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RELATIVE COHOMOLOGY AND VOLUME FORMS

J.-P. FRANCOISE

Université de Paris-Sud, Mathématiques,
Orsay, France

We consider here the classification of couples (f, ω), where f is a germ of a function $f: (C^n, 0) \to (C, 0)$, and ω is a germ of an n-form, modulo the group of germs of diffeomorphisms of $(C^n, 0)$.

For this classification, the relative cohomology of f appears to be the essential tool. E. Brieskorn and M. Sebastiani have proved that, in the isolated singularity case, these relative cohomology groups are free $C\{f\}$-modules of rank μ, the Milnor number of f. In the case where f is quasi-homogeneous, we establish the convergence of an effective algorithm which allows to compute the associated characteristic series.

This paper is a survey about questions which have been investigated previously ([F$_1$], [F$_2$], [F$_3$]).

The author expresses his gratitude to S. Denkowska and S. Łojasiewicz for inviting him to participate in the meeting on Singularities in Warsaw.

We will use the following notation:

$\mathcal{O}$: the local ring of germs of analytic functions at $0 \in C^n$.

$\mathfrak{m}$: its maximal ideal.

Ω^k: the $\mathcal{O}$-module of germs at $0 \in C^n$ of holomorphic k-forms.

$\mathrm{Diff}_0(n)$: the group of germs of diffeomorphisms tangent to the identity at the order two.

Γ_P: the orbit of $P \in \mathcal{O}$ under $\mathrm{Diff}_0(n)$.

$G = \Omega^n/dP \wedge d\Omega^{n-2}$ considered as a $C\{t\}$-module where t acts by multiplication by P.

$\Omega^n_* = \{\omega \in \Omega^n \mid \omega(0) \neq 0\}$.

We will call *volume forms* the elements of Ω^n_*.

All the coordinate systems are supposed to be holomorphic. By coordinate systems we actually always mean germs of coordinate systems.

1. The case of a Morse function

We assume here that P is a germ of a Morse function, hence there is a holomorphic coordinate system x around $0 \in C^n$ so that $P = \sum_{i=1}^{n} x_i^2$. Let Γ_P be the set $\{f \in \mathcal{O} \mid f \in (P) + \mathfrak{m}^3\}$.

J. Vey and independently V. Guillemin ([V], [G]) proved the following:

THEOREM 1.1. *Given $(f, \omega) \in \Gamma_P \times \Omega_*^n$, there is a system of local coordinates $y = (y_1, \ldots, y_n)$ such that f is a power series $\psi^*(P)$ where $P = \sum_{i=1}^{n} y_i^2$ and $\omega = dy_1 \wedge \ldots \wedge dy_n$.*

Such a coordinate system is not unique, but the series ψ^* is a characteristic of the couple (f, ω) and can be interpreted geometrically by means of integrals on vanishing cycles of the level hypersurfaces $P = t$.

Let us observe that in dimension two we get Birkhoff's statement on the local normal form of a quadratic Hamiltonian system in the group of analytic symplectic transformations. We know that such a normal form exists in the formal symplectic category in higher dimensions but a theorem of C. L. Siegel shows its generic divergence ([Si]).

So Theorem 1.1 implies, by contrast, that everything goes well in the volume-preserving case for the classification of generic germs of functions. It can be interpreted in the following way.

Let f be a Morse germ, x a system of coordinates around $0 \in C^n$, and $\omega = dx_1 \wedge \ldots \wedge dx_n$ the associated volume form. Then there is a coordinate system $y = (y_1, \ldots, y_n)$ so that $f = \psi(\sum_{i=1}^{n} y_i^2)$ and $dy_1 \wedge \ldots \wedge dy_n = dx_1 \wedge \ldots \wedge dx_n$. Here we will consider another version of this theorem.

THEOREM 1.2 *Given $(f, \omega) \in \Gamma_P \times \Omega_*^n$, there is a coordinate system $y = (y_1, \ldots, y_n)$ so that $f = \sum_{i=1}^{n} y_i^2$ and $\omega = \psi(P) dy_1 \wedge \ldots \wedge dy_n$ where $\psi(t) \in C\{t\}$ is a characteristic of the couple (f, ω).*

Such a theorem is, of course, no longer a statement about normal forms and it must be read in the following way: For each orbit of $\Gamma_P \times \Omega_*^n$ under $\mathrm{Diff}_0(n)$, there is a simple representative (which we call a *local model*) of the type $(P, \psi(P)dy)$. We will just indicate how Theorem 1.2 is a consequence of Theorem 1.1 since the result can be deduced as a corollary of our general method presented in Section 2.

Proof. 1.1 $\Leftrightarrow$ 1.2.

Let $(f, \omega) \in \Gamma_P \times \Omega_*^n$. Theorem 1.1 provides us with a local coordinate system $x = (x_1, \ldots, x_n)$ so that $f = \psi^*(P) = P + \ldots$ and $\omega = dx_1 \wedge \ldots \wedge dx_n$ with $P = \sum_{i=1}^{n} x_i^2$.

Let us write $\psi^*(P)^{1/2} = P^{1/2} u(P)$ with $u \in C\{t\}$, $u(0) = 1$. Then we define a new coordinate system by the relations $x_i' = x_i u(P)$. Clearly, in this new coordinate system we have

$$P' = \sum_{i=1}^{n} x_i'^2 = \Big(\sum_{i=1}^{n} x_i^2\Big) u^2(P) = \psi^*(P) = f.$$

Furthermore, the relations $x_i' = x_i u(P)$ turn into $x_i = x_i' v(P')$ with $v \in C\{t\}$, $v(0) = 1$. Hence,

$$\omega = dx_1 \wedge \ldots \wedge dx_n = [v(P')^n + 2v(P')^{n-1} v'(P') P'] \, dx_1' \wedge \ldots \wedge dx_n',$$

and

$$\omega = \psi(P') \, dx_1' \wedge \ldots \wedge dx_n',$$

and we get Theorem 1.2.

Now, conversely, if we have a coordinate system $x' = (x_1', \ldots, x_n')$ so that $f = P' = \sum_{i=1}^{n} x_i'^2$ and $\omega = \psi(P) \, dx_1' \wedge \ldots \wedge dx_n'$, then we solve the differential equation for w:

$$\frac{2}{n} t w'(t) + w(t) = \psi(t).$$

If we fix $w(0) = 1$, the variation of the constants leads to the following formula:

$$w(t) = t^{-n/2} \int_0^t \tfrac{1}{2} n \tau^{(n-2)/2} \psi(\tau) \, d\tau + 1$$

which is analytic in t.

Then, if we write $v = w^{1/n}$ and $x_i = x_i' v(P')$ in the coordinate system $x = (x_1, \ldots, x_n)$, we conclude that $f = P'$ is a power series in $P = \sum_{i=1}^{n} x_i^2$, and furthermore we have $\omega = dx_1 \wedge \ldots \wedge dx_n$, which is the statement of Theorem 1.1. □

2. The general isolated case

In this section, P will denote an element of $\mathcal{O}$ which has an isolated singularity at $0 \in C^n$. Let us denote by χ the Lie algebra of germs of holomorphic vector fields at $0 \in C^n$.

We wish to describe the simplest possible local model in each orbit of $\Gamma_P \times \Omega^n_*$ under the action of $\mathrm{Diff}_0(n)$.

Let us consider a couple $(f, \omega) \in \Gamma_P \times \Omega^n_*$. After a first change of local coordinates, we can assume that $f = P$ and write $\omega = a(x)\,dx$, $a(0) = a \neq 0$. Now in order to find a local model for the couple, we can try to reduce ω to a simple form with a diffeomorphism which preserves the function P.

So we need to understand better the action of $I(P)$, the isotropy subgroup of P for the group $\mathrm{Diff}_0(n)$. I am indebted to J. Martinet for the following

PROPOSITION 2.1. *Let* $\omega \in \Omega^n_*$. *The tangent space* $T(\omega)$ *at* ω *to the orbit of* ω *for the group* $I(P)$ *is* $dP \wedge d\Omega^{n-2}$.

Proof. By definition, $T(\omega)$ is the set of Lie derivatives $L_X \omega$, where X is an element of $\mathscr{I}(P) \subset \chi$, the Lie algebra of vector fields which preserve P:

$$T(\omega) = \{L_X \omega \mid X \in \chi,\ L_X P = 0\}.$$

Let $X \in \mathscr{I}(P)$. Then $dP \wedge i_X \omega = (L_X P)\omega = 0$, and the theorem of De Rham ([D]) implies the existence of $\eta \in \Omega^{n-1}$ such that $i_X \omega = dP \wedge \eta$; and by the formula of H. Cartan, we have

$$L_X \omega = dP \wedge d(-\eta).$$

Conversely, let η be an element of Ω^{n-1}. Since ω is a volume form, there is an element X of χ such that

$$i_X \omega = dP \wedge d(-\eta).$$

Actually $X \in \mathscr{I}(P)$ since we have

$$0 = dP \wedge i_X \omega = (L_X P)\omega,$$

and finally

$$L_X \omega = dP \wedge d\eta. \qquad \square$$

This proposition is not necessary in what follows but we can see from it why the module $G = \Omega^n / dP \wedge d\Omega^{n-1}$ is involved in our problem.

Now we use J. Moser's method for the conjugacy problem of forms ([Mo]).

THEOREM 2.1. *Let* (ω, ω') *be elements of* Ω^n_* *such that* $\omega - \omega' \in dP \wedge d\Omega^{n-2}$. *There is a* $\varphi \in I(P)$ *so that* $\varphi^* \omega' = \omega$.

Proof. Let us write $\omega - \omega' = dP \wedge d\eta$ and introduce the path $\omega_t = \omega + t(\omega' - \omega)$, $t \in [0, 1]$.

We look for a path of diffeomorphisms $\varphi_t \in \mathrm{Diff}_0(n)$ which satisfies

$$\varphi_t^* \omega_t = \omega \tag{1}$$

and comes from the integration of a path of vector fields X_t. The equation (1) implies that

$$\varphi_t^*(L_{X_t}\omega_t+\omega_t) = 0,$$

and so

$$L_{X_t}\omega_t = dP \wedge d\eta. \tag{2}$$

Let us then define X_t by

$$i_{X_t}\omega_t = -dP \wedge \eta; \tag{3}$$

such an X_t satisfies (2) and the family of diffeomorphisms φ_t that we get by integration of X_t, fixing $f_0 = \mathrm{Id}$, satisfies (1).

We remind the reader of the fact proved for instance in [To], that we can integrate X_t on a given ball for all t because X_t vanishes at $0 \in C^n$.

Now (3) implies that

$$0 = dP \wedge i_{X_t}\omega_t = (L_{X_t}\cdot P)\,\omega_t.$$

So $X_t \in \mathscr{J}(P)$ and the φ_t obtained by integration preserve P. We now take $\varphi = \varphi_1$ to prove the result required. □

We will use

THEOREM 2.3. ([B], [Se], [Ma$_2$]). *The $C\{t\}$-module $G = \Omega^n/dP \wedge d\Omega^{n-2}$ is free of rank μ.*

After a choice of $\gamma_\alpha \in \Omega^n$ ($\alpha \in A$, A an appropriate index set so that the classes of γ_α modulo $dP \wedge d\Omega^{n-2}$ give a $C\{t\}$-basis of G), we can prove the following

COROLLARY. *Given a couple $(f, \omega) \in \Gamma_P \times \Omega_*^n$, there is a $\varphi \in \mathrm{Diff}_0(n)$ such that $\varphi^* f = P$ and*

$$\varphi^*\omega = \sum_{\alpha \in A} \psi_\alpha(P)\gamma_\alpha$$

where $\psi_\alpha(t) \in C\{t\}$.

We now consider the question of the unicity of the functions ψ_α of the corollary.

In the following, we use the notations:

$\mathrm{Diff}_0^\wedge(n)$: the group of formal transformations with n variables whose first order jet is the identity;

$\hat{\chi}(n)$: the Lie algebra of formal vector fields in n variables;

$\hat{\Omega}^k$: the formal completion of Ω^k;

$\hat{G}$: $\hat{\Omega}^n/dP \wedge \hat{\Omega}^{n-2}$ regarded as a $C[[t]]$-module.

An analytic function P is considered as an element of $\hat{\mathcal{O}}$, the ring of formal power series, through the injection $\mathcal{O} \hookrightarrow \hat{\mathcal{O}}$.

LEMMA 2.4. *Let φ be an element of $I(P)$; φ can be interpolated by a one-parameter group of formal transformations which preserve the function P.*

Proof. We want to construct a one-parameter group φ_t contained in $\mathrm{Diff}_0^{\wedge}(n)$ such that $\varphi_1 = \varphi$ and $\varphi_t^* P = P$.

We write $\varphi = \{\varphi_i;\ i = 1, \dots, n\}$, the component functions of φ, and

$$\varphi_i = x_i + \sum_j \sum_{|\beta| = j} \varphi_{i,\beta}\, x^\beta,$$

their Taylor developments at $0 \in C^n$. We construct $\varphi_t = \{\varphi_{t,i};\ i = 1, \dots, n\}$,

$$\varphi_{t,i} = x_i + \sum_j \sum_{|\beta| = j} \varphi_{i,\beta}(t)\, x^\beta,$$

as the solutions of the differential equation (cf. [St])

$$\varphi_t' = \varphi_0' \circ \varphi_t \tag{4}$$

with the boundary values $\varphi_0 = \mathrm{Id}$ and $\varphi_1 = \varphi$.

It we assume that the $\varphi_{i,\beta}(t)$ are known by induction up to $|\beta| = j \leqslant k-1$, then we can determine the $\varphi_{i,\beta}(t)$ with $|\beta| = k$. The equation (4) imposes a condition

$$\varphi_{i,\beta}'(t) = \varphi_{i,\beta}'(0) + f_{i,\beta}(t) \tag{5}$$

where the $f_{i,\beta}(t)$ are known by induction and vanish at 0.

The equation (5) can be integrated to give

$$\varphi_{i,\beta}(t) = \varphi_{i,\beta}'(0)\, t + \int_0^t f_{i,\beta}(\tau)\, d\tau.$$

We can check that $\varphi_{i,\beta}(0) = 0$ and we need to choose $\varphi_{i,\beta}'(0)$ such that

$$\varphi_{i,\beta}(1) = \varphi_{i,\beta}.$$

We then observe that the coefficients $\varphi_{i,\beta}(t)$ are polynomials in t, and from the fact that φ_t is an interpolation of φ, $\varphi_t^* P - P = 0$ for all integers t.

If we fix a number k, the homogeneous part of degree k of $\varphi_t^* P - P$ is a polynomial in t which must vanish for all integer values of t and consequently this polynomial must vanish identically. We deduce that $\varphi_t^* P = P$ for all real values of t. □

Let us denote by $\hat{X} \in \hat{\chi}$ the formal vector field which generates the group φ_t. From the preceding lemma we deduce that $L_{\hat{X}} \cdot P = 0$. We do not know if φ can be interpolated by an analytic one-parameter group but this has no importance because of the following

THEOREM 2.5. *$\hat{G}$ is isomorphic to $G \otimes_{C\{t\}} C[[t]]$.*

This is a theorem of Bloom and Brieskorn who proved it using the Hironaka desingularisation. In [Ma$_2$], B. Malgrange gave another proof using the regularity of the Gauss–Manin connection and his theorem on the analytical index.

In the special case where P is quasi-homogeneous, it is a simple consequence of our Section 3.

So if $[\gamma_\alpha]$ is a basis of G as a $C\{t\}$-module, then $[\gamma_\alpha]$ is a basis of $\hat{G}$ as a $C[[t]]$-module and any $\omega \in \hat{\Omega}^n$ can be written in a unique way as

$$\omega = \sum_{\alpha \in A} \hat{\psi}_\alpha(P)\gamma_\alpha + dP \wedge d\hat{\eta} \tag{6}$$

where $\hat{\psi}_\alpha \in C[[t]]$ and $\hat{\eta} \in \hat{\Omega}^{n-2}$.

We can prove

PROPOSITION 2.6. *Let $\varphi \in I(P)$ and $\omega \in \Omega^n$. There is an $\eta \in \Omega^{n-2}$ such that*

$$\omega - \varphi^* \omega = dP \wedge d\eta.$$

Proof. We interpolate φ by the one-parameter formal group $\varphi_t = \exp t\hat{X}$ constructed in Lemma 2.4. We get

$$\omega - \varphi^* \omega = \int_0^1 \frac{d}{dt} \varphi_t^* \omega \, dt = \int_0^1 \varphi_t^* (L_{\hat{X}} \omega) \, dt. \tag{7}$$

Since we have $L_X \hat{P} = 0$, we get

$$dP \wedge i_{\hat{X}} \omega = 0,$$

and De Rham's theorem implies that there is a $\hat{\varrho} \in \hat{\Omega}^{n-2}$ such that

$$i_{\hat{X}} \omega = dP \wedge \hat{\varrho}.$$

Now φ_t preserves P, so we have

$$\omega - \varphi^* \omega = dP \wedge d\left(\int_0^1 \varphi_t^* \hat{\varrho} \, dt\right) = dP \wedge d\hat{\eta}. \tag{8}$$

We can write $\omega - \varphi^* \omega = \sum_{\alpha \in A} \psi_\alpha(P)\gamma_\alpha + dP \wedge d\eta$, but this can be read as a decomposition in $\hat{\Omega}^n$. From (8) and the unicity of the decomposition (6) we have $\psi_\alpha = 0$ for all $\alpha \in A$, and

$$\omega - \varphi^* \omega = dP \wedge d\eta. \qquad \square$$

We can summarize the results in

THEOREM 2.7. *There is a bijection between* $\mathrm{Diff}_0^{\wedge}(n) \backslash \Gamma_P \times \Omega_*^n$ *and* $C\{t\}^\mu$; *the μ series ψ_α which are associated to the orbit of a couple $(f, \omega) \in \Gamma_P \times \Omega_*^n$ can be obtained in the following way: we choose a coordinate system in which f can be written as P and then we decompose in this system*

$$\omega = \sum_{\alpha \in A} \psi_\alpha(P)\gamma_\alpha + dP \wedge d\eta.$$

We have to prove that the functions ψ_α are effectively independent of the coordinate system that we use to write f as P, and only depend on $[\gamma_\alpha]$.

If we have two coordinate systems in which f can be written as P, then the diffeomorphism φ which transforms one system into another has to preserve P. In the first coordinate system, we write

$$[\omega] = \sum_{\alpha \in A} \psi_\alpha(P)[\gamma_\alpha].$$

In the second one, we have

$$[\varphi^* \omega] = \sum_{\alpha \in A} \psi'_\alpha(P)[\gamma_\alpha],$$

and Proposition 2.6 implies that

$$\psi_\alpha(P) = \psi'_\alpha(P), \quad \text{for all } \alpha \in A.$$

3. Algorithm of construction of the characteristic series for the quasi-homogeneous case

M. Sebastiani [Se] proved the conjecture of Brieskorn that if $P \in \mathcal{O}$ has an isolated singularity whose Milnor number is μ, then $G = \Omega^n/dP \wedge d\Omega^{n-2}$ is a free $C\{t\}$-module of rank μ where the action of t is the multiplication by P.

Here we give a new constructive proof of this result in the case where P is quasi-homogeneous, based on B. Malgrange's theorem on "privileged neighborhoods".

First of all we make precise some more notations.

Let

$$H = \sum_{i=1}^{n} m_i x_i \frac{\partial}{\partial x_i}, \qquad m = (m_1, \ldots, m_n) \in (Q_+^*)^n,$$

be the weight vector field of P. So we have

$$L_H \cdot P = H \cdot P = P.$$

We choose monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ $(\alpha \in A \subset N^n)$ whose classes modulo J_P, the Jacobian ideal of P, give a C-basis of $\mathcal{O}/J_P$.

Let $d^n x = dx_1 \wedge \ldots \wedge dx_n$ denote the standard volume form.

We can state

THEOREM 3.0. *The μ classes of $x^\alpha d^n x$ modulo $dP \wedge d\Omega^{n-2}$ give a $C\{t\}$-basis of the module G.*

We first prove that G is finitely generated as a $C\{t\}$-module.

3.1. G is finitely generated

3.1.1. *The formal case.* Let us denote in this section by $\hat{\mathcal{O}} = C[[x_1, \ldots, x_n]]$ the ring of formal series in the indeterminate $x = (x_1, \ldots, x_n)$, $\hat{\Omega}^k$

the $\hat{\mathcal{O}}$-module of k-forms with coefficients in $\hat{\mathcal{O}}$ and $\hat{J}_P$ the Jacobian ideal of P considered as a part of $\hat{\mathcal{O}}$.

The choice of monomials x^α ($x \in A \subset N^n$) whose classes modulo $\hat{J}_P$ give a C-basis of $\hat{\mathcal{O}}/\hat{J}_P$ determines a decomposition of $\hat{\Omega}^n$; for $f \in \hat{\mathcal{O}}$, we write

$$f\, d^n x = \sigma(f)\, d^n x + dP \wedge \pi = \sum_{\alpha \in A} \sigma_\alpha(f)\, x^\alpha d^n x + dP \wedge \pi \quad (\sigma_\alpha(f) \in C). \tag{9}$$

We begin with

LEMMA 3.1.1. *For* $H = \sum_{i=1}^n m_i x_i \frac{\partial}{\partial x_i}$ *write* $M = \sum_{i=1}^n m_i$. *Then* $H + M$: $F \mapsto H \cdot F + Mf$ *is a bijection of* $\hat{\mathcal{O}}$ *onto* $\hat{\mathcal{O}}$.

Proof. $(H+M)x^\beta = (\langle \beta, m\rangle + M)x^\beta$ and $\langle \beta, m \rangle + M$ is never zero. □

We now consider $\hat{G} = \hat{\Omega}^n / dP \wedge d\hat{\Omega}^{n-2}$ as a $C[[t]]$-module where the action of t is multiplication by P. Then we can state

PROPOSITION 3.1. $\hat{G}$ *is a* $C[[t]]$*-module of finite type and the classes of* $x^\alpha d^n x$ $(\alpha \in A)$ *give a system of generators.*

Proof. For every element $f \in \hat{\mathcal{O}}$, we have to find $\varphi_\alpha \in C[[t]]$ and $\xi \in \hat{\Omega}^{n-2}$ such that

$$f\, d^n x = \sum_{\alpha \in A} x^\alpha \varphi_\alpha(P)\, d^n x + dP \wedge d\xi.$$

In order to do this, we write

$$f\, d^n x = \sigma(f)\, d^n x + dP \wedge \pi_1$$

(where we choose a π_1 which is determined obviously modulo $dP \wedge \hat{\Omega}^{n-2}$ ([D])). Then we have

$$d\pi_1 = u_1\, d^n x;$$

and if $\varphi_1 = (H+M)^{-1} u_1$, we have

$$d\pi_1 = d(i_{\varphi_1 H}\, d^n x).$$

We then produce a ζ_2 such that

$$\pi_1 = i_{\varphi_1 H}\, d^n x + d\zeta_2,$$

and we have

$$f\, d^n x = \sigma(f)\, d^n x + dP \wedge i_{\varphi_1 H}\, d^n x + dP \wedge d\zeta_2,$$

$$f\, d^n x = \sigma(f)\, d^n x + P_{\varphi_1}\, d^n x + dP \wedge d\zeta_2.$$

We then write

$$\varphi_1\, d^n x = \sigma(\varphi_1)\, d^n x + dP \wedge \pi_2, \qquad d\pi_2 = u_2\, d^n x,$$

$$\varphi_2 = (H+M)^{-1} u_2, \qquad \pi_2 = i_{\varphi_2 H} d^n x + d\zeta_3$$

and so on ...

For the p-th iterate of this process, we get

$$f\, d^n x = \sum_{j=0}^{p} P^j \sigma(\varphi_j) d^n x + P^p dP \wedge \pi_{p+1} + dP \wedge d\Big(\sum_{j=1}^{p} P^{j-1} \zeta_{j+1}\Big)$$

where we write $\varphi_0 = F$.

It is easy to check the convergence in Krull's topology and we get the result. □

We have to emphasize the fact that at each step of the iteration process the choice of π_j has some arbitrariness. Anticipating somewhat on what will follow, let us note that if we identify $\hat{\Omega}^n$ with $\hat{\mathcal{O}}$ and $\hat{\Omega}^{n-1}$ with $\hat{\mathcal{O}}^n$, then the choice of π_j can be done when we have a section λ of u: $dP \wedge \ldots : \hat{\mathcal{O}}^n \to \hat{\mathcal{O}}$. In the analytic case, we need to precise the choice of a section.

3.1.2. The analytic case. For $r = (r_1, \ldots, r_n) \in \mathbf{R}_+^{*n}$ let us introduce the polycylinder

$$D(r) = \{x \in \mathbf{C}^n \mid |x_i| \leqslant r_i\}.$$

For $f = \sum_\beta a_\beta x^\beta \in \mathcal{O}$, we write

$$|f|_r = \sum_\beta |a_\beta| r^\beta.$$

We denote by $\mathcal{O}_r$ the set of elements of $\mathcal{O}$ such that $|f|_r < \infty$. For $f = (f_1, \ldots, f_p) \in \mathcal{O}_r^p$, we write

$$|f|_r = \sum_{i=1}^{p} |f_i|_r.$$

Let Ω_r^k denote the $\mathcal{O}_r$-module of exterior k-forms with coefficients in $\mathcal{O}_r$. We identify Ω_r^n (resp. Ω_r^{n-1}) with $\mathcal{O}_r$ (resp. with $\mathcal{O}_r^n$).

Let u be an $\mathcal{O}_r$-linear mapping from $\mathcal{O}_r^n$ to $\mathcal{O}_r$. A *section* λ of u is a $\mathbf{C}$-linear mapping from $\mathcal{O}_r$ to $\mathcal{O}_r^n$ such that $u = u\lambda u$.

We say that λ is *adapted* ([Ma$_1$]) *to* $D(r)$ if λ is a continuous map from the Banach space $\mathcal{O}_r$ to the Banach space $\mathcal{O}_r^n$, i.e. if there is a $C_r > 0$ such that

$$|\lambda f|_r \leqslant C_r |f|_r, \qquad \text{for all } f \in \mathcal{O}_r^n.$$

The theorem on the privileged neighborhoods of B. Malgrange has the following consequence:

PROPOSITION 3.2. *Given u, there is a section λ such that the set of $D(r)$ to which λ is adapted gives a fundamental system of neighborhoods of the origin.*

In the following, we choose such a section to write the iteration process that we described in Section 3.1.1. We now need to check the norms at each step of the computation.

LEMMA 3.1.2. *Let* $m = \min_{j=1,\dots,n} (m_j)$, $r_0 = \min_{j=1,\dots,n} (r_j)$. *If* $\varphi \in \mathcal{O}_r$ *and* $\pi \in \Omega_r^{n-1} \simeq \mathcal{O}_r^n$ *are such that* $d(i_{\varphi H} d^n x) = d\pi$, *then we have*

$$|\varphi|_r \leqslant \frac{1}{mr_0} |\pi|_r.$$

Proof. We can write with $d\hat{x}_j = dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_n$

$$\pi = \sum_{j=1}^{n} \pi_j \, d\hat{x}_j, \qquad \pi_j = \sum_{\beta} a_\beta^j a^\beta;$$

then we have

$$\varphi = \sum_{j=1}^{n} \sum_{\beta} \frac{\beta_j}{\langle \beta, m \rangle - m_j + M} a_\beta^j x^{\beta - I_j},$$

with $I_j = (0, \dots, 1, \dots, 0)$ where the 1 is at the j-th position. □

LEMMA 3.1.3. *Let us denote* $R = \sum_{j=1}^{n} r_j$ *and let* $\theta \in \Omega_r^{n-1}$ *be such that* $d\theta = 0$; *then there is a* $\zeta \in \Omega_r^{n-2}$ *such that* $\theta = d\zeta$ *and* $|\zeta|_r \leqslant R |\theta|_r$.

This is an obvious consequence of any proof of Poincaré's theorem. We can now state

THEOREM 3.2. *Let* $D(r)$ *be a polycylinder to which* λ *is adapted. Then there is a* r' *such that* $D(r') < D(r)$ *and for all* $f \in \mathcal{O}_r$ *there are* $\xi \in \Omega_{r'}^{n-2}$ *and* $\varphi_\alpha(P) \in \mathcal{O}_r$ *satisfying the equation*

$$f \, d^n x = \sum_{\alpha \in A} x^\alpha \varphi_\alpha(P) \, d^n x + dP \wedge d\xi.$$

Proof. At the first step of the iteration, we have

$$f \, d^n x = \sigma(f) \, d^n x + dP \wedge \pi_1 \qquad \text{with } |\pi_1|_r \leqslant C_r |f|_r;$$

then $d(i_{\varphi_1 H} d^n x) = d\pi_1$, and Lemma 3.1.2 gives

$$|\varphi_1|_r \leqslant \frac{C_r}{mr_0} |f|_r.$$

The relation $d\zeta_2 = i_{\varphi_1 H} d^n x - \pi_1$ and Lemma 3.1.3 then give

$$|\zeta_2|_r \leqslant R \left[1 + \frac{MR}{mr_0} \right] C_r |f|_r,$$

and so on. For the p-th iterate we get

$$|\varphi_j|_r \leqslant \left(\frac{C_r}{mr_0}\right)^j |f|_r,$$

$$|\zeta_{j+1}|_r \leqslant RC_r\left[1+\frac{MR}{mr_0}\right]\left[\frac{C_r}{mr_0}\right]^{j-1} |f|_r,$$

for $j = 1, \ldots, p$.

Let us choose r' so that $|P|_{r'} C_r/(mr_0) < 1$. Then since

$$\left|\sum_{j=1}^{p} P^{j-1}\zeta_{j+1}\right|_{r'} \leqslant \sum_{j=1}^{p} |P|_{r'}^{j-1} |\zeta_{j+1}|_{r'} \leqslant \sum_{j=1}^{p} |P|_{r'}^{j-1} |\zeta_{j+1}|_r$$

and

$$\left|\sum_{j=0}^{p} P^j \sigma(\varphi_j)\right|_{r'} \leqslant \sum_{j=0}^{p} |P|_{r'}^j |\varphi_j|_r,$$

the result of the theorem follows with, furthermore, the explicit estimations

$$|\xi|_{r'} \leqslant \frac{RC_r(1+MR/(mr_0))}{1-|P|_{r'} C_r/(mr_0)} |f|_r \quad \text{and} \quad |\varphi_\alpha(P)|_{r'} \leqslant \frac{|f|_r}{1-|P|_{r'} C_r/(mr_0)}. \quad \square$$

We deduce as a corollary that $G = \Omega^n/dP \wedge d\Omega^{n-2}$ is a $\mathbf{C}\{t\}$-module of finite type. We now prove that G has no torsion.

3.2. The weight vector field $H = \sum_{i=1}^{n} m_i x_i \frac{\partial}{\partial x_i}$ defines a graduation of the exterior algebra $\Omega = \bigoplus_{k \geqslant 0} \Omega^k$.

A form $\bar{\omega}$ is *quasi-homogeneous of weight* h if $L_H \bar{\omega} = h\bar{\omega}$. Given a differential form, we denote by $\bar{\omega} = \sum_h \bar{\omega}_h$ its decomposition into quasi-homogeneous components.

In particular, an element $f \in \mathcal{O}$ $(\sim \Omega^0)$ can be written as $f = \sum_h f_h$.

LEMMA 3.2.1. *Let f be an element of $\mathcal{O}$ such that $f\, d^n x \in dP \wedge d\Omega^{n-2}$. Then every quasi-homogeneous component f_h satisfies*

$$f_h\, d^n x \in dP \wedge d\Omega^{n-2}.$$

Proof. We write $f\, d^n x = dP \wedge d\eta$, and so we have

$$f_h\, d^n x = dP \wedge (d\eta)_{h-1+M};$$

but

$$(d\eta)_{h-1+M} = d(\eta_{h-1+M})$$

because the Lie derivative commutes with d. $\square$

PROPOSITION 3.3. *The module G is torsion free.*

Proof. Let $f \in \mathcal{O}$ be so that $Pfd^n x = dP \wedge d\eta$; we can write

$$dP \wedge (i_{fH} d^n x - d\eta) = 0,$$

and so ([D]) we have

$$i_{fH} d^n x - d\eta = dP \wedge \varrho \quad \text{and} \quad (H \cdot f + Mf) d^n x = dP \wedge d(-\varrho).$$

If we decompose f into its quasi-homogeneous components, we get

$$f_h d^n x = dP \wedge d\left(\frac{-\varrho h - 1 + M}{h + M}\right),$$

and from this we can deduce the existence of an analytic $(n-2)$-form

$$\xi = \sum_h -\varrho \frac{h - 1 + M}{h + M}$$

such that

$$f\, d^n x = dP \wedge d\xi. \qquad \square$$

3.3. A basis of G. Let $F = \dfrac{dP \wedge \Omega^{n-1}}{dP \wedge d\Omega^{n-2}}$ be considered as a $C\{t\}$-module; then clearly $tG \subset F$.

PROPOSITION 3.3. *We have the equality* $tG = F$.

Proof. Let us denote by $[\omega]$ the class in G of an element ω of Ω^n. We remark that the classes $[\omega_{\beta,i}]$, where $\omega_{\beta,i} = dP \wedge (x^\beta d^n x_i)$, generate F. A simple computation then leads to

$$t[\beta_i x^{\beta - 1_i}] = \langle \beta, m \rangle + \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j [\omega_{\beta,i}]. \qquad \square$$

We can deduce from this proposition that $G/tG = G/F$. But then

$$G/F \simeq \Omega^n / dP \wedge \Omega^{n-1} \simeq \mathcal{O}/J_P.$$

The fact that the classes $[x^\alpha d^n x]$ $(\alpha \in A)$ give a basis of G follows from Nakayama's lemma, so Theorem 3.0 is proved.

4. The non-isolated case (normal crossing)

How to extend the classification of couples functions-volume forms to the case where the function has a non-isolated singularity? Obviously, there are examples where the moduli space $\Omega^n / dP \wedge d\Omega^{n-2}$ is no longer a finitely generated $C\{t\}$-module.

But it is interesting to note first of all that we are in fact concerned with

$$G' = \Omega^n/\{d\eta \mid dP \wedge \eta = 0\},$$

and that this module is $C\{t\}$ finitely generated in the case where P is a normal crossing $P = x_1^{p_1} \ldots x_n^{p_n}$. We write in this case $p_i = p_i' d$ $(i = 1, \ldots, n)$, where d is the g.c.d. of the p_i, and $\omega_j = \prod_{i=1}^{n} x_i^{jp_i'-1} d^n x$ for $j = 1, \ldots, d$.

Given a couple $P = x_1^{p_1} \ldots x_n^{p_n}$ and $\omega = a(x) d^n x$ with $a(0) = 1$, we have:

THEOREM 4.1. *There is a diffeomorphism φ which preserves P and transforms ω into the model*

$$\varphi^* \omega = dx_1 \wedge \ldots \wedge dx_n + \sum_{j=1}^{n} \psi_j(P)\omega_j.$$

Let $\overline{\Omega^n}$ be the set of n-forms $d\eta$ such that $dP \wedge \eta = 0$. The same proof as in Theorem 2.1 gives the fact that if two volume forms ω and ω' are such that $\omega(0) = \omega'(0) = dx_1 \wedge \ldots \wedge dx_n$ and $\omega - \omega' \in \overline{\Omega^n}$, then there is a diffeomorphism φ so that $\varphi^* P = P$ and $\varphi^* \omega = \omega'$.

The above theorem is a consequence of

PROPOSITION 4.1. *The $C\{t\}$-module $G' = \Omega^n/\overline{\Omega^n}$ is free of rank d and the classes of the ω_j modulo $\overline{\Omega^n}$ give a basis of G' as a $C\{t\}$-module.*

Proof. Given $f \in \mathcal{O}$ and $\omega = f\, dx_1 \wedge \ldots \wedge dx_n$, we write $f = \sum_\alpha f_\alpha x^\alpha$ and we try to decompose ω into

$$f\, d^n x = d\eta + \sum_{j=1}^{d} \psi_j(P)\omega_j$$

with $d\eta \in \overline{\Omega^n}$.

We can treat each monomial separately, and so given a monomial $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, we need to know for which values of $\alpha = (\alpha_1, \ldots, \alpha_n)$ we can be sure to find $\eta_1, \ldots, \eta_n$ such that

$$x^\alpha = \frac{\partial \eta_1}{\partial x_1} + \ldots + \frac{\partial \eta_n}{\partial x_n},$$

$$0 = \frac{\partial x^p}{\partial x_1}\eta_1 + \ldots + \frac{\partial x^p}{\partial x_n}\eta_n.$$

We can look for $\eta_1, \ldots, \eta_n$ of the type

$$\eta_1 = A_1 x_1^{\alpha_1+1} \ldots x_n^{\alpha_n}, \quad \ldots, \quad \eta_n = A_n x_1^{\alpha_1} \ldots x_n^{\alpha_n+1}.$$

We then have to solve the system

$$(\alpha_1+1)A_1+\ \dots\ +(\alpha_n+1)A_n = 1$$
$$p_1 A_1+\ \dots\ +p_n A_n = 0.$$

Such a system has a solution if and only if $(\alpha_1+1, \dots, \alpha_n+1)$ is not a multiple of $(p_1, \dots, p_n)$. This case corresponds to

$$\alpha_1+1 = \lambda p_1 = (\lambda d)\, p'_1,$$
$$\dots\dots\dots\dots\dots$$
$$\alpha_n+1 = \lambda p_n = (\lambda d)\, p'_n.$$

λd is necessarily an integer and the different values $\lambda = j/d$, when j varies from 1 to d, correspond to the different monomials which give the d series $\psi_j(P)\,\omega_j$. □

In [F_3], I have considered the question of C^∞ conjugacy for the real normal crossing.

The problem of C^∞-conjugacy has been considered by Colin de Verdière and J. Vey ([C]) and V. Guillemin ([G]) for the Morse functions, and by C. Roche in general in his thesis ([R]).

References

[B] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. 2 (1970), 103–161.

[C] Y. Colin de Verdière et J. Vey, *Le lemme de Morse isochore*, Topology 18 (1979), 283–293.

[D] G. De Rham, *Sur la division des formes et des courants par une forme linéaire*, Comment. Math. Helv. 28 (1954), 346–352.

[F_1] J.-P. Francoise, *Modèle local simultané d'une fonction et d'une forme de volume*, Astérisque 59–60 (1978), 119–130.

[F_2] —, *Le théorème de M. Sebastiani pour une singularité quasi homogène isolée*, Ann. Inst. Fourier (Grenoble) 29 (1979), 247–254.

[F_3] —, *Réduction simultanée d'un croisement normal et d'un volume*, Bol. Soc. Brasil. Mat. 13 (1982), 79–83.

[G] V. Guillemin, *Band asymptotics in two dimensions*, Adv. in Math. 42 (1981), 248–282.

[Ma_1] B. Malgrange, *Frobenius à singularités I: codimension un*, Publ. Math. Inst. Hautes Études Sci. 46 (1976), 163–173.

[Ma_2] —, *Intégrales asymptotiques et monodromie*, Ann. École Nat. Sup. Méc. Nantes 4 (1974), 405–430.

[Mo] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120–121 (1965), 286–294.

[R] C. Roche, Thèse, Université de Grenoble, 1981.

[Se] M. Sebastiani, *Preuve d'une conjecture de Brieskorn*, Manuscripta Math. 2 (1970), 301–308.

[Si] C. L. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, Berlin–Heidelberg 1971.
[St] S. Sternberg, *Infinite Lie groups and formal aspects of dynamics*, J. Math. Mech. 10 (1961).
[To] J.-C. Tougeron, *Idéaux de fonctions différentiables*, Ergeb. Math. Grenzgeb. 71, Springer-Verlag, Berlin 1970.
[V] J. Vey, *Sur le lemme de Morse*, Invent. Math. 40 (1977), 1–9.

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