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DECOMPOSITIONS INTO SUBMANIFOLDS OF FIXED CODIMENSION *

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Introduction

Let G denote an upper semicontinuous (henceforth abbreviated as *usc*) decomposition of an $(n+k)$-manifold M into closed connected n-manifolds, or, more generally, into continua of that shape. What can be said about the decomposition space $B = M/G$? Is B an ANR? If so, under what additional conditions is the decomposition map $p\colon M \to B$ an approximate fibration? When is B a generalized manifold? These topics form the subject of this survey.

My own interest in this area can be traced to two sources. The first involves work by Liem [L], who proved that if G is a decomposition of an $(n+1)$-manifold M into compacta with the shape of S^n, then M/G is a 1-manifold (possibly with boundary if $n = 1$) and, for $n > 4$, $p\colon M \to M/G$ can be approximated by a locally trivial bundle map. The second concerns a substantial body of work by Coram and Duvall, initially that contained in [CD2], particularly the problems raised by Coram in [C]. The developers and chief exploiters of the concept of approximate fibration, they showed that maps from S^3 to S^2 whose point inverses all have the shape of S^1 are approximate fibrations over the complement of at most two points [CD2]. Their argument strongly used the knowledge that the given range is S^2; later Coram asked [C] exactly the sort of question addressed here — is the decomposition space B associated with a *usc* decomposition G of S^3 into continua having the shape of a circle necessarily homeomorphic to S^2?

In the past two years our understanding of these matters has progressed markedly. For the low codimensional cases, where k is small, the main results seem to be in place. When $k < 3$, the decomposition space B is necessarily a k-manifold, as long as certain orientablility requirements are met. When $k = 3$ B need not be a manifold but it must be finite-dimensional;

* This paper is in final form and no version of it will be submitted for publication elsewhere.

with this case the question of whether B is an ANR takes the center of the stage. Otherwise, for $k > 2$ there are scattered results providing conditions on M and G under which B is either an ANR or a generalized k-manifold, but vast pieces of this territory remain uncharted.

A word about approximate fibrations (see [CD1] for the definition) is in order. Of the maps prevalent in the category at hand, these are specially advantageous because they provide useful, computable interrelationships among the structures on M, B, and G, the elements of which necessarily all have the same shape. Coram and Duvall [CD1] have derived an exact homotopy sequence for an approximate fibration, like the one for Hurewicz fibrations.

What follows is a review of the current status of these matters, beginning with the most thoroughly understood case, where $k = 1$. Throughout G will denote a *usc* decomposition of an $(n+k)$-manifold M into continua having the shape of closed n-manifolds, B will denote the associated decomposition space, and $p\colon M \to B$ will denote the natural decomposition map. Typically g will be used to denote an arbitrary element of G. We emphasize that here an *n-manifold* is considered to be a separable metric space modelled on E^n, Euclidean n-space, and an *n-manifold with boundary*, a similar object modelled on the n-cell B^n. A *generalized n-manifold* is a finite dimensional ANR X such that $H_*(X, X-x)$ is isomorphic to $H_*(E^n, E^n - \text{point})$ for all points x in X.

1. The codimension one case

THEOREM 1.1 [D2]. *If G is an usc decomposition of an $(n+1)$-manifold M into continua having the shape of closed n-manifolds, then B is a 1-manifold, possibly with boundary.*

A decomposition element g goes to a boundary point of B iff g is 1-sided in M. Consequently, BdB will be empty when M and the elements of G are orientable.

PROPOSITION 1.2 [DH]. *If* BdB *is empty and M is noncompact (and connected), then the inclusion $g \to M$ induces homology isomorphisms $H_*(g) \to H_*(M)$ with all coefficient groups.*

Wildness is the barrier to obtaining better results. Anytime the elements of G are genuine manifolds locally flatly embedded in M, much more can be said.

THEOREM 1.3 [D2]. *If the elements of G are closed n-manifolds locally flatly embedded in M as 2-sided subsets, then $p\colon M \to B$ is an approximate fibration. Moreover, when M is noncompact, each inclusion $g \to M$ is a homotopy equivalence and M is homeomorphic to $g \times E^1$.*

In fact, p: $M \to B$ always can be approximated by locally trivial bundle maps when M (as in Theorem 1.3) is noncompact, but not, according to an example of Husch [H], when M is compact. Husch's example emphasizes the value of nontrivial h-cobordisms.

EXAMPLE 1.4 [D2]. A *usc* decomposition G of an orientable $(n+1)$-manifold M $(n > 4)$ into homology n-spheres, some nontrivial and others being S^n, such that B is an open interval.

The example takes too much effort to describe here. It depends on the existence of an acyclic map from a nontrivial homology n-sphere to S^n whose mapping cylinder can be embedded in an $(n+1)$-manifold. A significant feature of Example 1.4 is that its associated decomposition map p: $M \to B$ fails to be an approximate fibration.

One expects only a few $(n+1)$-manifolds M to admit such rigidly limited decompositions G into closed n-manifolds. The structure of those that do can be described as follows: M can be expressed as a locally finite union of compact manifolds with boundary W_j, having pairwise disjoint interiors and intersecting pairwise (if at all) in boundary components of each, where W_j is either a twisted I-bundle, an h-cobordism (including the case where W_j is the product of I and a closed n-manifold), or something called a laminated homology cobordism, by which we mean a compact $(n+1)$-manifold having two boundary components N_1, N_2 and equipped with a *usc* decomposition G_j into closed n-manifolds such that N_1, $N_2 \in G_j$. The word "homology" is used because, just as in Proposition 1.2, the inclusions $g \to W_j$ induce homology isomorphisms.

The structural situation will be even clearer when we know which compact manifolds with boundary are laminated homology cobordisms. The result below sheds some light.

THEOREM 1.5 [DT2]. *Suppose W is a compact $(n+1)$-manifold $(n > 4)$ with two boundary components N_1 and N_2 such that the inclusion $N_2 \to W$ is a homotopy equivalence and the kernel of the inclusion-induced $\pi_1(N_1) \to \pi_1(W)$ is the normal closure of a finitely generated perfect group. Then W is a laminated homology cobordism.*

Finally, it should be remarked that in this case upper semicontinuity is forced upon most (all?) partitions into closed n-manifolds. Every partition P of a closed $(n+1)$-manifold M into continua having the shape of closed n-manifolds is necessarily *usc* [D2]. With so much extra room in codimension two, upper semicontinuity becomes a crucial hypothesis. A particularly nice partition, not *usc*, of E^3 into round circles is exhibited in [S].

2. The codimension 2 case

THEOREM 2.1 [DW2]. *If G is a usc decomposition of an orientable $(n+2)$-manifold M into continua having the shape of closed, orientable n-manifolds, then $B = M/G$ is a 2-manifold. On the other hand, if M is nonorientable, then B is a 2-manifold with boundary.*

COROLLARY 2.2. *There is no usc decomposition of E^{n+2} $(n > 0)$ into closed, connected, orientable n-manifolds.*

The corollary above is just an eye-catching application of a more general structural result.

COROLLARY 2.3. *If G is a usc decomposition of a noncompact $(n+2)$-manifold M for which $H_1(M)$ is trivial, then M has exactly one end and at this end M fails to be homologically 1-connected.*

Should the elements of G be n-spheres, then p will be an approximate fibration (except for small n).

THEOREM 2.4 [DW1]. *Let G denote a usc decomposition of an $(n+2)$-manifold M into continua having the shape of S^n. Then for $n > 1$ the decomposition map $p\colon M \to B$ is an approximate fibration, while for $n = 1$, under the additional assumption that M is orientable, B contains a locally finite set F such that the restriction of p to $M - p^{-1}(F) \to B - F$ is an approximate fibration.*

COROLLARY 2.5. *There is no usc decomposition of S^{n+2} $(n > 1)$ into continua having the shape of S^n.*

Theorem 2.4 also combines with a result of Coram–Duvall [CD2] to give:

COROLLARY 2.6. *If G is a usc decomposition of an orientable 3-manifold M into continua having the shape of circles, then the decomposition map $p\colon M \to B$ can be approximated by Seifert fiber maps.*

Work of Quinn [Q] provides another structure theorem. See [Q] for the relevant definition.

COROLLARY 2.7. *If the $(n+2)$-manifold M $(n > 1)$ admits a usc decomposition into continua having the shape of S^n, then M is a topological block bundle over B and $p\colon M \to B$ can be approximated by bundle projections.*

An analytical thread running through all these arguments depends on a notion of local n-winding function introduced by Coram and Duvall [CD2], [CD3]. Given a point b in B, there exist neighborhoods U' and U of b in B such that the inclusion-induced homomorphism

$$s\colon \check{H}_n(p^{-1}b) \to H_n(p^{-1}U')$$

is an isomorphism onto the image of

$$s_U\colon H_n(p^{-1}U) \to H_n(p^{-1}U').$$

For any other point b' in U, the image of

$$s'\colon \check{H}_n(p^{-1}b') \to H_n(p^{-1}U')$$

is contained in the image of s_U. Hence,

$$s^{-1}s'\colon \check{H}_n(p^{-1}b') \to \check{H}_n(p^{-1}b)$$

is a well-defined homomorphism between copies of Z, implying that it amounts to multiplication by some nonnegative integer q'. Define the *n*-winding function $A\colon U \to Z$ by $A(b') = q'$.

The decomposition space B is fairly tractable at those points where the n-winding functions are locally constant. That the other points are so sparse is fundamentally the explanation why B is a manifold.

PROPOSITION 2.8. *If G is a usc decomposition of an orientable $(n+2)$-manifold M into continua having the shape of closed, orientable n-manifolds, then B contains a locally finite set F such that the n-winding functions on B are locally constant at points of $B-F$.*

EXAMPLE 2.9 [DW2]. A *usc* decomposition G of an orientable $(n+2)$-manifold M into closed, orientable n-manifolds, locally flatly embedded in M, such that the n-winding functions defined on B are locally constant but the elements of G are not pairwise homologically equivalent.

This should be contrasted with Proposition 1.2.

3. The codimension three case

Now B need not be a 3-manifold, nor even a generalized 3-manifold.

EXAMPLE 3.1: A *usc* decomposition G of an $(n+3)$-manifold M such that B is a nonmanifold generalized 3-manifold.

Let Z be any nonmanifold decomposition space associated with a cell-like *usc* decomposition of a 3-manifold. Then $\dim B = 3$ [KW], [W]. Let M be $Z \times S^n$ and G the decomposition of M into the n-spheres $z \times S^n$, $z \in Z$. By Edwards' Cell-like Approximation Theorem [E] M is an $(n+3)$-manifold when $n > 1$ (see [D1]); for $n = 1$ examples where $M = Z \times S^1$ is a 4-manifold exist, but it still is unknown whether each Z has this property. In any event, $B = M/G$ is naturally equivalent to Z.

EXAMPLE 3.2. A *usc* decomposition G of a 5-manifold M into 2-manifolds such that B is not a generalized 3-manifold.

Let T denote a torus and M the 5-manifold $T \times E^3$. Consider the decomposition G consisting of $T \times 0$ and the 2-spheres $p \times rS^2$, $p \in T$ and rS^2 the sphere of radius $r > 0$ in E^3 centered at the origin 0. Then B is equivalent to the open cone on T, which is not algebraicly like a manifold at the cone point.

The best available result demonstrates the finite dimensionality of B.

THEOREM 3.3 [D3]. *If G is a usc decomposition of an $(n+3)$-manifold M into continua having the shape of closed, orientable n-manifolds, then B is 3-dimensional.*

4. Results about arbitrary codimension

First we have a general fact describing the richness of approximate fibrations and in the second item an indication of the benefits attached to approximate fibrations.

THEOREM 4.1 [DH]. *If G is a usc decomposition of an $(n+k)$-manifold M into continua having the shape of closed n-manifolds such that B is finite-dimensional, then B contains a dense open subset U such that $p|p^{-1}(U) \to U$ is an approximate fibration.*

THEOREM 4.2 [DH]. *If p: $M \to B$ is an approximate fibration and B is finite dimensional, then B is a generalized k-manifold.*

Next is a basic technical result, established by a spectral sequence argument.

PROPOSITION 4.3 [DW3]. *Suppose G is a usc decomposition of an orientable $(n+k)$-manifold M into continua having the shape of closed, orientable n-manifolds such that B is finite dimensional. Then* $\dim B = k$.

Armed with the above, when the elements of G are homologically trivial in dimensions $1, \ldots, k$, one can apply the Vietoris–Begle Mapping Theorem plus some facts proved in [DS] to show that B is a generalized k-manifold. The next result relaxes the requirements concerning homological triviality a bit.

THEOREM 4.4 [DW3]. *Suppose $n \geqslant k \geqslant 2$ and G is a usc decomposition of an orientable $(n+k)$-manifold M into continua having the shape of closed n-manifolds with trivial Čech homology in dimensions $1, \ldots, k-1$, and suppose B is finite dimensional. Then B is a generalized k-manifold.*

This leads to an extension of Theorem 2.4.

COROLLARY 4.5. *Under the hypotheses of Theorem 4.4, if each $g \in G$ has Property* UV^1, *then p: $M \to B$ is an approximate fibration.*

COROLLARY 4.6. *There is no usc decomposition of S^{n+k} $(n \geqslant k \geqslant 2)$ into continua having the shape of S^n.*

With extra conditions to guarantee that the elements of G are properly aligned homologically, Theorem 4.4 can be extended to encompass the situation where the elements of G have the same (nontrivial) homology groups. Both of the following again depend heavily upon results from [DS].

THEOREM 4.7 [DW3]. *Suppose G is a usc decomposition of an orientable $(n+k)$-manifold M into continua having the shape of closed, orientable n-manifolds having pairwise isomorphic Čech homology groups in dimensions $1, \ldots, k-2$, and suppose B is a finite dimensional space on which the n-winding functions associated with $p\colon M \to B$ are locally constant. Then B is an* ANR.

THEOREM 4.8 [DW3]. *Under the hypotheses of Theorem 4.7, if the decomposition elements also have pairwise isomorphic Čech homology groups in dimension $k-1$, then B is a generalized k-manifold.*

5. Questions

(1) If G is a *usc* decomposition of an $(n+k)$-manifold M into continua having the shape of closed n-manifolds, is $B = M/G$ an ANR?

(2) What can be said about the structure of an $(n+k)$-manifold M admitting a *usc* decomposition into closed, connected n-manifolds?

(3) Suppose M is a noncompact $(n+1)$-manifold with a *usc* decomposition into closed, connected, 2-sided n-manifolds. Does M have the homotopy type of a closed n-manifold?

(4) If W is a compact $(n+1)$-manifold with boundary and N is a component of BdW such that the inclusion $N \to W$ is a homotopy equivalence, does W admit a decomposition into closed, connected n-manifolds?

(5) If G is a *usc* decomposition of an $(n+1)$-manifold M into pairwise homeomorphic closed n-manifolds and B is homeomorphic to E^1, is M equivalent to a product $g \times E^1$?

(6) Is there a *usc* decomposition of some Euclidean space E^{n+k} into closed n-manifolds?

(7) For which integers n and k does there exist a *usc* decomposition of S^{n+k} into n-spheres? Into closed, connected n-manifolds?

(8) Let G denote a *usc* decomposition of an orientable $(n+3)$-manifold M into continua having the shape of closed, orientable n-manifolds, and let F denote the set of points at which B fails to be a generalized 3-manifold. Is F locally finite?

(9) If G is a *usc* decomposition of an $(n+k)$-manifold M into continua having the shape of closed, orientable n-manifolds such that the local n-winding functions defined on B are locally constant, is B a generalized k-manifold?

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