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ON ZARISKI-REGULARITY, THE VANISHING OF Tor AND A UNIFORM ARTIN-REES THEOREM

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0. Introduction

Recently we have proved the following theorem of uniform Artin–Rees type:

THEOREM 1 [6]. *Let S be a $J-2$ ring. Let N be a finitely generated S-module, with M a submodule of N. Then there exists a positive integer k_0 such that, for all integers $k > k_0$ and for all maximal ideals $\mathfrak{m}$ of S,*

$$M \cap \mathfrak{m}^k N = \mathfrak{m}^{k-k_0}(M \cap \mathfrak{m}^{k_0} N).$$

Remark. In fact, the proof of Theorem 1 shows we need only suppose that $\operatorname{Reg}\operatorname{Spec}(S/P)$ be open, for all $P \in \operatorname{Spec} S$, rather than that S be a $J-2$ ring. Furthermore the proof actually establishes that, for all integers $k > k_0$ and for all prime ideals P of S,

$$(1) \qquad (M \cap P^k N)_P = \big(P^{k-k_0}(M \cap P^{k_0} N)\big)_P.$$

All rings in this paper are commutative, Noetherian and have an identity element.

This theorem provides a positive answer to a generalized version of a question of Eisenbud and Hochster [8], which arose in connection with their proof of a generalization of Zariski's Main Lemma on holomorphic functions. For a uniform Artin–Rees theorem of a different type, in the context of analysis, see [3, 9] (and the 1984 preprint mentioned in the latter). In fact, there is an overlap in some of the basic ideas in all these papers, and it may be worthwhile pursuing this connection. For example, we immediately deduce from Theorem 1 the following "uniform Chevalley estimate" (cf. [3]):

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COROLLARY. *Following the notation and hypotheses of the theorem above, for every nonegative integer r there exists a nonnegative integer s such that, for all maximal ideals* $\mathfrak{m}$ *of S,*

$$M \cap \mathfrak{m}^s N \subseteq \mathfrak{m}^r M.$$

The proof of Theorem 1 depends on the concept of what we shall here call Zariski-regularity (to be defined below); this is a development and generalization of Zariski's "key" to his proof of the Main Lemma on holomorphic functions [18, p. 193]. For comments on the history and background of these ideas and of the Main Lemma, see [15, 6] and references cited there.

The purpose of the present paper is to bring out certain general features of the notion of Zariski-regularity, and to give further properties of it (involving, in particular, the vanishing of a suitable Tor). Our hope is that this will suggest possible adaptations of the theory. (The implicit question raised in connection with [1, Corollary (6.3)] should be noted, though it may be that the basic considerations involved are of a different nature.) We also take the opportunity to discuss some extensions of our work, related matters and open questions.

1. Zariski-regularity and the vanishing of Tor

First we establish some notation. Let R be a (Noetherian) ring, and let N be a finitely generated R-module with M a submodule of N. Consider $P, Q \in \operatorname{Spec} R$ with $P \supseteq Q$. Set $A = R/Q$, $\bar{P} = P/Q \in \operatorname{Spec} A$, $B = \bigoplus_{r\geq 0} Q^r/Q^{r+1}$ and

$$\begin{aligned} L &= \bigoplus_{r\geq 0} Q^r(N/M)/Q^{r+1}(N/M) \\ &\approx \bigoplus_{r\geq 0} Q^r N/((M \cap Q^r N)+Q^{r+1} N) \quad \text{(as } B\text{-modules)} \\ &= \bigoplus_{r\geq 0} L^{(r)}, \quad \text{in an obvious notation.} \end{aligned}$$

Note that L (and so $\bigoplus_{r\geq 0} L^{(r)}$) is a finitely generated B-module, with B a finitely generated algebra over the domain A.

DEFINITION. The prime ideal P is said to be *Zariski-regular* (with respect to $(M, N; Q)$) if

(i) for all $r \geq 0$, $L_{\bar{P}}^{(r)}$ is a free $A_{\bar{P}}$-module, and (ii) $\bar{P} \in \operatorname{Reg} \operatorname{Spec} A$.

Remarks. (a) As will be apparent from the proof of Theorem 2 below, (i) is equivalent to the following:

(i)′ $L_{\bar{P}}$ is a flat $A_{\bar{P}}$-module.

Thus P is Zariski-regular (with respect to $(M, N; Q)$) if and only if N/M is normally flat along Q at P and P corresponds to a smooth point on the variety defined by Q.

(b) We find it convenient to add quite mild hypotheses to ensure that the above concept is flexible and tractable. These extra hypotheses, which will stand from now on, are as follows:

(iii) $\operatorname{Reg}\operatorname{Spec} A$ is open;
(iv) N_P is a free R_P-module and
(v) $(Q^r/Q^{r+1})_P$ is a free A_P-module, for all $r \geqslant 0$.

Condition (iii) can be ensured by assuming, for example, that R be a $J-2$ (or, more strongly, an excellent) ring (cf. § 0); note also that A is a domain. As for condition (iv), we remark that in the proof of Theorem 1 we reduce to the case where N itself is a free module, while in Zariski's original paper [18] we are in the case $N = R$ right from the beginning. Finally, as for condition (v), this is ensured in [18] since there R is regular, being a polynomial ring (see (ii) together with [13, Theorem 36 (4) and Theorem 27 (ii)]). More generally, in [6], (v) is ensured by the use of generic flatness. In fact, the proof of Theorem 2 below shows that

$$\{\bar{P} \in \operatorname{Spec} A \mid (Q^r/Q^{r+1})_P \quad \text{is a free } A_P\text{-module, for all } r \geqslant 0\}$$

is a (nonempty) open subset of $\operatorname{Spec} A$, and that (v) is equivalent to R being normally flat along Q at P.

Thus, in the light of Remark (a), conditions (ii) and (v) are equivalent to the condition that P be Zariski-regular with respect to $(0, R; Q)$.

Our first result establishes the ubiquity of Zariski-regular primes (in the presence of standing hypothesis (iii) only). Given the remark on [15, p. 651, l. 15], Theorem 2 is a generalization of an infinite version of [18, Corollary 2], and it sharpens part of the argument in the proof of Theorem 1 (cf. [6]). We use the notation already established.

THEOREM 2. *The set*

$$\{\bar{P} \in \operatorname{Spec} A \mid P \textit{ is Zariski-regular with respect to } (M, N; Q)\}$$

is a nonempty open subset of $\operatorname{Spec} A$.

Proof. As above, consider $P \in \operatorname{Spec} R$ with $P \supseteq Q$. Then

$$\begin{aligned} L_P \text{ is a flat } A_P\text{-module} &\leftrightarrow \text{each } L_P^{(r)} \text{ is a flat } A_P\text{-module} \\ &\leftrightarrow \text{each } L_P^{(r)} \text{ is a free } A_P\text{-module.} \end{aligned}$$

(See [7, (6.10.1)].)

The result follows easily from [11, Theorem 1, p. 188] (together with the standing hypothesis (iii)).

We now take a homological tack. It is well-known that, in a local situation, the freedom of a finitely generated module can be tested by the vanishing of the associated Tor_1-functor at the residue field (cf. [17, Proposition 20, p. 92]). A deeper result is that in a regular local situation, the freedom of a finitely generated module can be tested by the vanishing of the associated Tor_1-functor at a finitely generated module of depth zero (cf. [12, Corollary 6], and its proof). This leads to the next result, which provides a generalization (together with its converse) of [18, Theorem 1] (see the remarks following the proof).

THEOREM 3. *Consider* $P \in \operatorname{Spec} R$ *with* $P \supseteq Q$, *where* $\bar{P} \in \operatorname{Reg}\operatorname{Spec} A$. *Then, for a given nonnegative integer* r, *the following are equivalent*:

(i) $L_P^{(r)}$ *is a free* A_P*-module*;

(ii) $M \cap Q^r N \cap P^{r+1} N = P(M \cap Q^r N) + M \cap Q^{r+1} N$, *after localising at* P, *and*

(iii) *there exists* $s > r$ *such that*

$$((M \cap Q^r N) + Q^{r+1} N) \cap P^s N = P^{s-r}(M \cap Q^r N) + Q^{r+1} N, \tag{2}$$

after localising at P.

Moreover, if any one of these conditions holds, then (2) *holds for all* $s > r$, *after localising at* P.

Proof. (i) $\Rightarrow$ (ii) We essentially follow the argument in [6]. Consider the short exact sequence of R/Q-modules:

$$0 \to \frac{(M \cap Q^r N) + Q^{r+1} N}{Q^{r+1} N} \to \frac{Q^r N}{Q^{r+1} N} \to L^{(r)} \to 0.$$

Apply $- \otimes_{R/Q} R/P$ and localise at P.

Injectivity is preserved on the left, and this is easily seen to yield that

$$M \cap PQ^r N \subseteq P(M \cap Q^r N) + Q^{r+1} N,$$

after localising at P. Applying $- \bigcap M_P$ across both sides gives

$$M \cap PQ^r N \subseteq P(M \cap Q^r N) + M \cap Q^{r+1} N, \tag{3}$$

after localising at P. By [16, Lemmas 1.3 and 1.1 (ii)], $PQ^r = P^{r+1} \cap Q^r$, after localising at P (see our standing hypothesis (v)). Moreover, by our standing hypothesis (iv),

$$(P^{r+1} \cap Q^r) N = P^{r+1} N \cap Q^r N$$

after localising at P. Hence (3) yields that

$$M \cap Q^r N \cap P^{r+1} N = P(M \cap Q^r N) + M \cap Q^{r+1} N, \tag{4}$$

after localising at P, since it is clear that the R.H.S. of (4) is already contained in the L.H.S.

(ii) $\Rightarrow$ (iii) This is easy to see: put $s = r+1$, and recall that $Q \subseteq P$.

(iii) $\Rightarrow$ (i) An obvious adaptation of the reverse of the argument employed in the proof of (i) $\Rightarrow$ (ii) shows that

$$(2) \Rightarrow (\mathrm{Tor}_1^A(A/\bar{P}^{s-r}, L^{(r)}))_P = 0.$$

By [12, Corollary 6], the latter implies that $L_P^{(r)}$ is a free A_P-module.

Finally, suppose that (i) holds, and consider an arbitrary integer $s > r$. Repeat the argument of the proof of (i) $\Rightarrow$ (ii), only this time apply $-\otimes_{R/Q}(R/(P^{s-r}+Q))$ in place of $-\otimes_{R/Q} R/P$. The result follows.

COROLLARY (cf. [18, Theorem 1]). *Let P be Zariski-regular with respect to $(M, N; Q)$. Then, given integers $s > r \geqslant 0$,*

$$M \cap Q^r N \cap P^s N = P^{s-r}(M \cap Q^r N) + M \cap Q^{r+1} N \cap P^s N, \tag{5}$$

after localising at P.

Proof. By equation (2) in the statement of Theorem 3,

$$M \cap Q^r N \cap P^s N \subseteq P^{s-r}(M \cap Q^r N) + Q^{r+1} N,$$

after localising at P. Apply $-\bigcap (M \cap P^s N)_P$ across both sides, and the result easily follows.

Remarks. (a) Note that the particular case of (5) where $s = r+1$ appears in (ii) of Theorem 3.

(b) In his proof of his Main Lemma [18], Zariski relies on (5) in the case where $N = R$ is a polynomial ring over a field, M is a prime ideal, s is arbitrary and r successively takes the values $1, 2, \ldots, s-1$. In the proof of Theorem 1 [6], we need (5) in the situation where s is arbitrary and r successively takes the values $0, 1, 2, \ldots, s-1$.

2. Further aspects

(a) An obvious question to ask is whether Theorem 1 continues to hold for classes of ideals other than maximal ideals. In particular, does equation (1) of § 0 continue to hold if we drop the localisation at the prime P? Continuing with the notation there, let $J = \bigoplus_{k > k_0} (M \cap P^k N)/P^{k-k_0}(M \cap P^{k_0} N)$. Then J is a finitely generated module over the graded ring $\bigoplus_{r \geqslant 0} P^r$. Consider $T := R \setminus P$ as a multiplicatively closed subset of degree 0 elements in the ring $\bigoplus_{r \geqslant 0} P^r$. By equation (1), $T^{-1} J = 0$ so $J_t = 0$ for some $t \in T$. In particular, if P is of co-rank 1 in R, we have

$$M \cap P^k N = P^{k-k_0}(M \cap P^{k_0} N), \qquad k > k_0,$$

at all maximal ideals except possible the finite set of maximals containing (P, t).

Unfortunately, at present this is all we have to say about the above questions.

(b) Suppose that k_0 is a uniform Artin–Rees bound over all maximal ideals $\mathfrak{m}$ of the ring R with respect to the finitely generated module N and submodule M (cf. Theorem 1). By the usual proof of the Artin–Rees lemma (cf. [2, Proposition 10.9], say), this is equivalent to there being a uniform bound k_0 on the degrees of the generators of the $C := \bigoplus_{r \geqslant 0} \mathfrak{m}^r$-module $H := \bigoplus_{r \geqslant 0} (M \cap \mathfrak{m}^r N)$, as $\mathfrak{m}$ varies through the maximal ideals of R. Suppose further that the number of generators of $\mathfrak{m}$ has a uniform bound, as $\mathfrak{m}$ varies. (Thus these hypotheses are satisfied if R is affine, say.) We now wish to show that there is also a uniform bound on the number of generators of the C-module H, as $\mathfrak{m}$ varies.

Clearly, it suffices to prove that the R-module $\bigoplus_{t=0}^{k_0}(M \cap \mathfrak{m}^t N)$ has a uniform bound on the number of generators, as $\mathfrak{m}$ varies. Suppose we consider a typical maximal ideal $\mathfrak{m}$ in the stratum of Max Spec R defined by a prime ideal Q, the stratification being given by Zariski-regularity; thus $\mathfrak{m}$ is Zariski-regular with respect to $(M, N; Q)$ (for details see [6]). By the corollary to Theorem 3,

$$M \cap Q^r N \cap \mathfrak{m}^s N = \mathfrak{m}^{s-r}(M \cap Q^r N) + M \cap Q^{r+1} N \cap \mathfrak{m}^s N \tag{6}$$

for all integers r, s with $s > r \geqslant 0$, after localising at $\mathfrak{m}$; hence in fact (6) holds absolutely for such r and s. Thus each R-module $M \cap \mathfrak{m}^t N$, $0 \leqslant t \leqslant k_0$, has a filtration of length $t+1$ with factors which are homomorphic images of $\mathfrak{m}^{t-r}(M \cap Q^r N)$, $0 \leqslant r \leqslant t$ (recall that $\mathfrak{m} \supseteq Q$, by definition). Letting $\mathfrak{m}$ vary through the maximal ideals in this stratum, for which Q is fixed, we see that the desired result holds on this particular stratum of Max Spec R. Since Max Spec R is covered by a finite number of strata, the full result follows.

(c) Finally we wish to note how some of the ideas met above can be used to simplify and extend some results in [4]. We now drop our standing notation and begin afresh.

In algebraic language, [4, Proposition 1.7] considers the following situation: let A be a Noetherian domain, and let $B = \bigoplus_{n \geqslant 0} B_n$ be a finite homogeneous ring over A which is flat as an A-module. Let $P \in \operatorname{Spec} A$, and let $k(P)$ denote the residue field at P. We wish to consider the Hilbert function

$$H_P(n) = \dim_{k(P)} B_n \otimes_A k(P).$$

Since B_P is a flat A_P-module it follows as in the proof of Theorem 2 that $(B_n)_P$ is a free A_P-module. Hence it follows easily from [10, II, Lemma 8.9] that for each $n \geqslant 0$

$$H_P(n) = \dim_{k(P)} B_n \otimes_A k(P) = \dim_K B_n \otimes_A K$$

is independent of P; here K denotes the quotient field of A. Thus the V of [4, Proposition 1.7] in fact equals Spec A. (Note that, by their hypotheses, it suffices to treat the local situation.)

This approach can be used to simplify and clarify other aspects of [4 § 1].

(d) Further refinements of the theory are given in [5].

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