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INFORMATION BASED COMPLEXITY AND OPERATOR EQUATIONS

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We are interested in the intrinsic difficulty (or *complexity*) of computing an approximate solution of the linear operator equation $Lu = f$. Practical examples of such problems include the cases where L is a known partial differential or integral operator. Problems of the form $Lu = f$ are typically solved under the constraint that only *partial* information about f is available, such as the values of a finite number of inner products, or the values of f at a finite number of points. It is of interest to determine when algorithms which are in wide use are *optimal* algorithms, i.e., algorithms which produce an approximation with minimal cost. We are especially interested in determining conditions which are necessary and sufficient for the finite element method (FEM) to be optimal. For the cases of elliptic partial differential equations and of Fredholm integral equations of the second kind, we describe such a condition, in the form of an inequality involving the order of the problem and the degree of the finite element subspace. Suppose this inequality is violated; is the non-optimality of the FEM inherent in the information used by the FEM, or is it because the FEM uses this information in a non-optimal manner? The latter is the case; there always exists an algorithm using this information which is optimal. We also discuss the situation in which the information used by the finite element method (which consists of inner products) is not available. Suppose that the only admissible information about f consists of evaluations of f. In the case of the Fredholm problem of the second kind, this information is optimal; moreover, a finite element method in which the inner products are approximated by quadrature rules is an optimal algorithm. However, there exist elliptic problems of positive order for which this new information is non-optimal. In addition, we consider the solution of ill-posed problems, such as the Fredholm integral equation of the first kind. We show that in the worst-case setting, there is no algorithm

whose error is finite. In the average-case setting, we show that finite-error algorithms exist if and only if the solution operator is "bounded on the average." When this happens, we find optimal algorithms and optimal information, and show that the minimal error goes to zero as the amount of information goes to infinity.

1. An overview

Information-based complexity (as described in Traub et al. [27]–[29]) is a new approach to the synthesis and analysis of algorithms. This approach has been designed especially for problems which cannot be solved exactly with finite cost. This includes many of the important problems of applied mathematics, such as ordinary differential equations, partial differential equations, and integral equations.

A rich source of such problems is the approximate solution of linear operator equations. We are given a known linear transformation L of function spaces; we wish to find, for some f, an ε-approximation to the solution u of the problem $Lu = f$. Here, by an *ε-approximation*, we mean an element u_ε such that $\|u - u_\varepsilon\| < \varepsilon$. Furthermore, this ε-approximation is to be found with minimal cost. In order to precisely define cost, we will need a model of computation; this will be specified later. Hence our goal is to find

(1) the *ε-complexity* $\mathrm{COMP}(\varepsilon)$, which is defined to be the minimal cost of finding an ε-approximation, and

(2) an *optimal algorithm* φ_ε, that is, an algorithm which procedures an ε-approximation and whose cost is $\mathrm{COMP}(\varepsilon)$.

In order to do this, we must know something about f. The knowledge that we have about f is called the *information Nf*. Most often, this information Nf consists of a finite number of linear functionals of f. For example, if f belongs to a Hilbert space, then Nf might consist of a finite number of inner products of f, while if f belongs to a space of continuous functions, then Nf might consist of a finite number of evaluations of f.

As a rule, f will belong to an infinite-dimensional space, since f will typically be defined on a region containing an uncountable number of points. But the information which is known about each f is finite. Hence, *the information Nf does not uniquely determine the right-hand side f.* (For instance, if the information consists of the values of f at a finite set $\{x_1, \ldots, x_n\}$ of points, then for each f there will be an infinite class of functions $\tilde{f}$ such that $\tilde{f}(x_i) = f(x_i)$ for $1 \leqslant i \leqslant n$.) Since there are many f which yield the same information, we say that the information N is *partial*. The lack of injectivity in N allows one to find a (sharp) lower bound $r(N)$ on the error of algorithms using N. (For reasons of geometry, $r(N)$ is called the *radius of information*; see Traub and Woźniakowski [28]), Chapter 1. This implies that

one can find an ε-approximation iff

$$r(N) < \varepsilon.$$

It is desirable to find an algorithm φ_N using N whose error equals $r(N)$. This means that φ_N has minimal error among all algorithms using N. Moreover, one can often determine *optimal information* for each n, i.e., information N_n^* involving n linear functionals such that $r(N_n^*)$ equals the nth *minimal radius of information* $r(n)$ (i.e., the minimal value of $r(N)$ over all N consisting of n linear functionals). Since N_n^* is nth optimal information, we see that the algorithm $\varphi_{N_n^*}$ (which has minimal error among all algorithms using the information N_n^*) has minimal error among all algorithms using information involving N linear functionals.

As promised above, we now define an abstract model of computation. We proceed in two steps. Our first step is to assume that we are given a class of linear functionals such that any functional in that class is computable. (This is sometimes called an *oracle model* of computation.) For example, if the function space to which f belongs is a Hilbert space, we might choose the class of linear functionals to be inner products. If the function space is a space of continuous functions over some fixed domain, we might choose function evaluations as our class of linear functionals.

Furthermore, we assume that the cost of evaluating a linear functional is fixed. This assumption is made only for the sake of simplicity. It can also be justified by the following example. Suppose we have a procedure for computing the value of a function at a point; then the assumption of fixed cost essentially means that we charge for the number of times the procedure is invoked. The cost of evaluating a linear functional will be denoted by c throughout this paper.

Note that we can now determine the minimal number of linear functionals needed to find an ε-approximation. Since $r(n)$ measures the nth minimal radius of information, the answer is given by

$$m(\varepsilon) = \inf\{n\colon\ r(n) < \varepsilon\}.$$

Note that this gives a lower bound on the ε-complexity of the problem, namely that

$$\text{COMP}(\varepsilon) \geqslant cm(\varepsilon). \tag{1.1}$$

This lower bound is an intrinsic property of the problem. For example, if we find that $m(\varepsilon)$ is unacceptably large, then we *cannot* find an ε-approximation with a cost that we are willing to pay.

Our second step in defining the model of computation deals with the cost of combining information. Suppose we have found information N_ε, consisting of $m(\varepsilon)$ linear functionals, which is strong enough to compute an ε-

approximation. Let φ_ε be an algorithm whose error equals $r(N_\varepsilon)$, i.e., the error of φ_ε is less than ε. How hard is it to implement this algorithm? The answer clearly depends on the form of φ_ε. In many cases, we can (fortunately) prove that φ_ε is *linear*, i.e., a linear combination of the functionals making up the information. More precisely, suppose that the optimal information is of the form

$$N_\varepsilon f = \begin{bmatrix} \lambda_1(f) \\ \vdots \\ \lambda_n(f) \end{bmatrix} \quad \text{for} \quad n = m(\varepsilon).$$

Then there exists functions $g_1, \ldots, g_n$ such that

$$\varphi_\varepsilon(N_\varepsilon f) = \sum_{j=1}^{n} \lambda_j(f) g_j.$$

Since the functions $g_1, \ldots, g_n$ are independent of f, they may be precomputed in advance. If we agree to do this precomputation, then the evaluation of $\varphi_\varepsilon(N_\varepsilon f)$ at a point requires at most n scalar multiplications and $n-1$ scalar additions. Hence, the total cost of using φ_ε to find an ε-approximation is at most

$$(c+2)\,m(\varepsilon)-1. \tag{1.2}$$

Comparing this result with (1.1), we see that φ_ε is optimal to within a constant additive factor. Furthermore, since $c \gg 1$ in all practical situations (i.e., evaluation of a linear functional is much harder than an arithmetic operation), the ε-complexity $\mathrm{COMP}(\varepsilon)$ is essentially equal to $cm(\varepsilon)$ and is achieved by the algorithm φ_ε.

Although this information-based approach is appealing, there are at least two reasons why it may be less popular than one might expect. First of all, it can be difficult to determine optimal algorithms and information for specific problems. Furthermore, even in situations where optimal algorithms and information can be determined, they must compete with other techniques which have been used for a long time. This is despite the fact that the criteria by which these methods were selected may have been ad hoc, having nothing to do with the (perhaps) more basic goal of finding an algorithm which solves the problem to within the desired error and which has minimal cost.

For this reason, it is especially gratifying to find that a "standard" algorithm is optimal. When this happens, we are not faced with the problem of trying to apply a general technique for constructing an optimal algorithm to a situation in which the calculations involved in constructing this algorithm may be expensive. In addition, since a standard technique is now

shown to be nearly optimal, we do not have to overcome a user's natural resistance to abandoning a tried and true method for a novel one.

However, we should point out that classical algorithms are not always optimal. For example, it has been shown that Gauss quadrature is not an optimal algorithm for integrating certain families of analytic functions. Moreover, the penalty for using Gauss quadrature instead of the optimal algorithm for ε-approximation is unbounded as $\varepsilon \to 0$. (See Kowalski et al. [18] for details.)

In this paper, we will restrict our attention to regularly elliptic partial differential equations and to Fredholm integral equations. We shall distinguish between *well-posed* and *ill-posed* problems. A well-posed problem is one in which a small change in the input data yield at most a small change in the output, whereas an ill-posed problem is one in which this is not the case. (This terminology dates back to Hadamard [16].) We will initially be concerned with well-posed problems, so that we shall first consider the cases of elliptic partial differential equations and of the Fredholm problem of the second kind. Later on, we shall turn to the Fredholm problem of the first kind, which is an ill-posed problem.

The *finite element method* (FEM) is a very popular algorithm for such well-posed problems in differential and integral equations (see Babuška and Aziz [4], Ciarlet [7], and Oden and Reddy [20]). The *finite element information* (FEI) used by the FEM consists of inner products $(f, s_1), \ldots, (f, s_n)$, where $\{s_1, \ldots, s_n\}$ is a basis for a piecewise polynomial space of degree k. For each of the problems $Lu = f$ discussed in this paper, we give a simple condition which is necessary and sufficient for the FEM to be an almost-optimal algorithm. This condition depends on the degree k of the finite element subspaces being used and the smoothness r of the right-hand side f. (More precisely, we assume that an a priori bound is known for the Sobolev r-norm of f.)

We find that the ε-complexity of an elliptic problem of $2m$th order is

$$\Theta(\varepsilon^{-N/(r+m)}) \quad \text{as} \quad \varepsilon \to 0,$$

and the FEM is optimal for this problem iff

$$k \geq 2m-1+r.$$

The ε-complexity of a Fredholm integral equation of the second kind is

$$\Theta(\varepsilon^{-1/r}) \quad \text{as} \quad \varepsilon \to 0,$$

and the FEM is optimal iff

$$k \geq r-1.$$

For the elliptic PDE, we see that the ε-complexity goes to infinity very quickly (as ε goes to zero) if r is close to $-m$; for the Fredholm integral

equation, this happens if r is close to zero. This behavior is intrinsic to the problem, and there is nothing that can be done about it.

Next, suppose the condition for optimality is violated. Is the non-optimality of the FEM inherent in the *finite element information* (FEI) it uses, or is it because the FEM uses FEI in a non-optimal manner? We show that the latter is the case; there always exists an algorithm using FEI (called the *spline algorithm*) which is optimal.

Of course, in order for the finite element method described above to be defined, one must be able to exactly calculate the inner products comprising the finite element information. This means that for any f, the exact values of inner products of f with certain piecewise polynomials must be available. This is often an unrealistic assumption. It is usually more reasonable to assume that any f can be evaluated at any point in its domain. If this is the case, one can define a *finite element method with quadrature* (FEMQ) using roughly as many function evaluations as the FEM uses inner products, where the integrals appearing in the FEM are approximated by a quadrature rule. It is then reasonable to ask when the FEMQ is optimal. It turns out that the FEMQ is optimal for the Fredholm problem of the second kind iff

$$k \geqslant r-1;$$

that is, the FEMQ is optimal for the Fredholm problem precisely when the FEM is optimal for this problem. However, the FEMQ is not optimal for all elliptic problems of positive order. For instance, in the case of a second-order elliptic problem in one dimension, one can show that the ε-complexity increases from $\Theta(\varepsilon^{-1/(r+1)})$ to $\Theta(\varepsilon^{-1/r})$ if the only information allowed consists of evaluation of functions at a point. Hence, the FEMQ is non-optimal for such problems precisely because it uses non-optimal information.

The preceding paragraphs describe the situation for well-posed problems in differential and integral equations. However, many important problems arising in the physical sciences and in engineering are ill-posed. Examples of such ill-posed problems include

(i) inversion of the Laplace transform, whether the "usual" transform defined over $[0, \infty)$ or the finite transform (whose inversion is discussed in Dunn [11]),

(ii) Fujita's equation relating molecular weight distribution to the steady-state concentration or optical density in a centrifuged sample (see Gehatia and Wiff [14]),

(iii) solving the heat equation backwards in time (see Seidman [25]), and

(iv) remote sensing (see Twomey [30]).

Since a small change in the problem element may cause a large change in the solution, we see that the reason for the ill-posedness of the problem is that we are trying to approximate an unbounded solution operator. Perhaps

the most important source of such problems is the *Fredholm problem of the first kind*, which is the case of a solution operator S which is the inverse of a compact operator L. Note that all of the problems mentioned above are Fredholm problems of the first kind.

Our main result regarding ill-posed problems is that they cannot be solved in the worst-case setting described above. However, they *can* be solved in an average-case setting. More precisely, there is no algorithm for solving an ill-posed problem whose (worst-case) error is finite. (This result should be compared to the result of Pour-El and Richards [21].)

We now outline the structure of this paper. In Section 2, we consider the case of a two-point boundary-value problem. Although this is a very simple example it does allow us to illustrate the key points mentioned above. In Section 3, we describe the results for the case of an elliptic partial differential equation. In Section 4, we discuss the situation of a Fredholm problem of the second kind. In Section 5, we discuss ill-posed problems. Finally, in Section 6, we briefly describe some open problems and discuss directions for future research.

2. A two-point boundary value problem

To give the reader the flavor of our results, it will be helpful to look at a model problem. On the one hand, the description of this problem does not require a great deal of technical background; on the other hand, the results that have been obtained for this problem are typical of those for more general elliptic problems. To this end, we consider the weak solution u of the two-point boundary value problem

$$\begin{aligned} -u''(x)+u(x) &= f(x) \quad \text{for} \quad 0 < x < 1, \\ u'(0) &= u'(1) = 0. \end{aligned} \tag{2.1}$$

In order to explain what is meant by a weak solution, we have to use some standard terminology about Sobolev spaces. Let $I = [0, 1]$. For any non-negative integer l, we define the *Sobolev l-inner product* $(\cdot, \cdot)_l$ and the *Sobolev l-norm* $\|\cdot\|_l$ by

$$(v, w)_l = \sum_{i=0}^{l} \int_I v^{(i)}(x)\, w^{(i)}(x)\, dx \tag{2.2}$$

and

$$\|v\|_l = \sqrt{(v, v)_l} = \sqrt{\sum_{i=0}^{l} \int_I (v^{(i)}(x))^2\, dx}, \tag{2.3}$$

respectively. (When $l = 0$, it is customary to omit the zero subscript.) Then the Sobolev space $H^l(I)$ is defined to be the closure in $L_2(I)$ of the set of all

$C^\infty(I)$ functions whose Sobolev l-norm is finite. (Note that $H^0(I) = L_2(I)$.) When l is a negative integer, the Sobolev l-norm is defined by duality, i.e.,

$$\|v\|_l = \sup_{w \in C_0^\infty(I)} |(v, w)|/\|w\|_{-l} \tag{2.4}$$

(with $0/0 = 0$). When l is not an integer, there is a technique known as "Hilbert space interpolation" which may be used to define the Sobolev l-norm; see Butzer and Berens [6].

We are now ready to describe the weak form of problem (2.1). Define a bilinear form B on $H^1(I)$ by

$$B(v, w) = \int_I (v'(x)w'(x) + v(x)w(x))\,dx \qquad \forall v, w \in H^1(I). \tag{2.5}$$

Then we seek an element $u \in H^1(I)$ such that

$$B(u, v) = (f, v) \qquad \forall v \in H^1(I). \tag{2.6}$$

Since $B(v, v) = \|v\|_1^2$, one can use the Lax–Milgram Lemma of functional analysis (see e.g., Schechter [23]) to see that for any $f \in H^{-1}(I)$, there exists a unique $u \in H^1(I)$ such that (2.6) holds; we write $u = Sf$ to indicate this fact. Using an integration by parts, it is easy to see that any solution u of (2.1) is also the solution of (2.6); moreover, if the solution of (2.6) is sufficiently smooth, it is also a solution of (2.1).

As indicated in Section 1, we only know (for each f) information consisting of the values of a finite number of linear functionals at f. It is often assumed that this information is of the form

$$N_n f = \begin{bmatrix} (f, s_1) \\ \vdots \\ (f, s_n) \end{bmatrix}. \tag{2.7}$$

Here, $\{s_1, \ldots, s_n\}$ is a basis for a *finite element subspace* $\mathcal{S}_n$ of $H^1(I)$ having dimension n and degree k. That is, we subdivide the interval I into equal subintervals. Then $\mathcal{S}_n$ consists of the space of continuous functions which, when restricted to each of these subintervals, are polynomials of degree k. Such functions are called *splines*, and $\mathcal{S}_n$ is sometimes called a *spline space*. (See e.g., Schultz [24].)

Assuming this information N_n is available, we can now define an algorithm for approximating the weak solution Sf of the boundary-value problem with right-hand side f. For each n, this approximation u_n has the form

$$u_n(x) = \sum_{j=1}^{n} \alpha_j s_j(x), \tag{2.8}$$

the coefficients $\alpha_1, \ldots, \alpha_n$ being chosen so that

$$B(u_n, s_i) = (f, s_i) \quad (1 \leqslant i \leqslant n). \tag{2.9}$$

Note that u_n depends on f only through the information N_n. We indicate this fact by writing $u_n = \varphi_n(N_n f)$. The algorithm φ_n is referred to as the *finite element method* (FEM) defined by $\mathcal{S}_n$. Clearly, the FEM is an algorithm which uses the *finite element information* (FEI) given by (2.7).

Why should one use the finite element method? The FEM has a long and noble history, with roots stretching back to Galerkin [13] and Courant [9]. Moreover, the FEM is easily implemented. Formula (2.9) may be expressed as the solution of an $n \times n$ linear system. The coefficient matrix of this system is banded, with a bandwidth that is independent of n (although it does depend on k). Hence, the coefficients of (2.8) may be found in time which is proportional to n.

However, the FEM was derived by an ad hoc criterion (i.e., the projection of the weak solution into a spline space). Our basic goal is to find ε-approximations as cheaply as possible. What does the one have to do with the other?

To answer this question, we must decide how to measure the error of the FEM. In order to do this, we recall two results for the FEM. As above, we write $u = Sf$ for the actual solution and $u_n = \varphi_n(N_n f)$ for the approximate solution produced by the FEM using $\mathcal{S}_n$. The first result is that the FEM gives a best approximation in $\mathcal{S}_n$. That is,

$$\|Sf - \varphi_n(N_n f)\|_1 = \|u - u_n\|_1 = \inf_{s \in \mathcal{S}_n} \|u - s\|_1. \tag{2.10}$$

The second result is a bound on the $H^1(I)$-error in approximating the solution. That is, if $u \in H^{r+2}(I)$ for some $r \geqslant -1$, then there exists a positive constants C, depending only on k and r, such that

$$\|u - u_n\|_1 \leqslant C n^{-\mu} \|u\|_{r+2}, \tag{2.11}$$

where

$$\mu = \min\{k, r+1\}. \tag{2.12}$$

This bound may be expressed in terms of f, rather than u, by using the "shift theorem," which states that if $f \in H^r(I)$, then $u = Sf \in H^{r+2}(I)$; moreover, there exists a positive constant C, depending only on r, such that

$$C^{-1} \|Sf\|_{r+2} \leqslant \|f\|_r \leqslant C \|Sf\|_{r+2}. \tag{2.13}$$

Then (2.11)–(2.13) yield that for any $r \geqslant -1$, there is a positive constant C such that for every positive integer n,

$$\|Sf - \varphi_n(N_n f)\|_1 \leqslant C n^{-\mu} \|f\|_r \quad \forall f \in H^r(I). \tag{2.14}$$

The estimate (2.14) may be rephrased as follows. Let

$$F = \{f \in H^r(I): \|f\|_r \leq 1\} \tag{2.15}$$

denote the unit ball of $H^r(I)$. The *error* $e(\varphi_n, N_n)$ of the finite element method is defined to be

$$e(\varphi_n, N_n) = \sup_{f \in F} \|Sf - \varphi_n(N_n f)\|_1. \tag{2.16}$$

Then

$$e(\varphi_n, N_n) \leq Cn^{-\mu}. \tag{2.17}$$

We now ask whether this estimate is sharp, and whether we can find an algorithm using finite element information which is better than the finite element method, in the sense of having smaller error. Here, we define the *error* of an arbitrary algorithm φ using N_n by (2.16), except that now φ_n is replaced by φ; that is,

$$e(\varphi, N_n) = \sup_{f \in F} \|Sf - \varphi(N_n f)\|_1. \tag{2.18}$$

Recall (from Section 1) that the radius of information

$$r(N_n) = \inf_{\varphi} e(\varphi, N_n) \tag{2.19}$$

denotes the minimal error over all algorithms using the finite element information N_n. We then have the following result from Werschulz [33]:

THEOREM 2.1. (1) *The error of the FEM is given by*

$$e(\varphi_n, N_n) = \Theta(n^{-\mu}) \quad \text{as} \quad n \to \infty,$$

where $\mu = \min\{k, r+1\}$.

(2) *The radius of FEI is given by*

$$r(N_n) = \Theta(n^{-(r+1)}) \quad \text{as} \quad n \to \infty.$$

Remark 2.1. We briefly describe a linear algorithm φ_n^* using the finite element information N_n such that $e(\varphi_n^*, N_n) = r(N_n)$, i.e., having minimal error among all algorithms using FEI. Let S^*: $H^1(I) \to H^r(I)$ denote the Hilbert space adjoint of S, i.e.,

$$(Sg, v)_1 = (g, S^* v)_r \quad \forall g \in H^r(I), \ \forall v \in H^1(I).$$

Recalling that s_i is the ith basis function for $\mathcal{S}_n$, we let

$$t_j = SS^* s_j \quad (1 \leq j \leq n).$$

$$u_n^*(x) = \sum_{j=1}^{n} \alpha_j t_j(x),$$

where the coefficients $\alpha_1, \ldots, \alpha_n$ are chosen so that

$$B(u_n^*, s_i) = (f, s_i) \quad (1 \leqslant i \leqslant n).$$

Since u_n^* depends on f only through the finite element information N_n, we write $u_n^* = \varphi_n^*(N_n f)$. The algorithm φ_n^* is called the *spline algorithm* using N_n. From the results of Traub and Woźniakowski [28], Chapter 4, this spline algorithm φ_n^* has minimal error among all algorithms using N_n. ∎

Hence, we conclude from Theorem 2.1 that the FEM makes (almost) optimal use of its information iff $k \geqslant r+1$, i.e., we have a condition which is necessary and sufficient to insure that the FEM is the best method using FEI. We now ask whether there is any information which is better than finite element information. Recalling that $r(n)$ measures the nth minimal radius of information, this question is answered by the following result (see Werschulz [32] and [33] for details):

THEOREM 2.2. *The n-th minimal radius of information is given by*

$$r(n) = \Theta(n^{-(r+1)}) \quad \text{as} \quad n \to \infty. \ ∎$$

Hence we see that finite element information always yields the smallest possible error.

Of course, all of the previous discussion is based on the assumption that finite element information is available. That is, we have assumed that the inner products $(f, s_i) = \int_I f(x) s_i(x)\,dx$ (for $1 \leqslant i \leqslant n$) can be calculated for *any* $f \in F$. This is often an unrealistic assumption. It is often more reasonable to assume that $f(x)$ can be evaluated for any $f \in F$ and at any $x \in [0, 1]$. Of course, this assumption itself is built on the more basic assumption that $f(x)$ is defined for any $f \in F$ and any point $x \in [0, 1]$. By the Sobolev imbedding theorem (Adams [1]) and the fact that the class F of all right-hand sides f is the unit ball of $H^r(I)$, we must now assume that

$$r > \tfrac{1}{2}, \tag{2.20}$$

since otherwise $f(x)$ will not be defined for every f and x.

If we allow the evaluation of any right-hand side at any point in its domain, one can then approximate the integrals required by the FEM via a quadrature rule. We describe such a finite element method with quadrature based on the spline space $\mathcal{S}_n$ described above. Recall that the interval I has been subdivided into equal subintervals $I_1, \ldots, I_l$. On each subinterval I_j, let $x_{1,j}, \ldots, x_{k,j}$ and $\omega_{1,j}, \ldots, \omega_{k,j}$ respectively denote the nodes and weights of a Gauss quadrature rule, so that

$$\int_{I_j} w(x)\,dx \approx \sum_{i=1}^{k} \omega_{i,j}\, w(x_{i,j}), \tag{2.21}$$

with equality if w is polynomial of degree $2k-1$. We now rewrite the set of nodes as $\{x_1, \ldots, x_n\}$ (with $0 < x_1 < \ldots < x_n < 1$) and the set of weights as $\{\omega_1, \ldots, \omega_n\}$, with ω_j being the weight corresponding to the node x_j. For each i, define the linear functional λ_i approximating the inner product $(\cdot, s_i)$ by

$$\lambda_i(f) = \sum_{j=1}^{n} \omega_j f(x_j) s_i(x_j). \tag{2.22}$$

Note that we can compute $\lambda_1(f), \ldots, \lambda_n(f)$ from the *standard information*

$$\tilde{N}_n f \doteq \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}. \tag{2.23}$$

(The information $\tilde{N}_n$ is called "standard" information, because it is more usual to assume that we can evaluate a function at a point than it is to assume that we can evaluate more general linear functionals, such as inner products with finite element basis functions.)

We are now able to define an algorithm, using the standard information $\tilde{N}_n$, for approximating the weak solution Sf of the boundary-value problem with right-hand side f. For each n, this approximation $\tilde{u}_n$ has the form

$$\tilde{u}_n(x) = \sum_{j=1}^{n} \tilde{\alpha}_j s_j(x), \tag{2.24}$$

the coefficients $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ being chosen so that

$$B(\tilde{u}_n, s_i) = \lambda_i(f) \quad (1 \leqslant i \leqslant n). \tag{2.25}$$

That is, we replace the integrals appearing in the definition of the FEM by quadratures. Note that $\tilde{u}_n$ depends of f only through the standard information $\tilde{N}_n$. We indicate this fact by writing $\tilde{u}_n = \tilde{\varphi}_n(\tilde{N}_n f)$. The algorithm $\tilde{\varphi}_n$ is referred to as the *finite element method with quadrature* (FEMQ) defined by $\mathcal{S}_n$.

How good is the FEMQ? For the sake of exposition, we restrict our attention to the case where

$$k \geqslant r+1. \tag{2.26}$$

The results in Chapter 4.1 of Ciarlet [7] may be used to see that there is a positive constant C such that for every positive integer n,

$$\|Sf - \tilde{\varphi}_n(\tilde{N}_n f)\|_1 \leqslant Cn^{-r}\|f\|_r \quad \forall f \in H^r(I). \tag{2.27}$$

That is, the error of the FEMQ satisfies

$$e(\tilde{\varphi}_n, \tilde{N}_n) \leqslant Cn^{-r}. \tag{2.28}$$

This estimate indicates that the FEMQ may be worse than the FEM based on the same spline space. That is, since $k \geq r+1$, the error of the FEM is $\Theta(n^{-(r+1)})$; this should be compared to the $O(n^{-r})$ estimate of the FEMQ's error given by (2.28). Can this degradation in error (when going from the FEM to the FEMQ) be avoided? The answer, as found in Werschulz [32] is "no." More precisely, let

$$\tilde{r}(n) = \inf\{r(\tilde{N}_n)\colon\ x_1, \ldots, x_n \in I\} \tag{2.29}$$

denote the *n-th minimal radius of standard information.* That is, $\tilde{r}(n)$ is the minimal error among *all* algorithms using standard information consisting of function evaluations at *any* n points in the interval I. We then have

THEOREM 2.3. *The n-th minimal radius of standard information is*

$$\tilde{r}(n) = \Theta(n^{-r}) \quad as \quad n \to \infty. \quad \blacksquare$$

This tells us that (2.28) is a sharp estimate of the error of the FEMQ. Moreover, although the FEMQ is a linear algorithm, this result shows that it has almost minimal error among all algorithms using standard information. Moreover, the loss when going from the FEM to the FEMQ is due to the fact that standard information is weaker than finite element information; that is, the minimal error among all algorithms using standard information is greater than the minimal error among all algorithms using FEI.

We now seek to translate these results concerning minimal error into results which say when the FEM is (almost) an optimal algorithm; that is, we seek to determine when the cost of using the FEM for ε-approximation equals the problem complexity COMP(ε) (at least to within a constant factor). In order to do this, we let

$$\begin{aligned}\text{FEM}(\varepsilon) = \inf\{&\text{cost}(\varphi_n)\colon\ \varphi_n \text{ is a FEM using FEI} \\ &N_n \text{ such that } e(\varphi_n, N_n) < \varepsilon\}\end{aligned} \tag{2.30}$$

denote the minimal cost of using the FEM to compute an ε-approximation. We then have the following result from Werschulz [33]:

THEOREM 2.4.

(1) $$\text{COMP}(\varepsilon) = \Theta(\varepsilon^{-1/(r+1)}) \quad as \quad \varepsilon \to 0.$$

(2) $$\text{FEM}(\varepsilon) = \Theta(\varepsilon^{-1/\mu}) \quad as \quad \varepsilon \to 0, \quad where\ \mu = \min\{k, r+1\}. \quad \blacksquare$$

These results may be viewed in two different lights. If we take an optimistic viewpoint, we see that the FEM is optimal (to within a constant) for all r satisfying $r \leq k-1$. That is, if we choose k large, then the FEM is nearly optimal for a wide range of r. Moreover, the FEM itself does not depend on r. Such results are important, since it may be difficult to determine the exact smoothness of a given right-hand side f.

If we choose a pessimistic viewpoint, we can say that the FEM is not optimal whenever we know that $r > k-1$. In this case, the spline algorithm φ_n^* using finite element information N_n (with n sufficiently large) is an optimal algorithm for this problem. More precisely, let

(2.31) $\mathrm{SPLINE}(\varepsilon) = \inf\{\mathrm{cost}(\varphi_n^*)\colon \varphi_n^* \text{ is a spline algorithm using FEI } N_n \text{ such that } e(\varphi_n^*, N_n) < \varepsilon\}.$

That is, $\mathrm{SPLINE}(\varepsilon)$ is the minimal cost of finding an ε-approximation with the spline algorithm using FEI. Then Theorem 2.4 and the results of Section 1 yield that

(2.32) $$\mathrm{SPLINE}(\varepsilon) = \Theta(\mathrm{COMP}(\varepsilon)) = \Theta(\varepsilon^{-1/(r+1)}) \quad \text{as} \quad \varepsilon \to 0.$$

Moreover, if we let denote the penalty for using the FEM (rather than an optimal algorithm), we see that

(2.33) $$\mathrm{pen}(\varepsilon) = \Theta(\varepsilon^{-\lambda}) \quad \text{as} \quad \varepsilon \to 0,$$

where

(2.34) $$\lambda = \frac{1}{k} - \frac{1}{r+1},$$

so that

(2.35) $$\lim_{\varepsilon \to 0} \mathrm{pen}(\varepsilon) = +\infty.$$

Hence, the asymptotic penalty for using the FEM instead of an optimal algorithm when k is too small is unbounded.

3. Elliptic boundary value problems

In the previous section, we discussed the situation of a simple two-point boundary value problem. In this section, we describe how these results may be extended to general elliptic problems in several dimensions. The results are all taken from Werschulz [34]. The notation used is the standard notation for multi-indices and for Sobolev spaces, inner products, and norms, found in Ciarlet [7]. As before, fractional- and negative-order Sobolev spaces are defined by Hilbert-space interpolation and duality, respectively.

Let $\Omega \subset \boldsymbol{R}^N$ be a bounded, simply connected, C^∞ region. Define the uniformly strongly elliptic operator

$$Lv = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta} D^\beta v),$$

with real coefficients $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$ such that $a_{\alpha\beta} = a_{\beta\alpha}$. In order to have

appropriate boundary conditions, let

$$B_j v = \sum_{|\alpha| \leq q_j} b_{j\alpha} D^\alpha v \qquad (0 \leq j \leq m-1),$$

where $b_{j\alpha} \in C^\infty(\partial\Omega)$ are real-valued and

$$0 \leq q_0 \leq \ldots \leq q_{m-1} \leq 2m-1.$$

We assume that $\{B_j\}_{j=0}^{m-1}$ is a normal family of operators which covers L on $\partial\Omega$. To make the boundary-value problem be self-adjoint, we let

$$m^* = \min\{j:\ q_j \geq m\},$$

and require that

$$\{q_j\}_{j=0}^{m^*-1} \cup \{2m-1-q_j\}_{j=m^*}^{m-1} = \{0, \ldots, m-1\}.$$

(See Chapter 3 of Babuška and Aziz [4] and Chapter 5 of Oden and Reddy [20] for further definitions and illustrative examples.)

We are interested in solving the *elliptic boundary-value problem*:

For $f \in H^r(\Omega)$, where $r \geq -m$, find u: $\bar{\Omega} \to \boldsymbol{R}$ such that

$$\text{(3.1)} \qquad \begin{aligned} &(1) && Lu = f \quad \text{in } \Omega \\ &(2) && B_j u = 0 \quad \text{on } \partial\Omega \quad (0 \leq j \leq m-1). \end{aligned}$$

Let

$$H_E^m(\Omega) = \{v \in H^m(\Omega):\ B_j v = 0\ (0 \leq j \leq m^*-1)\}$$

denote the space of $H^m(\Omega)$-functions satisfying the essential boundary conditions. We define a symmetric, continuous bilinear form B on $H_E^m(\Omega)$ by

$$B(v, w) = \sum_{|\alpha|,|\beta| \leq m} \int_\Omega a_{\alpha\beta} D^\alpha v D^\beta w.$$

We assume that B is *weakly* $H_E^m(\Omega)$-coercive (see p. 310 of Oden and Reddy [20]). Since B is symmetric, this means that there exists a positive constant γ such that

For any nonzero $v \in H_E^m(\Omega)$, there exists nonzero $w \in H_E^m(\Omega)$ such that

$$\text{(3.2)} \qquad |B(v, w)| \geq \gamma \|v\|_m \|w\|_m.$$

We now define the *variational boundary problem* as follows. Let $r \geq -m$. We wish to solve the following problem:

For $f \in H^r(\Omega)$, find $u = Sf \in H_E^m(\Omega)$ such that

$$\text{(3.3)} \qquad B(u, v) = (f, v) = \int_\Omega fv \qquad \forall v \in H_E^m(\Omega).$$

From the Generalized Lax–Milgram Theorem (Theorem 5.2.1 of Babuška and Aziz [4]), S: $H^r(\Omega) \to H_E^m(\Omega)$ is a well-defined bounded linear transformation.

As in the previous section, we assume that we know information consisting of the values of a finite number of linear functionals of any right-hand side f. Once again, it is often assumed that this information is of the form

$$N_n f = \begin{bmatrix} (f, s_1) \\ \vdots \\ (f, s_n) \end{bmatrix}, \tag{3.4}$$

where $\{s_1, \ldots, s_n\}$ is a basis for a *finite element subspace* $\mathcal{S}_n$ of $H_E^m(\Omega)$ having dimension n and degree k. We assume that the family $\{\mathcal{S}_n\}_{n=1}^{\infty}$ of finite element subspaces is *quasi-uniform* (Oden and Reddy [20], p. 272). Of course, since the region Ω is C^∞, we must make an additional assumption about the boundary elements to guarantee that $\mathcal{S}_n \subseteq H_E^m(\Omega)$ in the situation where (3.1) is not a Neumann problem. (For instance, we may use curved elements as in Ciarlet and Raviart [8].)

We now seek to approximate, for a right-hand side f, the solution Sf of the problem (3.1). This approximation u_n will be found by requiring that $u_n \in \mathcal{S}_n$ (so that u_n has the form (2.8)) satisfy

$$B(u_n, s_i) = (f, s_i) \quad (1 \leqslant i \leqslant n). \tag{3.5}$$

Since u_n depends on f only through the information $N_n f$, we write $u_n = \varphi_n(N_n f)$. The algorithm φ_n is once again called the *finite element method* (FEM) using the *finite element information* (FEI) N_n which is defined by $\mathcal{S}_n$.

We may now ask whether or not the FEM is optimal for this problem. In order to do this, we must first decide how to measure the error. Following the ideas of the previous section, let us measure the error of an algorithm φ using information N by

$$e(\varphi, N) = \sup_{f \in F} \|Sf - \varphi(Nf)\|_l. \tag{3.6}$$

Here, F is the unit ball of $H^r(\Omega)$, i.e.,

$$F = \{f \in H^r(\Omega): \|f\|_r \leqslant 1\}, \tag{3.7}$$

where r must be chosen so that

$$r \geqslant -m. \tag{3.8}$$

Note that we are now measuring error in the Sobolev l-norm. In what follows, we will require that

$$0 \leqslant l \leqslant m \quad \text{and} \quad k \geqslant 2m - 1 - l. \tag{3.9}$$

(See Remark 3.1 of Werschulz [33] for further discussion.)

Standard results (see Babuška and Aziz [4] or Oden and Reddy [20] for details) yield a bound on the error of the FEM of the form

$$e(\varphi_n, N_n) \leqslant Cn^{-(\mu+m-l)/N}, \tag{3.10}$$

where

$$\mu = \min\{k+1-m, m+r\}. \tag{3.11}$$

Once again, we ask whether the estimate (3.10) is sharp, and whether we can find an algorithm using FEI whose error is smaller than that of the FEM. The answer is given by the following result from Werschulz [34]:

THEOREM 3.1. *The error of the FEM is given by*

$$e(\varphi_n, N_n) = \Theta(n^{-(\mu+m-l)/N}) \quad \text{as} \quad n \to \infty,$$

where μ *is given by* (3.11). *The radius of FEI is given by*

$$r(N_n) = \Theta(n^{-(r+2m-l)/N}) \quad \text{as} \quad n \to \infty. \quad \blacksquare$$

Remark 3.1. As in Section 2, the spline algorithm φ_n^* is a linear algorithm using the finite element information N_n which has minimal error among *all* algorithms using FEI. (See Traub and Woźniakowski [28] Chapter 4, for details.) ■

Hence, we conclude from Theorem 3.1 that the FEM makes (almost) optimal use of its information iff $k \geqslant 2m-1+r$. As before, we now ask whether there is any information which is better than finite element information. The answer to this question is given by the following result from Werschulz [34]:

THEOREM 3.2. *The n-th minimal radius of information is given by*

$$r(N) = \Theta(n^{-(r+2m-l)/N}) \quad \text{as} \quad n \to \infty.$$

Once again, finite element information always yields the smallest possible error.

As before, we translate the results on minimal-error algorithms into results on complexity. As in the previous section, we let $\mathrm{COMP}(\varepsilon)$ denote the (intrinsic) ε-complexity of the problem; $\mathrm{FEM}(\varepsilon)$ denotes the complexity of using the FEM to find an ε-approximation. We then have the following result from Werschulz [34]:

THEOREM 3.3. (1) $\mathrm{COMP}(\varepsilon) = \Theta(\varepsilon^{-N/(r+2m-l)})$ *as* $\varepsilon \to 0$.

(2) $\mathrm{FEM}(\varepsilon) = \Theta(\varepsilon^{-N/(\mu+m-l)})$ *as* $\varepsilon \to 0$, *where* $\mu = \min\{k+1-m, m+r\}$. ■

Viewed in an optimistic light, this result says that the FEM is optimal (to within a constant) for all r satisfying $r \leqslant k+1-2m$. The pessimistic interpretation of this result is that the FEM is non-optimal whenever $r > k$

$+1-2m$. In this latter case, one can show (as in the previous section) that the asymptotic penalty for using the FEM, rather than the spline algorithm using the same finite element information as the FEM uses, is unbounded.

4. The Fredholm problem of the second kind

In the two previous sections, we dealt with boundary-value problems for elliptic differential equations. In this section, we consider an integral equation, namely the Fredholm problem of the second kind. There is a vast literature dealing with the numerical solution of these problems. See, e.g., the books Anderssen et al. [2], Atkinson [3], Baker [5], Delves and Walsh [10], Golberg [15], and te Riele [22], as well as the survey article Ikebe [17].

The complexity results in this section are all taken from Werschulz [35]. Although we will describe these results in a Hilbert space setting (i.e., error is measured in the L_2-norm), the results of Werschulz [35] are established in an L_p-setting, where $p \in (1, \infty]$. These results in Werschulz [35] include, as a special case, the results of Emel'yanov and Il'in [12], which appear to be the first results on optimal algorithms for the Fredholm problem of the second kind.

Let I denote the unit interval $[0, 1]$ and let r be a non-negative integer. Let $k: I \times I \to \mathbf{R}$ be a function such that $\partial_1^j k$ is continuous for $0 \leq j \leq r$, where ∂_i^j denotes the jth partial derivative with the ith variable. Define a linear operator $K: L_2(I) \to L_2(I)$ by

$$(Kv)(x) = \int_I k(x, y)\, v(y)\, dy. \tag{4.1}$$

Then K is compact. We also assume that 1 is not an eigenvalue of K. Set

$$L = I - K. \tag{4.2}$$

Then L is an invertible bounded linear operator on $L_2(I)$, i.e., L has a bounded inverse on $L_2(I)$.

We are interested in solving the *Fredholm problem of the second kind*:

For $f \in H^r(I)$, find $u = Sf \in L_2(I)$ such that

$$Lu = f. \tag{4.3}$$

By the remarks above, $S: H^r(I) \to L_2(I)$ is a well-defined bounded linear transformation.

As always, we only know (for each f), information consisting of the values of a finite number of linear functionals at f. It is often assumed that

this information is of the form

$$N_n f = \begin{bmatrix} (f, s_1) \\ \vdots \\ (f, s_n) \end{bmatrix}, \tag{4.4}$$

where $\{s_1, \ldots, s_n\}$ is a basis for a *finite element subspace* $\mathcal{S}_n$ of $L_2(I)$ having dimension n and degree k. That is, we subdivide the interval I into equal subintervals; then S_n is the space of functions whose restriction to each of the subintervals is a polynomial of degree k. Note that the only difference between $\mathcal{S}_n$ as defined here and as defined in Section 2 is that interelement continuity was imposed in Section 2, but that no such requirement is made in this section.

For a right-hand side f, we seek to approximate the solution Sf of problem (4.3). An approximation u_n will be chosen by requiring that $u_n \in \mathcal{S}_n$ (so that once again, u_n has the form (2.8)) and that

$$(Lu_n, s_i) = (f, s_i) \quad (1 \leqslant i \leqslant n). \tag{4.5}$$

Since u_n depends on f through the information $N_n f$, we write $u_n = \varphi_n(N_n f)$. Once again, the algorithm φ_n is called the *finite element method* (FEM) using the *finite element information* (FEI) N_n which is defined by $\mathcal{S}_n$.

Does the FEM make optimal use of finite element information? Is FEI optimal information? In order to answer these questions, we must once again specify how to measure the error of an algorithm. Let us agree to measure the error of an algorithm φ using information N by

$$e(\varphi, N) = \sup_{f \in F} \|Sf - \varphi(Nf)\|_0, \tag{4.6}$$

where F is once again the unit ball of $H^r(I)$, i.e.,

$$F = \{f \in H^r(I): \|f\|_r \leqslant 1\}. \tag{4.7}$$

That is, we measure error in the norm of $L_2(I) = H^0(I)$.

We then have the following results of Werschulz [35]:

THEOREM 4.1. (1) *The error of the FEM is given by*

$$e(\varphi_n, N_n) = \Theta(n^{-\mu}) \quad \text{as} \quad n \to \infty,$$

where $\mu = \min\{k+1, r\}$.

(2) *The radius of FEI is given by*

$$r(N_n) = \Theta(n^{-r}) \quad \text{as} \quad n \to \infty.$$

(3) *The n-th minimal radius of information is given by*

$$r(n) = \Theta(n^{-r}) \quad \text{as} \quad n \to \infty. \quad \blacksquare$$

Thus we see that the FEM makes (almost) optimal use of its information iff $k \geq r-1$. However, FEI is *always* optimal information (at least to within a constant factor). Furthermore, there exists a linear algorithm using FEI, called the *spline algorithm*, which has minimal error among all algorithms using FEI. From this, we see that the spline algorithm using finite element information N_n has (almost) minimal error among all algorithms using information involving n linear functionals.

Of course, the previous discussion has based on the assumption that the inner products required by the FEM are available. That is, we assume that for any $f \in F$ and for any finite element basis function s_i, we are able to compute $\int_I f(x)s_i(x)\,dx$, Often, this is not the case. As in Section 2, it is more common to assume that we can evaluate $f(x)$ for any $f \in F$ and at any $x \in I$. Of course, we must assume that $r > \frac{1}{2}$ in order for $f(x)$ to be defined; since r is a non-negative integer, this means that we now must assume that $r \geq 1$.

If we now allow the evaluation of $f(x)$ for any f and any x, we can then describe a finite element method with quadrature for this problem. The details are almost the same as those in Section 2, except that we now require the points $x_1, \ldots, x_n$ of evaluation to be the nodes of a piecewise $(k+1)$-point Gauss quadrature rule. Thus we once again have an approximation λ_i of the linear functional $(\cdot, s_i)$ (which is now exact for piecewise polynomials of degree $2k+1$). As before, we can compute $\lambda_1(f), \ldots, \lambda_n(f)$ from the *standard information*

$$\tilde{N}_n f = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}. \tag{4.8}$$

Since we are going to replace integrals by quadratures in the right-hand side, there is one further place where this replacement can be done. Recall that the approximation produced by the FEM has the form (2.8), i.e., a linear combination of the finite element basis functions. The vector α of weights is found by solving a linear system of the form $G\alpha = \beta$, where $\beta_i = (f, s_i)$ and $G_{ij} = (Ls_j, s_i)$. So far, we have decided to replace β by $\tilde{\beta}$, where $\tilde{\beta}_i = \lambda_i(f)$. In practice, the inner products appearing in the coefficient matrix G would also be replaced by quadratures (using the same nodes and weights as those used in approximating the inner products on the right-hand side). This leads to a new bilinear form B_n such that

$$B_n(v, w) \approx (Lv, w). \tag{4.9}$$

We are now ready to define a finite element method with quadrature. For a right-hand side f, an approximation $\tilde{u}_n \in \mathcal{S}_n$ is chosen such that

$$B_n(\tilde{u}_n, s_i) = \lambda_i(f) \quad (1 \leq i \leq n). \tag{4.10}$$

Since $\tilde{u}_n$ depends on f through the standard information $\tilde{N}_n$, we write $\tilde{u}_n = \tilde{\varphi}_n(\tilde{N}_n f)$. The algorithm $\tilde{\varphi}_n$ is said to be the *finite element method with quadrature* (FEMQ) defined by $\mathcal{S}_n$.

How good is the FEMQ? From Werschulz [35], we find

THEOREM 4.2. *The error of the FEMQ satisfies*

$$e(\tilde{\varphi}_n, N_n) = \Theta(n^{-\mu}) \quad \text{as} \quad n \to \infty,$$

where $\mu = \min\{k+1, r\}$. ∎

Thus the FEMQ is as good as the FEM. Moreover, when $k \geq r-1$, the FEMQ has almost minimal error.

Finally, we translate these minimal-error results into results on complexity. As before, COMP(ε) denotes the intrinsic ε-complexity of the Fredholm problem, while FEM(ε) and FEMQ(ε) denote the complexity of using the FEM and the FEMQ (respectively) to find an ε-approximation. We then have the following result from Werschulz [35]:

THEOREM 4.3. *Let* $\mu = \min\{k+1, r\}$. *Then*:

(1) $\text{COMP}(\varepsilon) = \Theta(\varepsilon^{-1/r})$ *as* $\varepsilon \to 0$.

(2) $\text{FEM}(\varepsilon) = \Theta(\varepsilon^{-1/\mu})$ *as* $\varepsilon \to 0$.

(3) $\text{FEMQ}(\varepsilon) = \Theta(\varepsilon^{-1/\mu})$ *as* $\varepsilon \to 0$. ∎

As in the previous sections, this theorem can be viewed in either an optimistic or a pessimistic light. The good news is that both the FEM and the FEMQ are optimal (again, to within a constant) whenever $r \leq k+1$. However, the bad news is that the FEM and FEMQ are non-optimal whenever $r > k+1$. When $r > k+1$, one can once again show that the asymptotic penalty for using FEM or FEMQ, rather than the spline algorithm using FEI (which is optimal) is unbounded.

5. Ill-posed problems

In this section, we consider the case of an ill-posed problem. We report the results which will appear in Werschulz [36]. We are given normed linear spaces F_1 and F_2 (where F_1 is infinite-dimensional), as well as an unbounded linear operator

$$S\colon D \subset F_1 \to F_2,$$

D being the domain of the operator. In the *worst-case setting*, our goal is to approximate Sf for all f in a given class F_0 of *problem elements*. In what follows, we will take F_0 to be the set of all elements of D whose norm is at most one

$$F_0 = \{f \in D\colon \|f\| \leq 1\}.$$

As we have noted in previous sections of this paper, we do not really "know" the problem elements f when trying to determine Sf. Instead, our sole knowledge Nf of f is generally given by the values of n functionals at f, i.e.,

$$Nf = \begin{bmatrix} \lambda_1(f) \\ \vdots \\ \lambda_n(f) \end{bmatrix}.$$

Thus, *information* is an operator

$$N\colon F_0 \to \mathbf{R}^n.$$

Such information is used by an *algorithm*, which is a mapping

$$\varphi\colon D_\varphi \to F_2,$$

where D_φ denotes the domain of φ. We wish $\varphi(Nf)$ to be defined for any problem element $f \in F_0$, and so we require that $N(F_0) \subset D_\varphi$. We measure the quality of an algorithm φ using information N by its (worst-case) *error*

$$e(\varphi, N) = \sup_{f \in F_0} \|Sf - \varphi(Nf)\|.$$

In what follows, we asume (for simplicity) that N is *continuous linear information*, i.e., that the functionals $\lambda_1, \ldots, \lambda_n$ of which N consists are continuous linear functionals. We then have

THEOREM 5.1. *There is no algorithm using linear continuous information whose error is finite. That is, for any continuous linear information N and for any algorithm φ using N,*

$$e(\varphi, N) = \infty. \quad \blacksquare$$

Actually, things are even worse than this. In Werschulz [36], we show that the class of permissible functionals of which the information consists can be extended to include non-continuous linear functionals, linear functionals chosen adaptively, and certain classes of nonlinear functionals.

Hence, there is no hope for solving such problems in the worst case. However, they can sometimes be solved in an *average-case setting*, as studied in (e.g.) Wasilkowski and Woźniakowski [31].

Once again, we are given a linear unbounded solution operator

$$S\colon D \subset F_1 \to F_2,$$

where F_1 and F_2 are now separable Hilbert spaces. However, we now assume that S is a *closed* linear operator. In addition, we are now given a complete Gaussian measure μ (with zero mean element and covariance operator S_μ), defined on the Borel field $\mathcal{B}(F_1)$, which is the σ-field generated by the open subsets of F_1. (See Kuo [19] and Skorohod [26] for details and further background material concerning measures on Hilbert spaces.)

In what follows, we assume that the domain D of the solution operator is of full measure, i.e., that

$$\mu(D) = 1.$$

This is a reasonable assumption, since we are interested in average behavior over the domain D of the solution operator S, i.e., we require μ to be a probability measure over D.

We now consider information and algorithms for solving such ill-posed problems in the average setting. As before, *information* N of *cardinality* at most n will be a bounded linear operator

$$N\colon F_1 \to \mathbf{R}^n;$$

that is, we only consider linear continuous information. Thus, we only consider information operators having the form

$$Nf = \begin{bmatrix} (f, f_1) \\ \vdots \\ (f, f_n) \end{bmatrix} \quad \forall f \in F_1$$

where $f_1, \ldots, f_n$ are elements of F_1. An *algorithm* φ *using* N is then a mapping

$$\varphi\colon N(F_1) \to F_2$$

such that $\|\varphi(N\cdot)\|^2$ is μ-measurable. Given an algorithm φ using N, its (average) *error* $e^{\text{ave}}(\varphi)$ is then defined to be

$$e^{\text{ave}}(\varphi, N) = \Big[\int_D \|Sf - \varphi(Nf)\|^2 \mu(df)\Big]^{1/2}.$$

Clearly, the error of any algorithm is well-defined; it may be infinite.

Given information N, we seek the minimal average error among all algorithms using that information. This is given by the *average radius of information*

$$r^{\text{ave}}(N) = \inf\{e^{\text{ave}}(\varphi, N)\colon \varphi \text{ using } N\}.$$

An algorithm φ^* using N is said to be an *optimal (average) error algorithm* using N if

$$e^{\text{ave}}(\varphi^*, N) = r^{\text{ave}}(N).$$

For such information N, we now construct the spline algorithm, which will be a linear algorithm of central importance. We assume without loss of generality that card $N = n$, so that $f_1, \ldots, f_n$ are linearly independent. Hence, we may also assume without any loss of generality that

$$(S_\mu f_i, f_j) = \int_{F_1} (f, f_i)(f, f_j)\mu(df) = \delta_{ij} \quad (1 \leqslant i, j \leqslant n).$$

Define P: $F_1 \to F_1$ by

$$Pf = \sum_{i=1}^{n} (f, f_i) S_\mu f_i \qquad \forall f \in F_1;$$

Pf is said to be the *spline interpolating* $f \in F_1$. Since $Pf \in S_\mu^{1/2}(F_1) \subset D$, the *spline algorithm* φ^* given by

$$\varphi^*(Nf) = SPf = \sum_{i=1}^{n} (f, f_i) SS_\mu f_i$$

is well-defined for every $f \in F_1$. Of course, φ^* is a linear algorithm using N.

We then have

THEOREM 5.2. *The spline algorithm φ^* is an optimal error algorithm. Moreover,*

$$e^{\text{ave}}(\varphi^*, N) = r^{\text{ave}}(N) = \Big[\int_D \|Sf\|^2 \mu(df) - \sum_{i=1}^{n} \|SS_\mu f_i\|^2\Big]^{1/2}.$$

Hence

$$r^{\text{ave}}(N) < \infty \qquad \textit{iff} \qquad \int_D \|Sf\|^2 \mu(df) < \infty. \qquad \blacksquare$$

This theorem may be rephrased as follows. Let us say that S is *bounded on the average* if

$$\int_D \|Sf\|^2 \mu(df) < \infty.$$

Then, for any information N, *the average radius of information $r(N)$ is finite if and only if S is bounded on the average.* In other words, there are no finite-error algorithms unless S is bounded on the average. Furthermore, when this occurs, we see that the radius is finite, and that the spline algorithm is an optimal error algorithm which is linear.

Assuming that S is bounded on the average, we can determine nth optimal information in terms of the spectrum of the operator $(SS_\mu^{1/2})^*(SS_\mu^{1/2})$. The spline algorithm using that information is an nth minimal error algorithm. For further information, see Werschulz [36].

6. Summary, open problems, and future directions

In the previous sections, we discussed how information-based complexity has been applied to the ε-approximation of certain differential and integral equations. For well-posed problems, we found that if the degree of the spline space is properly chosen, depending on the smoothness of the right-hand side f and the order of the operator L, then the classical FEM is optimal.

Although the implementation of the FEM is not trivial, much work has been done on this area, and this problem is well-understood. When the FEM was non-optimal, we found that the fault lay with the fact that the FEM used its information in a non-optimal manner. In fact, one could always find a method, using the same finite element information that the FEM used, which was optimal. On the other hand, we found that for ill-posed problems, there is no algorithm with finite error in the worst-case. However, for the average case, we found that the problem is solvable, provided the solution operator is bounded on the average.

What else can we expect from the application of this approach to the solution of differential and integral equations? In this section, we describe a few possible areas of attack. The list is by no means intended to be exhaustive.

The first thing to note is that for the operator equations $Lu = f$ considered in this paper, the norms used to measure both the smoothness of a right-hand side f and the error in the approximation of u were Hilbert Sobolev norms. These are by no means the only norms of interest. For instance, we might need a good pointwise approximation of u, so that an L_∞ error estimate is required. Analogously, the smoothness of f might be measured by a non-Hilbert Sobolev norm. Hence, we are interested in determining the ε-complexity of $Lu = f$, where f is in the unit ball of one Sobolev space, and the error is measured in the norm of another Sobolev space. In particular, it is important to know under what circumstances the FEM is optimal for such problems.

The reader has probably noticed a certain similarly in the results of the previous sections. Of course, Section 2 is merely a special case of Section 3; however, no such easy relation exists between the partial differential equation in Section 3 and the integral equation in Section 4. Is there a common framework which ties these areas together? If so, is there a common explanation of the results of Section 3 and 4? What further results can one obtain from this common viewpoint?

Note that all of the results of this paper were expressed using Θ-notation. Except for the case of simple model problems, we do not known the value of the Θ-constants. The important problem of determining these constants (or even of determining explicit bounds for them) should be investigated. We suspect that this problem will be quite difficult.

The next item of interest is to consider nonlinear problems, such as

$$-u''(x) = f(x, u(x), u'(x)) \quad \text{for} \quad 0 < x < 1 \tag{5.1}$$

(subject to some boundary conditions). Under what conditions on f can we find an ε-approximation for positive ε? When is the FEM an optimal algorithm? If the FEM is non-optimal, can we find an easily-implemented algorithm that is optimal?

The results reported in this paper depend highly on the fact that these problems admit "shift theorems" which relate smoothness in f and smoothness in u. For example, the fact that shift theorems hold for certain elliptic problems (see Babuška and Aziz [4], Chapter 3 and Oden and Reddy [20], Chapter 8) was used in establishing the results described in this paper. There are a number of situations (such as problems with shocks) in which shift theorems do *not* hold. What does the information-based approach have to say about such problems?

Finally, note that all of the results of this paper were given in terms of a worst-case setting, under the assumption that the information was free of error. One should also determine the situation for an average-case setting (in which the worst-case error is replaced by an average-case error), as well as an asymptotic setting (in which we are interested in optimizing the rate of convergence for a fixed right-hand side). Furthermore, since it is generally unrealistic to assume that the information is error-free, it will be important to determine what happens when the information is contaminated by error.

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