array}
$$

Diagram 1

defining $\check{F}(T^{-m}P^{\cdot}_{m})$ (see Diagram 1), whose horizontal differentials are given by the operator $\sum_{i=0}^{n+1}(\wedge e_i)\otimes\mathrm{id}\otimes(\cdot x_i)$ whereas the vertical ones by $\sum_{i=0}^{n+1}\mathrm{id}\otimes(\wedge e_i)\otimes(\cdot x_i)$. All nonzero rows but the lowest one are exact and all nonzero columns are exact except in the lowest nonzero place. Consequently, using the explicit formula for $\check{F}$ (see Remark 1(i)) we can show by an elementary induction "from left to right" that

$$H^i(\check{F}(T^{-m}P^{\cdot}_{-m})) = \begin{cases} (\Lambda^0)^*\otimes\Lambda^*_{(m)}\otimes S^0 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

This finishes the proof.

References

[Ba] D. Baer, *Tilting sheaves in representation theory of algebras*, Manuscripta Math. 60 (1987), 323–348.

[Be] A. A. Beilinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems of linear algebra*, Funktsional. Anal. i Prilozhen. 12 (3) (1978), 68–89 (in Russian).

[BGG] I. N. Bernshtein, I. M. Gel'fand and S. I. Gel'fand, *Algebraic bundles over* $\mathbf{P}^n$ *and problems of linear algebra*, ibid., 66–67 (in Russian).

[BG] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.

[Bu] R.-O. Buchweitz, *The comparison theorem*, in: Lecture Notes in Math. 1273, Springer, 1987, 96–116.

[G] S. I. Gel'fand, *Sheaves on* $\mathbf{P}^n$ *and problems of linear algebra*, appendix to Russian transl. of: Ch. Okonek, M. Schneider and H. Spindler, *Vector Bundles on Complex Projective Spaces*, 278–305 (in Russian).

[H] D. Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. 62 (1987), 339–389.

[Ha] R. Hartshorne, *Residues and Dualities*, Lecture Notes in Math. 20, Springer, 1966.

[K1] M. M. Kapranov, *The derived category of coherent sheaves on Grassmann manifolds*, Funktsional. Anal. i Prilozhen. 17 (2) (1983), 78-79 (in Russian).

[K2] —, *The derived category of coherent sheaves on a quadric*, ibid. 20 (2) (1986), 67 (in Russian).

[R] J. Rickard, *Morita theory for derived categories*, preprint.

[V] J. L. Verdier, *Catégories dérivées*, in: Lecture Notes in Math. 569, Springer, 1977, 262–311.