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Smoothness of the attractor of almost all solutions of a delay differential equation

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Abstract

Let a C^1-function $f : \mathbb{R} \to \mathbb{R}$ be given which satisfies $f(0) = 0, f'(\xi) < 0$ for all $\xi \in \mathbb{R}$, and $\sup f < \infty$ or $-\infty < \inf f$. Let $C = C([-1, 0], \mathbb{R})$. For an open-dense set of initial data the phase curves $[0, \infty) \to C$ given by the solutions $[-1, \infty) \to \mathbb{R}$ to the negative feedback equation

$$x'(t) = -\mu x(t) + f(x(t-1)), \quad \text{with } \mu > 0,$$

are absorbed into the positively invariant set $S \subset C$ of data $\phi \neq 0$ with at most one sign change. The global attractor A of the semiflow restricted to $\bar{S}$ is either the singleton $\{0\}$ or it is given by a Lipschitz continuous map a with domain pA in a 2-dimensional subspace $L \subset C$ and range in a complementary subspace Q; pA is homeomorphic to the closed unit disk in $\mathbb{R}^2$. We show that a is in fact C^1-smooth.

1. Introduction

Result and method. The equation

$$x'(t) = -\mu x(t) + f(x(t-1)) \tag{1}$$

with $\mu > 0$ and a function $f : \mathbb{R} \to \mathbb{R}$ is the simplest model for a system governed by delayed feedback and decay. In case

$$f(0) = 0 \quad \text{and} \quad \xi f(\xi) < 0 \quad \text{for all } \xi \neq 0$$

there is a rest state given by $\xi = 0$, and the feedback is negative with respect to this rest state. The hypothesis in the present paper is the stronger condition that

$$f \text{ is } C^1\text{-smooth}, \quad f(0) = 0 \quad \text{and} \quad f'(\xi) < 0 \quad \text{for all } \xi \neq 0,$$

and that f is bounded from below or bounded from above.

In [12] it is shown that in this case there is an open and dense set of initial data ϕ in the phase space

$$C = C([-1,0],\mathbb{R}) \quad \text{with} \quad \|\phi\| = \max_{-1 \le t \le 0} |\phi(t)|$$

so that the solution $x^\phi : [-1,\infty) \to \mathbb{R}$ given by $x^\phi|[-1,0] = \phi$ is eventually slowly oscillating in the sense that there exists $t_\phi \ge -1$ so that all zeros of x^ϕ in $[t_\phi,\infty)$ are farther apart than the delay 1 in the equation. The phase curves

$$[0,\infty) \ni t \mapsto x_t \in C, \quad x_t(s) = x(t+s),$$

of such solutions $x = x^\phi$ enter the set

$$S = \{\phi \in C \setminus \{0\} : \begin{array}{l} \text{there are } z \in [-1,0] \text{ and } j \in \{0,1\} \\ \text{with } (-1)^j \phi(s) \le 0 \text{ for } s \in [-1,z] \\ \text{and } 0 \le (-1)^j \phi(s) \text{ for } s \in [z,0]\} \end{array}$$

of data with at most one sign change, which is positively invariant under the semiflow

$$F : [0,\infty) \times C \ni (t,\phi) \mapsto x_t^\phi \in C$$

of equation (1). The position of S in C can be described in terms of the linearization of F at the stationary point $0 \in C$. The generator of the C_0-semigroup of the operators

$$D_2F(t,0), \quad t \ge 0,$$

has a leading pair of eigenvalues, and the associated reellified generalized eigenspace $L \subset C$ satisfies $\dim L = 2$ and $L \subset \overline{S}$ while the reellified generalized eigenspace Q defined by the remaining spectrum is disjoint from S.

The restricted semiflow on the closure $\overline{S} = S \cup \{0\}$ has a global attractor $A \subset \overline{S}$. Every $\phi \in A \setminus \{0\}$ uniquely determines a solution $x = x(\phi)$ which is defined on $\mathbb{R}$, satisfies

$x_0 = \phi$, is bounded, and slowly oscillating in the sense that all zeros are farther apart than the delay 1. The attractor A consists of the segments x_t of all solutions of this type, and of the stationary point $0 \in C$. In [20] the inclusion $A - A \subset \bar{S}$ is derived and used to show that the attractor A can be written as a map a from a subset $L_a \subset L$ into Q,

$$A = \{\chi + a(\chi) : \chi \in L_a\}.$$

In case $A \neq \{0\}$ the domain L_a is a neighbourhood of 0 in L and homeomorphic to the closed unit disk in $\mathbb{R}^2$, and ∂L_a is the projection of the orbit in C of a slowly oscillating periodic solution. The periodic orbits on A project into simple closed curves in the plane L which are nested and contain $0 \in L$ in the interior. For the aperiodic solutions with segments in A a Poincaré–Bendixson theorem holds. An a-priori estimate of the form

$$c\|q(\phi - \psi)\| \leq \|p(\phi - \psi)\| \quad \text{for } \phi, \psi \text{ in } A \tag{2}$$

with the projection $p : C \to C$ onto L along Q and $q = \mathrm{id} - p$ shows in [20] that the map a is Lipschitz continuous.

The present paper proves that in case $A \neq \{0\}$ the map a is continuously differentiable. The precise result is stated as Theorem 2.3.1 below. The proof is long and involved. Partial results were obtained earlier in [17, 19]. The main result of [17] implies that a is C^1-smooth on an open neighbourhood of $0 \in L$ provided the stationary point is linearly unstable, i.e., the real parts of the leading pair of eigenvalues are positive. The results in [19] yield that a is C^1-smooth on open annuli given by unstable sets of unstable hyperbolic periodic orbits in A. It is not hard to see, however, that there are cases where A is nontrivial with the stationary point linearly stable. Furthermore, there is at least one periodic orbit in A which is not hyperbolic and unstable, namely the orbit projecting onto the boundary ∂L_a.

The starting point of the smoothness proof is the simple fact that the phase curves $\mathbb{R} \ni t \mapsto x_t \in C$ of bounded slowly oscillating solutions x on $\mathbb{R}$ are C^1-smooth. Using this foliation of A into smooth curves it is not very difficult to show that a is C^1-smooth in a neighbourhood of a point $p\phi, \phi \in A \setminus \{0\}$, provided there exist $t > 1$ and a C^1-curve $\alpha : (-1, 1) \to C$ on A so that α and the phase curve $s \mapsto x(\phi)_s$ intersect transversally at $\alpha(0) = x(\phi)_{-t}$.

Such transversals are rather easily found for wandering points $\phi \in A \setminus \{0\}$, i.e., for points ϕ so that the solution $x(\phi)$ is not periodic. Due to the Poincaré–Bendixson theorem from [20] the α-limit set of $x(\phi)$ is either the stationary point or a periodic orbit. Suppose it is a hyperbolic periodic orbit. Then ϕ belongs to an unstable set as studied in [19], and a piece of a one-dimensional local unstable manifold of a Poincaré map on a hyperplane transversal to the periodic orbit yields a suitable curve α.

More difficult is the construction of smooth curves on A which transversally intersect periodic orbits which are attracting or stable. Consider a slowly oscillating periodic solution y with minimal period $\omega > 0$. Set $\phi = y_0$. Let a closed hyperplane $Z \subset C$ be given so that $\phi + Z$ is transversal to the phase curve $t \mapsto y_t$ at $t = \omega$, i.e.,

$$\phi' = y'_\omega = D_1F(\omega, \phi)1 \notin Z.$$

Assume for simplicity that the orbit $\{y_t : t \in \mathbb{R}\}$ projects into the open kernel L_a°.

A technique from [20] yields an injective continuous curve

$$d : (-\delta_e, \delta_i) \to C \quad \text{with} \quad d(0) = \phi$$

which parametrizes the intersection $A \cap (\phi + Z)$ in a neighbourhood of ϕ and has the property that $d((0, \delta_i))$ projects into the interior of the simple closed planar curve $[0, \omega] \ni t \mapsto py_t \in L$ while $d((-\delta_e, 0))$ projects into the exterior. Suppose that no value $d(s), s \neq 0$, is on a periodic orbit. This can be achieved if the periodic orbit is attracting. The smoothness of A at wandering points can then be used to show that both restrictions $d|(-\delta_e, 0)$, $d|(0, \delta_i)$ are C^1-smooth, with $d'(s) \neq 0$ for $s \neq 0$. In order to have smoothness of d at 0 and transversality one needs that the tangent vectors $d'(s) \in Z, s \neq 0$, converge to a nonzero limit vector as $0 \neq s \to 0$. The proof of convergence, in case of an attracting periodic orbit and for a reparametrization of d, is based on the following considerations.

(i) The invariance properties of A imply that the trace of d is locally positively invariant for the Poincaré map P_Z defined in a neighbourhood of ϕ in $\phi + Z$. Therefore, if $d'(0)$ exists, then necessarily

$$DP_Z(\phi)d'(0) \in \mathbb{R}d'(0),$$

and $d'(0)$ is an eigenvector or zero.

(ii) The inclusion $A - A \subset \bar{S}$ implies $d'(s) \in \bar{S}$ for $s \neq 0$, and $d'(0) \in \bar{S}$ if d is differentiable at 0.

(iii) Floquet theory for slowly oscillating periodic solutions of equation (1) yields a radius $\varrho \in (0, 1)$ so that the reellified generalized eigenspace $C_>$ given by the Floquet multipliers $\lambda \in \mathbb{C}$ of the periodic solution y with $|\lambda| > \varrho$ satisfies

$$\dim C_> = 2 \quad \text{and} \quad C_> \subset \bar{S}$$

while the reellified generalized eigenspace $C_<$ of the remaining spectrum of the monodromy operator $D_2F(\omega, \phi)$ satisfies

$$C_< \cap S = \emptyset.$$

The fact that 1 is a Floquet multiplier with eigenvector $D_1F(\omega, \phi)1 = y'_\omega = \phi'$ implies

$$C_> = \mathbb{R}\phi' \oplus \mathbb{R}\phi_*$$

for some unit vector $\phi_* \in C_>$, and it is not hard to see that the choice $Z = C_< \oplus \mathbb{R}\phi_*$ yields a Poincaré map for which the nontrivial multiples of ϕ_* are the only eigenvectors of $DP_Z(\phi)$ in S. Let λ_* be the eigenvalue of $DP_Z(\phi)$ associated with ϕ_*. Then $|\lambda_*| > \varrho$, in fact,

$$\lambda_* \in (\varrho, \infty)$$

as will be shown in Subsection 2.4 below.

(iv) In case the fixed point ϕ of P_Z is hyperbolic and attractive, i.e., $\varrho < \lambda_* < 1$, trajectories $(\phi_n)_{n=0}^{\infty}$ of P_Z in, say, $d((0, \delta_i))$ converge to $d(0)$ as $n \to \infty$. Then $\phi_n = d(s_n)$ with $s_n \in (0, \delta_i)$, and a special case of the desired convergence property would be that the sequence of the tangent vectors $d'(s_n)$ has a limit as $n \to \infty$. Observe that

$$d'(s_n) \in \bar{S} \setminus \{0\} = S \subset C \setminus C_<$$

This suggests using an inclination lemma in order to show that at least the slopes of $d'(s_n)$ with respect to the decomposition $C = C_> \oplus C_<$ tend to 0, or equivalently, that

$$\frac{1}{\|d'(s_n)\|} d'(s_n) \to \{\phi_*, -\phi_*\} \quad \text{as } n \to \infty.$$

An analysis of this approach shows that it requires a new a-priori estimate of the form

$$c\|d'(s)\| \le \|p_* d'(s)\| \quad \text{for } s \ne 0 \tag{3}$$

with the projection $p_* : Z \to Z$ onto $\mathbb{R}\phi_*$ along $C_<$. In other terms, the vectors $d'(s)$, which are tangent to the set A at $d(s)$ and in Z, should belong to a certain cone which contains the most unstable direction for $DP_Z(\phi)$ and is disjoint from the complementary space $C_<$.

(v) An estimate of the form (3) follows rather easily from another a-priori estimate which generalizes (2) in the sense that the projection p is replaced by the projection onto $C_>$ along $C_<$ associated with a slowly oscillating periodic solution, and $\phi - \psi \in A - A \subset \overline{S}$ is replaced by elements from a larger subset of $\overline{S}$.

Organization of the paper. Section 2 contains facts about slowly oscillating solutions, the set S, the attractor A, Floquet multipliers of slowly oscillating periodic solutions, Poincaré maps, local invariant manifolds, and curves in a plane. For proofs of results which are presented without reference, see [17, 20]. In addition to the local Poincaré maps on the special hyperplanes

$$y_0 + (C_< \oplus \mathbb{R}\phi_*)$$

mentioned before, a global return map P as in [20] and in many earlier papers on slowly oscillating periodic solutions is discussed; for data $\phi \in C$ so that $\phi(-1) = 0 < \phi(0)$, $[-1, 0] \ni t \mapsto e^{\mu t}\phi(t) \in \mathbb{R}$ is increasing, and x^ϕ has a first and second zero $z_1(\phi)$ and $z_2(\phi)$ in $(0, \infty)$, P is given by

$$P(\phi) = F(z_2(\phi) + 1, \phi).$$

The map P will be useful in the proof that the map a is smooth in a neighbourhood of $0 \in L$.

In Section 3 the a-priori estimate of the form (3) is derived. The proof of the generalization of (2) is modelled after the proofs of variants of (2) in [16, 17, 4]; it is technically more complicated.

In Section 4 sufficient conditions for smoothness of a are given in terms of the existence of smooth curves on A which are transversal to the semiflow. Furthermore, smoothness at wandering points is established.

Section 5 deals with the construction of curves on A which intersect or end at periodic orbits. In Subsection 5.3 smoothness at wandering points and the a-priori estimate of Subsection 3.2 are used to obtain in certain cases C^1-curves whose tangents have a limit at the periodic orbit. In Subsection 5.4 a curve on A is constructed which connects the stationary point to a periodic orbit and on which the global return map P is conjugate to a strictly monotone interval map. Differentiability of this curve will not be needed.

Section 6 completes the proof that a is smooth at projected periodic orbits, using the results of Subsection 5.3 and local invariant manifolds of Poincaré maps.

In Section 7 it is shown that a is smooth at $0 \in L$, with $Da(0) = 0$. Ingredients of the proof are local invariant manifolds of the semiflow at $0 \in C$, the curve from Subsection 5.4, and a lemma on inclinations of tangent spaces $T_{\chi+a(\chi)}A$ for χ close to $0 \in L$ and a smooth in a neighbourhood of χ.

Related work. I. Tereščák announced a proof that attractors of certain semilinear parabolic initial boundary value problems are contained in smooth finite-dimensional manifolds [15]. Concerning Floquet theory and Poincaré–Bendixson theorems for delay differential equations, see the work of J. Mallet-Paret and G. Sell [10, 11].

Terminology and notation. For a subset M of a topological space the closure, boundary, and the set of inner points are denoted by $\overline{M}, \partial M, M^\circ$, respectively.

A *curve* γ is a continuous map defined on an open interval in $\mathbb{R}$. Its range, or trace, is often written $|\gamma|$. The interior and the exterior of a simple closed curve in a 2-dimensional vector space over $\mathbb{R}$ are denoted by $\mathrm{int}(\gamma)$ and $\mathrm{ext}(\gamma)$, respectively.

If X and Y are Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ then $L_c(X,Y)$ stands for the Banach space of linear continuous maps from X into Y.

If M is a subset of a Banach space X over $\mathbb{R}$ and $x \in M$ then T_xM denotes the set of all tangent vectors of M at x, i.e. the set of all $v \in X$ so that there exists a differentiable curve $\gamma : (-1,1) \to X$ with

$$\gamma(0) = x, \quad |\gamma| \subset M, \quad v = \gamma'(0) = D\gamma(0)1.$$

Observe that $0 \in T_xM$, $\mathbb{R}T_xM \subset T_xM$, and that $T_xM + T_xM \not\subset T_xM$ is possible. If f is a differentiable map from an open subset U of X into a Banach space Y over $\mathbb{R}$, if $M \subset U$ and $f(M) \subset N \subset Y$ then

$$Df(x)T_xM \subset T_{f(x)}N \quad \text{for all } x \in M,$$

by the chain rule.

The word "solution" (of a delay differential equation) always refers to a real- or complex-valued function while the word "phase curve" is reserved for the associated curves of the form $t \mapsto x_t$ with values in the space of initial data. The word "trajectory" is used if a set X, a map $f : M \to X, M \subset X$, and a sequence

$$(x_j)_{j\in J}, \quad J = \mathbb{Z} \cap I, \quad I \subset \mathbb{R} \text{ an interval},$$

are given so that

$$x_{j+1} = f(x_j) \quad \text{for all } j \in J \text{ with } j+1 \in J.$$

Reference in subsection (n.m) to equation (c) from another subsection (a.b) is made using the label (a.b.c), analogously for propositions, lemmas, corollaries, and theorems.

2. The delay differential equation and its attractor of almost all solutions

2.1. The delay differential equation. Let a C^1-function $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(0) = 0, \quad f'(x) < 0 \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad -\infty < \inf f \text{ or } \sup f < \infty.$$

Let $\mu > 0$. A *solution* of the equation (1.1.1)

$$x'(t) = -\mu x(t) + f(x(t-1))$$

is either a continuous real function x which is defined on an interval $[t_0 - 1, \infty), t_0 \in \mathbb{R}$, and is differentiable and satisfies equation (1.1.1) for all $t > 0$, or a differentiable real function x which satisfies equation (1.1.1) for all $t \in \mathbb{R}$. Solutions $x : [t_0 - 1, t_1) \to \mathbb{R}$ with $t_0 < t_1$ or $x : [t_0 - 1, \infty) \to \mathbb{R}$ of more general equations

$$x'(t) = g(t, x(t-1)) \tag{1}$$

given by functions $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}, t_0 \in \mathbb{R}$, or $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined analogously.

Let C denote the space of continuous real functions on the interval $[-1, 0]$, equipped with the norm given by

$$\|\phi\| = \max_{t \in [-1,0]} |\phi(t)|.$$

Every $\phi \in C$ extends to a uniquely determined solution x^ϕ on the interval $[-1, \infty)$. This is most easily seen using the variation-of-constants formulae

$$x(t) - x(n) = e^{-\mu(t-n)} x(n) + \int_n^t e^{-\mu(t-s)} f(x(s-1))\, ds$$

for $n \in \mathbb{N}_0$, $n \le t \le n+1$. Solutions depend continuously on the initial data in the sense that given $\phi \in C$, $\varepsilon > 0$, and $t_0 \ge 0$, there exists $\delta > 0$ so that for all $\psi \in C$ with $\|\psi - \phi\| \le \delta$ and for all $t \in [-1, t_0]$,

$$|x^\psi(t) - x^\phi(t)| < \varepsilon.$$

For a function $y : D \to \mathbb{R}$ and for $t \in \mathbb{R}$ with $[t-1, t] \subset D$, the segment $y_t : [-1, 0] \to \mathbb{R}$ is defined by

$$y_t(s) = y(t+s).$$

The relations

$$F(t, \phi) = x_t^\phi, \quad \phi \in C,\ t \ge 0,$$

define a continuous semiflow $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Each map $F(t, \cdot) : C \to C,\ t \ge 0$, is injective; if solutions $x : [-1, \infty) \to \mathbb{R}$ and $y : [-1, \infty) \to \mathbb{R}$ satisfy $x_t = y_t$ for some $t \ge 0$ then $x_s = y_s$ for all $s \ge 0$. All maps $F(t, \cdot)$, $t \ge 1$, are compact in the sense that for every bounded set $B \subset C$ the set $\overline{F(t, B)}$ is compact. For every $\phi \in C$ the ω-limit set

$$\omega(\phi) = \{\psi \in C : \text{there exists a sequence } (t_n)_{n=0}^\infty \text{ in } [0, \infty) \text{ so that } t_n \to \infty \text{ and } F(t_n, \phi) \to \psi \text{ as } n \to \infty\}$$

is nonempty, compact, connected; for each $\psi \in \omega(\phi)$ there exists a solution $x = x(\psi)$ which is defined on $\mathbb{R}$ and satisfies $x_0 = \psi$. Note that due to injectivity of the maps $F(t, \cdot)$, $x(\psi)$ is uniquely determined. The set $\omega(\phi)$ is invariant in the sense that

$$x(\psi)_t \in \omega(\phi) \quad \text{for all } \psi \in \omega(\phi),\ t \in \mathbb{R}.$$

Similarly, every solution $x : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) which is bounded at $-\infty$ has a nonempty α-limit set

$$\alpha(x) = \{\psi \in C : \text{there exists a sequence } (t_n)_{n=-\infty}^{0} \text{ in } \mathbb{R} \\ \text{so that } t_n \to -\infty \text{ and } x_{t_n} \to \psi \text{ as } n \to -\infty\},$$

which is compact and connected and has the same invariance properties as the ω-limit sets.

Each map $F(t,\cdot),\ t \geq 0$, is C^1-smooth, and for all $t \geq 0,\ \phi \in C,\ \psi \in C$,

$$D_2F(t,\phi)\psi = v_t^{\psi}$$

where $v = v^{\psi}$ is the solution of the variational equation along $x = x^{\phi}$,

$$v'(t) = -\mu v(t) + f'(x(t-1))v(t-1) \tag{2}$$

with initial condition $v_0 = \psi$.

Each map $D_2F(t,\phi),\ t \geq 0$ and $\phi \in C$, is injective, and all maps $D_2F(t,\phi),\ t \geq 1$ and $\phi \in C$, are compact. For $t > 1$ and $\phi \in C$ the partial derivative $D_1F(t,\phi)$ exists, and

$$D_1F(t,\phi)1 = x_t' \quad \text{for } x = x^{\phi}$$

where $x_t' = (x_t)'$. The map $D_1F : (1,\infty)\times C \to L_c(\mathbb{R}, C)$ is continuous, and the restriction $F|(1,\infty)\times C$ is C^1-smooth. If $x : [t_0 - 1, \infty) \to \mathbb{R}$ is a solution of equation (1.1.1) and $t \geq s > t_0 + 1$ then

$$x_t' = D_2F(t-s, x_s)x_s'.$$

Let $\phi \in C$, $t > 1$, and let Z be a closed hyperplane in C such that

$$D_1F(t,\phi)1 \notin Z.$$

Z is the nullspace of a linear continuous functional $\phi^* : C \to \mathbb{R}$. An application of the Implicit Function Theorem to the equation $\phi^*(F(s,\psi) - F(t,\phi)) = 0$ with the solution $s = t,\ \psi = \phi$ shows that there exist an open neighbourhood U of ϕ, $\varepsilon > 0$ with $1 < t - \varepsilon$, and a C^1-map $\tau : U \to \mathbb{R}$ so that

$$\tau(\phi) = t, \quad \tau(U) \subset (t-\varepsilon, t+\varepsilon),$$
$$F(\tau(\psi),\psi) \in F(t,\phi) + Z \quad \text{for all } \psi \in U,$$

and for all $\psi \in U$ and $s \in (t-\varepsilon, t+\varepsilon)$,

$$\tau(\psi) = s \quad \text{if and only if} \quad F(s,\psi) \in F(t,\phi) + Z.$$

Moreover,

$$D_1F(\tau(\psi),\psi)1 \notin Z \quad \text{for all } \psi \in U.$$

The map τ is called a *stopping time*, and the C^1-map

$$I : U \ni \psi \mapsto F(\tau(\psi),\psi) \in C$$

is called an *intersection map*. Obviously, $I(U) \subset F(t,\phi) + Z$. The derivatives of I satisfy

$$DI(\psi) = p_{\xi} \circ D_2F(\tau(\psi),\psi), \quad \psi \in U,$$

with the projection $p_{\xi} : C \to C$ along $\mathbb{R}\xi,\ \xi = D_1F(\tau(\psi),\psi)1$, onto $Z = T_{I(\psi)}(F(t,\phi) + Z)$; for each linear continuous functional $\phi^* : C \to \mathbb{R}$ with $Z = (\phi^*)^{-1}(0)$ and for all $\chi \in C$,

$$p_{\xi}\chi = \chi - \frac{\phi^*(\chi)}{\phi^*(\xi)}\xi.$$

The linear operators $T(t) = D_2F(t,0),\ t \geq 0$, which are given by the solutions of the linearization

$$x'(t) = -\mu x(t) - \alpha x(t-1), \tag{3}$$

$\alpha = -f'(0) > 0$, of equation (1.1.1) at the zero solution $\mathbb{R} \ni t \mapsto 0 \in \mathbb{R}$, form a strongly continuous semigroup. The spectrum σ of its generator is discrete and consists of eigenvalues; it is given by the characteristic equation

$$\lambda + \mu + \alpha e^{-\lambda} = 0.$$

There exists a *leading pair* $\Lambda_0 = \{\lambda_0, \widetilde{\lambda}_0\} \subset \sigma$, i.e.,

$$\mathrm{Re}(\widetilde{\lambda}_0) \leq \mathrm{Re}(\lambda_0) \quad \text{and} \quad \mathrm{Re}(\widetilde{\lambda}_0) > \max\{\mathrm{Re}(\lambda) : \lambda \in \sigma,\ \lambda_0 \neq \lambda \neq \widetilde{\lambda}_0\}.$$

Furthermore,

$$\Lambda_0 \subset \mathbb{R} + i(-\pi, \pi) \quad \text{and} \quad \sigma \setminus \Lambda_0 \subset \{\lambda \in \mathbb{C} : 2\pi < |\,\mathrm{Im}(\lambda)|\}.$$

The reellified generalized eigenspace $L \subset C$ given by the spectral set Λ_0 has dimension 2. Let Q denote the reellified generalized eigenspace given by the complementary spectral set $\sigma \setminus \Lambda_0$, and let $p : C \to C$ denote the projection along Q onto L; set $q = \mathrm{id} - p$.

Depending on α and μ, either both λ_0 and $\widetilde{\lambda}_0$ are real and $\lambda_0 < 0$, or $0 < \mathrm{Im}(\lambda_0)$ and $\widetilde{\lambda}_0 = \overline{\lambda}_0$. In particular,

$$0 < \mathrm{Im}(\lambda_0) \quad \text{and} \quad \widetilde{\lambda}_0 = \overline{\lambda}_0 \quad \text{in case } 0 \leq \mathrm{Re}(\lambda_0).$$

Set $u_0 = \mathrm{Re}(\lambda_0)$, $v_0 = \mathrm{Im}(\lambda_0)$.

The subsequent properties of curves in L are needed later.

PROPOSITION 1. *Let $g : [a,b] \to L$ be a simple closed C^1-curve. Suppose $\gamma = g'(a)$ and $\chi \in L$ are linearly independent. Then there exist $\varepsilon > 0$, $k \in \{0,1\}$, $q \in (0,1)$ so that*

$$g(a) + (0,\varepsilon)(-1)^k\chi \subset \mathrm{int}(g), \quad g(a) + (0,\varepsilon)(-1)^{k+1}\chi \subset \mathrm{ext}(g),$$
$$\emptyset = |g| \cap \{g(a) + x\gamma + y\chi : |x| < q|y|,\ -\varepsilon < y < 0\}.$$

If $h : (c,d) \to L$ is a C^1-curve with $c < 0 < d$, $h(0) = g(a)$, $h'(0) = (-1)^k\chi$ then there exists $\varepsilon' \in (0,\varepsilon)$ with $h((0,\varepsilon')) \subset \mathrm{int}(g)$, $h((-\varepsilon',0)) \subset \mathrm{ext}(g)$.

PROPOSITION 2. *Let $g : [a,b] \to L$ be a simple closed C^1-curve. Suppose $\gamma = g'(a)$ and $\chi \in L$ are linearly independent. Let $h : (c,d) \to L$ be a continuous curve with $c < 0 < d$, $h(0) = g(a)$. Assume that the restrictions $h|(c,0), h|(0,d)$ are C^1-smooth, $h((c,0)) \subset \mathrm{ext}(g)$, $h((0,d)) \subset \mathrm{int}(g)$,*

$$h'(s) \to \chi \quad \text{as } 0 > s \to 0 \quad \text{and} \quad h'(s) \to (-1)^j\chi \quad \text{as } 0 < s \to 0$$

for some $j \in \{0,1\}$. Then $j = 0$, and h is C^1-smooth.

Proof. Choose ε and q according to Proposition 1, and choose $\overline{\varepsilon} \in (0,1)$ with $\overline{\varepsilon}/(1-\overline{\varepsilon}) < q$. Assume $j = 1$. The equations

$$h(s) - g(a) = t(s)\gamma + u(s)\chi \quad \text{for } c < s < 0,$$
$$h(s) - g(a) = v(s)\gamma + \omega(s)\chi \quad \text{for } 0 < s < d,$$

define real C^1-functions t, u on $(c, 0)$ and v, w on $(0, d)$ which satisfy

$$t(s) \to 0,\ u(s) \to 0,\ t'(s) \to 0,\ u'(s) \to 1 \qquad \text{as } 0 > s \to 0,$$
$$v(s) \to 0,\ \omega(s) \to 0,\ v'(s) \to 0,\ \omega'(s) \to -1 \quad \text{as } 0 < s \to 0.$$

It follows that there exists $\delta \in (0, \min\{d, -c\})$ so that

$$|u(s)| < \varepsilon, \quad |u(s) - s| \le \overline{\varepsilon}|s|, \quad |t(s)| \le \overline{\varepsilon}|s| \quad \text{for } -\delta < s < 0,$$
$$|w(s)| < \varepsilon, \quad |w(s) + s| \le \overline{\varepsilon}|s|, \quad |v(s)| \le \overline{\varepsilon}|s| \quad \text{for } 0 < s < \delta.$$

Hence

$$|t(s)| \le \frac{\overline{\varepsilon}}{1 - \overline{\varepsilon}}|u(s)| \quad \text{and} \quad -\varepsilon < u(s) < 0 \quad \text{for } -\delta < s < 0,$$
$$|v(s)| \le \frac{\overline{\varepsilon}}{1 - \overline{\varepsilon}}|w(s)| \quad \text{and} \quad -\varepsilon < \omega(s) < 0 \quad \text{for } 0 < s < \delta.$$

The convex set $\{g(a) + x\gamma + y\chi : |x| < q|y|,\ -\varepsilon < y < 0\}$ is disjoint from $|g|$ and contains points of $\mathrm{ext}(g)$ as well as points of $\mathrm{int}(g)$, which implies a contradiction. ∎

2.2. Slowly oscillating solutions. A function $x : D \to M,\ D \subset \mathbb{R}$, is called *slowly oscillating* if $|z - z'| > 1$ for every pair of zeros $z \ne z'$. It is called *eventually slowly oscillating* if there exists $t \in \mathbb{R}$ so that $[t, \infty) \subset D$ and $x|[t, \infty)$ is slowly oscillating. The main result in [12] implies that the set

$$E = \{\phi \in C : x^\phi \text{ eventually slowly oscillating}\}$$

is open and dense in C.

Segments x_t of slowly oscillating solutions belong to the set

$$S = \{\phi \in C \setminus \{0\} : \text{there exist } n \in \{0, 1\} \text{ and } z \in [-1, 0] \text{ so that } (-1)^n\phi(t) \le 0 \text{ on } [-1, z], 0 \le (-1)^n\phi(t) \text{ on } [z, 0]\}$$

of data with at most one sign change, which satisfies

$$\bar{S} = S \cup \{0\}, \quad \mathbb{R}\bar{S} = \bar{S}, \quad L \subset \bar{S}, \quad S \cap Q = \emptyset. \tag{1}$$

A useful observation is that the scaled differences $t \mapsto e^{\mu t}(x(t) - y(t))$ of solutions x, y of equation (1.1.1) which are defined on some interval $[t_0 - 1, \infty),\ t_0 \in \mathbb{R}$, solve equation (2.1.1) with $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ given by

$$g(t, \delta) = e^{\mu t}(f(e^{-\mu(t-1)}\delta + y(t-1)) - f(y(t-1))) = e^{\mu t} \int_{y(t-1)}^{e^{-\mu(t-1)}\delta + y(t-1)} f'(\xi)\, d\xi$$

so that the negative feedback condition

$$\delta g(t, \delta) < 0 \tag{2}$$

for $t \ge t_0,\ 0 \ne \delta \in \mathbb{R}$, is satisfied. If $\phi \in C$ and $v : [-1, \infty) \to \mathbb{R}$ is a solution of the variational equation along x^ϕ then the scaled function $t \to e^{\mu t}v(t)$ is a solution of equation (2.1.1) with

$$g(t, \delta) = e^\mu f'(x^\phi(t-1))\delta \quad \text{for } t \ge 0,\ \delta \in \mathbb{R},$$

and (2) holds for $t \ge 0,\ 0 \ne \delta \in \mathbb{R}$.

Proposition 1. *Let* $t_0 \in \mathbb{R}$, $t_1 > t_0$. *Suppose* $g : [t_0, t_1) \times \mathbb{R} \to \mathbb{R}$ *is continuous*, (2) *holds for* $t_0 \le t < t_1$ *and* $0 \ne \delta \in \mathbb{R}$, *and* $x : [t_0 - 1, t_1) \to \mathbb{R}$ *is a solution of equation* (2.1.1) *with* $x_{t_0} \in S$. *Then* $x_t \in S$ *for all* $t \in [t_0, t_1)$. *In case* $t_0 + 4 \le t_1$ *there exists* $t \in [t_0, t_0 + 4)$ *so that* x_t *has no zero, and* $x|[t-1, t_1)$ *is slowly oscillating.*

Proof. Compare the proofs of Remark 6.1 and Proposition 6.1 in [17]. ∎

Corollary 1. (i) *Let* $t_0 \in \mathbb{R}$. *For every pair* $x : [t_0 - 1, \infty) \to \mathbb{R}$, $y : [t_0 - 1, \infty) \to \mathbb{R}$ *of solutions of equation* (1.1.1) *with* $x_{t_0} - y_{t_0} \in S$,

$$x_t - y_t \in S \quad \textit{for all } t \ge t_0$$

and there exists $t \in [t_0, t_0 + 4]$ *so that* $x_t - y_t$ *has no zero. The restriction* $(x-y)|[t-1, \infty)$ *is slowly oscillating.*

(ii) *Let* $\phi \in C$. *If* $v : [-1, \infty) \to \mathbb{R}$ *is a solution of the variational equation along* x^ϕ *with* $v_0 \in S$ *then* $v_t \in S$ *for all* $t \ge 0$, *and there exists* $t \in [0, 4]$ *so that* v_t *has no zero*; *the restriction* $v|[t-1, \infty)$ *is slowly oscillating.*

Proof. Compare the proofs of Remark 6.1 and Proposition 6.1 in [17]. ∎

The convex cone

$$K = \{\phi \in C : \phi(-1) = 0,\ t \mapsto e^{\mu t}\phi(t) \text{ increasing},\ 0 < \phi(0)\}$$

in the closed hyperplane

$$H = \{\phi \in C : \phi(-1) = 0\} = \mathrm{ev}^{-1}(0) \quad \text{for } \mathrm{ev} : C \ni \phi \mapsto \phi(-1) \in \mathbb{R}$$

satisfies $K \subset S, \overline{K} = K \cup \{0\}$.

Proposition 2. (i) *Suppose* $\phi \in C$ *has no zero*, $z > 0$, $x^\phi(t) \ne 0$ *for* $0 < t < z$ *and* $x^\phi(z) = 0$. *Then* $x^\phi_{z+1} \in K \cup (-K)$.

(ii) *Suppose* $\phi \in K \cup (-K)$, *or* ϕ *has no zero. Either* $x^\phi(t) \ne 0$ *for all* $t > 0$, $|x^\phi|$ *is decreasing on* $(0, \infty)$, *and* $x^\phi(t) \to 0$ *as* $t \to \infty$, *or the zeros of* x^ϕ *in* $(0, \infty)$ *are all simple and form a strictly increasing sequence of points* $z_n(\phi)$, $n \in I(\phi) = \mathbb{N} \cap [1, n(\phi))$ *with* $n(\phi) \in \mathbb{N} \cup \{\infty\}$. *In the last case,*

$$F(z_n(\phi) + 1, \phi) \in (-1)^n K \quad \textit{for all } n \in I(\phi) \textit{ if } \phi \in K,$$
$$F(z_n(\phi) + 1,\ \phi) \in (-1)^{n+1} K \quad \textit{for all } n \in I(\phi) \textit{ if } \phi \in -K.$$

(iii) *Let* $\phi \in K \cup (-K)$. *Suppose* x^ϕ *has positive zeros. In case* $\phi \in K$, $(x^\phi)'(t) < 0$ *on* $(0, z_1(\phi))$ *and* $0 > x^\phi(t) \ge \min_{[0, \|\phi\|]} f$ *for* $z_1(\phi) < t < z_1(\phi) + 1$. *In case* $\phi \in -K$, $(x^\phi)'(t) > 0$ *on* $(0, z_1(\phi)$ *and* $0 < x^\phi(t) \le \max_{[-\|\phi\|, 0]} f$ *for* $z_1(\phi) < t < z_1(\phi) + 1$.

(iv) *Suppose* $\phi \in K \cup (-K)$, x^ϕ *has zeros in* $(0, \infty)$, *and* $n(\phi) > 2$. *Then there exists a neighbourhood* N *of* ϕ *in* C *so that each* x^ψ, $\psi \in N \cap (K \cup (-K))$, *has zeros in* $(0, \infty)$ *with* $n(\psi) > 2$, *and the map*

$$N \cap (K \cup (-K)) \ni \psi \mapsto z_2(\psi) \in \mathbb{R}$$

is continuous.

(v) *In case* $u_0 \ge 0$, *the zeros of each* x^ϕ, $\phi \in K \cup (-K)$, *in* $(0, \infty)$ *are unbounded, and for every bounded subset* $B \subset K \cup (-K)$ *there exists* $b > 0$ *with*

$$z_2(\phi) < b \quad \textit{for all } \phi \in B.$$

P r o o f. For (i) and (ii), compare the proof of Proposition 3.2 in [20]. Proof of (iii) in case $\phi \in K$ and x^ϕ has positive zeros: For $0 < t < z_1(\phi)$, $(x^\phi)'(t) = -\mu x^\phi(t) + f(x^\phi(t-1)) \le -\mu x^\phi(t) < 0$, and for $z_1(\phi) < t < z_1(\phi) + 1$,

$$(x^\phi)'(t) > f(x^\phi(t-1)) \ge \min_{[-1,z_1(\phi)]} f(x^\phi(s)) = \min_{[0,\|\phi\|]} f.$$

For (iv), compare part 1 of the proof of Proposition 9.2 in [17]. For (v), compare the proof of Proposition 6.3 in [17]. ■

Consider the return map $P : K \cup \{0\} \cup (-K) \to K \cup \{0\} \cup (-K)$ given by

$$P(\phi) = \begin{cases} F(z_2(\phi)+1, \phi) & \text{in case } \phi \neq 0, \text{ and } x^\phi \text{ has positive zeros with } n(\phi) > 2, \\ 0 & \text{otherwise.} \end{cases}$$

The domain of P is closed.

PROPOSITION 3. (i) *P is continuous, and $\overline{P(K \cup \{0\} \cup (-K))}$ is compact.*

(ii) *Suppose $J \subset K \cup \{0\} \cup (-K)$ is a subset so that for every $\phi \in J \setminus \{0\}$,*

$$0 \neq \phi(t) \quad \textit{for } -1 < t \le 0,$$

and x^ϕ has positive zeros with $n(\phi) > 2$. Then the restriction $P|J$ is injective.

P r o o f. Assertion (i) follows by arguments as in the proof of Proposition 3.4 in [20]. Let ϕ, ψ in J be given with $P(\phi) = P(\psi)$. If $\phi = 0$ then the equation $0 = P(\phi) = P(\psi)$, the properties of J and the definition of P altogether yield $\psi = 0$. In case $0 \neq \phi \in K$ one has $K \ni P(\phi) = P(\psi)$. It follows that $\psi \in K$, and $F(z_2(\phi)+1, \phi) = F(z_2(\psi)+1, \psi)$.

Proof of $z_2(\phi) = z_2(\psi)$: In case $z_2(\psi) < z_2(\phi)$ the injectivity of the maps $F(t, \cdot)$, $t \ge 0$, gives $F(z_2(\phi) - z_2(\psi), \phi) = \psi \in K$. On the other hand, the hypothesis on J and Proposition 2(ii) yield $F(t, \phi) \in C \setminus K$ for all $t \in (0, z_2(\phi)+1)$, a contradiction.

The injectivity of $F(z_2(\phi)+1, \cdot) = F(z_2(\psi)+1, \cdot)$ implies $\phi = \psi$. The proof in case $0 \neq \phi \in -K$ is analogous. ■

The next results concern slowly oscillating solutions on $\mathbb{R}$.

PROPOSITION 3. *If $x : \mathbb{R} \to \mathbb{R}$ is a slowly oscillating solution of equation* (1.1.1) *which is bounded on $(-\infty, 0]$ then $\inf x^{-1}(0) = -\infty$. The zeros of x are all simple and form a strictly increasing sequence $(z_n(x))_{n \in I(x)}$, $I(x) = \{n \in \mathbb{Z} : n < n(x)\}$ with $n(x) \in \mathbb{Z} \cup \{\infty\}$, so that $x_{z_n(x)+1} \in K \cup (-K)$ for all $n \in I(x)$.*

P r o o f. See Proposition 3.1(ii) in [20], and use Proposition 2. ■

PROPOSITION 4. (i) *For every slowly oscillating solution $x : \mathbb{R} \to \mathbb{R}$ of equation* (1.1.1) *which is bounded on $(-\infty, 0]$ there exists a strictly increasing sequence $(t_n)_{n=-\infty}^0$ in $\mathbb{R}$ with $t_n \to -\infty$ as $n \to -\infty$ so that $(x_{t_n})_{n=-\infty}^0$ is a trajectory of P in K.*

(ii) *For every bounded trajectory $(\phi_n)_{n=-\infty}^0$ of P in $K \cup (-K)$ there exist a bounded slowly oscillating solution $x : \mathbb{R} \to \mathbb{R}$ of eq.* (1.1.1) *and a strictly increasing sequence $(t_n)_{n=-\infty}^0$ in $\mathbb{R}$ with $t_n \to -\infty$ as $n \to -\infty$ so that $\phi_n = x_{t_n}$ for all integers $n \le 0$.*

P r o o f. To prove (i) observe that for integers n with $n+1 < n(x)$ and $j \in \{0,1\}$ with $x_{z_n(x)+1} \in (-1)^j K$, the simplicity of $z_{n+1}(x) > z_n(x) + 1$ yields $x_{z_{n+1}(x)+1} \in (-1)^{j+1} K$. Fix some integer $n < n(x)$ with $x_{z_n(x)+1} \in K$, and set $t_j = z_{n+2j}(x) + 1$, for integers $j \le 0$.

Proof of (ii): The hypothesis that $(\phi_n)_{n=-\infty}^0$ is a trajectory of P in $K\cup(-K)$ implies that for every integer $j<0$, x^{ϕ_j} has positive zeros, $n(\phi_j)>2$, and $\phi_{j+1}=P(\phi_j)=F(z_2(\phi_j)+1,\phi_j)$. Consider the sequence in $\mathbb{R}$ given by $t_0=0$ and $t_j=t_{j+1}-(z_2(\phi_j)+1)$ for integers $j<0$. The equations $x(t)=x^{\phi_j}(t-t_j)$ for $t_j<t\le t_{j+1}$, $j\in-\mathbb{N}$, determine a solution $x:\mathbb{R}\to\mathbb{R}$ of equation (1.1.1) with $x_{t_j}=\phi_j\in K\cup(-K)$ for all integers $j\le 0$, and Proposition 2(ii) guarantees that x is slowly oscillating. It remains to show that x is bounded. Observe that parts (iii) and (i) of Proposition 2 imply that in case $\inf f>-\infty$ each x^{ϕ}, $\phi\in K\cup(-K)$, with positive zeros and $2<n(\phi)$ is bounded on $[z_2(\phi),\infty)$ by $c=\max_{[\inf f,0]}f-\inf f$. For every $t\in\mathbb{R}$ there exists an integer $j\le 0$ so that $t_j\le t$, $x(t)=x^{\phi_{j-1}}(t-t_{j-1})$, $t-t_{j-1}\ge t_j-t_{j-1}=z_2(\phi_{j-1})+1$, and $\phi_{j-1}\in K\cup(-K)$. Therefore $|x(t)|\le c$. The proof in case $\sup f<\infty$ is analogous. ∎

Proposition 4(ii) shows in particular that nonzero fixed points ϕ of P define slowly oscillating periodic solutions of equation (1.1.1) with period $\omega=z_2(\phi)+1$.

The next result generalizes the fact that phase curves of bounded slowly oscillating solutions enter and remain in a cone containing the linear space L, which consists of segments of slowly oscillating solutions of the linear equation (2.1.3), and 0.

Proposition 5. *Let $r>0$. There exists $c(r)>0$ with the following property. If $x:[t_0-1,\infty)\to\mathbb{R}$, $y:[t_0-1,\infty)\to\mathbb{R}$ are solutions of equation* (1.1.1) *so that x_0-y_0 has no zero and $|x(t)|\le r$, $|y(t)|\le r$ on $[t_0-1,\infty)$, then*

$$c(r)\|x_t-y_t\|\le\|p(x_t-y_t)\|\quad \text{for all } t\ge t_0+2.$$

Proof. See the proof of Proposition 7.1 in [17], and correct the argument in case B2 according to the arguments in case II, subcase 2, in the proof of Lemma 5 in [16], or see the arguments in case (D)(iv) in the proof of Proposition XV.4.2 in [4] with $|x(t-1)|>|x(t)|$ instead of $\ldots=\ldots$ in line 3, page 396, and "all" instead of "some" in line 7, page 396. ∎

2.3. The attractor of eventually slowly oscillating solutions. An *attractor* M_∞ of a continuous semiflow $\Phi:[0,\infty)\times M\to M$ on a complete metric space M is a compact set $M_\infty\subset M$ which is invariant in the sense that

$$\Phi(t,M_\infty)=M_\infty\quad\text{for all } t\ge 0$$

and which attracts all bounded sets in the sense that for every bounded set $B\subset M$ and for every neighbourhood N of M_∞ there exists $t_{BN}\ge 0$ with

$$\Phi(t,B)\subset N\quad\text{for all } t\ge t_{BN}.$$

This definition is equivalent to the definition of compact global attractors used in [5]; see Chapter XVI in [4]. In case all maps $\Phi(t,\cdot):M\to M$, $t\ge 0$, are injective it is also equivalent to the definition given in Chapter 4 of [20]. Attractors contain all ω-limit sets, in particular, all stationary points and periodic orbits. In Chapter 4 of [20] it is shown that the restricted semiflow

$$F_S:[0,\infty)\times\overline{S}\ni(t,\phi)\mapsto F(t,\phi)\in\overline{S}$$

has an attractor which is denoted by A.

PROPOSITION 1. (i) *$\phi \in A$ if and only if either $\phi = 0$, or there exist a bounded slowly oscillating solution $x : \mathbb{R} \to \mathbb{R}$ of equation* (1.1.1) *and $t \in \mathbb{R}$ with $\phi = x_t$.*

(ii) *If $x : \mathbb{R} \to \mathbb{R}$ is a solution of equation* (1.1.1) *so that $x|(-\infty, 0]$ is bounded and if there is a sequence $(t_n)_{n=-\infty}^{0}$ with $t_n \to -\infty$ as $n \to -\infty$ and $x_{t_n} \in S$ for all integers $n \le 0$ then $x_t \in A$ for all $t \in \mathbb{R}$.*

(iii) *$A \cap H = A \cap (K \cup \{0\} \cup (-K))$.*

(iv) *$P(A \cap K) = A \cap K, P(A \cap (-K)) = A \cap (-K), P(A \cap H) = A \cap H$.*

(v) *Suppose that for every $\phi \in A \cap (K \cup \{0\} \cup (-K))$ the solution x^{ϕ} has positive zeros, and $n(\phi) \ge 2$. Then the map $A \cap H \ni \phi \mapsto P(\phi) \in A \cap H$ is bijective.*

Proof. 1. *Proof of* (i): Proposition 4.1 of [20] yields that 0 and all segments x_t of bounded slowly oscillating solutions $x : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) belong to A. Conversely, if $\phi \in A$ then the same proposition shows that there are a bounded solution $x : \mathbb{R} \to \mathbb{R}$ with $x_s \in \overline{S}$ for all $s \in \mathbb{R}$, and $t \in \mathbb{R}$ with $\phi = x_t$. In case $\phi \ne 0$, the injectivity of the maps $F(t, \cdot)$, $t \ge 0$, implies $x_s \in \overline{S} \setminus \{0\} = S$ for all $s \in \mathbb{R}$, and Proposition 2.2.2(i), (ii) guarantees that x is slowly oscillating.

2. *Proof of* (ii): Suppose $\inf f > -\infty$. Corollary 2.2.1 yields that x is slowly oscillating. Apply Proposition 2.2.3. In case $n(x) < \infty$, Proposition 2.2.2(ii) implies $x(t) \to 0$ as $t \to \infty$. In case $n(x) = \infty$, Proposition 2.2.2(iii) gives

$$0 \ge x(t) > \inf f \quad \text{for } z_n(x) < t < z_{n+1}(x),\ n \in \mathbb{Z},\ x'(z_n(x)) < 0,$$

and consequently

$$0 \le x(t) < \max_{[\inf f, 0]} f \quad \text{for } z_n(x) < t < z_{n+1}(x),\ n \in \mathbb{Z},\ 0 < x'(z_n(x)).$$

It follows that x is bounded. Apply assertion (i). The proof in case $\sup f < \infty$ is analogous.

3. *Proof of* (iii): For $0 \ne \phi \in A \cap H$, consider $t \in \mathbb{R}$ and a solution x according to assertion (i), and apply Proposition 2.2.2(i) to x_s with $s < t - 1$ so that x has no zero on $[s - 1, t - 1)$. It follows that $\phi \in K \cup (-K)$.

4. *Proof of* $A \cap K \subset P(A \cap K)$: Let $\psi \in A \cap K$. Assertion (i) and Proposition 2.2.3 show that there exist $t \in \mathbb{R}$ and a bounded slowly oscillating solution $x : \mathbb{R} \to \mathbb{R}$ and an integer $n < n(x)$ with $\psi = x_t$ and $t - 1 = z_n(x)$. It follows that $\phi = x_{z_{n-2}(x)+1}$ belongs to $A \cap K$, $z_2(\phi) + 1 = z_n(x) - z_{n-2}(x)$, and $P(\phi) = F(z_2(\phi) + 1, \phi) = F(z_n(x) - z_{n-2}(x), x_{z_{n-2}(x)+1}) = \psi$.

5. *Proof of* (v): Part (i) and Proposition 2.2.3 combined guarantee that for $\phi \in A \cap (K \cup (-K))$ and for $-1 < t \le 0$, $\phi(t) \ne 0$. Apply Proposition 2.2.3(ii). ∎

For $\phi \in A$ let $x(\phi)$ denote the uniquely determined solution of equation (1.1.1) which is defined on $\mathbb{R}$ and satisfies $x(\phi)_0 = \phi$.

PROPOSITION 2. *The map $F_A : \mathbb{R} \times A \ni (t, \phi) \mapsto x(\phi)_t \in A$ is a continuous flow.*

Proof. See Proposition 4.3 in [20]. ∎

PROPOSITION 3. *For every $\phi \in A$ the curve $\mathbb{R} \ni t \mapsto x(\phi)_t \in C$ is C^1-smooth, with $x(\phi)'_t = D(s \mapsto x(\phi)_s)(t)1$ for all $t \in \mathbb{R}$. The map $A \ni \phi \mapsto \phi' \in C$ is continuous, and for all $\phi \in A \setminus \{0\}$, $\phi' \ne 0$. The map $\mathbb{R} \times A \ni (t, \phi) \mapsto x(\phi)'_t \in C$ is continuous.*

Proof. The equations $x(\phi)_{t+h} - x(\phi)_t = F(2+h, x(\phi)_{t-2}) - F(2, x(\phi)_{t-2})$ for $t \in \mathbb{R}$ and $|h| < 1$ yield the differentiability of the curve, and the formula

$$D(s \mapsto x(\phi)_s)(t)1 = D_1 F(2, x(\phi)_{t-2})1 = F(2, x(\phi)_{t-2})' = (x(\phi)_t)'.$$

Continuity of the derivative of the curve follows from

$$x(\phi)_t' = -\mu x(\phi)_t + f \circ x(\phi)_{t-1} = -\mu F_A(t,\phi) + f \circ (F_A(t-1,\phi)).$$

The last equation shows in case $t = 0$ that the map $A \ni \phi \mapsto \phi' \in C$ is continuous. $\phi' = 0$ implies $0 = -\mu\phi + f \circ x(\phi)_{-1}$, hence $\mu\phi(0) = f(x(\phi)(-1)) = f(\phi(-1)) = f(\phi(0))$, and therefore $\phi(0) = 0$, and $\phi = 0$. ∎

In Chapter 7 of [20] it is shown that

$$A - A \subset \overline{S}. \tag{1}$$

This inclusion and the relations $S \cap Q = \emptyset$, or equivalently, $0 \notin pS$, are used in [20] to obtain a map $a : pA \to Q$ so that

$$A = \{\chi + a(\chi) : \chi \in pA\}.$$

An a-priori estimate as in Proposition 2.2.5 yields that the map a is Lipschitz continuous. The next result is a first indication that a is even better.

Proposition 4. *For every pair of differentiable curves* $\gamma : (-1,1) \to C$, $\varrho : (-1,1) \to C$ *with* $\gamma(0) = \varrho(0)$ *and* $\gamma([0,1)) \cup |\varrho| \subset A$,

$$\mathbb{R}\gamma'(0) + \mathbb{R}\varrho'(0) \subset \overline{S}.$$

In particular, $T_\phi A + T_\phi A \subset \overline{S}$.

Proof. Let $r, s \in \mathbb{R}$. Then

$$r\gamma'(0) = \lim_{h\to 0} \frac{r}{h}(\gamma(h) - \gamma(0)), \quad -s\varrho'(0) = \lim_{h\to 0} \frac{-s}{h}(\varrho(h) - \varrho(0)).$$

In case $r > 0 \neq s$,

$$r\gamma'(0) + s\varrho'(0) = \lim_{0<h\to 0} \frac{1}{h}[(\gamma(rh) - \gamma(0)) - (\varrho(-sh) - \varrho(0))] = \lim_{0<h\to 0} \frac{1}{h}(\gamma(rh) - \varrho(-sh)).$$

Recall (1) and $\mathbb{R}\overline{S} \subset \overline{S}$. It follows that $r\gamma'(0) + s\varrho'(0) \in \overline{S}$. In case $r < 0 \neq s$, use

$$r\gamma'(0) + s\varrho'(0) = \lim_{0>h\to 0} \frac{1}{h}[(\gamma(rh) - \gamma(0)) - (\varrho(-sh) - \varrho(0))].$$

The proof in case $r = 0$ or $s = 0$ is similar and simpler. ∎

Note that Propositions 4 and 3 combined yield

$$\phi' \in S \quad \text{and} \quad 0 \neq p\phi' \quad \text{for every } \phi \in A \setminus \{0\}. \tag{2}$$

In case the attractor A is nontrivial, i.e., $A \neq \{0\}$, there are periodic orbits in A. For a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) with zero sequence $(z_n(y))_{n=-\infty}^{\infty}$, the minimal period $\omega = \omega(y) > 0$ is given by $z_{n+2}(y) - z_n(y)$, and the orbit parametrization $\eta : [0,\omega] \ni t \mapsto y_t \in C$ is a simple closed C^1-curve with $|\eta| \subset A$. The projected parametrization $p \circ \eta$ is a simple closed curve, with $0 \in \mathrm{int}(p \circ \eta)$; for any other slowly oscillating periodic solution $\widetilde{y} : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) with orbit parametrization $\widetilde{\eta}$, either

$$\widetilde{y} = y(t + \cdot) \quad \text{for some } t \in \mathbb{R},$$

or

$$|\eta| \cap |\widetilde{\eta}| = \emptyset, \quad \text{and} \quad |p \circ \eta| \subset \operatorname{int}(p \circ \widetilde{\eta}) \quad \text{or} \quad |p \circ \widetilde{\eta}| \subset \operatorname{int}(p \circ \eta).$$

In case $A \neq \{0\}$ there exists a slowly oscillating periodic solution y^b of equation (1.1.1), with $y^b(-1) = 0 < (y^b)'(-1)$, so that

$$pA = \operatorname{int}(p \circ \eta^b) \cup |p \circ \eta^b|,$$

with the orbit parametrization η^b of y^b. For every $\phi \in A \setminus \{0\}$ so that $x(\phi)$ is not periodic,

$$\alpha(x(\phi)) \cap \omega(\phi) = \emptyset,$$

and each limit set is either the singleton $\{0\}$ or the orbit $|\eta|$ of a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ of eq. (1.1.1). For proofs, see [20].

In the present paper we prove the following result on smoothness.

THEOREM 1. *In case $A \neq \{0\}$ the restriction $a|\operatorname{int}(p \circ \eta^b)$ is C^1-smooth, and for every $\chi \in \partial pA = |p \circ \eta^b|$ there exist an open neighbourhood N of χ in L and a C^1-map $a_N : N \to Q$ with*

$$a|pA \cap N = a_N|pA \cap N.$$

Assume from now on $A \neq \{0\}$. The next propositions on periodic orbits in A are used in parts of the proof of Theorem 1.

PROPOSITION 5. *Let $y : \mathbb{R} \to \mathbb{R}$ be a slowly oscillating periodic solution with orbit parametrization η. Let $\phi \in A$. If $p\phi \in \operatorname{int}(p \circ \eta)$ $[\ldots \in \operatorname{ext}(p \circ \eta)]$ then $pF_A(t, \phi) \in \operatorname{int}(p \circ \eta)$ $[\ldots \in \operatorname{ext}(p \circ \eta)]$ for all $t \in \mathbb{R}$.*

Proof. Let $\phi \in A$, $p\phi \in \operatorname{int}(p \circ \eta)$. Suppose $pF_A(t, \phi) \in L \setminus \operatorname{int}(p \circ \eta)$ for some $t \in \mathbb{R}$. Then there exists $s \in \mathbb{R}$ with $pF_A(s, \phi) \in \partial(\operatorname{int}(p \circ \eta)) = |p \circ \eta|$, and $pF_A(s, \phi) = py_t$ for some $t \in \mathbb{R}$. Hence $F_A(s, \phi) = pF_A(s, \phi) + a(pF_A(s, \phi)) = py_t + a(py_t) = y_t$, and consequently $\phi = F_A(-s, y_t) = y_{t-s}$, or $p\phi \in |p \circ \eta|$, which yields a contradiction to $p\phi \in \operatorname{int}(p \circ \eta)$. ∎

Incidentally, note that for every slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ and for all $n \in \mathbb{Z}$,

$$H \cap |\eta| = \{y_{z_n(y)+1}, y_{z_{n+1}(y)+1}\} = (K \cup (-K)) \cap |\eta| \tag{3}$$

and

$$y_{z_n(y)+1} \in K \quad \text{if and only if} \quad y_{z_{n+1}(y)+1} \in -K. \tag{4}$$

PROPOSITION 6. *Suppose*

$$0 < I = \inf\{\|py_t\| : y \text{ is a slowly oscillating periodic solution of equation (1.1.1)},\ t \in \mathbb{R}\}.$$

Then there exist a slowly oscillating periodic solution y of equation (1.1.1) and $t \in \mathbb{R}$ so that $I = \|py_t\|$. Let η denote the orbit parametrization of y. For every $\phi \in A \setminus \{0\}$ with

$p\phi \in \mathrm{int}(p \circ \eta)$, *either*

$$\alpha(x(\phi)) = \{0\} \quad and \quad \omega(\phi) = |\eta|$$

or

$$\alpha(x(\phi)) = |\eta| \quad and \quad \omega(\phi) = \{0\}.$$

Proof. 1. There is a sequence of slowly oscillating periodic solutions $y^{(n)}$, $n \in \mathbb{N}$, with minimal periods ω_n and $y_0^{(n)} \in K$ for all integers n, and there is a sequence of reals $t_n \in [0, \omega_n)$, $n \in \mathbb{N}$, so that

$$\|py_{t_n}^{(n)}\| \to I \quad \text{as } n \to \infty.$$

All $y_0^{(n)}$ belong to the compact set $A \cap \overline{K}$, and a subsequence $(y_0^{(n_j)})_{j=1}^{\infty}$ converges to some $\phi \in A \cap \overline{K}$, with $\|p\phi\| = \lim_{j\to\infty} \|py_0^{(n_j)}\| \geq I > 0$. It follows that $\phi \in \overline{K} \setminus \{0\} = K$, and the continuity of P gives

$$P(\phi) = P(\lim_{j\to\infty} y_0^{(n_j)}) = \lim_{j\to\infty} P(y_0^{(n_j)}) = \lim_{j\to\infty} y_0^{(n_j)} = \phi,$$

so that $y = x(\phi)$ is a slowly oscillating periodic solution, with minimal period

$$\omega = z_2(\phi) + 1 = z_2(\lim_{j\to\infty} y_0^{(n_j)}) + 1 = \lim_{j\to\infty} z_2(y_0^{(n_j)}) + 1,$$

according to Proposition 2.2.2(iv). Hence $\omega_{n_j} \to \omega$ as $j \to \infty$, and the sequence $(t_{n_j})_{j=1}^{\infty}$ is bounded. A subsequence of points $s_k = t_{n_{j_k}}$, $k \in \mathbb{N}$, converges to some $t \in [0, \omega + 1]$. Using continuous dependence on initial data on the interval $[0, \omega + 1]$ and the equations

$$y_{s_k}^{(n_{j_k})} - y_t = (y_{s_k}^{(n_{j_k})} - y_{s_k}) + (y_{s_k} - y_t), \quad k \in \mathbb{N},$$

one finds

$$y_{s_k}^{(n_{j_k})} \to y_t \quad \text{as } k \to \infty, \quad py_{s_k}^{(n_{j_k})} \to py_t \quad \text{as } k \to \infty,$$

and consequently, $I = \|py_t\|$.

2. Let η denote the orbit parametrization of y. Let $\phi \in A \setminus \{0\}$, $p\phi \in \mathrm{int}(p \circ \eta)$.

Claim: $x(\phi)$ is not periodic.

Proof: Suppose the slowly oscillating solution $x(\phi)$ is periodic. Let ξ denote its orbit parametrization. Then $|p \circ \xi| \subset \mathrm{int}(p \circ \eta)$, according to Proposition 5. It follows that $\mathrm{ext}(p \circ \eta) \subset \mathrm{ext}(p \circ \xi)$, and

$$|p \circ \xi| \cap \{py_t\} = \emptyset.$$

Moreover,

$$|p \circ \xi| \cap [0, 1)py_t = \emptyset$$

since otherwise $\|px_s\| < \|py_t\| = I$, contrary to the definition of I. Choose a convex open neighbourhood N of py_t in L so that $N \cap |p \circ \xi| = \emptyset$. Then N contains points in $\mathrm{ext}(p \circ \eta) \subset \mathrm{ext}(p \circ \xi)$. It follows that $0 \in \mathrm{int}(p \circ \xi)$ can be connected by a continuous curve in $L \setminus |p \circ \xi|$ to points in $\mathrm{ext}(p \circ \xi)$, which yields a contradiction.

3. Suppose $\{0\} \neq \alpha(x(\phi)) \neq |\eta|$. Then there exists a slowly oscillating periodic solution $\widetilde{x}$ of equation (1.1.1) with orbit parametrization $\widetilde{\eta}$ so that $\alpha(x(\phi)) = |\widetilde{\eta}|$. By Proposition 5, $px(\phi)_t \in \mathrm{int}(p \circ \eta)$ for all $t \in \mathbb{R}$. This yields

$$|p \circ \widetilde{\eta}| = p\alpha(x(\phi)) \subset \overline{\mathrm{int}(p \circ \eta)} = \mathrm{int}(p \circ \eta) \cup |p \circ \eta|.$$

In case $p \circ \widetilde{\eta}(s) \in \mathrm{int}(p \circ \eta)$ for some s one obtains a contradiction to the result of the claim in part 2. In case $p \circ \widetilde{\eta}(s) \in |p \circ \eta|$ for some s one obtains $|\eta| = |\widetilde{\eta}| = \alpha(x(\phi))$, contrary to the assumption. It follows that $\alpha(x(\phi)) = \{0\}$ or $\alpha(x(\phi)) = |\eta|$.

The proof for $\omega(\phi)$ is analogous. ∎

2.4. Floquet multipliers of slowly oscillating periodic solutions and adapted Poincaré maps. Let $y : \mathbb{R} \to \mathbb{R}$ be a slowly oscillating periodic solution of equation (1.1.1) with minimal period $\omega > 2$ and orbit parametrization $\eta : [0, \omega] \ni t \mapsto y_t \in C$. The associated monodromy operator

$$Y = D_2 F(\omega, y_0)$$

is compact. The nonzero points λ in the spectrum Σ of Y are called *Floquet multipliers*; each of them is an isolated point of Σ and an eigenvalue with finite-dimensional generalized eigenspace in the complexification of the space C. They are real or occur in complex conjugate pairs. The derivative y' is a solution of the variational equation along $y|[-1, \infty)$, with $y'(t) \neq 0$ for some $t \in [-1, 0]$ since otherwise $y(-1) = y(0)$ and $0 = -\mu y(0) + f(y(-1)) = -\mu y(0) + f(y(0))$, hence $0 = y(0) = y(-1)$ and y would not be slowly oscillating. It follows that

$$Y y_0' = y_0' \neq 0,$$

and 1 is a Floquet multiplier. The solution y is called *hyperbolic* if the generalized eigenspace of the Floquet multiplier 1 has dimension 1, and if

$$|\lambda| \neq 1 \quad \text{for all Floquet multipliers } \lambda \neq 1.$$

The proofs of the results on slowly oscillating solutions and on the attractor A of the restricted semiflow F_S which are recalled in Subsection 2.3 do not make use of Floquet multipliers. The proof of Theorem 2.3.1 in the present paper, however, relies on a-priori results about them. Such results were derived in [3, 18, 8] for equation (1.1.1) with $\mu = 0$. In the sequel they are extended to the case $\mu > 0$.

Proposition 1. *Let $a : \mathbb{R} \to \mathbb{R}$ be continuous, and $b = e^{\mu} a$. A function $v : [t_0 - 1, \infty) \to \mathbb{R}$ ($v : \mathbb{R} \to \mathbb{R}$) is a solution of the equation*

$$v'(t) = -\mu v(t) + a(t) v(t-1) \tag{1}$$

if and only if the function w given by $w(t) = e^{\mu t} v(t)$ on the domain of v is a solution of the equation

$$w'(t) = b(t) w(t-1). \tag{2}$$

The zeros of v and w coincide, and a zero of v is simple if and only if the zero of w is simple. The map $I : C \to C$ given by $(I\phi)(t) = e^{\mu t}\phi(t)$ is a topological isomorphism, with $IS = S$. For every $s \geq 0$ the maps $A_s : C \to C$ and $B_s : C \to C$ given by $A_s \phi = v_s$ where $v : [-1, \infty) \to \mathbb{R}$ is the solution of equation (1) with $v_0 = \phi$, and $B_s \psi = w_s$ where $w : [-1, \infty) \to \mathbb{R}$ is the solution of eq. (2) with $w_0 = \psi$, satisfy

$$B_s \circ I = e^{\mu s} I \circ A_s = I \circ e^{\mu s} A_s. \tag{3}$$

The proof is left as an exercise.

Consider $a : \mathbb{R} \to \mathbb{R}$ given by $a(t) = f'(y(t-1))$, $s = \omega$. Then the function $b = e^{\mu} a$ is continuous, negative, and has period ω. Let $B = B_\omega$, and observe that $Y = A_\omega$. The results in Section 5 of [8] show that there exists $\beta > 0$ so that the spectrum $\Sigma(B)$ of B is disjoint from $\{\lambda \in \mathbb{C} : |\lambda| = \beta\}$, and the reellified generalized eigenspaces $C_{B<}$, $C_{B>}$ associated with the spectral subsets $\Sigma(B)_< = \{\lambda \in \Sigma(B) : |\lambda| < \beta\}$, $\Sigma(B)_> = \{\lambda \in \Sigma(B) : \beta < |\lambda|\}$, respectively, satisfy

$$C = C_{B<} \oplus C_{B>}, \quad C_{B<} \cap S = \emptyset, \quad C_{B>} \subset \overline{S}, \quad \dim C_{B>} = 2.$$

The equations (3) imply that the spectra $\Sigma(B)$ and $\Sigma(e^{\mu\omega} Y)$ coincide, and that for every spectral subset the associated reellified generalized eigenspaces C_B and C_μ for B and $e^{\mu\omega} Y$, respectively, satisfy

$$C_B = I C_\mu .$$

Furthermore, $\Sigma(e^{\mu\omega} Y) = e^{\mu\omega}\Sigma$, and $\Sigma_* \subset \Sigma$ is a spectral subset if and only if $e^{\mu\omega}\Sigma_*$ is a spectral subset of $\Sigma(e^{\mu\omega} Y)$; given a spectral subset $\Sigma_* \subset \Sigma$ the associated reellified generalized eigenspaces C_* of Σ_* and Y, and $C_{*\mu}$ of $e^{\mu\omega}\Sigma_*$ and $e^{\mu\omega} Y$ coincide. Set

$$\varrho = e^{-\mu\omega}\beta .$$

Corollary 1. *For all $\lambda \in \Sigma$, $|\lambda| \neq \varrho$, and $\varrho < 1$. The reellified generalized eigenspaces $C_<, C_>$ associated with the spectral subsets $\{\lambda \in \Sigma : |\lambda| < \varrho\}$ and $\{\lambda \in \Sigma : \varrho < |\lambda|\}$, respectively, satisfy*

$$C = C_< \oplus C_>, \quad C_< \cap S = \emptyset, \quad C_> \subset \overline{S}, \quad \dim C_> = 2.$$

Proof. All assertions except the inequality for ϱ are immediate from the preceding remarks. The inclusion $A - A \subset \overline{S} = \mathbb{R}\overline{S}$ yields $\overline{S} \ni \lim_{t \to 0} \frac{1}{t}(y_t - y_0) = y_0' \neq 0$. Corollary 2.2.1(ii) and periodicity imply that y' is slowly oscillating. Therefore the reellified generalized eigenspace of the Floquet multiplier 1 contains the element $y_0' \in S$, and the assumption $\varrho > 1$ would imply a contradiction to $C_< \cap S = \emptyset$. ∎

Note the analogy with the properties of the spaces Q and L of Subsection 2.1. Corollary 1 leaves the following possibilities.

(4) The Floquet multiplier 1 has multiplicity 2, and $|\lambda| < 1$ for all $\lambda \in \Sigma \setminus \{1\}$,

or

(5) y is hyperbolic, and there exists a real Floquet multiplier $\lambda_* \in \Sigma \setminus \{1\}$ with $\varrho < |\lambda_*|$.

Proposition 2. *In case* (5), $0 < \lambda_*$.

Proof. 1. Suppose (5) holds. There exists $\phi \in C_>$ with $Y\phi = \lambda_* \phi \neq 0$. It follows that there is a solution $v : \mathbb{R} \to \mathbb{R}$ of equation (1) with $v_{n\omega} = \lambda_*^n \phi$ for all integers n. Observe that for every $t \geq 0$,

$$\begin{aligned} v_{t+\omega} &= D_2 F(t+\omega, y_0)\phi = D_2 F(t, F(\omega, y_0)) D_2 F(\omega, y_0)\phi \\ &= D_2 F(t, y_0) Y \phi = \lambda_* D_2 F(t, y_0)\phi = \lambda_* v_t . \end{aligned}$$

In particular,

$$v(t+\omega) = \lambda_* v(t) \quad \text{for all } t \geq -1. \tag{6}$$

The solutions $rv + sy', r, s$ in $\mathbb{R}$, of equation (1) form a linear subspace V of $\mathbb{R}^{\mathbb{R}}$ with

$$C_> = \{x_0 : x \in V\}, \quad \dim V = 2, \quad x_{n\omega} \in C_> \subset \bar{S} \quad \text{for all integers } n.$$

Using Corollary 2.2.1(ii) one deduces that each $x \in V \setminus \{0\}$ is slowly oscillating. For $0 \neq x \in V$ all zeros are simple since otherwise,

$$0 = x'(t) = -\mu x(t) + a(t)x(t-1) = a(t)x(t-1)$$

for some zero t gives $x(t) = 0 = x(t-1)$, which contradicts the fact that x is slowly oscillating.

The properties of y imply that the zeros of $y' \in V$ are given by a strictly increasing sequence $(q_n)_{n=-\infty}^{\infty}$. Applications of Lemmas 1, 2 of [18] to the space of all solutions

$$w : \mathbb{R} \ni t \mapsto e^{\mu t}x(t) \in \mathbb{R}, \quad x \in V,$$

of equation (2) yield that for every $x \in V\setminus\{0\}$ the zeros form a strictly increasing sequence $(t_{n,x})_{n=-\infty}^{\infty}$, and that for x and u in $V \setminus \{0\}$ and for all $n \in \mathbb{Z}$ with $t_{n,x} < t_{n,u} < t_{n+1,x}$,

$$t_{n+1,x} < t_{n+1,u}. \tag{7}$$

Set $t_n = t_{n,v}$, for all integers n. Choose $j \in \mathbb{Z}$ with $-1 \leq q_j$. Set $n = \max\{m \in \mathbb{Z} : t_m \leq q_j\}$. There exists a positive integer k with $q_j + \omega = q_{j+2k}$ since y' has period ω, and all zeros of y' are simple.

2. Suppose $t_n < q_j$. Then $v(q_j) \neq 0$, $q_j < t_{n+1}$, and consequently $q_{j+m} < t_{n+1+m} < q_{j+(m+1)}$ for all nonnegative integers m. In particular, $q_{j+2k} \in (t_{n+2k}, t_{n+2k+1})$, and the equations

$$0 \neq \operatorname{sign}(v(q_j)) = \operatorname{sign}(v'(t_n)) = \operatorname{sign}(v'(t_{n+2k}))$$

(due to the simplicity of the zeros)

$$= \operatorname{sign}(v(q_{j+2k})) = \operatorname{sign}(v(q_j + \omega)), \quad v(q_j + \omega) = \lambda_* v(q_j)$$

yield $\lambda_* > 0$.

3. Suppose $t_n = q_j$. Then $v(q_j + \omega) = \lambda_* v(q_j) = 0 = y'(q_j) = y'(q_j + \omega)$. Suppose $v(t) = 0$ for some $t \in (q_m, q_{m+1})$, $m \leq j + 2k - 1$. Then repeated application of (7) yields $v(q_j + \omega) = v(q_{j+2k}) \neq 0$, a contradiction.

It follows that $v^{-1}(0) \cap (-\infty, q_j + \omega] \subset \{q_m : m \leq j + 2k\} = (y')^{-1}(0) \cap (-\infty, q_j + \omega]$. Analogously one gets $(y')^{-1}(0) \cap (-\infty, q_j + \omega] \subset v^{-1}(0) \cap (-\infty, q_j + \omega]$. Hence

$$v^{-1}(0) \cap [q_j, q_j + \omega] = \{q_{j+m} : m = 0, \dots, 2k\},$$

and

$$0 \neq \operatorname{sign}(v'(q_j + \omega)) = \operatorname{sign}(v'(q_{j+2k})) = \operatorname{sign}(v'(q_j)) \quad \text{(by simplicity).}$$

Now (6) yields $0 < \lambda_*$. ■

Let $p_< : C \to C$ and $p_> : C \to C$ denote the projections along $C_>$ onto $C_<$, and along $C_<$ onto $C_>$, respectively. By $C_< \cap S = \emptyset$,

$$0 \notin p_> S \tag{8}$$

in analogy to the relation $0 \notin pS$ which follows from $Q \cap S = \emptyset$ in (2.2.1).

In the second part of this section Floquet multipliers are used to construct adapted Poincaré maps. In case (4) holds choose a unit vector $\phi_* \in C_> \setminus \mathbb{R}y'_0$; in case (5) holds choose a unit eigenvector $\phi_* \in C_>$ of λ_*. In both cases, ϕ_* and y_0' are linearly independent, and

$$C_> = \mathbb{R}\phi_* \oplus \mathbb{R}y_0'. \tag{9}$$

Proposition 3. *$p\phi_*$ and py_0' are linearly independent.*

Proof. For real r, s with $rp\phi_* + spy_0' = 0$, the relations $0 = p(r\phi_* + sy_0')$ and $r\phi_* + sy_0' \in C_> \subset \overline{S}$ yield $r\phi_* + sy_0' \in Q \cap \overline{S} = \{0\}$, and linear independence of ϕ_* and y_0' gives $r = 0 = s$. ∎

Set $C_* = \mathbb{R}\phi_*,\ C_y = C_< \oplus C_*$, and let $p_* : C_y \to C_y$ and $p^< : C_y \to C_y$ denote the projections along $C_<$ onto C_* and along C_* onto $C_<$, respectively. Then

$$\text{for every } \phi \in C_y,\ p_>\phi = p_*\phi \text{ and } p_<\phi = p^<\phi. \tag{10}$$

Set $H_y = y_0 + C_y$, and observe that $F(\omega, y_0) = y_\omega = y_0 \in H_y$,

$$D_1F(\omega, y_0)1 = y_\omega' = y_0' \notin C_y = T_{y_0}H.$$

For $\phi = y_0,\ t = \omega,\ Z = C_y$, consider an open neighbourhood U of y_0 in C, $\varepsilon > 0$, and a stopping time $\tau : U \to \mathbb{R}$ as in Subsection 2.1.

Proposition 4. *There exist an open neighbourhood U_y of y_0 in U, $\varepsilon_y \in (0, \varepsilon)$, and $t_y \in [0, \omega)$ with the following properties*:

- $\|F(t, \psi)\| \le \max_{t\in[0,\omega]} |y(t)| + 1$ *for all* $t \in [0, \omega + \varepsilon]$, $\psi \in U_y$.
- $F(t_y, \psi) \in S$ *for all* $\psi \in U_y$.
- $\tau(U_y) \subset (\omega - \varepsilon_y, \omega + \varepsilon_y)$.
- $|\eta| \cap (H_y \cap U_y) = \{y_0\}$.
- $\psi' \notin C_y$ *for all* $\psi \in H_y \cap U_y \cap A$.
- $F(s, \psi) \notin H_y$ *for all* $\psi \in H_y \cap U_y \cap A$ *and all* $s \in (0, 2\varepsilon_y)$.

Proof. 1. For $\omega \ne s \in (\omega - \varepsilon, \omega + \varepsilon)$, $F(s, y_0) \notin H_y$. By periodicity, $F(s, y_0) \notin H_y$ for $0 < s < \varepsilon$. The set $\eta([\varepsilon, \omega - \varepsilon])$ is compact and does not contain y_0. Choose an open neighbourhood U_{y1} of y_0 in U so small that $U_{y1} \cap \eta([\varepsilon, \omega - \varepsilon]) = \emptyset$. Then

$$|\eta| \cap (H_y \cap U_{y1}) = \{y_0\}.$$

2. Recall $y_0' = y_\omega' \notin C_y$. Proposition 2.3.3 yields an open neighbourhood U_{y2} of y_0 in U_{y1} so that $\psi' \notin C_y$ for all $\psi \in U_y \cap A$.

3. The closed hyperplane C_y is the nullspace of a continuous linear functional $\phi^* : C \to \mathbb{R}$. For every $\psi \in U_{y2} \cap A$ and $s \in \mathbb{R}$,

$$F_A(s, \psi) \in H_y \quad \text{is equivalent to} \quad h_\psi(s) = 0$$

where $h_\psi : \mathbb{R} \to \mathbb{R}$ is defined by

$$h_\psi(s) = \phi^*(F_A(s, \psi) - y_0).$$

Proposition 2.3.3 shows that each $h_\psi, \psi \in U_{y2} \cap A$, is C^1-smooth, with

$$h_\psi'(s) = \phi^*(x(\psi)_s') \quad \text{for all } s \in \mathbb{R},$$

and that the map $\mathbb{R} \times (U_{y2} \cap A) \ni (s, \psi) \mapsto h_\psi'(s) \in \mathbb{R}$ is continuous. Observe $h_{y_0}'(0) = \phi^*(y_0') = \phi^*(y_\omega') \ne 0$ since $y_\omega' \in C \setminus C_y$. It follows that there exist an open neighbourhood

U_{y3} of y_0 in U_{y2} and $\varepsilon_y \in (0,\varepsilon)$ so that

$$h'_\psi(s) \neq 0 \quad \text{for all } s \in [0, 2\varepsilon_y),\ \psi \in U_{y3} \cap A.$$

Consequently,

$$h_\psi(s) = h_\psi(s) - h_\psi(0) = \int_0^s h'_\psi(\widetilde{s})\, d\widetilde{s} \neq 0,$$

or $F_A(s,\psi) \notin H_y$, for $0 < s < 2\varepsilon_y$ and $\psi \in U_{y3} \cap A$.

3. There exists $t \in [0,\omega)$ so that y_t has no zero. Choose an open neighbourhood U_y of y_0 in U_{y3} so that

$$|x^\psi(s) - y(s)| < 1 \quad \text{for all } \psi \in U_y,\ s \in [-1, \omega + \varepsilon],$$
$$F(t,\psi)(s) \neq 0 \quad \text{for all } \psi \in U_y,\ s \in [-1, 0],$$

and $\tau(U_y) \subset (\omega - \varepsilon_y, \omega + \varepsilon_y)$. Set $t_y = t$. ■

Set $N_y = H_y \cap U_y,\ \tau_y = \tau|U_y$, and consider the C^1-map

$$P_y : N_y \ni \psi \mapsto F(\tau_y(\psi), \psi) \in H_y.$$

COROLLARY 2. *The restriction $P_y|N_y \cap A$ and all derivatives $DP_y(\psi),\ \psi \in N_y \cap A$, are injective.*

Proof. 1. Suppose $P_y(\psi) = P_y(\chi)$ for ψ and χ in $N_y \cap A$. In case $\tau_y(\chi) < \tau_y(\psi)$ the injectivity of the maps $F(t,\cdot),\ t \geq 0$, yields

$$F(\tau_y(\psi) - \tau_y(\chi), \psi) = \chi \in H_y.$$

contrary to $0 < \tau_y(\psi) - \tau_y(\chi) < 2\varepsilon_y$. In the same way the inequality $\tau_y(\psi) < \tau_y(\chi)$ is excluded. Therefore $\tau_y(\chi) = \tau_y(\psi)$, and the injectivity of $F(\tau_y(\psi), \cdot)$ implies $\psi = \chi$.

2. Let $\psi \in N_y \cap A$. Suppose $0 = DP_y(\psi)\chi$ for some $\chi \in C_y$. Then $0 = p_\xi D_2 F(\tau_y(\psi), \psi)\chi$ where $p_\xi : C \to C$ is the projection onto C_y along $\mathbb{R}\xi$,

$$\xi = D_1 F(\tau_y(\psi), \psi)1 = x(\psi)'_{\tau_y(\psi)} = D_2 F(\tau_y(\psi), \psi)\psi'.$$

The formula for p_ξ in Subsection 2.1 gives

$$D_2 F(\tau_y(\psi), \psi)\chi \in \mathbb{R} D_2 F(\tau_y(\psi), \psi)\psi'.$$

The injectivity of $D_2 F(\tau_y(\psi), \psi)$ yields $\chi \in \mathbb{R}\psi'$. It follows that $\chi \in \mathbb{R}\psi' \cap C_y = \{0\}$. ■

The choice of the hyperplane $C_y = C_< \oplus C_*$ and the formula

$$DP_y(y_0)\chi \in p_\xi Y \chi$$

with the projection $p_\xi : C \to C$ onto C_y along $\mathbb{R}\xi$, $\xi = D_1 F(\omega, y_0)1 = y'_\omega = y'_0$, yield

$$DP_y(y_0)\chi = Y\chi \quad \text{for all } \chi \in C_< \subset C_y.$$

In particular, $DP_y(y_0)C_< \subset C_<$, and the spectrum of the map $A_< : C_< \ni \chi \mapsto DP_y(y_0)\chi \in C_<$ coincides with the spectral set $\sigma_< \subset \Sigma$.

PROPOSITION 5. (i) *In the hyperbolic case* (5),

$$DP_y(y_0)\phi_* = \lambda_* \phi_* \quad \textit{and} \quad \sigma_< = \{\lambda \in \Sigma : |\lambda| < \min\{1, \lambda_*\}\}.$$

(ii) *If* (4) *holds then*

$$DP_y(y_0)\phi_* = \phi_* \quad \textit{and} \quad \sigma_< = \{\lambda \in \Sigma : |\lambda| < 1\}.$$

Proof. 1. If (5) holds then $DP_y(y_0)\phi_* = p_\xi Y\phi_* = p_\xi\lambda_*\phi_* = \lambda_*\phi_*$, as $\lambda_*\phi_* \in C_y$.

2. Suppose (4) holds. Then 1 is the only point in the spectrum of the map $Y_> : C_> \ni \chi \mapsto Y\chi \in C_>$.

Claim: $Y\phi_* = \phi_* + ry_0'$ for some $r \in \mathbb{R}$.

Proof: $Y\phi_* = s\phi_* + ry_0'$ with real r, s implies

$$(Y - s\cdot \mathrm{id})(Y - \mathrm{id})\phi_* = (Y - \mathrm{id})(Y - s\cdot \mathrm{id})\phi_* = (Y - \mathrm{id})ry_0' = 0.$$

Either $(Y-\mathrm{id})\phi_* = 0$, or $(Y-\mathrm{id})\phi_* \neq 0$. In the second case s is an eigenvalue of $Y_>$, therefore $s = 1$.

3. It follows that $DP_y(y_0)\phi_* = p_\xi Y\phi_* = p_\xi(\phi_* + y_0') = p_\xi\phi_* = \phi_*$ since $\phi_* \in C_y$. ■

PROPOSITION 6 (Trajectories of P_y and solutions of equation (1.1.1)). (i) *If $x : \mathbb{R} \to \mathbb{R}$ is a solution of equation* (1.1.1) *so that $x_t \to |\eta|$ as $t \to -\infty$ then there exists a strictly increasing sequence $(t_j)_{j=-\infty}^0$ with $t_j \to -\infty$ as $j \to -\infty$ so that $x_{t_j} \in N_y$, $P_y(x_{t_j}) = x_{t_{j+1}}$ for all integers $j \le -1$, and $x_{t_j} \to y_0$ as $j \to -\infty$.*

(ii) *If $(\phi_j)_{j=-\infty}^0$ is a trajectory of P_y then there exist a bounded slowly oscillating solution $x : \mathbb{R} \to \mathbb{R}$ of equation* (1.1.1) *and a strictly increasing sequence $(t_j)_{j=-\infty}^0$ with $t_j \to -\infty$ as $j \to -\infty$ and $\phi_j = x_{t_j}$ for all integers $j \le 0$.*

Proof. 1. *Claim*: There exist $\delta \in (0, \omega/3)$ and an open neighbourhood N_δ of y_0 in $N_y \subset H_y$ such that $F(t,\psi) \notin N_\delta$ for all $\psi \in N_\delta$ and $t \in (\delta, \omega + 3\delta) \setminus \{\tau_y(\psi)\}$.

Proof: Set $\delta = \varepsilon_y/5$. The properties of N_y and ε_y imply $y_t \notin H_y$ for $t \in (\omega - 3\delta, \omega + 3\delta) \setminus \{\omega\}$. By periodicity, $y_t \notin H_y$ for $t \in (0,\delta)$. The relation

$$\{y_0\} \cap \{y_t : \delta \le t \le \omega - 3\delta\} = \emptyset$$

yields disjoint open neighbourhoods U' of y_0 in U_y and V of $\{y_t : \delta \le t \le \omega - 3\delta\}$ in C. The compactness of $[\delta, \omega - 3\delta]$ gives an open neighbourhood U_δ of y_0 in U' so small that $F(t,\psi) \in V$ for every $\psi \in U_\delta$ and $t \in [\delta, \omega - 3\delta]$. Set $N_\delta = U_\delta \cap H_y$. For $\psi \in N_\delta$ and $\delta < t \le \omega - 3\delta$, $F(t,\psi) \in V \subset C \setminus U' \subset C \setminus N_\delta$. The inclusion $N_\delta \subset N_y$ and the choice of δ yield $F(t,\psi) \in C \setminus H_y \subset C \setminus N_\delta$ for $\psi \in N_\delta$, $\omega - 3\delta < t < \omega + 3\delta$, $t \neq \tau_y(\psi)$.

2. *Proof of* (i): Choose an open neighbourhood U' of y_0 in U_y so small that $F(\tau_y(\psi),\psi) \in N_\delta$ and $\omega - \delta < \tau_y(\psi) < \omega + \delta$ for all $\psi \in U'$. For every $t \in [0,\omega)$ there exists an open neighbourhood U_t of y_t in C so that $F(\omega - t, U_t) \subset U'$. Set $V = \bigcup_{[0,\omega)} U_t$. Observe that for every $\psi \in V$ there exists $s \in [0,\omega)$ so that $F(\tau_y(F(s,\psi)) + s, \psi) \in N_\delta$. Choose $u \in \mathbb{R}$ so that for $t \le u$, x_t belongs to the neighbourhood V of $|\eta|$. It follows that for every $t \le u$ there exists $s \in (\omega - \delta, 2\omega + \delta)$ with $x_{t+s} \in N_\delta$. Choose $t_{-1} \le u$ with $x_{t-1} \in N_\delta$, set $u_{-2} = t_{-1} - 2\omega - 2\delta$, and choose $s \in (\omega - \delta, 2\omega + \delta)$ with $x_{u_{-2}+s} \in N_\delta$. Observe $\delta < t_{-1} - (u_{-2} + s) < \omega + 3\delta$. The last inequalities and the relations $N_\delta \ni x_{t_{-1}} = F(t_{-1} - (u_{-2}+s), x_{u_{-2}+s}), x_{u_{-2}+s} \in N_\delta$ yield

$$t_{-1} - (u_{-2} + s) = \tau_y(x_{u_{-2}+s}),$$

hence $P_y(x_{u_{-2}+s}) = x_{t_{-1}}$. Set $t_{-2} = u_{-2} + s$. Proceed by induction.

3. *Proof of* (ii): Set $t_0 = 0$ and consider the sequence given by $t_j = t_{j-1} + \tau_y(\phi_{j-1})$ for integers $j \le 0$. By Proposition 4, $0 < \omega - \varepsilon_y < \tau_y(\phi_j) < \omega + \varepsilon_y$ for all $j \le 0$. It follows

that there is a solution $x:\mathbb{R}\to\mathbb{R}$ of equation (1.1.1) with

$$x_{t_j}=\phi_j \quad \text{for all } j\le 0,$$

and

$$\|x_{t_j+t}\|=\|F(t,\phi_j)\|\le \sup_{s\in\mathbb{R}}|y(s)|+1 \quad \text{for all } t\in[0,\omega+\varepsilon],\ j\le 0,$$

so that the restriction $x|(-\infty,0]$ is bounded. Moreover,

$$x_{t_j+t_y}=F(t_y,\phi_j)\in S \quad \text{for all } j\le 0.$$

Use Corollary 2.2.1 to show that x is slowly oscillating. Use Proposition 2.2.2 and the boundedness property of f to show that the restriction $x|[0,\infty)$ is bounded. ∎

The invariance property of A yields

(11) $$P_y(N_y\cap A)\subset A.$$

Proposition 2.3.5 implies that for every slowly oscillating periodic solution $\widetilde{y}$ of eq. (1.1.1) with orbit parametrization $\widetilde{\eta}$ and

(12) for every $\phi\in N_y\cap A$ with $p\phi\in \operatorname{int}(p\circ\widetilde{\eta})$ $[\in \operatorname{ext}(p\circ\widetilde{\eta})]$,
$$pP_y(\phi)\in \operatorname{int}(p\circ\widetilde{\eta}) \quad [\in \operatorname{ext}(p\circ\widetilde{\eta})].$$

It is convenient to restate Proposition 3.5 of [8] on derivatives of iterates of P_y. If $(\phi_j)_{j=0}^n$ is a finite trajectory of P_y and if $\chi\in T_{\phi_0}H_y=C_y$ then

(13) $$DP^n(\phi_0)\chi=p_\xi D_2F\Big(\sum_{j=0}^{n-1}\tau_y(\phi_j),\phi_0\Big)\chi$$

with the projection $p_\xi: C\to C$ onto C_y along $\mathbb{R}\xi$, $\xi=D_1F(\sum_{j=0}^{n-1}\tau_y(\phi_j),\phi_0)1$.

2.5. Local invariant manifolds. This subsection contains the results on local stable, center, and unstable manifolds for the semiflow F at the stationary point 0 and for the adapted Poincaré maps P_y of the preceding subsection at the fixed point y_0 which will be used in the proof of Theorem 2.3.1.

PROPOSITION 1. (i) *In case $u_0<0$ there exist an open neighbourhood W^s of 0 in C and constants $c\ge 1$, $k\in[0,1)$ so that for all $\phi\in W^s$ and all integers $n\ge 0$,*

$$\|F(n,\phi)\|\le ck^n\|\phi\|.$$

(ii) *In case $u_0=0$ there exist a C^1-map $w^c: L\to Q$ with $w^c(0)=0$ and $Dw^c(0)=0$, and an open neighbourhood N of 0 in C so that $W^c=\{\chi+w^c(\chi):\chi\in L\}$ has the following properties*:

(1) *If $x:\mathbb{R}\to\mathbb{R}$ is a solution of equation (1.1.1) with $x_t\in N$ for all $t\le 0$ then $x_0\in W^c$.*

(2) *If $\phi\in W^c$, $t\ge 0$, and $F(s,\phi)\in N$ for all $s\in[0,t]$, then $F(t,\phi)\in W^c$.*

P r o o f. 1. Part (i) is a standard result on linearized stability for the map $F(1,\cdot)$.

2. Let $u_0 = 0$. We recall the description of the construction of W^c in Section 6 of [20], which is based on [4], and indicate the modifications which are needed to derive assertion (ii).

2.1 (*Dual semigroups*). The elements $\phi^\odot \in C^*$ for which the adjoints of the operators $T(t)$ define a continuous curve

$$[0,\infty) \ni t \mapsto T(t)^*\phi^\odot \in C^\odot$$

form a positively invariant subspace $C^\odot$, which is called the *sun subspace*. The operators

$$T^\odot(t) : C^\odot \ni \phi^\odot \to T(t)^*\phi^\odot \in C^\odot, \quad t \geq 0,$$

constitute a C_0-semigroup on $C^\odot$. Using this last semigroup one defines the space $C^{\odot\odot} \subset C^{\odot *}$. The space C is *sun-reflexive* with respect to the original C_0-semigroup in the sense that there exists a norm-preserving isomorphism of C onto $C^{\odot\odot}$.

There is an isomorphism between $C^{\odot *}$ and $\mathbb{R} \times L^\infty(-1,0;\mathbb{R})$. Let $r^{\odot *} \in C^{\odot *}$ denote the preimage of $(1,0)$.

For a given continuous function $\widetilde{g} : \mathbb{R} \to C^{\odot *}$ and real $a \leq b$ the weak-star integral

$$\int_a^b T^\odot(b-t)^*\widetilde{g}(t)\,dt \in C^{\odot *}$$

is defined by

$$\Big(\int_a^b T^\odot(b-t)^*\widetilde{g}(t)\,dt\Big)(x^\odot) = \int_a^b (T^\odot(b-t)^*\widetilde{g}(t))(x^\odot)\,dt$$

for $x^\odot \in X^\odot$.

If $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and if $x : \mathbb{R} \to \mathbb{R}$ is a solution of the equation

$$x'(t) = -\mu x(t) - \alpha x(t-1) + g(t) \tag{3}$$

with $\alpha = -f'(0)$, then the curve $u : \mathbb{R} \ni t \mapsto x_t \in C$ is a solution of the integral equation

$$u(t) = T(t-s)u(s) + \int_s^t T^\odot(t-\tau)^*(g(\tau)r^{\odot *})\,d\tau, \quad t \geq s; \tag{4}$$

this last equation is in fact an equation between elements of $C^{\odot *}$ where the isomorphism $C \cong C^{\odot\odot}$ and the inclusion map $C^{\odot\odot} \to C^{\odot *}$ are omitted. Conversely, if $u : \mathbb{R} \to C$ satisfies (4) then $x : \mathbb{R} \ni t \mapsto u(t)(0) \in \mathbb{R}$ is a solution of equation (3), and $x_t = u(t)$ for all $t \in \mathbb{R}$.

2.2 (*Solutions slowly growing at infinity*). Fix $\eta > 0$ with $u_1 < -\eta < 0 = u_0$. For a given Banach space E over $\mathbb{R}$, let $BC^\eta(\mathbb{R}, E)$ denote the space of continuous maps $u : \mathbb{R} \to E$ so that

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|}\|u(t)\| < \infty,$$

and consider the norm $\|\cdot\|_\eta$ on $BC^\eta(\mathbb{R}, E)$ which is given by the last expression. For each $\widehat{F} \in BC^\eta(\mathbb{R}, C^{\odot *})$ there exists a unique solution

$$u = \widehat{K}(\widehat{F}) \in BC^\eta(\mathbb{R}, C)$$

of the integral equation

$$u(t) = T(t-s)u(s) + \int_s^t T^{\odot}(t-\tau)^* \widehat{F}(\tau)\, d\tau, \quad t \geq s, \tag{5}$$

with $pu(0) = 0$. The solution map $\widehat{K} : BC^{\eta}(\mathbb{R}, C^{\odot *}) \to BC^{\eta}(\mathbb{R}, C)$ is linear and continuous.

2.3 (*Modified equations*, *center manifold*). There are sequences of open intervals I_n and C^1-functions $r_n : \mathbb{R} \to \mathbb{R}$ with compact supports, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, $0 \in I_n$,

$$\begin{aligned} f(\xi) &= f'(0)\xi + r_n(\xi) \quad \text{for all } \xi \in I_n, \\ |r_n(\xi) - r_n(\xi')| &\leq \frac{1}{n}|\xi - \xi'| \quad \text{for all } \xi, \xi' \text{ in } \mathbb{R}. \end{aligned} \tag{6}$$

The equations

$$R_n(u)(t) = r_n(u(t)(-1))r^{\odot *}$$

define substitution operators

$$R_n : BC^{\eta}(\mathbb{R}, C) \to BC^{\eta}(\mathbb{R}, C^{\odot *}), \quad n \in \mathbb{N},$$

with Lipschitz constants L_n so that $L_n \to 0$ as $n \to \infty$. The hypothesis $u_0 = 0$ implies that there is a constant $M \geq 1$ with $\|T(t)\| \leq M$ for all $t \geq 0$. Fix an integer $n \geq 1$ so that

$$\frac{M}{n}\|r^{\odot *}\| < \eta, \quad L_n \|\widehat{K}\| < \frac{1}{2}.$$

The operators $T(t),\ t \geq 0$, induce a group of isomorphisms $T_L(t) : L \to L,\ t \in \mathbb{R}$, which are uniformly bounded. For every $\chi \in L$ there is a unique solution $u = u(\chi) \in BC^{\eta}(\mathbb{R}, C)$ of the equation

$$u = T_L(\cdot)\chi + \widehat{K}(R_n(u)).$$

Define

$$W^c = \{u(\chi)(0) : \chi \in L\}.$$

There exists a C^1-map $w^c : L \to Q$ with $w^c(0) = 0$ and $Dw^c(0) = 0$ so that

$$W^c = \{\chi + w^c(\chi) : \chi \in L\}.$$

Choose an open neighbourhood N of 0 in C so small that $\phi(t) \in I_n$ for every $\phi \in N$ and for all $t \in [-1, 0]$.

2.4. Now (1) follows by arguments as in the proof of Proposition 6.4 in [20].

2.5. *Proof of* (2): Let $\phi \in W^c,\ t \geq 0$, and $F(s, \phi) \in N$ for all $s \in [0, t]$. There exist $u \in BC^{\eta}(\mathbb{R}, C)$ and $\chi \in L$ so that

$$u(0) = \phi \quad \text{and} \quad u = T_L(\cdot)\chi + \widehat{K}(R_n(u)).$$

For all real $\widetilde{t} \geq s$,

$$\begin{aligned} u(\widetilde{t}) &= T_L(\widetilde{t})\chi + (u(\widetilde{t}) - T_L(\widetilde{t})\chi) \\ &= T_L(\widetilde{t})\chi + T(\widetilde{t} - s)(u(s) - T_L(s)\chi) + \int_s^{\widetilde{t}} T^{\odot}(\widetilde{t} - \tau)^* R_n(u)(\tau)\, d\tau \end{aligned}$$

$$= T(\widetilde{t}-s)u(s) + \int_s^{\widetilde{t}} T^{\odot}(\widetilde{t}-\tau)^* R_n(u)(\tau)\,d\tau$$
$$= T(\widetilde{t}-s)u(s) + \int_s^{\widetilde{t}} T^{\odot}(\widetilde{t}-\tau)^* r_n(u(\tau)(-1))r^{\odot *}\,d\tau.$$

The remarks in part 2.1 show that $x : \mathbb{R} \ni s \mapsto u(s)(0) \in \mathbb{R}$ is a solution of the modified equation

$$x'(s) = -\mu x(s) - \alpha x(s-1) + r_n(x(s-1)), \tag{7}$$

with $x_s = u(s)$ for all $s \in \mathbb{R}$. Recall (6) and the choice of N. It follows that $x_s = F(s,\phi)$ for $s \in [0,t]$. In particular, $x_t = F(t,\phi)$. Consider the solution $\widetilde{x} = x(t+\cdot)$ of eq. (7), and $\widetilde{u} : \mathbb{R} \ni s \mapsto \widetilde{x}_s \in C$. It remains to show that $\widetilde{u}(0) \in W^c$. Now, $\widetilde{u}$ is in $BC^{\eta}(\mathbb{R}, C)$ since for every $s \in R$,

$$e^{-\eta|s|}\|\widetilde{u}(s)\| = e^{-\eta|t+s-t|}\|u(t+s)\| \le e^{\eta t}e^{-\eta|t+s|}\|u(t+s)\| \le e^{\eta t}\|u\|_{\eta}.$$

The remarks in parts 2.1 and 2.4 yield

$$\widetilde{u}(\widetilde{t}) = T(\widetilde{t}-s)\widetilde{u}(s) + \int_s^{\widetilde{t}} T^{\odot}(\widetilde{t}-s)^* r_n(\widetilde{u}(\tau)(-1))r^{\odot *}\,d\tau$$
$$= T(\widetilde{t}-s)\widetilde{u}(s) + \int_s^{\widetilde{t}} T^{\odot}(\widetilde{t}-\tau)^* R_n(\widetilde{u})(\tau)\,d\tau$$

for $\widetilde{t} \ge s$. The function $\widetilde{u} - T_L(\cdot)p\widetilde{u}(0)$ belongs to $BC^{\eta}(\mathbb{R}, C)$ since $T_L(\cdot)p\widetilde{u}(0)$ is bounded. Note $p[\widetilde{u}(0) - T_L(0)p\widetilde{u}(0)] = 0$. For $\widetilde{t} \ge s$,

$$\widetilde{u}(\widetilde{t}) - T_L(\widetilde{t})p\widetilde{u}(0) = T(\widetilde{t}-s)[\widetilde{u}(s) - T_L(s)p\widetilde{u}(0)] + \int_s^{\widetilde{t}} T^{\odot}(\widetilde{t}-\tau)^* R_n(\widetilde{u})(\tau)\,d\tau.$$

It follows that $\widetilde{u} - T_L(\cdot)p\widetilde{u}(0) = \widehat{K}(R_n(\widetilde{u}))$, or $\widetilde{u}(0) \in W^c$. ∎

COROLLARY 1. *If there exists a solution $x : \mathbb{R} \to \mathbb{R}$ of eq.* (1.1.1) *with $x_0 \ne 0$ and $x_t \to 0$ as $t \to -\infty$ then $u_0 \ge 0$.*

Proof. Assume $u_0 < 0$. There exists $t \le 0$ with $x_s \in W^s$ and $\|x_s\| \le 1$ for all $s \le t$. For every integer $n \ge 0$,

$$\|x_t\| = \|F(n, x_{t-n})\| \le ck^n\|x_{t-n}\| \le ck^n.$$

Therefore $x_t = 0$, and $x_0 = F(-t, x_t) = 0$ contrary to the hypothesis $x_0 \ne 0$. ∎

PROPOSITION 2. *Let $u_0 = 0$, and consider W^c as in Proposition* 1(ii). *There exists an open neighbourhood U of 0 in C so that the set $X = W^c \cap H \cap U$ is a one-dimensional submanifold of C with*

$$T_0X = L \cap H, \quad X \subset K \cup \{0\} \cup (-K), \quad P(X) \subset W^c,$$

and $P|X$ is injective.

Proof. 1. As in the proof of Proposition 9.1 in [17] one finds that the inclusion map $i_H : H \ni \phi \mapsto \phi \in C$ is transversal to W^c at $\phi = 0$, and $\dim H \cap L = 1$. Corollary 17.2 of

[1] shows that there is an open neighbourhood U_1 of 0 in C so that $X_1 = W^c \cap H \cap U_1 = i_H^{-1}(W^c) \cap U_1$ is a one-dimensional C^1-submanifold of C, with $T_0 X_1 \subset T_0 W^c \cap H = L \cap H$. It follows that $T_0 X_1 = L \cap H$.

2. As in the proof of Proposition 6.4 in [17] one finds an open neighbourhood U_2 of 0 in U_1 so that for every $\phi \in U_2 \cap W^c$ with $\phi \neq 0$ there exists $t \in [0,2]$ so that $F(t,\phi)$ has no zero.

3. By Proposition 2.2.2(v) the zeros of each x^ϕ, $\phi \in K \cup (-K)$, are unbounded, and there exists $b \geq 2$ with $z_2(\phi) + 1 < b$ for all $\phi \in K \cup (-K)$ with $0 < \|\phi\| < 1$.

4. Continuous dependence on initial data close to the stationary point and property (2) (local positive invariance of W^c) yield an open neighbourhood U_3 of 0 in $U_2 \cap \{\phi \in C : \|\phi\| < 1\}$ with

$$F(t,\phi) \in W^c \quad \text{for all } \phi \in U_3 \cap W^c,\ t \in [0,b].$$

5. The linear map $D_2F(2,0) = T(2)$ defines an isomorphism of $L = T_0 W^c$ onto itself, and $F(\{2\} \times (U_3 \cap W^c)) \subset W^c$. It follows that there exist open neighbourhoods U_4 of 0 in U_3 and U_5 of 0 in U_1 so that $F(2,\cdot)$ maps $U_4 \cap W^c$ onto $U_5 \cap W^c$.

Claim: $X_1 \cap U_5 \subset K \cup \{0\} \cup (-K)$, and $0 \neq \phi(s)$ for all $\phi \in X_1 \cap U_5 \setminus \{0\}$, $s \in (-1,0]$.

Proof: Let $\phi \in X_1 \cap U_5 \subset U_5 \cap W^c$ with $\phi \neq 0$. There exist $\psi \in U_4 \cap W^c$ and $t \in [0,2]$ with $\phi = F(2,\psi)$ so that $F(t,\psi)$ has no zero. Since $\phi \in H$, or $\phi(-1) = 0$, each $F(s,\psi)$ with $1 \leq s \leq 2$ has a zero. Therefore $0 \leq t < 1$. Proposition 2.2.2(ii) gives $\phi \in K \cup (-K)$, and $\phi(s) \neq 0$ for $-1 < s \leq 0$.

6. Parts 3, 4 and 5 combined yield

$$P(\phi) = F(z_2(\phi) + 1, \phi) \in W^c$$

for all $\phi \in X_1 \cap U_5 \cap U_3$ with $\phi \neq 0$. Proposition 2.2.3, applied to $J = X_1 \cap U_5$, shows that $P|X_1 \cap U_5$ is injective. Set $U = U_5 \cap U_3$. ■

Let $y : \mathbb{R} \to \mathbb{R}$ be a slowly oscillating periodic solution of equation (1.1.1), and consider $C_<, C_*, H_y, N_y$, and the adapted Poincaré map $P_y : N_y \to H_y$ with fixed point y_0 as in Subsection 2.4.

Proposition 3. (i) *In case* (2.4.5) *holds with* $\lambda_* < 1$ *there exists an open neighbourhood* W_s *of* y_0 *in* N_y *so that for every* $\phi \in W_s$ *there is a trajectory* $(\phi_n)_{n=0}^{\infty}$ *of* P_y *with* $\phi_0 = \phi$ *and*

$$\phi_n \to y_0 \quad \text{as } n \to \infty.$$

(ii) *In case* (2.4.5) *holds with* $\lambda_* > 1$ *there exist an open neighbourhood* C_{*u} *of* 0 *in* C_* *and a* C^1*-map* $w_u : C_{*u} \to C_<$ *with* $w_u(0) = 0$ *and* $Dw_u(0) = 0$ *so that for every* $\phi \in W_u = y_0 + \{\chi + w_u(\chi) : \chi \in C_{*u}\}$ *there is a trajectory* $(\phi_n)_{n=-\infty}^{0}$ *of* P_y *with* $\phi_0 = \phi$ *and*

$$\phi_n \to y_0 \quad \text{as } n \to -\infty.$$

There exists an open neighbourhood N_u *of* y_0 *in* N_y *so that* $P_y(W_u \cap N_u) \subset W_u$, *and* $\phi_0 \in W_u$ *for every trajectory* $(\phi_n)_{n=-\infty}^{0}$ *of* P_y *in* N_u.

(iii) *In case* (2.4.4) *holds there exist an open neighbourhood* C_{*c} *of* 0 *in* C_*, *a* C^1*-map* $w_c : C_{*c} \to C_<$ *with* $w_c(0) = 0$ *and* $Dw_c(0) = 0$, *and an open neighbourhood* N_c *of* y_0 *in*

N_y so that $P_y(\phi) \in W_c$ for every $\phi \in W_c = y_0 + \{\chi + w_c(\chi) : \chi \in C_{*c}\}$ with $P_y(\phi) \in N_c$, and $\phi_0 \in W_c$ for every trajectory $(\phi_n)_{n=-\infty}^0$ of P_y in N_c.

Proof. Assertion (i) is a standard result on linearized stability. For (ii), see the standard result on local unstable manifolds at hyperbolic fixed points for C^1-maps in Banach spaces, e.g. Theorem 3.1 in [6] and Theorem 2.7 in [13]. Assertion (iii) can be shown modifying the proof of Theorem 2 in Chapter V of [7] on existence of and attraction to local center manifolds, and using the arguments in Section 4 of [2] on smoothness. For details, see the report [9]. ∎

COROLLARY 2. (i) *If there is a trajectory $(\phi_n)_{n=-\infty}^0$ of P_y in $N_y \setminus \{y_0\}$ with $\phi_n \to y_0$ as $n \to -\infty$ then* (2.4.4) *holds*, *or* (2.4.5) *holds with $\lambda_* > 1$.*

(ii) *If* (2.4.5) *holds with $\lambda_* > 1$ then $W_u \subset A$.*

Proof. 1. *Proof of* (i): In case (2.4.5) holds with $\lambda_* < 1$, the spectrum of $A_y = DP_y(y_0)$ is contained in the closed disk with radius $\lambda_* = \sup_{n\in\mathbb{N}_0} \|A_y^n\phi\|^{1/n}$ and center $0 \in \mathbb{C}$. The norm on C_y given by $\|\phi\|_y = \sup_{n\in\mathbb{N}_0} \|A_y^n\phi\|^{1/n}$ is equivalent to the restriction of $\|\cdot\|$ to C_y, and for all $\phi \in C_y$, $\|A_y\phi\|_y \le \lambda_*\|\phi\|_y$. It follows that there is a bounded neighbourhood N of y_0 in N_y so that for every $\phi \in N$, $P_y(\phi) \in N$ and $\|P_y(\phi) - y_0\|_y \le \frac{\lambda_*+1}{2}\|\phi - y_0\|$. Choose an integer $n \le 0$ with $\phi_j \in N$ for all integers $j \le n$. Then

$$0 < \|\phi_n - y_0\|_y \le \left(\frac{\lambda_* + 1}{2}\right)^k \|\phi_{n-k} - y_0\|$$

for all integers $k \ge 0$, which yields a contradiction.

2. *Proof of* (ii): Use Proposition 3(ii), Proposition 2.4.6(ii), and Proposition 2.3.1(i). ∎

The next result concerns continuous maps in Banach spaces and trajectories in one-dimensional graphs.

LEMMA 1. *Let $h : U \to X$ be a continuous map on a subset U of a Banach space X, with fixed point z. Suppose there are closed subspaces E, E^c of X with $X = E \oplus E^c$, $\dim E = 1$, and there are an open neighbourhood E_w of 0 in E and a continuous map $w : E_w \to E^c$ so that*

$$W = z + \{x + w(x) : x \in E_w\} \subset U.$$

Assume that $h|W$ is injective. Let $x_e \in E \setminus \{0\}$, $\varepsilon > 0$ with $(-\varepsilon, \varepsilon)x_e \subset E_w$, and $\delta > 0$ be such that the injective curve

$$c : (-\varepsilon, \varepsilon) \ni t \mapsto z + tx_e + w(tx_e) \in X$$

and h satisfy $h(c((-\delta, \delta))) \subset c((-\varepsilon, \varepsilon))$. Suppose there exists a trajectory $(x_n)_{n=-\infty}^0$ of h in $c((-\delta, \delta))$. Then for each $s_0 \in \mathbb{R}$ with $|s_0| \le |c^{-1}(x_0)|$ and $\mathrm{sign}(s_0) = \mathrm{sign}(c^{-1}(x_0))$ there is a trajectory $(y_n)_{n=-\infty}^0$ of h in $c((-\delta, \delta))$ with $y_0 = c(s_0)$ and $|c^{-1}(y_n)| \le |c^{-1}(x_n)|$ for all integers $n \le 0$.

Remark. The curve c defines a homeomorphism onto the open subset $c((-\varepsilon, \varepsilon))$ of W.

Proof. Let $\mathrm{pr} : X \to X$ denote the projection onto E along E^c. Then $c((-\varepsilon, \varepsilon)) = \{x \in W : \mathrm{pr}(x-z) \in (-\varepsilon, \varepsilon)x_e\}$ is an open subset of W, and the inverse $c^{-1} : c((-\varepsilon, \varepsilon)) \to$

$\mathbb{R}$ of c is continuous since it is given by

$$c^{-1}(x)x_e = \mathrm{pr}(x - z) \quad \text{for all } x \in c((-\varepsilon, \varepsilon)).$$

The transformed map

$$h_c : (-\delta, \delta) \ni t \mapsto c^{-1}(h(c(t))) \in \mathbb{R}$$

with $h_c(0) = 0$ is continuous and injective, hence strictly increasing or strictly decreasing. The equations $x_n = c(t_n),\ n \in \mathbb{N}_0$, define intervals

$$I_n = [\min\{t_n, 0\}, \max\{t_n, 0\}] \subset (-\delta, \delta)$$

with $h_c(I_{n-1}) = I_n$ for all integers $n \le 0$. It follows that for every $s_0 \in I_0$ there is a trajectory $(s_n)_{n=-\infty}^{0}$ of h_c with $s_n \in I_n$ for all integers $n \le 0$. Set $y_n = c(s_n)$, for all integers $n \le 0$. ■

3. A-priori estimates

3.1. Nonautonomous equations. Consider $t_0 \in \mathbb{R}$ and continuous functions $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ which satisfy the negative feedback condition (2.2.2) for all $t \ge t_0$ and $\delta \neq 0$. The aim of this subsection is an a-priori estimate which expresses that certain slowly oscillating solutions of equation (2.1.1) do not decay too fast, for g in a set of functions given by a growth condition.

Set

$$K_0 = \{\phi \in C : \phi(-1) = 0,\ \phi \text{ increasing},\ 0 < \phi(t) \text{ for all } t \in (-1, 0]\},$$
$$M = K_0 \cup \{\phi \in C : \phi \text{ decreasing},\ 0 < \phi(t) \text{ for all } t \in [-1, 0]\},$$
$$S_m = M \cup (-M).$$

Obviously, the set S_m of monotone data is contained in S.

PROPOSITION 1 (Entering S_m, starting in S_m). *Let $t_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (2.2.2) *for all $t \ge t_0$ and $\delta \neq 0$. Let $x : [t_0 - 1, \infty) \to \mathbb{R}$ be a solution of equation* (2.1.1). *Set $\phi = x_{t_0}$.*

(i) *If $\phi \in S$ then $x_t \in S$ for all $t \ge 0$, and there exists $t \in [t_0, t_0 + 3]$ with $x_t \in S_m$.*

(ii) *If $\phi \in S_m$ then either* $\mathrm{sign}(x(t)) = \mathrm{sign}(x(t_0)) = -\mathrm{sign}(x'(t))$ *for all $t > 0$, or there exists a zero $z > t_0$ of x with* $\mathrm{sign}(x'(t)) = -\mathrm{sign}(x(t_0))$ *for all $t \in (t_0, z + 1)$, and $x'(z + 1) = 0$. The solution x is slowly oscillating. If $z \in (t_0, \infty)$ is a zero then* $\mathrm{sign}(x'(t)) = \mathrm{sign}(x(z + 1))$ *for all $t \in [z, z + 1)$.*

Proof. 1. For the first assertion in (i), see Proposition 2.2.1. The assertion (ii) is a consequence of (2.2.2) for all $t \ge t_0$ and $\delta \neq 0$.

2. *Proof of the second assertion in* (i): Let $\phi \in S$. Suppose $0 \le x(t)$ for all $t \in [t_0 - 1, t_0]$. Then $x'(t) \le 0$ in $(t_0, t_0 + 1]$, and $x'(s) < 0$ for some $s \in (t_0, t_0 + 1]$. If $0 < x(t_0 + 1)$ then $x_{t_0+1} \in M$. If $x(t_0 + 1) \le 0$ then there exist zeros $z' \le z$ in $[t_0, t_0 + 1]$ so that

$$z' = t_0,\ z < t_0 + 1, x(t) < 0 \quad \text{in } (z, t_0 + 1),$$

or

$$t_0 < z',\ 0 < x(t) \quad \text{in } (t_0, z'),\ z = t_0 + 1,$$

or

$$t_0 < z',\ 0 < x(t) \quad \text{in } (t_0, z'), \quad z < t_0 + 1,\ x(t) < 0 \quad \text{in } (z, t_0 + 1].$$

It follows that $x_{z+1} \in -K_0$.

If there exists $z \in (-1, 0)$ with $0 \le x(t)$ for all $t \in [z, t_0]$ and $x(t) \le 0$ for all $t \in [t_0 - 1, z]$ then $0 \le x(t)$ for all $t \in [z, z+1]$, and as above one finds $t \in [z+1, z+3]$ with $x_t \in S_m$. Now it is obvious how to proceed in the remaining cases. ∎

Let real $a \in (0,1)$, $b > 1$ be given, and set

$$I = [-1 - 1/(2b), 0].$$

Observe $I \subset (-2, 0]$. Consider the set D_1 of continuous functions $\psi : I \to \mathbb{R}$ so that

(1) $\psi|[-1/(2b), 0]$ is differentiable with $|\psi'(t)| \ge a|\psi(t-1)|$ for $-1/(2b) \le t \le 0$,

(2) ψ is strictly decreasing with $0 < \psi(-1 - 1/(2b))$.

Let D_2 denote the set of continuous functions $\psi : I \to \mathbb{R}$ with property (1) so that there exists $m \in I^\circ$ with

(3) $0 < \psi(m)$,

(4) $\psi|(m - 1 - 1/b, m] \cap I$ is strictly increasing and $\psi|[m, 0]$ is strictly decreasing,

(5) $m < -1/(2b)$ and $0 < \psi(t)$ for all $t \in [-1 - 1/(2b), m)$, or $-1/(2b) \le m$ and $\psi(m-1) = 0$,

(6) $\psi(t) \ge \psi(m)(1 - b(t - m))$ for all $t \in [m, m + 1/b] \cap I$.

Set $D_> = D_1 \cup D_2$, and $D = D_> \cup (-D_>)$. Observe that $\psi_0 \in S$ for every $\psi \in D$.

For a map $x : M \to \mathbb{R}$ and $t \in \mathbb{R}$ with $[t - 1 - 1/(2b), t] \subset M$ define $x^t : I \to \mathbb{R}$ by $x^t(s) = x(t + s)$ for all $s \in I$.

Proposition 2 (Entering D). *Let $t_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (2.2.2) *for all $t \ge t_0$ and $\delta \neq 0$, and assume*

$$a|\delta| \le |g(t,\delta)| \le b|\delta| \quad \textit{for all } t \ge t_0,\ \delta \in \mathbb{R}. \tag{7}$$

Let $x : [t_0 - 1, \infty) \to \mathbb{R}$ be a solution of equation (2.1.1) *with $x_{t_0} \in S_m$. Then all the shifted restrictions x^t, $t \ge t_0 + 2$, belong to the set D.*

Proof. 1. Let $t \ge t_0 + 2$. Set $\psi = x^t$. The restriction $\psi|[-1/(2b), 0]$ is differentiable with

$$|\psi'(s)| = |x'(t+s)| = |g(t+s, x(t+s-1))| \ge a|x(t+s-1)| = a|\psi(s-1)|$$

for $-1/(2b) \le t \le 0$, so that (1) holds.

2. Suppose $x(t - 1 - 1/(2b)) = 0$. Set $z = t - 1 - 1/(2b)$. Then $z > t_0$, and Proposition 1(ii) yields $x'(z) \neq 0$. If $0 < x'(z)$ then $0 < x'(s)$ for all $s \in [z, z+1)$ and $x'(s) < 0$ for $z + 1 < s < z + 2$. For $s \in [z+1, z+1+1/b]$,

$$|x'(s)| \le b|x(s-1)| \le b|x(z+1)|,$$

hence

$$x(s) \ge x(z+1)(1 - b(s - (z+1))).$$

Therefore $\psi \in D_2 \subset D$. In case $x'(z) < 0$ one finds $\psi \in -D_2 \subset D$.

3. In case $x(t-1-1/(2b)) > 0$ and $x'(s) < 0$ for all $s \in (t-1-1/(2b), t)$ one has $\psi \in D_1 \subset D$.

4. Suppose $x(t-1-1/(2b)) > 0$ and $0 \le x'(s_0)$ for some $s_0 \in (t-1-1/(2b), t)$. Then $x'(t-1/(2b)) = g(t, x(t-1/(2b)-1)) < 0$, and there is a zero m of x' between $t-1/(2b)$ and s_0. Using (2.2.2) one gets $x(m-1) = 0$. Observe that

$$m - 1 > t - 1 - 1/(2b) - 1 \ge t_0 - 1/(2b) > t_0 - 1.$$

The hypothesis $x_{t_0} \in S_m$ yields $0 < m - 1$.

4.1. We prove $x'(s) \ne 0$ for all $s \in (t-1-1/(2b), m)$: Suppose $x'(m_0) = 0$ and $t-1-1/(2b) < m_0 < m$. As above one finds $x(m_0-1) = 0$ and $0 < m_0-1$. Proposition 1(ii) implies

$$0 < |x(s)| \le |x(m_0)| \quad \text{for } m_0 - 1 < s \le m_0.$$

It follows that for $m_0 \le s \le m_0 + 1/b$,

$$|x'(s)| \le b|x(s-1)| \le b|x(m_0)|,$$

and therefore

$$|x(s)| \ge |x(m_0)|(1 - b(s - m_0)).$$

Consequently, $x(s) \ne 0$ for $m_0 - 1 < s < m_0 + 1/b$. Condition (2.2.2) yields $x'(s) \ne 0$ for $m_0 < s < m_0 + 1 + 1/b$, contrary to $x'(m) = 0$ and $m_0 < m < t < m_0 + 1 + 1/(2b)$.

4.2. Suppose $x'(s) < 0$ for all $s \in (t-1-1/(2b), m)$. By Proposition 1(ii), $x'(s) < 0$ for all $s \in [m-1, m)$ and $x(s) < 0$ for $m - 1 < s \le m$. As before it follows that

$$x(s) \le x(m)(1 - b(s - m)) \quad \text{for } m \le s \le m + 1/b,$$

and $x(s) < 0$ for $m - 1 < s < m + 1/b$. The last inequality implies

$$0 < x'(s) \quad \text{for } m < s < m + 1 + 1/b,$$

and one has $\psi \in -D_2 \subset D$.

4.3. If $x'(s) > 0$ for all $s \in (t-1-1/(2b), m)$, then $\psi \in D_2 \subset D$.

5. Now it is obvious how to proceed in the remaining case $x(t-1-1/(2b)) < 0$. ■

Corollary 1. *Let $t_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (2.2.2) *for all $t \ge t_0$ and $\delta \ne 0$, and assume* (7). *If $x : [t_0 - 1, \infty) \to \mathbb{R}$ is a solution of equation* (2.1.1) *with $x_{t_0} \in S$ then $x^t \in D$ for all $t \ge t_0 + 5$.*

Proof. Use Propositions 1(i) and 2. ■

Proposition 3 (Invariance of D). *Let $t_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (2.2.2) *for all $t \ge t_0$ and $\delta \ne 0$, and assume* (7). *If $\psi \in D$ and a solution $x : [t_0 - 1, \infty) \to \mathbb{R}$ of equation* (2.1.1) *satisfy $x_{t_0} = \psi|[-1, 0]$ then $x^{t_0+1} \in D$.*

Proof. 1. The restriction $x^{t_0+1}|[-1/(2b), 0]$ is differentiable with

$$|(x^{t_0+1})'(t)| = |x'(t_0 + 1 + t)| \ge a|x(t_0 + t)| = a|x^{t_0+1}(t - 1)|$$

for $-1/(2b) \le t \le 0$, so that (1) is satisfied.

2. Suppose $\psi \in D_1$, $0 \le \psi(0)$. Then $0 < x(t_0 - 1/(2b))$, and $x'(t) < 0$ for $t_0 < t < t_0 + 1$, therefore $x^{t_0+1} \in D_1 \subset D$.

3. Suppose $\psi \in D_1$, $\psi(0) < 0$. Let ζ denote the unique zero of ψ in I°. If $-1-1/(2b) < \zeta \le -1$, then $0 < x'(t)$ for $t_0 < t \le t_0 + 1$, and x has a strict local minimum at t_0. For $t_0 < t \le t_0 + 1/b$,

$$|x'(t)| \le b|x(t-1)| \le b|x(t_0)|.$$

Hence

$$x(t) \le x(t_0)(1 - b(t - t_0)).$$

It follows that $x^{t_0+1} \in -D_2 \subset D$. If $-1 < \zeta < 0$, then $0 < x(t)$ for $t_0 - 1 \le t < t_0 + \zeta$ and $x(t) < 0$ for $t_0 + \zeta < t \le t_0$. Therefore $x'(t) < 0$ for $t_0 < t < t_0 + 1 + \zeta$, $0 < x'(t)$ for $t_0 + 1 + \zeta < t \le t_0 + 1$, and x has a strict local minimum at $t_0 + 1 + \zeta \in (t_0, t_0 + 1]$. Observe that x is strictly decreasing on $[t_0 - 1, t_0 + 1 + \zeta)$ and strictly increasing on $(t_0 + 1 + \zeta, t_0 + 1]$, with $x(t_0 + 1 + \zeta - 1) = \psi(\zeta) = 0$. As before one finds

$$x(t) \le x(t_0 + 1 + \zeta)(1 - b(t - (t_0 + 1 + \zeta))) \quad \text{for } t_0 + 1 + \zeta \le t \le t_0 + 1 + \zeta + 1/b,$$

and it follows that $x^{t_0+1} \in -D_2 \subset D$.

4. Suppose $\psi \in D_2$, $0 \le \psi(0)$. Consider $m \in I^\circ$ so that (3)–(6) hold. Then $0 < \psi(t)$ for $t \in (m-1, 0) \cap I$, hence $x'(t) < 0$ for $t_0 < t < t_0 + 1$.

4.1. If $m < -1/(2b)$ and $0 < \psi(t)$ for $-1 - 1/(2b) \le t < m$, then

$$x(t_0 - 1/(2b)) = \psi(-1/(2b)) > 0,$$

and x is strictly decreasing on $[t_0 - 1/(2b), t_0]$, hence $x^{t_0+1} \in D_1 \subset D$.

4.2. If $-1/(2b) \le m$ and $\psi(m-1) = 0$, then x is strictly increasing and positive on $[t_0 - 1/(2b), t_0 + m]$ and strictly decreasing on $[t_0 + m, t_0 + 1]$. For $t_0 + m \le t \le t_0$,

$$x(t) = \psi(t - t_0) \ge \psi(m)(1 - b(t - t_0 - m)) = x(t_0 + m)(1 - b(t - (t_0 + m))).$$

For $t_0 < t \le t_0 + m + 1/b$, we have $t < t_0 + 1$ and

$$\begin{aligned} 0 > x'(t) &= g(t, x(t-1)) = g(t, \psi(t - t_0 - 1)) \\ &\ge -b\psi(t - t_0 - 1) \ge -b\psi(m) = -bx(t_0 + m), \end{aligned}$$

therefore

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} x'(s)\,ds \ge x(t_0) - bx(t_0 + m)(t - t_0) \\ &\ge x(t_0 + m)\{(1 - b(t_0 - (t_0 + m))) - b(t - t_0)\} \\ &= x(t_0 + m)(1 - b(t - (t_0 + m))). \end{aligned}$$

It follows that $x^{t_0+1} \in D_2 \subset D$.

5. Suppose $\psi \in D_2$, $\psi(0) < 0$. Consider $m \in I^\circ$ as in part 4. There is a unique zero ζ of ψ in $(m, 0)$, and

$$\zeta \ge m + 1/b > -1,$$

$$0 < \psi(t) \quad \text{for } -1 \le t < \zeta,$$

$$\psi(t) < 0 \quad \text{for } \zeta < t \le 0.$$

This implies $x'(t) < 0$ for $t_0 < t < t_0 + 1 + \zeta$ and $0 < x'(t)$ for $t_0 + 1 + \zeta < t \le t_0 + 1$, and $m_\zeta = t_0 + 1 + \zeta \in (t_0, t_0 + 1)$ is a strict local minimum of x. The solution x is negative and strictly decreasing in $(m_\zeta - 1, m_\zeta]$ and strictly increasing in $[m_\zeta, m_\zeta + 1]$. For $m_\zeta \le t \le m_\zeta + 1/b$,

$$|x'(t)| \le b|x(t-1)| \le b|x(m_\zeta)|.$$

Therefore

$$x(t) \le x(m_\zeta)(1 - b(t - m_\zeta)).$$

The inequality $m+1/b \le \zeta < 0$ yields $m < -1/(2b)$. It follows that x is strictly decreasing on the interval $[t_0 - 1/(2b), t_0]$, and one obtains $x^{t_0+1} \in -D_2 \subset D$.

6. It is now obvious how to proceed in the remaining case $\psi \in -D_<$. ∎

Set

$$c(a,b) = \frac{a}{8b} \min\left\{\frac{1}{4}, \frac{a(2b-1)}{2b}\right\}.$$

PROPOSITION 4. *Let $t_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (2.2.2) *for all $t \ge t_0$ and $\delta \ne 0$, and assume* (7). *Let $\psi \in D$ and $x : [t_0 - 1, \infty) \to \mathbb{R}$ be a solution of equation* (2.1.1) *with $x_{t_0} = \psi|[-1, 0]$. Then*

$$c(a,b)\|x_{t_0}\| \le \|x_{t_0+1}\|.$$

Proof (Compare the proof of Lemma 5 in [16]). 1. If $t_0 \le t \le t' \le t_0 + 1$ and if x has no zero in $[t-1, t'-1]$ then

$$2\|x_{t_0+1}\| \ge a(t'-t) \min_{[t-1,t'-1]} |x(s)| \tag{8}$$

since $0 < a|x(s-1)| \le |g(s, x(s-1))|$ for $t \le s \le t'$ and

$$\begin{aligned}
2\|x_{t_0+1}\| \ge |x(t') - x(t)| &= \Big|\int_t^{t'} g(s, x(s-1))\, ds\Big| \\
&= \int_t^{t'} |g(s, x(s-1))|\, ds \ge a \int_{t-1}^{t'-1} |x(s)|\, ds.
\end{aligned}$$

The following cases are possible.

A. There is a strict local extremum m of x in $(t_0 - 1, t_0)$.

A.1. $t_0 \le m + 1/(2b)$.
A.2. $m + 1/(2b) < t_0$ and $|x(m)| = \|x_{t_0}\|$.
A.3. $m + 1/(2b) < t_0$ and $|x(m)| < \|x_{t_0}\|$.

B. The solution x is strictly monotone on $(t_0 - 1, t_0)$.

B.1. $|x(t_0)| = \|x_{t_0}\|$.
B.2. $|x(t_0)| < \|x_{t_0}\| = |x(t_0 - 1)| = |\psi(-1)|$, and there exists $s \in (-1 - 1/(2b), -1)$ with $|\psi(-1)| > |\psi(s)|$.
B.3. $|x(t_0)| < \|x_{t_0}\| = |x(t_0 - 1)| = |\psi(-1)| \le |\psi(s)|$ for all $s \in [-1 - 1/(2b), -1]$.

2. In case A.1 the properties of ψ yield $|x(t_0)| \ge |x(m)|/2 = \|x_{t_0}\|/2$. Therefore $\|x_{t_0+1}\| \ge |x(t_0)| \ge \|x_{t_0}\|/2$. In case A.2 the properties of ψ imply

$$|x(m)|/2 \le |x(s)| \quad \text{for } m \le s \le m + 1/(2b),$$

and (8) yields

$$2\|x_{t_0+1}\| \geq \frac{a}{2b} \min_{[m,m+1/(2b)]} |x(s)| \geq \frac{a}{4b}|x(m)| = \frac{a}{4b}\|x_{t_0}\|.$$

In case A.3 the properties of ψ imply that there is a zero of x in (m, t_0), and $\|x_{t_0}\| = |x(t_0)|$. Therefore $\|x_{t_0+1}\| \geq |x(t_0)| = \|x_{t_0}\|$.

3. In case B.1, $\|x_{t_0+1}\| \geq |x(t_0)| = \|x_{t_0}\|$.

4. In case B.2 and $x(t_0 - 1) > 0$, it follows that x is strictly decreasing in $(t_0 - 1, t_0)$. Furthermore, there exists $m \in (-1 - 1/(2b), -1]$ so that $\psi(m) \geq \psi(-1) > 0$ and $\psi(u) \geq \psi(m)(1 - b(u - m))$ for $m \leq u \leq m + 1/b$. For $t_0 + m + 1/(2b) \leq t \leq t_0 + m + 3/(4b)$, one obtains $t \in [t_0 - 1, t_0]$,

$$t - t_0 \in \left[m + \frac{1}{2b}, m + \frac{3}{4b}\right] \subset [t_0 - 1, t_0],$$

and

$$x(t) = \psi(t - t_0) \geq \frac{\psi(m)}{4} \geq \frac{\psi(-1)}{4} = \frac{x(t_0 - 1)}{4} > 0.$$

Using (8) one finds

$$2\|x_{t_0+1}\| \geq a\frac{1}{4b} \cdot \frac{x(t_0 - 1)}{4} = \frac{a}{16b}\|x_{t_0}\|.$$

In case B.2 and $x(t_0 - 1) < 0$ the same estimate holds.

5. In case B.3 and $0 < x(t_0 - 1)$, as before, x is strictly decreasing on $(t_0 - 1, t_0)$. The function ψ has no zero in $[-1 - 1/(2b), -1]$. Using (1) one finds that ψ' has no zero in $[-1/(2b), 0]$, and

$$\left|x(t_0) - x\left(t_0 - \frac{1}{2b}\right)\right| = \left|\int_{t_0 - 1/(2b)}^{t_0} x'(t)\, dt\right| = \left|\int_{-1/(2b)}^{0} \psi'(s)\, ds\right| = \int_{-1/(2b)}^{0} |\psi'(s)|\, ds$$

$$\geq a \int_{-1-1/(2b)}^{-1} |\psi(s)|\, ds \geq \frac{a}{2b}|\psi(-1)| = \frac{a}{2b}|x(t_0 - 1)| = \frac{a}{2b}\|x_{t_0}\|.$$

Therefore

$$\frac{a}{4b}\|x_{t_0}\| \leq |x(t_0)| \quad (\leq \|x_{t_0+1}\|),$$

or

$$\frac{a}{4b}\|x_{t_0}\| \leq \left|x\left(t_0 - \frac{1}{2b}\right)\right|.$$

If the last inequality and $|x(t_0)| < (a/(4b))\|x_{t_0}\|$ hold then the fact that x is strictly decreasing on $(t_0 - 1, t_0)$ implies $x(t_0 - 1/(2b)) > 0$, and

$$x(t) \geq x\left(t_0 - \frac{1}{2b}\right) \geq \frac{a}{4b}\|x_{t_0}\| \quad \text{for } t_0 - 1 \leq t \leq t_0 - \frac{1}{2b}.$$

Using (8) one finds

$$2\|x_{t_0+1}\| \geq a\left(1 - \frac{1}{2b}\right)\frac{a}{4b}\|x_{t_0}\|.$$

Altogether,

$$\|x_{t_0+1}\| \geq a\left(1 - \frac{1}{2b}\right)\frac{a}{8b}\|x_{t_0}\|$$

in case B.3 with $0 < x(t_0 - 1)$. The same estimate holds in case B.3 with $x(t_0 - 1) < 0$.

6. Observe that

$$c(a,b)=\min\left\{\frac{1}{2},\frac{a}{8b},1,\frac{a}{32b},\frac{a^2}{8b}\left(1-\frac{1}{2b}\right)\right\}. \ \blacksquare$$

It is also convenient to state here the following simple result.

PROPOSITION 5. *Let $t_0\in\mathbb{R}$ and $g:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a continuous function which satisfies* (2.2.2) *for all $t\ge t_0$ and $\delta\ne 0$, and assume* (7). *For every solution $x:[t_0-1,\infty)\to\mathbb{R}$ of equation* (2.1.1) *and for every $t\in[t_0,t_0+1]$,*

$$\|x_t\|\le(b+1)\|x_{t_0}\|.$$

Proof. Use

$$\begin{aligned}|x(t)|&=\Big|x(t_0)+\int\limits_{t_0}^{t}g(s,x(s-1))\,ds\Big|\\&\le\|x_{t_0}\|+b(t-t_0)\max_{[t_0-1,t-1]}|x(s)|\quad\text{for } t_0\le t\le t_0+1. \ \blacksquare\end{aligned}$$

3.2. Vectors tangent to the attractor and to domains of adapted Poincaré maps. Let $y:\mathbb{R}\to\mathbb{R}$ be a slowly oscillating periodic solution of equation (1.1.1) with minimal period $\omega>2$, monodromy operator Y and adapted Poincaré map $P_y:N_y\to H_y$ as in Subsection 2.4. The main result of the present subsection says that certain vectors in the sets $T_\phi A\cap T_\phi N_y$, for ϕ close to y_0, belong to a cone which contains the subspace $C_*\subset C_y=T_\phi N_y$ and is disjoint from $C_<\setminus\{0\}$. Fix reals $a\in(0,1)$, $b>1$, so that

$$-b<e^\mu\min f'\circ y\le e^\mu\max f'\circ y<-a,$$

and let D denote the set of functions $\psi:[-1-1/(2b),0]\to\mathbb{C}$ associated with a and b as in the preceding section. Recall the decomposition $C=C_<\oplus C_>$ and the projection $p_>:C\to C$ onto $C_>$ along $C_<$.

PROPOSITION 1. *There exists a constant $c(y)>0$ so that for every $\psi\in D$ and $\phi\in C$ with*

$$\psi(t)=e^{\mu t}\phi(t)\quad\textit{for } -1\le t\le 0, \tag{1}$$

we have

$$c(y)\|\phi\|\le\|p_>\phi\|.$$

Proof. 1. Let n denote the smallest integer in $[\omega,\infty)$. We prove

$$e^{-\mu(1+\omega)}\frac{c(a,b)^n}{1+b}\|\phi\|\le\|Y\phi\|$$

for all $\psi\in D$ and $\phi\in C$ satisfying (1). Recall $Y\phi=v_\omega$ where $v:[-1,\infty)\to\mathbb{R}$ is the solution of the variational equation along y with $v_0=\phi$. The function $x:[-1,\infty)\ni t\mapsto e^{\mu t}v(t)\in\mathbb{R}$ is a solution of equation (2.1.1) with

$$g(t,\delta)=e^\mu f'(y(t-1))\delta\quad\text{for } t\ge 0 \text{ and } \delta\in\mathbb{R},$$

so that

$$\delta g(t,\delta)<0\quad\text{for all } t\ge 0 \text{ and } \delta\ne 0,$$

and

$$a|\delta| \le |g(t,\delta)| \le b|\delta| \quad \text{for all } t \ge 0 \text{ and } \delta \in \mathbb{R}.$$

Proposition 3.1.3 yields $x^j \in D$ for all $j \in \{1,\dots,n\}$. Proposition 3.1.4 yields

$$c(a,b)\|x_j\| \le \|x_{j+1}\| \quad \text{for all } j \in \{0,\dots,n-1\}.$$

Therefore $c(a,b)^n\|x_0\| \le \|x_n\|$. By Proposition 3.1.5,

$$\|x_n\| \le (b+1)\|x_\omega\|.$$

Use $e^{-\mu}\|\phi\| \le \|x_0\|$ and $\|x_\omega\| \le e^{\mu\omega}\|v_\omega\|$ to complete the proof.

2. The set $M = \{Y\phi : \|\phi\| = 1, \text{ and there exists } \psi \in D \text{ with (1)}\}$ is contained in S, according to Corollary 2.2.1, and $\overline{M}$ is compact. Part 1 of the proof shows that $\overline{M}$ is bounded away from 0. Therefore $\overline{M} \subset \overline{S} \setminus \{0\} = S$. By (2.4.8),

$$\begin{aligned} 0 &< \min_{\overline{M}} \|p_>\phi\| \\ &\le \inf\{\|p_> Y\phi\| : \|\phi\| = 1, \text{ and there exists } \psi \in D \text{ with (1)}\} \\ &= \inf\{\|Y p_>\phi\| : \|\phi\| = 1, \text{ and there exists } \psi \in D \text{ with (1)}\} \\ &\le \|Y\| \inf\{\|p_>\phi\| : \|\phi\| = 1, \text{ and there exists } \psi \in D \text{ with (1)}\}. \end{aligned}$$

Set

$$c(y) = \frac{1}{\|Y\|} \min_{\overline{M}} \|p_>\phi\|. \ \blacksquare$$

Recall the projection $p_* : C_y \to C_y$ onto C_* along $C_<$.

PROPOSITION 2. *There exists an open neighbourhood N^y of y_0 in N_y so that for every differentiable curve $\gamma : J \to C$ with $|\gamma| \subset A \cap H_y$ and for every $s \in J$ with $\gamma(s) \in N^y$ one has*

$$P_y(\gamma(s)) \in N_y, \quad P_y(P_y(\gamma(s))) \in N_y,$$

and the vector $\chi = DP_y^3(\gamma(s))\gamma'(s) \in T_{P_y^3(\gamma(s))}A \cap C_y$ satisfies $c(y)\|\chi\| \le \|p_*\chi\|$.

Proof. Choose an open neighbourhood N^y of y_0 in N_y so small that for every $\phi \in N^y$, one has

$$\begin{gathered} P_y(\phi) \in N_y, \quad P_y(P_y(\phi)) \in N_y, \\ 2 < \tau_y(P_y^j(\phi)) < \omega + 1 \quad \text{for } j = 0,1,2, \\ -b < e^{\mu} f'(x^{\phi}(t-1)) < -a \quad \text{for } 0 \le t \le 3\omega + 3. \end{gathered}$$

Set

$$u = \sum_{j=0}^{2} \tau_y(P_y^j(\gamma(s))) \in (6, 3\omega+3).$$

Consider a differentiable curve $\gamma : J \to C$ with $|\gamma| \subset A \cap H_y$, and $s \in J$ with $\gamma(s) \in N^y$. Then

$$\gamma'(s) \in T_{\gamma(s)}A \cap T_{\gamma(s)}N^y = T_{\gamma(s)}A \cap C_y.$$

Using (2.4.11) one finds that $\chi = DP_y^3(\gamma(s))\gamma'(s)$ belongs to $T_{P_y^3(\gamma(s))}A \cap C_y$. Set $x = x(\gamma(s))$. According to (2.4.13) there exists $r \in \mathbb{R}$ so that

$$\chi = D_2F(u,\gamma(s))\gamma'(s) - r x'_u,$$

or $\chi = v_u$, with the solution $v : [-1, \infty) \to \mathbb{R}$ of the variational equation (2.1.2) along x given by the initial condition $v_0 = \gamma'(s) - rx'_0$. Proposition 2.3.4 yields $v_0 \in \overline{S}$.

Assume $\chi \neq 0$. Then $v_u \neq 0$, hence $v_0 \neq 0$, and consequently $v_0 \in \overline{S} \setminus \{0\} = S$. The function $z : [-1, \infty) \ni t \mapsto e^{\mu t} v(t) \in \mathbb{R}$ is a solution of the equation

$$z'(t) = e^{\mu} f'(x(t-1)) z(t-1)$$

with $z_0 \in S$. Define $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by

$$g(t, \delta) = \begin{cases} e^{\mu} f'(x(t-1))\delta & \text{for } 0 \le t \le 3\omega + 3 \text{ and } \delta \in \mathbb{R}, \\ e^{\mu} f'(x(3\omega + 3 - 1))\delta & \text{for } 3\omega + 3 < t \text{ and } \delta \in \mathbb{R}. \end{cases}$$

Then g is continuous, and (2.2.2) for $t \ge 0$ and $\delta \neq 0$ and (3.1.7) are satisfied. The solution $d : [-1, \infty) \to \mathbb{R}$ of equation (2.1.1) with $d_0 = z_0 \in S$ satisfies $d^u \in D$, according to Corollary 3.1.1. For $-1 \le t \le 3\omega + 3$, $d(t) = z(t)$. In particular, $z^u \in D$. Proposition 1 gives

$$c(y)\|v_u\| \le \|p_> v_u\|.$$

Use (2.4.10) and $\chi = v_u$. ∎

4. Transversals on the attractor and smoothness

4.1. A sufficient condition for smoothness

PROPOSITION 1. *Let $\phi \in A$.*

(i) *If there exist $t > 1$ and a C^1-curve $\alpha : (-1, 1) \to C$ with $|\alpha| \subset A$, $\alpha(0) = x(\phi)_{-t}$ so that $\alpha'(0)$ and $x(\phi)'_{-t}$ are linearly independent then $p\phi \in (pA)^\circ$, and*

(1) *there is an open neighbourhood N of $p\phi$ in $(pA)^\circ$ so that $a|N$ is C^1-smooth.*

(ii) *If $\phi = y^b_s$ for some $s \in \mathbb{R}$, and if there exist $t > 1$ and a C^1-curve $\alpha : (-1, 1) \to C$ with $\alpha([0, 1)) \subset A, \alpha(0) = y^b_{s-t}$ so that $\alpha'(0)$ and $(y^b_{s-t})'$ are linearly independent then*

(2) *there are an open neighbourhood N of $p\phi$ in L and a C^1-map $a_N : N \to Q$ with $a|N \cap pA = a_N|N \cap pA$.*

P r o o f. 1. *Proof of* (i): Set $x = x(\phi)$. Let $\varepsilon \in (0, t-1)$. The C^1-map

$$h : (-\varepsilon, \varepsilon) \times (-1, 1) \ni (s, u) \mapsto pF(t + s, \alpha(u)) \in C$$

satisfies

$$Dh(0,0)(s,r) = pD_1F(t, x_{-t})s + pD_2F(t, x_{-t})D\alpha(0)r = pD_2F(t, x_{-t})[sx'_{-t} + r\alpha'(0)]$$

for all real s, r. Using Proposition 2.3.4, the inclusion $D_2F(t, x_{-t})\overline{S} \subset \overline{S} = S \cup \{0\}$, $0 \notin pS$, and linear independence of x'_{-t} and $\alpha'(0)$ one finds that $Dh(0,0)$ is injective. It follows that there exist $\delta \in (0, \varepsilon)$ and an open neighbourhood N of $p\phi = h(0,0)$ in L with $h((-\delta, \delta) \times (-\delta, \delta)) = N$ so that there is a C^1-inverse $h_N^{-1} : N \to \mathbb{R}^2$ of $h|(-\delta, \delta) \times (-\delta, \delta)$. The relation $|\alpha| \subset A$ and the invariance properties of A yield $N \subset pA$. For every $\chi \in N$,

$$\chi = h(h_N^{-1}(\chi)) = pF(t + (h_N^{-1}(\chi))_1, \alpha((h_N^{-1}(\chi))_2)),$$

and $F(\ldots) \in A$. Consequently,

$$a(\chi) = qF(t + (h_N^{-1}(\chi))_1, \alpha((h_N^{-1}(\chi))_2)),$$

and it becomes obvious that $a|N$ is C^1-smooth.

2. The proof of (ii) is analogous and leads to a C^1-map

$$a_N : N \ni \chi \mapsto qF(t + (h_N^{-1}(\chi))_1, \alpha((h_N^{-1}(\chi))_2)) \in Q$$

which coincides with a on $N \cap pA$. ■

Corollary 1. *Let $\phi \in A \setminus \{0\}$. If there exist $t > 1$ with $px(\phi)_{-t} \in (pA)^\circ$ and an open neighbourhood N_0 of $px(\phi)_{-t}$ in $(pA)^\circ$ so that $a|N_0$ is C^1-smooth then $p\phi \in (pA)^\circ$, and there is an open neighbourhood N of $p\phi$ in $(pA)^\circ$ so that $a|N$ is C^1-smooth.*

Proof. Set $\psi = x(\phi)_{-t}$. Observe $\psi \in A \setminus \{0\}$. By (2.3.2), $p\psi' \neq 0$. Choose $\chi \in L \setminus \mathbb{R}p\psi'$ and $\delta > 0$ with $p\psi + (-\delta, \delta)\chi \subset N_0$. Consider the curve

$$\alpha : (-1, 1) \ni r \mapsto p\psi + r\delta\chi + a(p\psi + r\delta\chi) \in C.$$

Apply Proposition 1. ■

4.2. Smoothness at wandering points

Theorem 1. *Let $\phi \in A \setminus \{0\}$ be such that $x(\phi)$ is not periodic. Then $p\phi \in (pA)^\circ$, and there exists an open neighbourhood N of $p\phi$ in $(pA)^\circ$ so that $a|N$ is C^1-smooth.*

Proof. 1. Set $x = x(\phi)$.

2. Suppose $\alpha(x)$ is the orbit in C of a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$. Recall the space $C_y = C_* \oplus C_<$ and the hyperplane $H_y = y_0 + C_y$. Consider the Poincaré map $P_y : N_y \to H_y$ associated with y. Proposition 2.4.6 shows that there is a strictly increasing sequence $(t_j)_{j=-\infty}^0$ in $(-\infty, -1)$ with $t_j \to -\infty$ as $j \to -\infty$ so that the points $\phi_j = x_{t_j}$, $j \in -\mathbb{N}_0$, form a trajectory of P_y which converges to the fixed point y_0 as $j \to -\infty$. Corollary 2.5.2 yields that either (2.4.4) holds, or (2.4.5) holds with $\lambda_* > 1$. Proposition 2.5.3(ii), (iii) implies that there exist an open neighbourhood C_{**} of 0 in C_*, a C^1-map $w : C_{**} \to C_<$ with $w(0) = 0$, $Dw(0) = 0$, and

$$W = y_0 + \{\chi + w(\chi) : \chi \in C_{**}\} \subset N_y,$$

and an integer $k \le 0$ with $\phi_j \in W$ for all integers $j \le k$. Furthermore, there is an open neighbourhood N_0 of y_0 in N_y so that

$$P_y(W \cap N_0) \subset W.$$

Corollary 2.4.2 yields $\varepsilon > 0$ with $(-\varepsilon, \varepsilon)\phi_* \subset C_{**}$ so that the restriction of P_y to the open subset $W_\varepsilon = \{\phi \in W : \|p_*(\phi - y_0)\| < \varepsilon\}$ of W is injective. The C^1-curve

$$\zeta : (-\varepsilon, \varepsilon) \ni s \mapsto y_0 + s\phi_* + w(s\phi_*) \in C$$

defines a homeomorphism onto W_ε. There exists $\delta > 0$ with

$$P_y(\zeta((-\delta, \delta))) \subset W_\varepsilon = \zeta((-\varepsilon, \varepsilon)),$$

and there is an integer $k_1 \le k$ with $\phi_j \in \zeta((-\delta, \delta))$ for all integers $j \le k_1$. Observe that $\zeta^{-1}(\phi_j) \to 0$ as $j \to -\infty$, and $\phi_j \neq y_0$ for all integers $j \le 0$, since x is not periodic.

There are integers $j \le k_1$ and $n < j$ so that the preimages

$$s_j = \zeta^{-1}(\phi_j), \quad s_n = \zeta^{-1}(\phi_n) \quad \text{in } (-\delta, \delta)$$

satisfy

$$\text{sign}(s_n) = \text{sign}(s_j) \quad \text{and} \quad 0 < |s_n| < |s_j|;$$

Lemma 2.5.1 guarantees that for each $s \in [\min\{s_j, 0\}, \max\{s_j, 0\}] = I$ there is a trajectory $(\psi_m)_{m=-\infty}^0$ of P_y with $\psi_0 = \zeta(s)$ and $\psi_m \to y_0$ as $m \to -\infty$. Choose $\delta_1 > 0$ with $(s_n - \delta_1, s_n + \delta_1) \subset I$. The C^1-curve

$$\alpha : (-1, 1) \ni s \mapsto \zeta(s_n + s\delta_1) \in C$$

satisfies

$$\alpha(0) = \zeta(s_n) = \phi_n = x_{t_n}.$$

Recall $t_n < -1$. Propositions 2.4.6 and 2.3.1(i) combined yield $|\alpha| \subset A$. The vectors

$$\alpha'(0) = \delta_1\phi_* + Dw(s_n\phi_*)\delta_1\phi_* \in (C_* \setminus \{0\}) + C_< \subset C_y$$

and

$$x'_{t_n} = D_1F(\tau_y(x_{t_{n-1}}), x_{t_{n-1}})1 \in C \setminus C_y$$

are linearly independent. Apply Proposition 1.

3. If $\alpha(x)$ is not the orbit in C of a slowly oscillating periodic solution then $\alpha(x) = 0$. Corollary 2.5.1 gives $u_0 \ge 0$ in this case.

4. In case $\alpha(x) = \{0\}$ and $u_0 > 0$, Proposition 6.3 of [20] shows that ϕ is contained in the submanifold $W = F([0, \infty) \times W_0)$ of Theorem 8.1 of [17]. $W \setminus \{0\}$ consists of segments of bounded slowly oscillating solutions $\widetilde{x} : \mathbb{R} \to \mathbb{R}$, hence $W \subset A$. The set pW is open in L, and there exists a C^1-map $w : pW \to Q$ with

$$W = \{\chi + w(\chi) : \chi \in pW\}.$$

Consequently, $p\phi \in pW \subset pA$, and the restriction $a|pW = w$ is C^1-smooth.

5. In case $\alpha(x) = \{0\}$ and $u_0 = 0$, consider the center manifold W^c of Proposition 2.5.1(ii) and the neighbourhood U of 0 in C, and the one-dimensional C^1-submanifold

$$X = W^c \cap H \cap U \subset K \cup \{0\} \cup (-K)$$

of Proposition 2.5.2. The equation $T_0X = L \cap H$ implies that there exist a complementary subspace E^c in C, $\varepsilon > 0$, an open neighbourhood V of 0 in U, and a C^1-map $v : \{\chi \in L \cap H : \|\chi\| < \varepsilon\} \to E^c$ with $v(0) = 0$, $Dv(0) = 0$, so that

$$X \cap V = \{\chi + v(\chi) : \chi \in L \cap H, \ \|\chi\| < \varepsilon\}.$$

Choose a unit vector $\psi \in L \cap H$. The C^1-curve

$$\zeta : (-\varepsilon, \varepsilon) \ni r \mapsto r\psi + v(r\psi) \in C$$

defines a homeomorphism onto $X \cap V$. The restriction $P|X \cap V$ is injective, and there exists $\delta \in (0, \varepsilon)$ so that

$$P(\zeta((-\delta, \delta))) \subset W^c \cap (K \cup \{0\} \cup (-K)) \cap V \subset X \cap V = \zeta((-\varepsilon, \varepsilon));$$

there is an open neighbourhood V_δ of 0 in V with $\zeta((-\delta, \delta)) = X \cap V_\delta$. The property (2.5.1) of W^c yields $t \le 0$ with $x_s \in W^c$ for all $s \le t$. Proposition 2.2.4(ii) shows that there is a strictly increasing sequence $(t_n)_{n=-\infty}^0$ in $\mathbb{R}$ with $t_n \to -\infty$ as $n \to -\infty$ so that the sequence $(x_{t_n})_{n=-\infty}^0$ is a trajectory of P in K. Of course, $x_{t_n} \to 0$ as $n \to -\infty$. It follows that there is an integer $n \le 0$ with $t_n < -1$ and

$$x_{t_j} \in W^c \cap K \cap V_\delta \subset X \cap V_\delta = \zeta((-\delta,\delta))$$

for all integers $j \le n$. Set $r_j = \zeta^{-1}(x_{t_j})$ for all integers $j \le n$. Recall that $x_s \neq 0$ for all $s \in \mathbb{R}$. There exist integers $j \le n$ and $k < j$ with $\operatorname{sign}(r_k) = \operatorname{sign}(r_j)$ and $0 < |r_k| < |r_j| < \delta$. Lemma 2.5.1 shows that for every

$$r \in I = [\min\{r_j, 0\}, \max\{r_j, 0\}] \subset (-\delta,\delta)$$

there exists a trajectory $(\psi_m)_{m=-\infty}^j$ of P with $\psi_j = \zeta(r)$ and $\psi_m \to 0$ as $m \to -\infty$. Choose $\delta_1 > 0$ with $(r_k - \delta_1, r_k + \delta_1) \subset I$. The C^1-curve

$$\alpha : (-1,1) \ni r \mapsto \zeta(r_k + r\delta_1) \in C$$

satisfies $\alpha(0) = \zeta(r_k) = x_{t_k}$. Recall $t_k < t_n < -1$. Propositions 2.2.4(ii) and 2.3.1(i) combined yield $|\alpha| \subset A$. Since $|\alpha| \subset X \subset H$, we have $\alpha'(0) \in H$. The relations

$$\alpha'(0) = D\zeta(r_k)\delta_1 = \delta_1\psi + Dv(r_k\psi)\delta_1\psi, \quad 0 \neq \delta_1\psi \in L \cap H, Dv(r_k\psi)\delta_1\psi \in E^c$$

yield $\alpha'(0) \neq 0$. The simplicity of the zeros of the bounded slowly oscillating solution x (see Proposition 2.2.3) and $x_{t_k} \in K$ give $x'_{t_k}(-1) = x'(t_k - 1) \neq 0$, or $x'_{t_k} \notin H$. It follows that $\alpha'(0)$ and x'_{t_k} are linearly independent. Apply Proposition 1. ∎

COROLLARY 1. *Let y be a slowly oscillating periodic solution of equation* (1), *with orbit parametrization η.*

(i) *If for every $\chi \in \operatorname{int}(p \circ \eta) \setminus \{0\}$ the solution $x(\chi + a(\chi))$ is not periodic then the restriction $a|(\operatorname{int}(p \circ \eta) \setminus \{0\})$ is C^1-smooth.*

(ii) *If $\widetilde{y}$ is another slowly oscillating periodic solution of equation* (1), *with orbit parametrization $\widetilde{\eta}$ and $|p \circ \widetilde{\eta}| \subset \operatorname{int}(p \circ \eta)$, and if for every $\chi \in \operatorname{ext}(p \circ \widetilde{\eta}) \cap \operatorname{int}(p \circ \eta)$ the solution $x(\chi + a(\chi))$ is not periodic then the restriction $a|(\operatorname{ext}(p \circ \widetilde{\eta}) \cap \operatorname{int}(p \circ \eta))$ is C^1-smooth.*

5. Curves on the attractor emanating from periodic orbits and connecting the stationary point to a periodic orbit

5.1. From lines in the plane L to curves on the graph A which are transversal to the flow. This subsection contains minor modifications of results from Chapter 8 of [20] which prepare the construction of curves on A in the next subsections. The curves will pass through or begin at periodic orbits, or connect the stationary point to a periodic orbit. They will be needed for the application of Theorem 4.2.1 and Corollary 4.2.1.

PROPOSITION 1. (i) *Let $\varrho \in pA \setminus \{0\}$, $t \in \mathbb{R}$, and let Z be a closed hyperplane in C. If $F_A(t, \varrho + a(\varrho))' \in C \setminus Z$ then there exist an open neighbourhood N of ϱ in L, $\varepsilon > 0$, and a continuous map $\sigma : N \cap pA \to (t-\varepsilon, t+\varepsilon)$ with $\sigma(\varrho) = t$ such that for every $\widetilde{\varrho} \in N \cap pA$ and $s \in (t - \varepsilon, t + \varepsilon)$,*

$$F_A(s, \widetilde{\varrho} + a(\widetilde{\varrho})) \in F_A(t, \varrho + a(\varrho)) + Z \quad \textit{is equivalent to} \quad s = \sigma(\widetilde{\varrho}).$$

If in addition $t > 1$ then there exist N, ε, and σ as before, and there are an open neighbourhood V of $\varrho + a(\varrho)$ in C and a C^1-map $\widehat{\sigma} : V \to (t-\varepsilon, t+\varepsilon)$ with $\widehat{\sigma}(\varrho + a(\varrho)) = t$ so that

$$\sigma(\widetilde{\varrho}) = \widehat{\sigma}(\widetilde{\varrho} + a(\widetilde{\varrho})) \quad \textit{for every } \widetilde{\varrho} \in N \cap pA.$$

(ii) *Let $\phi \in A \setminus \{0\}$, $t \in \mathbb{R}$, and $\chi \in L \setminus \{0\}$. If $p[F_A(t,\phi)'] \in L \setminus \mathbb{R}\chi$ then there exist an open neighbourhood U of ϕ in C, $\varepsilon > 0$, and a continuous map $\sigma : U \cap A \to (t-\varepsilon, t+\varepsilon)$ with $\sigma(\phi) = t$ so that for every $\widetilde{\phi} \in U \cap A$ and $s \in (t-\varepsilon, t+\varepsilon)$, $pF_A(s, \widetilde{\phi}) \in pF_A(t,\phi) + \mathbb{R}\chi$ is equivalent to $s = \sigma(\widetilde{\phi})$.*

Proof. Proceed as in the proof of Proposition 8.2 of [20]. In case $F(t, \varrho + a(\varrho))' \in C \setminus Z$ and $t > 1$, construct σ as the composition of a C^1-map $\widehat{\sigma}$ from an open neighbourhood V of $\varrho + a(\varrho)$ into $(t-\varepsilon, t+\varepsilon)$ for which

$$\widetilde{\phi} \in V, \ |s-t| < \varepsilon, \ F_A(s, \widetilde{\phi}) \in F_A(t, \varrho + a(\varrho)) + Z$$

is equivalent to $s = \widehat{\sigma}(\widetilde{\phi})$, with the homeomorphism $p(A \cap V) \ni \widetilde{\varrho} \mapsto \widetilde{\varrho} + a(\widetilde{\varrho}) \in A \cap V$. ∎

Corollary 1. *Let $\varrho \in pA \setminus \{0\}$, $t \in \mathbb{R}$, $\chi \in L \setminus \{0\}$, and let Z be a closed hyperplane in C such that*

$$p[(\varrho + a(\varrho))'] \in L \setminus \mathbb{R}\chi \quad \textit{and} \quad F_A(t, \varrho + a(\varrho))' \in C \setminus Z.$$

Then there exist open neighbourhoods N of ϱ in L and U of $F_A(t, \varrho + a(\varrho))$ in C, $\varepsilon > 0$, and a continuous map $\sigma : N \cap pA \to (t-\varepsilon, t+\varepsilon)$ with $\sigma(\varrho) = t$ such that the map

$$h : N \cap pA \cap (\varrho + \mathbb{R}\chi) \ni \widetilde{\varrho} \mapsto F_A(\sigma(\widetilde{\varrho}), \widetilde{\varrho} + a(\widetilde{\varrho})) \in C$$

defines a homeomorphism onto $U \cap A \cap (F_A(t, \varrho + a(\varrho)) + Z)$. If in addition $t > 1$ then there exist N, U, ε, and σ as before, and there are an open neighbourhood V of $\varrho + a(\varrho)$ in C and a C^1-map $\widehat{\sigma} : V \to (t-\varepsilon, t+\varepsilon)$ with

$$\sigma(\widetilde{\varrho}) = \widehat{\sigma}(\widetilde{\varrho} + a(\widetilde{\varrho})) \quad \textit{for every } \widetilde{\varrho} \in N \cap pA.$$

5.2. Arcs emanating from periodic orbits. Let $y : \mathbb{R} \to \mathbb{R}$ be a slowly oscillating periodic solution of equation (1.1.1) with minimal period $\omega > 2$ and orbit parametrization $\eta : [0, \omega] \ni t \mapsto y_t \in C$. This subsection contains the construction of curves on A which begin or end at the periodic orbit $|\eta|$ and which are C^1-smooth provided they do not intersect other periodic orbits. Consider a closed hyperplane Z in C with $y_0' \in C \setminus Z$. By (2.3.2), $py_0' \neq 0$. Recall Proposition 2.1.1. Choose $\chi \in L$ so that χ and py_0' are linearly independent, and

$$py_0' + (0, s)\chi \subset \operatorname{int}(p \circ \eta) \setminus \{0\} \quad \text{for some } s > 0.$$

An application of Corollary 5.1.1 to $\varrho = py_0$, $t = \omega$, and Z yields open neighbourhoods N of py_0 in L, U of $y_0 = y_\omega$ in C, a map $\widehat{\sigma}$, and a homeomorphism h mapping the set $N \cap pA \cap (py_0 + \mathbb{R}\chi)$ onto the subset $U \cap A \cap (y_0 + Z)$ of A. For $s \in \mathbb{R}$ with $py_0 + s\chi \in pA$ define

$$\phi_s = py_0 + s\chi + a(py_0 + s\chi) \quad \text{and} \quad x^{(s)} = x(\phi_s).$$

There exists $\delta_i > 0$ so that

$$py_0 + (0, \delta_i]\chi \subset (\mathrm{int}(p \circ \eta) \setminus \{0\}) \cap N, \quad py_0 + [-\delta_i, 0)\chi \subset \mathrm{ext}(p \circ \eta),$$
$$\chi \text{ and } p\phi_s' \text{ are linearly independent for every } s \in (0, \delta_i]$$

(use the continuity of the map $A \ni \phi \mapsto \phi' \in C$ guaranteed by Proposition 2.3.3), and

$$D_1F(\widehat{\sigma}(\phi_s), \phi_s)1 = h(py_0 + s\chi)' \in C \setminus Z \quad \text{for every } s \in (0, \delta_i].$$

The continuous map $d : [0, \delta_i) \to C$ given by

$$d(s) = h(py_0 + s\chi) = F_A(\widehat{\sigma}(\phi_s), \phi_s)$$

is injective and has a continuous inverse $d^{-1} : d([0, \delta_i)) \to \mathbb{R}$. In case

$$|\eta| \cap |\eta^b| = \emptyset, \tag{1}$$

or equivalently, $|p \circ \eta| \subset \mathrm{int}(p \circ \eta^b)$, there exists $\delta_e > 0$ so that

$$py_0 + (-\delta_e, 0)\chi \subset \mathrm{ext}(p \circ \eta) \cap \mathrm{int}(p \circ \eta^b), \quad py_0 + (0, \delta_e)\chi \subset \mathrm{int}(p \circ \eta),$$
$$\chi \text{ and } p\phi_s' \text{ are linearly independent for every } s \in (-\delta_e, 0),$$

and

$$D_1F(\widehat{\sigma}(\phi_s), \phi_s)1 = h(py_0 + s\chi)' \in C \setminus Z \quad \text{for every } s \in (-\delta_e, 0).$$

The continuous map $d_e : (-\delta_e, 0] \to C$ given by

$$d_e(s) = h(py_0 + s\chi) = F_A(\widehat{\sigma}(\phi_s), \phi_s)$$

is injective and has a continuous inverse $d_e^{-1} : d((-\delta_e, 0]) \to \mathbb{R}$.

PROPOSITION 1. (i) *For every $\delta \in (0, \delta_i]$ there exists an open neighbourhood U_δ of y_0 in C with*

$$d((0, \delta)) = \{\phi \in U_\delta \cap A \cap (y_0 + Z) : p\phi \in \mathrm{int}(p \circ \eta)\}.$$

If no solution $x^{(s)}$, $0 < s < \delta$, is periodic then the restriction $d|(0, \delta)$ is C^1-smooth, and $Dd(s) \neq 0$ for all $s \in (0, \delta)$.

(ii) *Suppose* (1) *holds, and $\delta \in (0, \delta_e]$. Then there exists an open neighbourhood U_δ of y_0 in C with*

$$d_e((-\delta, 0)) = \{\phi \in U_\delta \cap A \cap (y_0 + Z) : p\phi \in \mathrm{ext}(p \circ \eta)\}.$$

If no solution $x^{(s)}, -\delta < s < 0$, is periodic then the restriction $d|(-\delta, 0)$ is C^1-smooth, and $Dd_e(s) \neq 0$ for all $s \in (-\delta, 0)$.

Proof. 1. Let $\delta \in (0, \delta_i]$. There is an open neighbourhood U_δ of y_0 in U with

$$h(N \cap pA \cap (py_0 + (-\delta, \delta)\chi)) = U_\delta \cap A \cap (y_0 + Z).$$

Let $\phi \in U_\delta \cap A \cap (y_0 + Z)$ with $p\phi \in \mathrm{int}(p \circ \eta)$. By Proposition 2.3.5, $px(\phi)_t \in \mathrm{int}(p \circ \eta)$ for all $t \in \mathbb{R}$. There exists $s \in (-\delta, \delta)$ with

$$\phi = h(py_0 + s\chi) = F_A(\widehat{\sigma}(\phi_s), \phi_s).$$

It follows that $y_0 + s\chi = p\phi_s \in \mathrm{int}(p \circ \eta)$. Consequently, $s \in (0, \delta)$.

Conversely, let $s \in (0, \delta)$. Then $p\phi_s = y_0 + s\chi \in \mathrm{int}(p \circ \eta)$, and therefore $pd(s) = pF_A(\widehat{\sigma}(\phi_s), \phi_s) \in \mathrm{int}(p \circ \eta)$, with

$$d(s) = h(py_0 + s\chi) \in U_\delta \cap A \cap (y_0 + Z)$$

2. Let $\delta \in (0, \delta_i]$ and suppose no solution $x^{(s)}$, $0 < s < \delta$, is periodic. Theorem 4.2.1 and the last statement in Corollary 5.1.1 combined imply that $d|(0,\delta)$ is C^1-smooth. Assume $Dd(s) = 0$ for some $s \in (0, \delta)$. The chain rule and the formula for the derivatives of intersection maps in Subsection 2.1 give

$$0 = p_\xi D_2 F(\widehat{\sigma}(\phi_s), \phi_s)[\chi + Da(py_0 + s\chi)\chi]$$

where $p_\xi : C \to C$ is the projection onto Z along $\xi = D_1 F(\widehat{\sigma}(\phi_s), \phi_s)1$. The formula for p_ξ yields

$$D_2 F(\widehat{\sigma}(\phi_s), \phi_s)[\chi + Da(py_0 + s\chi)\chi] \in \mathbb{R}\xi = \mathbb{R} D_2 F(\widehat{\sigma}(\phi_s), \phi_s)\phi_s'.$$

Because of the injectivity of $D_2 F(\widehat{\sigma}(\phi_s, \phi_s)$ one finds $\chi + Da(py_0 + s\chi)\chi \in \mathbb{R}\phi_s'$, which implies a contradiction to the fact that χ and $p\phi_s'$ are linearly independent.

3. The proof of (ii) is analogous. ∎

5.3. Smooth ends at periodic orbits. This subsection prepares the proof that the map a is smooth at projected periodic orbits. Let $y : \mathbb{R} \to \mathbb{R}$ be a slowly oscillating periodic solution of equation (1.1.1) with minimal period $\omega > 2$ and orbit parametrization η as in the preceding subsection. For

$$Z = C_y = C_< \oplus C_*$$

consider the maps d and d_e constructed in Subsection 5.2, and recall the properties of the adapted Poincaré map $P_y : N_y \to H_y$, $H_y = y_0 + C_y$.

PROPOSITION 1. *There exists $\delta_j \in (0, \delta_i]$ with $P_y(d((0,\delta_j))) \subset d((0,\delta_i))$. If (5.2.1) holds then there exists $\delta_f \in (0, \delta_e)$ with*

$$P_y(d_e((-\delta_f, 0))) \subset d_e((-\delta_e, 0)).$$

Proof. Let $\delta = \delta_i$. Consider a neighbourhood U_δ of y_0 in C as in Proposition 5.2.1(i). There exist an open neighbourhood U of y_0 in U_δ with $P_y(U \cap H_y) \subset U_\delta$, and $\delta_j \in (0, \delta_i)$ so that $d([0, \delta_j)) \subset U$. For $0 < s < \delta_j$, $d(s) \in U \cap A \cap H_y$ and $pd(s) \in \text{int}(p \circ \eta)$. Therefore $P_y(d(s)) \in U_\delta \cap A \cap H_y$ and $pP_y(d(s)) \in \text{int}(p \circ \eta)$. Proposition 5.2.1(i) yields $P_y(d(s)) \in d((0, \delta_i))$. The proof of the second statement is analogous. ∎

The transformed map

$$P_j : (0, \delta_j) \ni s \mapsto d^{-1}(P_y(d(s))) \in (0, \delta_i)$$

is continuous. Corollary 2.4.2 imples that it is injective. Note that $P_j(s) \to 0$ as $s \to 0$. It follows that P_j is strictly increasing.

If (5.2.1) holds then also the transformed map

$$P_f : (-\delta_f, 0) \ni s \mapsto d_e^{-1}(P_y(d_e(s))) \in (-\delta_e, 0)$$

is continuous, injective, and strictly increasing, with $P_f(s) \to 0$ as $s \to 0$.

The next two propositions relate the attraction and repulsion properties of $0 \in \mathbb{R}$ for the interval maps P_j and P_f to the smoothness of d and d_e and to the stability properties of the fixed point y_0 of P_y. Attraction implies smoothness for restrictions of d and d_e to open intervals with endpoint 0.

The main result of the present subsection is that in the cases of attraction the normalized tangent vectors

$$\frac{1}{\|d'(s)\|}d'(s) \quad \text{and} \quad \frac{1}{\|d_e'(s)\|}d_e'(s)$$

have limits at $s = 0$, i.e., at the periodic orbit.

PROPOSITION 2. (i) *If*

$$\delta \in (0, \delta_j) \quad \textit{and} \quad P_j(s) < s \quad \textit{for all } s \in (0, \delta) \tag{1}$$

then the restriction $d|(0, \delta)$ *is* C^1*-smooth, and* $Dd(s) \neq 0$ *for all* $s \in (0, \delta)$.

(ii) *If*

$$\textit{for every } s \in (0, \delta_j) \textit{ there exists } \widetilde{s} \in (0, s) \textit{ with } \widetilde{s} \le P_j(\widetilde{s}) \tag{2}$$

then every neighbourhood of y_0 *in* N_y *contains a trajectory* $(\psi_n)_{n=-\infty}^0$ *of* P_y *with* $p\psi_n \in \operatorname{int}(p \circ \eta)$ *for all integers* $n \le 0$.

(iii) *If there are* $s \in (0, \delta_i)$ *and a trajectory* $(\psi_n)_{n=0}^\infty$ *of* P_y *with* $\psi_0 = d(s)$ *and* $\psi_n \to y_0$ *as* $n \to \infty$ *then there exists* $\delta > 0$ *with property* (1).

Proof. 1. *Proof of* (i): In view of Proposition 5.2.1(i) it remains to exclude the possibility that for some $s \in (0, \delta)$ the solution $x^{(s)}$ is periodic. Assume the periodicity. The solution $x(d(s))$ is a translate of $x^{(s)}$, hence periodic. Property (1) implies that there is a trajectory $(s_n)_{n=0}^\infty$ of P_j with $s_0 = s$ and $s_n \to 0$ as $n \to \infty$. The trajectory $(d(s_n))_{n=0}^\infty$ of P_y tends to y_0 as $n \to \infty$, and all $d(s_n)$ belong to the compact orbit $\{x(d(s))_t : t \in \mathbb{R}\}$. It follows that $y_0 = x(d(s))_t$ for some $t \in \mathbb{R}$, and $y = x(d(s))(t + \cdot)$. In particular, $d(s) = y_{-t} \in |\eta|$. As $0 < s < \delta_j$, $d(s) \in N_y$. Proposition 2.4.4 yields $d(s) = y_0$, and one arrives at $s = d^{-1}(y_0) = 0$, contrary to $s \in (0, \delta)$.

2. *Proof of* (ii): Let N be a neighbourhood of y_0 in N_y. Choose $\delta \in (0, \delta_j)$ with $d((0, \delta)) \subset N$. If there is a fixed point $s \in (0, \delta)$ of P_j then the fixed point $d(s)$ of P_y in N determines the desired trajectory. In the remaining case one obtains $s < P_j(s)$ for all $s \in (0, \delta)$, and each $s \in (0, \delta)$ determines a trajectory $(s_n)_{n=-\infty}^0$ of P_j in $(0, \delta)$ with $s_0 = s$ and $s_n \to 0$ as $n \to -\infty$. The points $d(s_n)$ form a trajectory of P_y in N with $pd(s_n) \in \operatorname{int}(p \circ \eta)$ for all integers $n \le 0$.

3. *Proof of* (iii): The hypothesis implies $p\psi_n \in \operatorname{int}(p \circ \eta)$ for all integers $n \ge 0$. According to Proposition 5.2.1(i) there is an open neighbourhood U_{δ_j} of y_0 in C with

$$d((0, \delta_j)) = \{\psi \in U_{\delta_j} \cap A \cap H_y : p\psi \in \operatorname{int}(p \circ \eta)\}.$$

It follows that there exists $k \in \mathbb{N}$ so that $\psi_n \in d((0, \delta_j))$ for all integers $n \ge k$. The preimages $s_n = d^{-1}(\psi_n)$, $n \ge k$, form a trajectory of P_j with $s_n \to 0$ as $n \to \infty$. Furthermore, $s_{n+1} < s_n$ for all integers $n \ge k$ since otherwise the fact that P_j is increasing would contradict $\lim_{n\to\infty} s_n = 0$.

Set $\delta = s_k$. Let $s \in (0, \delta)$ be given. There is an integer $n \ge k$ with $s_{n+1} \le s < s_n$. Consequently, $P_j(s) < P_j(s_n) = s_{n+1} \le s$. ■

PROPOSITION 3. *Suppose* (5.2.1) *holds.*

(i) *If*

$$\delta \in (0, \delta_f) \quad \textit{and} \quad s < P_f(s) \quad \textit{for all } s \in (-\delta, 0) \tag{3}$$

then the restriction $d_e|(-\delta,0)$ *is* C^1*-smooth, with* $Dd_e(s)\neq 0$ *for all* $s\in(-\delta,0)$.

(ii) *If*

$$\text{for every } s\in(-\delta_f,0) \text{ there exists } \widetilde{s}\in(s,0) \text{ with } P_j(\widetilde{s})\le\widetilde{s} \tag{4}$$

then every neighbourhood of y_0 *in* N_y *contains a trajectory* $(\psi_n)_{n=-\infty}^0$ *of* P_y *with* $p\psi_n\in\mathrm{ext}(p\circ\eta)$ *for all integers* $n\le 0$.

(iii) *If there are* $s\in(-\delta_e,0)$ *and a trajectory* $(\psi_n)_{n=0}^{\infty}$ *of* P_y *with* $\psi_0=d_e(s)$ *and* $\psi_n\to y_0$ *as* $n\to\infty$ *then there exists* $\delta>0$ *with property* (3).

Proof. Analogous to the proof of the preceding proposition. ■

The result on convergence of tangent vectors mentioned before is based on the a-priori estimate $c(y)\|\chi\|\le\|p_*\chi\|$ of Proposition 3.2.2.

PROPOSITION 4. (i) *Suppose* (1) *holds for some* $\delta>0$. *Then*

$$d'(P_j(s))\in\mathbb{R}DP_y(d(s))d'(s) \quad \text{for all } s\in(0,\delta), \tag{5}$$

and there exists $\delta_*\in(0,\delta)$ *with*

$$c(y)\|d'(s)\|\le\|p_*d'(s)\| \quad \text{for all } s\in(0,\delta_*). \tag{6}$$

(ii) *If* (5.2.1) *holds and if* (3) *holds for some* $\delta>0$ *then*

$$d_e'(P_j(s))\in\mathbb{R}DP_y(d_e(s))d_e'(s) \quad \text{for all } s\in(-\delta,0),$$

and there exists $\delta_*\in(0,\delta)$ *with*

$$c(y)\|d_e'(s)\|\le\|p_*d_e'(s)\| \quad \text{for all } s\in(-\delta_*,0). \tag{7}$$

Proof. 1. Suppose (1) holds for some $\delta>0$.

1.1. Let $s\in(0,\delta)$. Set $s_1=P_j(s)$. Then $0<s_1<s<\delta$. There exists $\varepsilon>0$ so that $\Delta=d((s_1-\varepsilon,s_1+\varepsilon))$ is a one-dimensional C^1-submanifold of C, with $T_{d(s_1)}\Delta=\mathbb{R}d'(s_1)$. P_j maps a neighbourhood I of s into $(s_1-\varepsilon,s_1+\varepsilon)$. Therefore $(P_y\circ(d|I))(I)\subset\Delta$, $P_y(d(s))=d(s_1)$, and $DP_y(d(s))d'(s)\in T_{d(s_1)}\Delta$. Using $d'(s)\neq 0$ and Corollary 2.4.2 one finds $DP_y(d(s))d'(s)\neq 0$, and (5) follows.

1.2. *Proof of* (6): Choose a neighbourhood N^y of y_0 in N_y as in Proposition 3.2.2, and $\varepsilon\in(0,\delta)$ with $d((0,\varepsilon))\subset N^y$. By (1), $P_j((0,\varepsilon))\subset(0,\varepsilon)$, and $P_y^3((0,\varepsilon))=(0,\delta_*)$ with $\delta_*=P_j^3(\varepsilon)\in(0,\delta)$.

Let $s\in(0,\delta_*)$. There exist s_0,s_1,s_2,s_3 in $(0,\varepsilon)$ with $s=s_3$ and $s_{k+1}=P_j(s_k)$ for $k\in\{0,1,2\}$. Property (5) shows that there are r_0,r_1,r_2 in $\mathbb{R}\setminus\{0\}$ so that

$$d'(s_{k+1})=r_kDP_y(d(s_k))d'(s_k) \quad \text{for } k\in\{0,1,2\}.$$

The chain rule yields

$$d'(s)=d'(s_3)=r_2r_1r_0D(P_y)^3(d(s_0))d'(s_0).$$

Set $r=r_2r_1r_0$. Proposition 3.2.2 gives

$$c(y)\|d'(s)\|=|r|c(y)\|D(P_y)^3(d(s_0))d'(s_0)\|\le|r|\cdot\|p_*D(P_y)^3(d(s_0))d'(s_0)\|=\|p_*d'(s)\|.$$

2. The proof of assertion (ii) is analogous. ■

For $\phi \in C_y \setminus C_<$, i.e., $p_*\phi \neq 0$, define the *inclination with respect to the decomposition* $C_y = C_< \oplus C_*$ by

$$\iota(\phi) = \|p^<\phi\|/\|p_*\phi\|.$$

Note that in case (1) holds for some $\delta > 0$, Proposition 2(i) and (6) combined imply

$$0 \neq p_* d'(s) \quad \text{for } 0 < s < \delta_*.$$

If (5.2.1) holds and if there exists $\delta > 0$ with property (3) then Proposition 3(i) and (5) combined yield

$$0 \neq p_* d_e'(s) \quad \text{for } -\delta_* < s < 0.$$

PROPOSITION 5 (Inclination lemma). (i) *Suppose* (1) *holds for some* $\delta > 0$. *Then there exists* $\delta_* \in (0,\delta)$ *with*

$$\iota(d'(s_k)) \to 0 \quad \textit{as } n \to \infty$$

for every sequence $(s_n)_{n=0}^\infty$ *in* $(0,\delta_*)$ *with* $s_n \to 0$ *as* $n \to \infty$.

(ii) *Suppose* (5.2.1) *holds*, *and there exists* $\delta > 0$ *with property* (3). *Then there is* $\delta_* \in (0,\delta)$ *so that*

$$\iota(d_e'(s_n)) \to 0 \quad \textit{as } n \to \infty$$

for every sequence $(s_n)_{n=0}^\infty$ *in* $(-\delta_*,0)$ *with* $s_n \to 0$ *as* $n \to \infty$.

Proof. 1. Suppose (1) holds for some $\delta > 0$.

1.1. Consider $\delta_* \in (0,\delta)$ as in Proposition 4(i). Let $(s_n)_{n=0}^\infty$ be a sequence in $(0,\delta_*)$ with $s_n \to 0$ as $n \to \infty$.

1.2. Set $A_y = DP_y(y_0)$. Then $A_y C_< \subset C_<$, and for all $\phi \in C_*$, $A_y\phi = \lambda_*\phi$, with $\lambda_* = 1$ in case (2.4.4) holds. Proposition 2.4.5 shows that there exists $\beta \in (0,\min\{1,\lambda_*\})$ with $|\lambda| < \beta$ for all λ in the spectrum of the map $A_{y<} : C_< \ni \phi \mapsto A_y\phi \in C_<$. There exist a norm $\|\cdot\|_y$ on C_y and $c_1 > 0,\ c_2 > 0$ with

$$c_1\|\phi\|_y \le \|\phi\| \le c_2\|\phi\|_y \quad \text{and} \quad \|A_y\phi\|_y \le \beta\|\phi\|_y$$

for all $\phi \in C_<$. For $\phi \in C_y$ with $p_*\phi \neq 0$ set

$$\iota_y(\phi) = \|p^<\phi\|_y/\|p_*\phi\|_y.$$

The nonlinear part

$$R : N_y - y_0 \ni \phi \mapsto P_y(\phi + y_0) - A_y\phi - y_0 \in C_y$$

of P_y at y_0 is C^1-smooth and satisfies $R(0) = 0,\ DR(0) = 0$, and $DP_y(\phi) = A_y\phi + R(\phi)$ for all ϕ in the open neighbourhood $N_y - y_0$ of 0 in C_y. Set

$$c = \frac{\|p^<\|_y}{c(y)} \cdot \frac{c_2}{c_1}$$

where $\|B\|_y = \sup_{\|\phi\|_y \le 1} \|B\phi\|_y$ for every continuous linear map $B : C_y \to C_y$. For $\phi \in N_y - y_0$ and $\psi \in C_y$ with $p_*\psi \neq 0$ and $\iota_y(\psi) \le c$ one finds

$$\begin{aligned}\|p^< DP_y(\phi + y_0)\psi\|_y &\le \|p^< A_y\psi\|_y + \|p^< DR(\phi)\|_y(1 + c)\|p_*\psi\|_y\\ &= \|A_y p^<\psi\|_y + \|p^< DR(\phi)\|_y(1 + c)\|p_*\psi\|_y\\ &\le \beta\|p^<\psi\|_y + \|p^< DR(\phi)\|_y(1 + c)\|p_*\psi\|_y\end{aligned}$$

and

$$\begin{aligned}\|p_*DP_y(\phi+y_0)\psi\|_y &\ge \lambda_*\|p_*\psi\|_y - \|p_*DR(\phi)\|_y(1+c)\|p_*\psi\|_y\\ &= (\lambda_* - \|p_*DR(\phi)\|_y(1+c))\|p_*\psi\|_y.\end{aligned}$$

1.3. Choose $a_0 \in (0,\delta_*)$. Then $P_j((0,a_0)) \subset (0,a_0)$, and there is a strictly decreasing trajectory $(a_n)_{n=0}^{\infty}$ of P_j in $(0,\delta_*)$ with $a_n \to 0$ as $n \to \infty$. Choose $\beta_0 \in (\beta/\lambda_*, 1)$ and $n_0 \in \mathbb{N}$ such that for all $s \in (0, a_{n_0}]$,

$$0 < \frac{\beta}{\lambda_* - \|p_*DR(d(s)-y_0)\|_y(1+c)} < \beta_0.$$

For integers $n \ge n_0$ set

$$\varepsilon_n = \sup_{(0,a_n]} \frac{\|p^{<}DR(d(s)-y_0)\|_y(1+c)}{\lambda_* - \|p_*DR(d(s)-y_0)\|_y(1+c)}.$$

Then $0 \le \varepsilon_{n+1} \le \varepsilon_n$ for all integers $n \ge n_0$, and $\varepsilon_n \to 0$ as $n \to \infty$.

1.4. *Claim*: For every $\varepsilon > 0$ there exists an integer $n_\varepsilon \ge n_0$ with

$$\iota_y(d'(s)) < \varepsilon \quad \text{for all } s \in [a_{n+1}, a_n], n \ge n_\varepsilon$$

Proof: Let $\varepsilon > 0$. Choose integers $m_\varepsilon \ge n_0$ and $k_\varepsilon \ge 1$ with

$$\begin{aligned}\varepsilon_m \frac{1}{1-\beta_0} &< \frac{\varepsilon}{2} \quad \text{for all integers } m \ge m_\varepsilon,\\ \beta_0^k c &< \frac{\varepsilon}{2} \quad \text{for all integers } k \ge k_\varepsilon.\end{aligned}$$

Set $n_\varepsilon = m_\varepsilon + k_\varepsilon$. Then $n_\varepsilon \ge 2$. Let $s \in [a_{n+1}, a_n], n \ge n_\varepsilon$. Recall that P_j is strictly increasing, with $[a_{k+2}, a_{k+1}] = P_j([a_{k+1}, a_k])$ for all integers $k \ge 0$. It follows that there exist $t_\nu \in [a_{\nu+1}, a_\nu]$ for $\nu \in \{0,\ldots,n\}$ with $s = t_n$ and $t_{\nu+1} = P_j(t_\nu)$ for all $\nu \in \{0,\ldots,n-1\}$. Hence

$$d(t_{\nu+1}) = P_y(d(t_\nu) \quad \text{for all } \nu \in \{0,\ldots,n-1\}.$$

According to (5) there exist $r_0,\ldots,r_{n-1}$ in $\mathbb{R}\setminus\{0\}$ with

$$d'(t_{\nu+1}) = r_\nu DP_y(d(t_\nu))d'(t_\nu) \quad \text{for all } \nu \in \{0,\ldots,n-1\}.$$

The inequality (6) yields $p_*d'(t_\nu) \ne 0$ and

$$\iota_y(d'(t_\nu)) = \frac{\|p^{<}d'(t_\nu)\|_y}{\|p_*d'(t_\nu)\|_y} \le \frac{\|p^{<}\|\|d'(t_\nu)\|}{\|p_*d'(t_\nu)\|} \cdot \frac{c_2}{c_1} \le c$$

for all $\nu \in \{0,\ldots,n\}$. For every $\nu \in \{m_\varepsilon,\ldots,n-1\}$, $s_\nu \le a_\nu \le a_{m_\varepsilon} \le a_{n_0}$, and the estimates in part 1.2 of the proof yield

$$\iota_y(d'(t_{\nu+1})) = \iota_y(DP_y(d(t_\nu))d'(t_\nu)) \le \beta_0\iota_y(d'(t_\nu)) + \varepsilon_\nu.$$

It follows that

$$\begin{aligned}\iota_y(d'(s)) = \iota_y(d'(t_n)) &\le \beta_0^{n-m_\varepsilon}\iota_y(d'(t_{m_\varepsilon})) + \varepsilon_{n-1} + \varepsilon_{n-2}\beta_0 + \ldots + \varepsilon_{m_\varepsilon}\beta_0^{n-1-m_\varepsilon}\\ &\le \beta_0^{n-m_\varepsilon}c + \varepsilon_{m_\varepsilon}\frac{1}{1-\beta_0} < \varepsilon.\end{aligned}$$

1.5. Let $\varepsilon > 0$. Consider $k = n_{c_1\varepsilon/c_2}$ as in the preceding claim. There exists $\nu \in \mathbb{N}$ so that for all integers $n \ge \nu$,

$$0 < s_n \le a_k, \quad s_n \in (a_{\kappa+1}, a_\kappa] \quad \text{for some } \kappa \ge k,$$

and

$$\iota(d'(s_n)) \le \frac{c_2}{c_1}\iota_y(d'(s_n)) < \varepsilon.$$

2. The proof of assertion (ii) is analogous. ∎

Suppose now that (1) holds for some $\delta > 0$. Then the trace $d((0,\delta))$ is reparametrized by arclength using the C^1-transformation

$$v : (0,\delta) \ni s \mapsto \int_{\delta/2}^{s} \|d'(u)\|\, du \in \mathbb{R}.$$

Obviously, $v'(s) > 0$ for all $s \in (0,\delta)$. The range $R = v((0,\delta))$ is an open interval, and the C^1-curve

$$\varrho : R \ni r \mapsto d(v^{-1}(r)) \in C$$

has the following properties:

$$\varrho(R) = d((0,\delta)) \subset \{\phi \in A : p\phi \in \operatorname{int}(p \circ \eta)\},$$
$$\|\varrho'(r)\| = 1 \quad \text{for all } r \in \mathbb{R},$$
$$p_*\varrho'(r) \ne 0 \quad \text{for all } r \in \mathbb{R} \text{ with } r < v(\delta_*),$$

where δ_* is given by Proposition 4(i), and

$$\varrho(r) \to y_0 \quad \text{and} \quad \iota(\varrho'(r)) = \iota(d'(v^{-1}(r))) \to 0 \quad \text{as } r \to \inf R.$$

Corollary 1. *There exists $k \in \{0,1\}$ with $\varrho'(r) \to (-1)^k \phi_*$ as $r \to \inf R$, and R is bounded from below.*

Proof. 1. The continuous map

$$(\inf R, v(\delta_*)) \ni r \mapsto \frac{1}{\|p_*\varrho'(r)\|} p_*\varrho'(r) \in \{\phi_*, -\phi_*\}$$

is constant. Let ϕ denote its value. For $\inf R < r < v(\delta_*)$,

$$\begin{aligned}\|\varrho'(r) - \phi\| &= \left\|\varrho'(r) - \frac{1}{\|p_*\varrho'(r)\|} p_*\varrho'(r)\right\| \le \|\varrho'(r) - p_*\varrho'(r)\| + \left|1 - \frac{1}{\|p_*\varrho'(r)\|}\right| \|p_*\varrho'(r)\| \\ &= \|p^<\varrho'(r)\| + |\|p_*\varrho'(r)\| - \|\varrho'(r)\|| \le 2\|p^<\varrho'(r)\| \\ &= 2\iota(\varrho'(r))\|p_*\varrho'(r)\| \le 2\iota(\varrho'(r))\|p_*\|,\end{aligned}$$

and it follows that $\varrho'(r) \to \phi$ as $r \to \inf R$.

2. Choose $r_0 \in R$ such that for $r \in (\inf R, r_0]$, $\|\varrho(r) - y_0\| \le 1$ and $\|\varrho'(r) - \phi\| < 1/2$. For such r,

$$\begin{aligned}2 \ge \|\varrho(r) - \varrho(r_0)\| &= \left\|\int_{r_0}^{r} \varrho'(s)\, ds\right\| \ge \left\|\int_{r_0}^{r} \phi\, ds\right\| - \left\|\int_{r_0}^{r} (\varrho'(s) - \phi)\, ds\right\| \\ &= r_0 - r - \left\|\int_{r_0}^{r} (\varrho'(s) - \phi)\, ds\right\| \ge r_0 - r - \frac{1}{2}(r_0 - r),\end{aligned}$$

or $r \ge r_0 - 4$. ∎

If (5.2.1) holds and if there exists $\delta > 0$ with property (3) then the trace $d_e((-\delta, 0))$ is reparametrized using the C^1-transformation

$$v_e : (-\delta, 0) \ni s \mapsto \int_{-\delta/2}^{s} \|d'_e(u)\|\, du \in \mathbb{R}$$

and the open interval $R_e = v_e((-\delta, 0))$; the C^1-curve $\varrho_e : R_e \ni r \mapsto d_e(v_e^{-1}(r)) \in C$ satisfies

$$\varrho_e(R_e) = d_e((-\delta, 0)) \subset \{\phi \in A : p\phi \in \mathrm{ext}(p \circ \eta)\},$$
$$\|\varrho'_e(r)\| = 1 \quad \text{for all } r \in R_e,$$
$$p_* \varrho'_e(r) \neq 0 \quad \text{for all } r \in R_e \text{ with } v_e(\delta_*) < r,$$

δ_* given by Proposition 4(ii),

$$\varrho_e(r) \to y_0 \quad \text{and} \quad \iota(\varrho'_e(r)) = \iota(d'_e(v_e^{-1}(r))) \to 0 \quad \text{as } r \to \sup R_e.$$

COROLLARY 2. *There exists $m \in \{0, 1\}$ with $\varrho'_e(r) \to (-1)^m \phi_*$ as $r \to \sup R_e$, and R_e is bounded from above.*

Proof. Analogous to the proof of the preceding corollary. ■

5.4. A curve on A connecting 0 in $\overline{K}$ to a periodic orbit. The subsequent construction will be used in the proof that the map a is smooth in a neighbourhood of the projected stationary point $0 \in L$. The general hypotheses throughout this subsection are that there is a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1), with minimal period $\omega > 2$ and orbit parametrization η, so that

(1) $$y_0 \in K, \text{ and no solution } x(\phi),\ 0 \neq p\phi \in \mathrm{int}(p \circ \eta), \text{ is periodic,}$$

and that

(2) $$\text{for each } x(\phi),\ 0 \neq \phi \in A, \text{ the zeros are not bounded from above.}$$

Consider the set

$$X = \{\phi \in A \cap H : \phi(0) \geq 0,\ p\phi \in \mathrm{int}(p \circ \eta) \cup |p \circ \eta|\}.$$

Using Proposition 2.3.1(iii) and equations (2.3.3) one finds

$$X = \{\phi \in A \cap K : p\phi \in \mathrm{int}(p \circ \eta) \cup |p \circ \eta|\} \cup \{0\}$$
$$= \{\phi \in A \cap K : p\phi \in \mathrm{int}(p \circ \eta)\} \cup \{0, y_0\},$$

and parts (iv) and (v) of Proposition 2.3.1 in combination with Proposition 2.3.5 show that the return map P defines a homeomorphism of the compact set X onto itself.

For

$$X_i = \{\phi \in A \cap K : p\phi \in \mathrm{int}(p \circ \eta)\}$$

one finds

$$X_i = \{\phi \in A \cap H : 0 < \phi(0),\ p\phi \in \mathrm{int}(p \circ \eta)\}$$

and

(3) $$P(X_i) = X_i.$$

PROPOSITION 1. *The set X_i is a one-dimensional C^1-submanifold of C.*

Proof. Consider the inclusion map $I : \{\phi \in H : 0 < \phi(0)\} \ni \phi \mapsto \phi \in C$ and the set $A_i = \{\phi \in A : 0 \neq p\phi \in \text{int}(p \circ \eta)\} = \{\chi + a(\chi) : 0 \neq \chi \in \text{int}(p \circ \eta)\}$, which is a 2-dimensional C^1-submanifold of C by Corollary 4.2.1(i). Obviously,

$$X_i = \{\phi \in H : 0 < \phi(0)\} \cap A_i = I^{-1}(A_i),$$

and in view of Corollary 17.2 of [1] it remains to show that I and X_i are transversal, i.e., for every $\phi \in X_i$ the preimage

$$DI(\phi)^{-1} T_{I(\phi)} A_i = H \cap T_\phi A_i$$

has a closed complementary subspace in H, and the image

$$DI(\phi) T_\phi \{\psi \in H : \psi(0) > 0\} = H$$

contains a closed complementary subspace of $T_{I(\phi)} A_i = T_\phi A_i$ in C.

To prove this, let $\phi \in H$ with $\phi(0) > 0$ and $I(\phi) \in A_i$ be given. Then $\phi(-1) = 0$, $F_A(t, \phi) \in A_i$ for all $t \in \mathbb{R}$, and $\phi'(-1) = x(\phi)'(-1) \neq 0$. It follows that

$$\phi' = D_1 F_A(0, \phi)1 \in T_\phi A_i \setminus H,$$

and therefore

$$C = H \oplus \mathbb{R}\phi'. \tag{4}$$

Furthermore, there exists $\chi \in (H \cap T_\phi A_i) \setminus \{0\}$ since otherwise (4) would yield $T_\phi A_i \subset \mathbb{R}\phi'$, which contradicts $\dim T_\phi A_i = 2$. Observe $T_\phi A_i = \mathbb{R}\chi \oplus \mathbb{R}\phi'$ and $\mathbb{R}\chi = H \cap T_\phi A_i$. There is a closed subspace H_χ of H with

$$H = H_\chi \oplus \mathbb{R}\chi = H_\chi \oplus DI(\phi)^{-1} T_{I(\phi)} A_i,$$
$$H_\chi \subset H = DI(\phi) T_\phi \{\psi \in H : \psi(0) > 0\}, \quad C = H_\chi \oplus T_\phi A_i. \blacksquare$$

The main result of the present subsection is that the set $X \supset X_i$ is the continuous injective image of a compact interval. The construction of the desired parametrization begins as in Subsection 5.2, with $Z = H$. Recall the relations $y_\omega \in H$, $y'_\omega = y'_0 \in C \setminus H$, and $py'_0 \neq 0$. Choose $\chi \in L \setminus \mathbb{R}py'_0$ with

$$py'_0 + (0, s)\chi \subset \text{int}(p \circ \eta) \setminus \{0\}$$

for some $s > 0$. An application of Corollary 5.1.1 to $\varrho = py_0$, $t = \omega$, and H yields open neighbourhoods N of py_0 in L, U of $y_0 = y_\omega$ in C, a map $\widehat{\sigma}$, and a homeomorphism h mapping $N \cap pA \cap (py_0 + \mathbb{R}\chi)$ onto the subset $U \cap A \cap H$ of A. For $s \in \mathbb{R}$ with $py_0 + s\chi \in pA$ define $\phi_s = py_0 + s\chi + a(py_0 + s\chi)$ and $x^{(s)} = x(\phi_s)$. There exist $\delta_i > 0$ and an open neighbourhood U_i of y_0 in $\{\phi \in U : \phi(0) > 0\}$ so that

$$py_0 + (0, \delta_i]\chi \subset (\text{int}(p \circ \eta) \setminus \{0\}) \cap N,$$
$$py_0 + [-\delta_i, 0)\chi \subset \text{ext}(p \circ \eta) \subset N,$$
$$h(pA \cap (py_0 + (-\delta_i, \delta_i)\chi)) = U_i \cap A \cap H,$$

and for every $s \in (0, \delta_i]$, χ and $p\phi'_s$ are linearly independent (use the continuity of the map $A \ni \phi \mapsto \phi' \in C$ guaranteed by Proposition 2.3.3). Note that $U_i \cap A \cap H = U_i \cap A \cap K$.

The continuous map $d : [0,\delta_i) \to C$ given by

$$d(s) = h(\phi_s) = F_A(\widehat{\sigma}(\phi_s), \phi_s)$$

is injective, with $d(0) = y_0$, and there is a continuous inverse $d^{-1} : d([0,\delta_i)) \to \mathbb{R}$.

COROLLARY 1. *The image $d([0,\delta_i))$ equals*

$$\{\phi \in U_i \cap A \cap H : p\phi \in \mathrm{int}(p \circ \eta)\},$$

and there exists $\delta \in (0,\delta_i)$ with $P(d((0,\delta))) = d((0,\delta_i))$.

Proof. 1. The first assertion is shown as in part 1 of the proof of Proposition 5.2.1.

2. Choose a neighbourhood U_j of y_0 in U_i with $P(U_j \cap K) \subset U_i$, and $\delta \in (0,\delta_i)$ with $d((0,\delta)) \subset U_j$. For $0 < s < \delta$, $d(s) \in U_j \cap A \cap H = U_j \cap A \cap K$. Consequently, $P(d(s)) \in U_i$. Also,

$$P(d(s)) \in A, \quad P(d(s)) \in K \subset H, \quad pP(d(s)) \in \mathrm{int}(p \circ \eta).$$

It follows that $P(d(s)) \in \{\phi \in U_i \cap A \cap H : p\phi \in \mathrm{int}(p \circ \eta)\} = d((0,\delta_i))$. ∎

The transformed return map

$$P_t : (0,\delta) \ni s \mapsto d^{-1}(P(d(s))) \in (0,\delta_i)$$

is continuous and injective, with $P_t(s) \to 0$ as $s \to 0$. It follows that P_t is strictly increasing. The hypothesis (1) excludes fixed points of the map P_t. Therefore

$$s < P_t(s) \quad \text{for all } s \in (0,\delta), \tag{5}$$

or

$$P_t(s) < s \quad \text{for all } s \in (0,\delta). \tag{6}$$

Choose $s_0 \in (0,\delta)$. If (5) holds then there is a strictly increasing trajectory $(s_j)_{j=-\infty}^{0}$ of P_t with $s_j \to 0$ as $j \to -\infty$, and P_t maps each interval $(s_{j-1}, s_j]$, $j < 0$, homeomorphically onto $(s_j, s_{j+1}]$. If (6) holds then there is a strictly decreasing trajectory $(s_j)_{j=0}^{\infty}$ of P_t with $s_j \to 0$ as $j \to \infty$, and P_t maps each interval $(s_{j+1}, s_j]$, $j \ge 0$, homeomorphically onto $(s_{j+2}, s_{j+1}]$.

PROPOSITION 2. *Let $\phi \in A \cap K$ with $p\phi \in \mathrm{int}(p \circ \eta)$.*

(i) *If (5) holds then the trajectory $(\phi_j)_{j=-\infty}^{\infty}$ of P given by $\phi_0 = \phi$ satisfies $\phi_j \to y_0$ as $j \to -\infty$ and $\phi_j \to 0$ as $j \to \infty$. There exist $j \in \mathbb{Z}$ and $s \in (s_{-1}, s_0]$ with $\phi_j = d(s)$.*

(ii) *If (6) holds then the trajectory $(\phi_j)_{j=-\infty}^{\infty}$ of P given by $\phi_0 = \phi$ satisfies $\phi_j \to 0$ as $j \to -\infty$ and $\phi_j \to y_0$ as $j \to \infty$. There exist $j \in \mathbb{Z}$ and $s \in (s_{-1}, s_0]$ with $\phi_j = d(s)$.*

Proof. 1. We have $\phi \in X_i \subset X$. Suppose (5) holds. There is a trajectory $(\phi_j)_{j=-\infty}^{\infty}$ of P in the compact set X, with $\phi_0 = \phi$ and $pF_A(t,\phi) \in \mathrm{int}(p \circ \eta)$ for all $t \in \mathbb{R}$. It follows that both sets $\alpha(x(\phi))$ and $\omega(\phi)$ belong to the compact set

$$\{\phi \in A : p\phi \in \mathrm{int}(p \circ \eta) \cup |p \circ \eta|\}$$

which contains 0 and $|\eta|$ but no other periodic orbit. Therefore

$$\alpha(x(\phi)) = |\eta| \quad \text{and} \quad \omega(\phi) = \{0\}, \quad \text{or} \quad \alpha(x(\phi)) = \{0\} \quad \text{and} \quad \omega(\phi) = |\eta|.$$

Suppose the last statement holds. Every subsequence of $(\phi_j)_{j=0}^{\infty}$ has a subsequence which converges to a point ψ in $A \cap H \cap |\eta|$ with $\psi(0) \ge 0$. Using (2.3.3) and $y_0 \in K$ one finds that $\phi_j \to y_0$ as $j \to \infty$. All ϕ_j belong to $A \cap H$, with $0 \neq p\phi_j \in \mathrm{int}(p \circ \eta)$. Corollary 1 shows that there exists an integer j_0 with $\phi_j \in d((0, \delta_i))$ for all $j \geq j_0$. Consequently, $d^{-1}(\phi_j) \to 0$ as $j \to \infty$, and there is an integer $j_1 \geq j_0$ with $0 < d^{-1}(\phi_j) < \delta$ for all $j \geq j_1$. The points $d^{-1}(\phi_j)$, $j \geq j_1$, form a trajectory of P_t, and one arrives at a contradiction to $s < P_t(s)$ for all $s \in (0, \delta)$.

2. It follows that $\alpha(x(\phi)) = |\eta|$ and $\omega(\phi) = \{0\}$. The last equation gives $\phi_j \to 0$ as $j \to \infty$. As in part 1 one finds $\phi_j \to y_0$ as $j \to -\infty$, and there exists an integer n so that $(d^{-1}(\phi_j))_{j=-\infty}^{n}$ is a trajectory of P_t which converges to 0 as $j \to -\infty$. There are integers $k \leq n$ with

$$0 < d^{-1}(\phi_k) \leq s_0$$

and $m \leq 0$ with

$$s_{m-1} < d^{-1}(\phi_k) \leq s_m.$$

Set $s = P_t^{-m}(d^{-1}(\phi_k))$, $j = k - m$. Then $s \in (s_{-1}, s_0]$ and $d(s) = P^{-m}(\phi_k) = \phi_j$.

3. The proof of assertion (ii) is analogous. ∎

In case (5) holds the restriction $d|[0, s_0]$ is extended to a map γ from a compact interval into C as follows. For $j \in \mathbb{N}$ set

$$s_j = s_0 + \sum_{\iota=1}^{j} 2^{-\iota},$$

and consider the affine map $a_j : \mathbb{R} \to \mathbb{R}$ given by

$$a_j(s_{j-1}) = s_{-1}, \quad a_j(s_j) = s_0.$$

Define $g = s_0 + 1$, $G = [0, g]$, and set

$$\begin{aligned}
&\gamma(s) = d(s) && \text{for } 0 \leq s \leq s_0,\\
&\gamma(s) = P^j(d(a_j(s))) && \text{for } s_{j-1} < s \leq s_j,\ j \in \mathbb{N},\\
&\gamma(g) = 0.
\end{aligned}$$

Then

$$P(\gamma(s_j)) = P^{j+1}(d(s_0)) = P^{j+1}(d(a_{j+1}(s_{j+1}))) = \gamma(s_{j+1}) \quad \text{for all } j \geq 0,$$

and Proposition 2 yields

$$\gamma(s_j) \to 0 \quad \text{as } j \to \infty. \tag{7}$$

Proposition 3. *The map γ is injective, and*

$$\gamma(G) = X, \quad \gamma((0, g)) = X_i, \quad \gamma([0, g)) = \{\phi \in A \cap K : p\phi \in \mathrm{int}(p \circ \eta) \cup |p \circ \eta|\}.$$

Proof. 1. *Proof of the equations*: Let $\phi \in X$. Then either $\phi = 0 = \gamma(g)$, or $\phi = y_0 = d(0) = \gamma(0)$, or $p\phi \in \mathrm{int}(p \circ \eta)$. In the last case there is a trajectory $(\phi_j)_{j=-\infty}^{\infty}$ of P in X with $\phi_0 = \phi$. Proposition 2(i) gives an integer j and $s \in (s_{-1}, s_0]$ with $\phi_j = d(s)$. In case $j < 0$,

$$\phi = \phi_0 = P^{-j}(\phi_j) = P^{-j}(d(s)) = P^{-j}(d(a_{-j}(a_{-j}^{-1}(s)))) = \gamma(a_{-j}^{-1}(s)) \in \gamma((0, g))$$

In case $j \geq 0$ there exists $r \in (s_{-j-1}, s_{-j}]$ with $P_t^j(r) = s$, and the equations

$$P^j(\gamma(r)) = P^j(d(r)) = d(P_t^j(r)) = d(s) = \phi_j = P^j(\phi_0)$$

imply $\phi = \phi_0 = \gamma(r) \in \gamma((0, g))$.

Conversely, let $\phi \in \gamma(G)$. Then either $\phi = \gamma(0) = y_0 \in X$, or $\phi = \gamma(g) = 0 \in X$, or $\phi \in \gamma((0,g))$. In the last case, either $\phi = \gamma(s)$ with $s \in (0, s_0]$, and therefore

$$\begin{aligned}\phi = d(s) &\in \{\psi \in U_i \cap A \cap H : p\psi \in \operatorname{int}(p \circ \eta)\} \quad \text{(see Corollary 1)}\\ &\subset X_i,\end{aligned}$$

or there are $j \in \mathbb{N}$ and $s \in (s_{j-1}, s_j]$ with

$$\phi = \gamma(s) = P^j(d(a_j(s))) \in P^j(d(0, s_0]) \subset P^j(X_i) \subset X_i.$$

2. *Proof of injectivity*: The restrictions of γ to the intervals $[0, s_0]$ and $(s_{j-1}, s_j]$, $j \in \mathbb{N}$, are all injective, and

$$\gamma(g) = 0 \notin \{\phi \in A \cap K : p\phi \in \operatorname{int}(p \circ \eta) \cup |p \circ \eta|\} = \gamma([0, g)).$$

Also,

$$\begin{aligned}\gamma(0) = y_0 &\notin \{\phi \in A \cap K : p\phi \in \operatorname{int}(p \circ \eta)\} \cup \{0\}\\ &= X_i \cup \{0\} = \gamma((0,g)) \cup \{\gamma(g)\} = \gamma((0,g]).\end{aligned}$$

Next, consider $s \in (0, s_0]$ and $t \in (s_{j-1}, s_j]$, with $j \in \mathbb{N}$. Then $\gamma(s) = d(s)$, $\gamma(t) = P^j(d(a_j(t)))$, and $a_j(t) \in (s_{-1}, s_0] \subset (0, s_0]$. There is an integer $k \le 0$ with $s_{k-1} < s \le s_k$, and there exists $r \in (s_{k-1-j}, s_{k-j}] \subset (0, s_0]$ with

$$s = P_t^j(r), \quad d(s) = P^j(d(r)).$$

Observe that $r \le s_{k-j} \le s_{-1} < a_j(t)$. The injectivity of d and of $P|X$ now yield

$$\gamma(s) = d(s) = P^j(d(r)) \ne P^j(d(a_j(t))) = \gamma(t).$$

Finally, consider arguments $s \in (s_{k-1}, s_k]$ and $t \in (s_{j-1}, s_j]$ with $0 < k \le j$. Then

$$\gamma(s) = P^k(d(a_k(s))), \quad \gamma(t) = P^j(d(a_j(t))),$$

and

$$a_k(s) \in (s_{-1}, s_0], \quad a_j(t) \in (s_{-1}, s_0].$$

There exists $r \in (s_{k-j-1}, s_{k-j}]$ with $P_t^{j-k}(r) = a_k(s)$. Observe that $0 < r \le s_{k-j} \le s_{-1} < a_j(t) \le s_0$. The injectivity of d and of $P|X$ now yield

$$\gamma(s) = P^k(d(a_k(s))) = P^k(d(P_t^{j-k}(r))) = P^j(d(r)) \ne P^j(d(a_j(t))) = \gamma(t). \blacksquare$$

Proposition 4. *The map γ is continuous.*

Proof. 1. Continuity at points in $[0, s_0)$ and (s_j, s_{j+1}), $j \ge 0$, is obvious.

2. Let $j \ge 0$ be an integer. Continuity at s_j from the left is obvious. To show continuity from the right, let $(t_k)_{k=1}^{\infty}$ be a sequence in $(s_j, s_{j+1}]$ with $t_k \to s_j$ as $k \to \infty$. Set $a_0 = \mathrm{id}_{\mathbb{R}}$. For every positive integer k,

$$\gamma(t_k) = P^{j+1}(d(a_{j+1}(t_k))) = P^j(P(d(a_j(a_j^{-1}(a_{j+1}(t_k)))))).$$

The points $r_k = a_j^{-1}(a_{j+1}(t_k)) \in (s_{j-1}, s_j]$, $k \in \mathbb{N}$, converge to s_{j-1} as $k \to \infty$. Therefore $a_j(r_k) \to s_{-1}$ as $k \to \infty$, and

$$\gamma(t_k) = P^j(P(a_j(r_k))) \to P^j(P(d(s_{-1}))) = P^j(d(s_0)) = P^j(d(a_j(s_j))) = \gamma(s_j).$$

3. *Proof that γ is continuous at g*:

3.1. By the previous proposition, γ is injective, and $\gamma((0,g)) = X_i$. Parts 1 and 2 of the proof imply that the map

$$\gamma_i : (0,g) \ni s \mapsto \gamma(s) \in X_i$$

is continuous.

Claim: The map γ_i is a homeomorphism.

Proof: The set $X_i = \gamma((0,g))$ is connected. It is not compact since $\gamma(s_j) \to \gamma(0) = y_0$ as $j \to -\infty$ and $y_0 \notin X_i$. Consequently, there exists a homeomorphism h from an open interval $(a,b) \subset \mathbb{R}$ onto the one-dimensional C^1-submanifold X_i (see e.g. 23.19 in Chapter VI of [14]). It follows that $h^{-1} \circ \gamma_i$ is a homeomorphism onto the interval (a,b), and

$$\gamma_i^{-1} = (\gamma_i^{-1} \circ h) \circ h^{-1} = (h^{-1} \circ \gamma_i)^{-1} \circ h^{-1}$$

is continuous.

3.2. Assume γ is not continuous at g. Then there are a sequence $(t_k)_{k=1}^{\infty}$ in G and $\varepsilon > 0$ so that $t_k \to g$ as $k \to \infty$ and $\|\gamma(t_k)\| \geq \varepsilon$ for $k \in \mathbb{N}$. The compactness of $X = \gamma(G)$ permits extracting a subsequence of points $u_j = t_{k_j}$, $j \in \mathbb{N}$, so that $(\gamma(u_j))_{j=1}^{\infty}$ converges to a point $\phi \in \gamma(G)$ with $\|\phi\| \geq \varepsilon$.

3.2.1. *Claim*: There exist a sequence $(r_m)_{m=1}^{\infty}$ in $(0,g)$ and $s \in (0,g)$ so that $r_m \to g$ and $\gamma(r_m) \to \gamma(s)$ as $m \to \infty$.

Proof: Observe that $\phi \neq 0 = \gamma(g)$. Assume $\phi = \gamma(0) = y_0$. Let $\varepsilon = \|y_0\|$. There exists $j_0 \in \mathbb{N}$ so that

$$\|\gamma(u_j) - y_0\| < \varepsilon/3 \quad \text{and} \quad \|\gamma(s_j)\| < \varepsilon/3 \quad \text{for all } j \geq j_0.$$

The compact interval I_j with endpoints u_j, s_j is contained in $(0,g)$. The continuity of $\gamma|(0,g)$ implies that there is a point $w_j \in I_j$ with

$$\|\gamma(w_j) - y_0\| \geq \varepsilon/3 \quad \text{and} \quad \|\gamma(w_j)\| \geq \varepsilon/3.$$

Note $w_j \to g$ as $j \to \infty$. As above one finds a subsequence $(w_{j_m})_{m=1}^{\infty}$ and a point $\psi \in X$ so that $\gamma(w_{j_m}) \to \psi$ as $m \to \infty$, with

$$\|\psi - y_0\| \geq \varepsilon/3 \quad \text{and} \quad \|\psi\| \geq \varepsilon/3.$$

Therefore $\psi \in X \setminus \{0, y_0\} = \gamma((0,g))$.

3.2.2. Recall $P(X_i) = X_i$, and $X_i = \gamma((0,g))$. It follows that there exists $t \in (0,g)$ with $P(\gamma(s)) = \gamma(t)$. Fix $\varepsilon \in (0, \min\{t, g-t\})$. Part 3.1 of the proof implies that there is an open neighbourhood U_t of $\gamma(t)$ in C with $\gamma((t-\varepsilon, t+\varepsilon)) = U_t \cap \gamma((0,g))$. By the continuity of P, there is an open neighbourhood U_s of $\gamma(s)$ in C with

$$P(U_s \cap \gamma((0,g))) \subset U_t \cap P(\gamma((0,g))) \subset U_t \cap \gamma((0,g)).$$

Choose $j \in \mathbb{N}$ so large that $t + \varepsilon < r_j$ and $\gamma(r_j) \in U_s$. Choose $k \in \mathbb{N}$ with $s_k > r_j$. The map

$$P_i : (0,g) \ni r \mapsto \gamma_i^{-1}(P(\gamma(r))) \in \mathbb{R}$$

is continuous, and

$$P_i(s_k) = s_{k+1} > s_k, \quad P_i(r_j) \in (t-\varepsilon, t+\varepsilon) \subset (-\infty, r_j).$$

It follows that there is a fixed point of P_i in (r_j, s_k), and P has a fixed point in $\gamma((0,g)) = X_i$. This implies a contradiction to the hypothesis (1) that no solution $x(\phi)$ with $0 \neq p\phi \in \mathrm{int}(p \circ \eta)$ is periodic. ∎

Corollary 2. *Assume* (1) *and* (2). *If* (5) *holds then there exist* $g > 0$ *and a continuous injective map* $\gamma : [0, g] \to C$ *with*

$$\gamma([0,g]) = \{\phi \in A \cap H : 0 \leq \phi(0),\ p\phi \in \mathrm{int}(p \circ \eta) \cup |p \circ \eta|\}$$

and $\gamma(0) = y_0, \gamma(g) = 0$.

Remark 1. If (1), (2) and (6) hold then a construction analogous to the one above yields the same existence result as in Corollary 2.

6. Smoothness at periodic orbits

6.1. Interior periodic orbits. Let $\phi \in A \setminus (\{0\} \cup |\eta^b|)$ be such that $x(\phi)$ is periodic. Then $p\phi \in (pA)^\circ$. The aim of the present subsection is to prove that (4.1.1) holds. Choose $t > 1$ and consider the slowly oscillating periodic solution

$$y : \mathbb{R} \ni s \mapsto x(\phi)(s - t) \in \mathbb{R}$$

of equation (1.1.1), with minimal period $\omega > 2$ and orbit parametrization $\eta : [0, \omega] \to C$. Consider the closed hyperplane $C_y = C_< \oplus C_*$, its translate $H_y = y_0 + C_y$, the adapted Poincaré map $P_y : N_y \to H_y$, and recall Proposition 2.4.5.

Proposition 1 (y hyperbolic and unstable). *If* (2.4.5) *is satisfied and if* $\lambda_* > 1$ *then* (4.1.1) *holds.*

Proof. Recall Proposition 2.5.3(ii). Choose $\varepsilon > 0$ with $(-\varepsilon, \varepsilon)\phi_* \subset C_{*u}$. The C^1-curve $\alpha : (-1, 1) \to C$ given by

$$\alpha(s) = y_0 + s\varepsilon\phi_* + w_u(s\varepsilon\phi_*)$$

has range in A (see Corollary 2.5.2(ii)), and $\alpha(0) = y_0$. The vectors $\alpha'(0) = \varepsilon\phi_* \in C_y$ and $x(\phi)'_{-t} = y'_0 \in C \setminus C_y$ are linearly independent. Apply Proposition 4.1.1(i). ∎

The proofs of property (4.1.1) in the remaining cases make use of the constructions in Subsection 5.3. The hypothesis $\phi \in C \setminus |\eta^b|$ implies that equation (5.2.1) holds. Recall the continuous maps $d : [0, \delta_i) \to C$ and $d_e : (-\delta_e, 0] \to C$ from Subsection 5.3, with values in A and $d(0) = y_0 = d_e(0)$.

Proposition 2 (y hyperbolic and stable). *If* (2.4.5) *is satisfied and if* $\lambda_* < 1$ *then* (4.1.1) *holds.*

Proof. Consider a neighbourhood W_s of y_0 in N_y as in Proposition 2.5.3(i). There exist $s \in (0, \delta_i)$ and $s_e \in (-\delta_e, 0)$ so that $d(s) \in W_s$ and $d_e(s_e) \in W_s$. Propositions 5.3.2(iii) and 5.3.3(iii) show that there exists $\delta \in (0, \min\{\delta_i, \delta_e\})$ so that the conditions (5.3.1) and (5.3.3) are satisfied. Propositions 5.3.2(i) and 5.3.3(i) yield that the restrictions $d|(0,\delta)$ and $d_e|(-\delta, 0)$ are C^1-smooth with all derivatives injective. The reparametrizations $\varrho : R \to C$ and $\varrho_e : R_e \to C$ of $d|(0,\delta)$ and $d_e|(-\delta, 0)$, respectively, which are constructed in the last part of Subsection 5.3, are C^1-smooth and have the following properties: R is bounded from below, R_e is bounded from above, $|\varrho| \cup |\varrho_e| \subset A$, $|p \circ \varrho| \subset \mathrm{int}(p \circ \eta)$, $|p \circ \varrho_e| \subset \mathrm{ext}(p \circ \eta)$, $\varrho(r) \to y_0$ as $r \to \inf R$, $\varrho_e(r) \to y_0$ as $r \to \sup R_e$; there exist j and k in $\{0,1\}$ so that

$$\varrho'(r) \to (-1)^j \phi_* \quad \text{as } r \to \inf R$$

and

$$\varrho_e'(r) \to (-1)^k \phi_* \quad \text{as } t \to \sup R_e.$$

Set $r_i = \inf R$, $r_e = \sup R_e$. Then the set

$$I = (R_e - r_e) \cup \{0\} \cup (R - r_i)$$

is an open interval, and the map $\alpha_I : I \to C$ given by

$$\begin{aligned} \alpha_I(r) &= \varrho_e(r + r_e) \quad \text{for } 0 > r \in I, \\ \alpha_I(0) &= y_0, \\ \alpha_I(r) &= \varrho(r + r_i) \qquad \text{for } 0 < r \in I, \end{aligned}$$

is continuous and has range in A. The restrictions $\alpha_I|I \cap (-\infty, 0)$ and $\alpha_I|I \cap (0, \infty)$ are C^1-smooth, with $\alpha_I'(r) \to (-1)^k \phi_*$ as $0 > r \to 0$ and $\alpha_I'(r) \to (-1)^j \phi_*$ as $0 < r \to 0$. Moreover,

$$\begin{aligned} p\alpha_I(r) &\in \mathrm{ext}(p \circ \eta) \quad \text{for } 0 > r \in I, \\ p\alpha_I(r) &\in \mathrm{int}(p \circ \eta) \quad \text{for } 0 < r \in I. \end{aligned}$$

Proposition 2.4.3 says that $p\phi_*$ and $(p \circ \eta)'(0) = py_0'$ are linearly independent. An application of Proposition 2.1.1 to $p \circ \eta$ and $p \circ \alpha_I$ yields $k = j$, and α_I is C^1-smooth, with $\alpha_I'(0) = (-1)^j \phi_*$. Choose $\varepsilon > 0$ with $(-\varepsilon, \varepsilon) \subset I$ and define a C^1-curve $\alpha : (-1,1) \to C$ by $\alpha(s) = \alpha_I(\varepsilon s)$. Then $|\alpha| \subset A$, $\alpha(0) = y_0 = x(\phi)_{-t}$, and $\alpha'(0) = \varepsilon(-1)^j \phi_*$ and $x(\phi)'_{-t} = y_0'$ are linearly independent. Apply Proposition 4.1.1(i). ■

PROPOSITION 3 (y not hyperbolic). *If* (2.4.4) *is satisfied then* (4.1.1) *holds.*

Proof. 1 (*A center manifold*). Consider a C^1-map $w_c : C_{*c} \to C_<$, the set $W_c = y_0 + \{\chi + w_c(\chi) : \chi \in C_{*c}\}$, and an open neighbourhood N_c of y_0 in N_y as in Proposition 2.5.3(iii). In the sequel an application of Lemma 2.5.1 is prepared. The continuity of P_y at y_0 and the local invariance property of W_c yield an open neighbourhood N_{cc} of y_0 in N_c with $P_y(W_c \cap N_{cc}) \subset W_c$. Corollary 2.4.2 implies that the derivative of the C^1-map

$$W_c \cap N_{cc} \ni \phi \mapsto P_y(\phi) \in W_c$$

at y_0 is an isomorphism of $T_{y_0} W_c$. It follows that there is a neighbourhood N_i of y_0 in N_{cc} so that the restriction $P_y|W_c \cap N_i$ is injective. Choose $\varepsilon_i > 0$ with $(-\varepsilon_i, \varepsilon_i)\phi_* \subset C_{*c}$ and

$$y_0 + s\phi_* + w_c(s\phi_*) \in N_i \quad \text{for } |s| < \varepsilon_i.$$

Recall again that $p\phi_*$ and py_0' are linearly independent (Proposition 2.4.3). Proposition 2.1.1 applies, and there exist $s > 0$ and $\psi \in \{-\phi_*, \phi_*\}$ with

$$py_0 + (0,s)p\psi \subset \mathrm{int}(p \circ \eta), \quad py_0 + (-s,0)p\psi \subset \mathrm{ext}(p \circ \eta),$$

and $\varepsilon \in (0, \varepsilon_i)$ so that the C^1-curve

$$\zeta : (-\varepsilon, \varepsilon) \ni s \mapsto y_0 + s\psi + w_c(s\psi) \in C$$

with $\zeta(0) = y_0$ and $\zeta'(0) = \psi$, $(p \circ \zeta)'(0) = p\psi$, satisfies

$$(p \circ \zeta)((0, \varepsilon)) \subset \operatorname{int}(p \circ \eta), \quad (p \circ \zeta)((-\varepsilon, 0)) \subset \operatorname{ext}(p \circ \eta).$$

The map ζ defines a homeomorphism onto $|\zeta|$, and there exists an open neighbourhood N_ε of y_0 in N_y with $\zeta((-\varepsilon, \varepsilon)) = W_c \cap N_\varepsilon$. The continuity of P_y at y_0 gives $\varepsilon' \in (0, \varepsilon)$ with

$$P_y(\zeta((-\varepsilon', \varepsilon'))) \subset W_c \cap N_\varepsilon = \zeta((-\varepsilon, \varepsilon)),$$

and there is an open neighbourhood N' of y_0 in N_ε with $\zeta((-\varepsilon', \varepsilon')) = W_c \cap N'$.

2 (*Branches of the center manifold on A*). Recall the transformed return maps $P_j : (0, \delta_j) \to (0, \delta_i)$ and $P_f : (-\delta_f, 0) \to (-\delta_e, 0)$ of Subsection 5.3.

2.1. If (5.3.2) holds then Proposition 5.3.2(ii) guarantees a trajectory $(\psi_n)_{n=-\infty}^0$ of P_y in $N' \cap N_c$ with $p\psi_n \in \operatorname{int}(p \circ \eta)$ for all integers $n \le 0$. Proposition 2.5.3(iii) yields $\psi_n \in W_c$ for all $n \le 0$. Hence

$$\psi_n \in W_c \cap N' \cap p^{-1}(\operatorname{int}(p \circ \eta)) = \zeta((0, \varepsilon')) \quad \text{for all } n \le 0.$$

In particular,

$$\psi_0 = \zeta(s_+) \quad \text{for some } s_+ \in (0, \varepsilon').$$

Lemma 2.5.1 shows that for every $s \in (0, s_+]$ there is a trajectory $(\chi_n)_{n=-\infty}^0$ of P_y in $\zeta((-\varepsilon', \varepsilon'))$ with $\chi_0 = \zeta(s)$ and $|\zeta^{-1}(\chi_n)| \le |\zeta^{-1}(\psi_n)|$ for all $n \le 0$. Propositions 2.4.6(ii) and 2.3.1(i) combined yield $\zeta((0, s_+)) \subset A$. The restriction $\alpha_+ = \zeta|(0, s_+)$ satisfies $|\alpha_+| \subset A$, $\alpha_+(s) \to y_0$ as $s \to 0$, $\alpha_+'(s) \to \psi$ as $s \to 0$.

2.2. If (5.3.4) holds then arguments analogous to those in part 2.1 show that there exist $\varepsilon_- > 0$ and a C^1-curve $\alpha_- : (-\varepsilon, 0) \to C$ with $|\alpha_-| \subset A$, $\alpha_-(s) \to y_0$ as $s \to 0$, $\alpha_-'(s) \to \psi$ as $s \to 0$.

2.3. If (5.3.1) holds then the arguments used in the proof of the preceding proposition yield an open interval R which is bounded from below, a C^1-curve $\varrho : R \to C$, and $\chi \in \{-\phi_*, \phi_*\}$, with the properties

$$|\varrho| \subset A, \quad |p \circ \varrho| \subset \operatorname{int}(p \circ \eta), \quad \varrho(r) \to y_0 \quad \text{and} \quad \varrho'(r) \to \chi \quad \text{as } r \to \inf R.$$

Set $r_i = \inf R$, $I_+ = R - r_i$, and consider the C^1-curve

$$\alpha_+ : I_+ \ni r \mapsto \varrho(r + r_i) \in C.$$

Then $|\alpha_+| \subset A$, $|p \circ \alpha_+| \subset \operatorname{int}(p \circ \eta)$, $\alpha_+(r) \to y_0$ and $\alpha_+'(r) \to \chi$ as $r \to 0$. Recall from part 1 of the proof that there exists $\delta > 0$ so that the C^1-curve

$$\beta : (-\delta, 0) \ni r \mapsto py_0 + rp\psi \in L$$

with $\psi \in \{-\phi_*, \phi_*\}$ satisfies $|\beta| \subset \operatorname{ext}(p \circ \eta)$. An application of Proposition 2.1.2 to $p \circ \eta$ and to the curve $\theta : (-\delta, r) \to L$ given by

$$r \in I_+, \quad \theta|(-\delta, 0) = \beta, \quad \theta(0) = py_0, \quad \theta|(0, r) = (p \circ \alpha_+)|(0, r),$$

yields $p\chi = p\psi (\ne 0)$, and consequently the unit vectors χ and ψ in $\{-\phi_*, \phi_*\}$ coincide. Hence $\alpha_+'(r) \to \psi$ as $r \to 0$.

2.4. If (5.3.3) holds then arguments as in part 2.3 show that there exist an open interval I_- with $\sup I_- = 0$ and a C^1-curve $\alpha_- : I_- \to C$ with $|\alpha_-| \subset A$, $\alpha_-(r) \to y_0$ and $\alpha_-'(r) \to \psi$ as $r \to 0$.

3. In every combination of one of the cases (5.3.1), (5.3.2) with one of the cases (5.3.3), (5.3.4) one obtains an open interval $I \ni 0$ and a C^1-curve $\alpha_I : I \to C$ with $|\alpha_I| \subset A$ and $\alpha_I(0) = y_0$ so that the unit vector $\alpha_I'(0) = \psi \in C_y$ and $x(\phi)'_{-t} = y_0' \in C \setminus C_y$ are linearly independent. Complete the proof of property (4.1.1) as in the proof of the preceding proposition. ■

Corollary 1. *Let $\phi \in A \setminus (\{0\} \cup |\eta^b|)$. If $x(\phi)$ is periodic then $p\phi \in (pA)^\circ$, and there is an open neighbourhood N of $p\phi$ in $(pA)^\circ$ so that $a|N$ is C^1-smooth.*

6.2. Smoothness at the boundary. Let $\phi \in |\eta^b|$. Choose $t > 1$. There is a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) with $\phi = y_t$. Let $\omega > 2$ denote the minimal period of y. Consider the orbit parametrization $\eta : [0, \omega] \to C$, the closed hyperplane $C_y = C_< \oplus C_*$, its translate $H_y = y_0 + C_y$, and the adapted Poincaré map $P_y : N_y \to H_y$ as in the preceding subsection.

Proposition 1. *Either* (2.4.4) *holds*, *or* (2.4.5) *holds with* $\lambda_* < 1$.

Proof. Assume (2.4.5) and $1 < \lambda_*$. An application of Proposition 4.1.1(i) as in the proof of Proposition 6.1.1 yields $p\phi \in (pA)^\circ$, which contradicts to the equation

$$pA = \mathrm{int}(p \circ \eta^b) \cup |p \circ \eta^b|. \ ■$$

Proposition 2. *There exist an open neighbourhood N of $p\phi$ in L and a C^1-map $a_N : N \to Q$ with $a|N \cap pA = a_N|N \cap pA$.*

Proof. 1. Recall the continuous map $d : [0, \delta_i) \to C$ of Subsection 5.3, with $d(0) = y_0$ and range in $A \cap H_y$.

2. Suppose (2.4.5) holds with $\lambda_* < 1$. Consider a neighbourhood W_s of y_0 in N_y as in Proposition 2.5.3(ii). There exists $s \in (0, \delta_i)$ with $d(s) \in W_s$. Proposition 5.3.2(iii) shows that there exists $\delta \in (0, \delta_i)$ so that (5.3.1) is satisfied. Proposition 5.3.2(i) yields that the restriction $d|(0, \delta)$ is C^1-smooth, with all derivatives injective. The reparametrization $\varrho : R \to C$ of $d|(0, \delta)$ constructed in the last part of Subsection 5.3 is C^1-smooth and has the following properties: R is bounded from below, $|\varrho| \subset A$, $|p \circ \varrho| \subset \mathrm{int}(p \circ \eta)$, $\varrho(r) \to y_0$ as $r \to \inf R$; there exists $j \in \{0, 1\}$ so that

$$\varrho'(r) \to (-1)^j \phi_* \quad \text{as } r \to \inf R.$$

Set $r_i = \inf R$, choose $r_0 \in R$, and apply Proposition 4.1.1(ii) to the C^1-curve $\alpha : (-1, 1) \to C$ given by

$$\begin{aligned} &\alpha(s) = \varrho(r_i + s(r_0 - r_i)) && \text{for } 0 < s < 1, \\ &\alpha(0) = y_0, \\ &\alpha(s) = y_0 + s(r_0 - r_i)(-1)^j \phi_* && \text{for } -1 < s < 0, \end{aligned}$$

which satisfies $\alpha'(0) = (r_0 - r_i)(-1)^j \phi_* \in C_y \setminus \{0\}$.

3. Suppose (2.4.4) holds. Consider a C^1-map $w_c : C_{*c} \to C_<$, the set $W_c = y_0 + \{\chi + w_c(\chi) : \chi \in C_{*c}\}$, and an open neighbourhood N_c of y_0 in N_y as in Proposition 2.5.3(iii). As in part 1 of the proof of Proposition 6.1.3 one obtains an open neighbourhood N_i of y_0 in N_c so that $P_y|W_c \cap N_i$ is injective, and there exist a unit vector $\psi \in \{-\phi_*, \phi_*\}$, $\varepsilon > 0$,

and $\varepsilon' \in (0,\varepsilon)$ with the following properties: $s\psi \in C_{*c}$ for all $s \in (-\varepsilon,\varepsilon)$; the C^1-curve $\zeta : (-\varepsilon,\varepsilon) \ni s \mapsto y_0 + s\psi + w_c(s\psi) \in C$ satisfies

$$\zeta(0) = y_0, \quad |\zeta| \subset N_i, \quad (p\circ\zeta)((0,\varepsilon)) \subset \mathrm{int}(p\circ\eta), \quad (p\circ\zeta)((-\varepsilon,0)) \subset \mathrm{ext}(p\circ\eta);$$
$$\zeta'(0) = \psi, \quad (p\circ\zeta)'(0) = p\psi, \quad P_y(\zeta((-\varepsilon',\varepsilon'))) \subset \zeta((-\varepsilon,\varepsilon)).$$

Furthermore, there are open neighbourhoods N_ε of y_0 in N_y and N' of y_0 in N_ε with

$$\zeta((-\varepsilon,\varepsilon)) = W_c \cap N_\varepsilon, \quad \zeta((-\varepsilon',\varepsilon')) = W_c \cap N'.$$

Recall the transformed Poincaré map $P_j : (0,\delta_j) \to (0,\delta_i)$ of Subsection 5.3. If (5.3.2) holds then one finds $s_+ \in (0,\varepsilon')$ with $\zeta((0,s_+)) \subset A$ as in part 2.1 of the proof of Proposition 6.1.3, and an application of Proposition 4.1.1(ii) to the C^1-curve $\alpha : (-1,1) \ni r \mapsto \zeta(rs_+) \in C$ yields the assertion. If (5.3.1) holds then Proposition 5.3.2(i) is applicable, and a curve α which satisfies the hypotheses of Proposition 4.1.1(ii) is constructed as in part 2 above. ∎

7. Smoothness at the stationary point

7.1. Cases of no attraction. Recall the leading real part u_0 of the eigenvalues of the generator of the linearization of the semiflow F at the stationary point $0 \in C$.

PROPOSITION 1. *In case* $u_0 > 0$,

(1) *there exists an open neighbourhood* L_0 *of* 0 *in* $(pA)^0$ *so that the restriction* $a|L_0$ *is* C^1*-smooth.*

Proof. In [17] it is proved that there are an open neighbourhood L_0 of 0 in L and a C^1-map $w : L_0 \to Q$ with $w(0) = 0$ so that for every $\chi \in L_0 \setminus \{0\}$ there is a bounded slowly oscillating solution $x : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) with $x_0 = \chi + w(\chi)$. Therefore $\chi + w(\chi) \in A$ for every $\chi \in L_0$, or $L_0 \subset pA$, and $w(\chi) = q(\chi + w(\chi)) = a(p(\chi + w(\chi))) = a(\chi)$ for all $\chi \in L_0$. ∎

Set

$$I = \inf\{\|py_t\| : t \in \mathbb{R}, \text{ and } y : \mathbb{R} \to \mathbb{R} \text{ is a slowly oscillating periodic solution of equation (1.1.1)}\}.$$

PROPOSITION 2. *If* $u_0 = 0$ *and* $I = 0$ *then* (1) *holds.*

Proof. Consider a C^1-map $w^c : L \to Q$, the set $W^c = \{\chi + w^c(\chi) : \chi \in L\}$, and an open neighbourhood N of 0 in C as in Proposition 2.5.1(ii). Recall that, due to the general assumption $A \neq \{0\}$, the stationary point 0 is an inner point of pA. The continuity of a at 0 yields $\varepsilon > 0$ with $\chi \in pA$ and $\chi + a(\chi) \in N$ for all $\chi \in L$ with $\|\chi\| < \varepsilon$. According to Propositions 2.3.1(iii) and 2.2.2(v) there exists $b > 1$ with $z_2(\phi) < b$ for all $\phi \in A \setminus \{0\}$ with $\phi(-1) = 0$. By continuous dependence on initial data, there is a neighbourhood N' of 0 in N so that

$$\|pF(t,\phi)\| < \varepsilon \quad \text{for all } \phi \in N',\ t \in [0, b+1].$$

The hypothesis $I = 0$ and the continuity of a at 0 imply that there exist a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1) and $t \in \mathbb{R}$ with $y_t = py_t + a(py_t) \in N'$.

There exists $s \in \mathbb{R}$ with $y_s \in A \cap K$, and $\omega = z_2(y_s) + 1 \in (2, b+1)$ is the minimal period of y. Note that

$$\|py_s\| < \varepsilon \quad \text{for all } s \in [t, t+b+1].$$

It follows that the orbit parametrization $\eta : [0, \omega] \ni s \mapsto y_s \in C$ satisfies

$$|p \circ \eta| \subset \{\chi \in L : \|\chi\| < \varepsilon\},$$

which in turn implies that the open neighbourhood $L_0 = \mathrm{int}(p \circ \eta)$ of 0 in pA is contained in the disk $\{\chi \in L : \|\chi\| < \varepsilon\}$. Let $\chi \in L_0$, $x = x(\chi + a(\chi))$. For every $t \in \mathbb{R}$, $px_t \in \mathrm{int}(p \circ \eta)$, and therefore $\|px_t\| < \varepsilon$, $x_t = px_t + a(px_t) \in N$. Hence $x_0 \in W^{\mathrm{c}}$, and

$$a(\chi) = q(\chi + a(\chi)) = qx_0 = w^{\mathrm{c}}(px_0) = w(\chi).$$

The equation $a|L_0 = w|L_0$ yields the assertion. ∎

The final result of this subsection concerns the case

$$u_0 = 0 \quad \text{and} \quad 0 < I. \tag{1}$$

The proof makes use of the curve constructed in Subsection 5.4 which connects 0 in $A \cap H$ to a periodic orbit. Recall first that in case $0 < I$ Proposition 2.3.6 guarantees the existence of a slowly oscillating periodic solution $y^i : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1), with minimal period $w > 2$ and orbit parametrization $\eta^i : [0, \omega] \to C$, so that for every $\phi \in A \setminus \{0\}$ with $p\phi \in \mathrm{int}(p \circ \eta^i)$, $x(\phi)$ is not periodic, and either

$$\alpha(x(\phi)) = \{0\} \quad \text{and} \quad \omega(\phi) = |\eta^i|, \tag{3}$$

or

$$\alpha(x(\phi)) = |\eta^i| \quad \text{and} \quad \omega(\phi) = \{0\}. \tag{4}$$

PROPOSITION 3. *If* $u_0 = 0$, *if* $0 < I$, *and if there exists* $\phi \in A \setminus \{0\}$ *with* $p\phi \in \mathrm{int}(p \circ \eta^i)$ *and property* (3) *then* (1) *holds.*

Proof. 1. There exists $t_i \in \mathbb{R}$ with $F_A(t_i, y_0^i) \in K$. Consider the translate $y : \mathbb{R} \ni t \mapsto y^i(t + t_i) \in \mathbb{R}$ and its orbit parametrization $\eta : [0, \omega] \to C$. The remarks preceding the proposition show that the general hypothesis (5.4.1) for the results of Subsection 5.4 is satisfied. Propositions 2.3.1(i), 2.2.3, and 2.2.2(i), (v) combined imply that also hypothesis (5.4.2) is satisfied. It follows that there exist a compact interval $G = [0, g]$, $g > 0$, and a continuous injective map $\gamma : G \to C$ with

$$\gamma(0) = y_0, \quad \gamma(g) = 0, \quad \gamma(G) = \{\psi \in A \cap K : p\psi \in \mathrm{int}(p \circ \eta)\} \cup \{0, y_0\},$$

and the return map P defines a homeomorphism of the compact set $\gamma(G)$ onto itself.

1.1. *Claim*: There is a trajectory $(s_n)_{n=-\infty}^{\infty}$ of the homeomorphism

$$P_\gamma : G \ni s \mapsto \gamma^{-1}(P(\gamma(s))) \in G$$

in $(0, g)$ with $s_n \to g$ as $n \to -\infty$.

Proof: Propositions 2.2.3(i) and 2.3.5 applied to the solution $x(\phi)$ yield a sequence $(t_n)_{n=-\infty}^{0}$ in $\mathbb{R}$ with $t_n \to -\infty$ as $n \to -\infty$ so that the points $\phi_n = x(\phi)_{t_n}$, $n \le 0$, form a trajectory of P in $A \cap K \cap p^{-1}(\mathrm{int}(p \circ \eta)) = \gamma((0, g))$. Use (5.4.3) to obtain a trajectory $(\phi_n)_{n=-\infty}^{\infty}$ of P in $\gamma((0, g))$ with $\phi_n \to 0$ as $n \to -\infty$. Set $s_n = \gamma^{-1}(\phi_n)$ for all integers n. As $\alpha(x(\phi)) = \{0\}$, $\phi_n \to 0$ as $n \to -\infty$. Consequently, $s_n \to g$ as $n \to -\infty$.

1.2. *Claim*: For every $s \in (0,g)$, $P_\gamma(s) < s$.

Proof: There are no fixed points of P_γ in $(0,g)$ since otherwise, $x = x(\gamma(s))$ with $s \in (0,g)$ and $P(\gamma(s)) = \gamma(s)$ would be a periodic solution with $0 \neq x_0 \in A$ and $px_0 \in \mathrm{int}(p \circ \eta) = \mathrm{int}(p \circ \eta^i)$, contradicting the properties of y^i. It follows that either the assertion is correct, or $s < P_\gamma(s)$ for all $s \in (0,g)$. In the last case, $s_n < s_0 < g$ for all $n \le 0$, which contradicts $\lim_{n\to-\infty} s_n = g$.

2. Consider a C^1-map $w^c : L \to Q$, the set $W^c = \{\chi + w^c(\chi) : \chi \in L\}$, and an open neighbourhood N of 0 in C as in Proposition 2.5.1(ii). Proposition 2.2.2(v) shows that there exists $b > 0$ with

$$z_2(\psi) < b \quad \text{for all } \psi \in A \cap (K \cup (-K)).$$

By continuous dependence on initial data, there is an open neighbourhood N' of 0 in N with $F([0,b+1] \times N') \subset N$. Choose an integer n with $\gamma((s_n,g]) \subset N'$.

Claim: For every $\psi \in \gamma((s_n,g])$ and all $t \le 0$, $x(\psi)_t \in W^c$.

Proof: For every $s \in (s_n,g]$ there is a trajectory $(r_j)_{j=-\infty}^{\infty}$ of P_γ with $r_0 = s$ and

$$s_n < s = r_0 \le r_j \le g \quad \text{for all integers } j \le 0.$$

It follows that for every $\psi \in \gamma((s_n,g])$ there is a trajectory $(\psi_j)_{j=-\infty}^{\infty}$ of P in $\gamma((s_n,g]) \subset N'$ with $\psi_0 = \psi$. For each $j \le -1$,

$$\psi_{j+1} = F(t,\psi_j) \quad \text{with } 2 < t < b+1,$$

and one obtains $x(\psi)_t \in N$ for all $t \le 0$. Use property (2.5.1).

3. There is an open neighbourhood N_n of 0 in N' with $\gamma((s_n,g]) = \gamma(G) \cap N_n$. The continuity of a at 0 and continuous dependence on initial data yields an open neighbourhood L_0 of 0 in $\mathrm{int}(p \circ \eta) \subset pA$ with

$$F(t, \chi + a(\chi)) \in N_n \quad \text{for all } \chi \in L_0,\ t \in [0,b+1].$$

Let $\chi \in L_0 \setminus \{0\}$. Recall that the zeros of $x = x(\chi + a(\chi))$ are not bounded from below. Proposition 2.2.2(i), (v) imply that $x_s \in K$ for some $s \in [0,b+1]$. By Proposition 2.3.5, $px_s \in \mathrm{int}(p \circ \eta)$. It follows that

$$x_s \in A \cap K \cap p^{-1}(\mathrm{int}(p \circ \eta)) \cap N_n \subset \gamma(G) \cap N_n = \gamma((s_n,g]).$$

The last claim yields $\chi + a(\chi) = x_0 \in W^c$, or

$$a(\chi) = q(\chi + a(\chi)) = w^c(p(\chi + a(\chi))) = w^c(\chi).$$

Using also $a(0) = 0 = w^c(0)$, one arrives at $a|L_0 = w^c|L_0$. ■

7.2. On the inclination of tangent spaces of the attractor close to the stationary point. The investigation of the smoothness of the map a close to $0 \in (pA)^\circ$ in the remaining cases employs an inclination lemma which is derived below. For $\psi \in C \setminus Q$ define the inclination with respect to the decomposition $C = L \oplus Q$ by

$$i(\psi) = \|q\psi\| / \|p\psi\|.$$

PROPOSITION 1. *There exists $c_A > 0$ with $i(\psi) \le c_A$ for all $\psi \in T_\phi A \setminus \{0\}, \phi \in A$.*

Proof. Set $r = \max_{\phi\in A} \|\phi\|$. Consider the constant $c(r) > 0$ of Proposition 2.2.5. Let $\phi \in A$, $\psi \in T_\phi A$. There is a differentiable curve $\alpha : (-1,1) \to C$ with $\alpha(0) = \phi$, $|\alpha| \subset A$,

$\alpha'(0) = \psi$. For $|h| < 1$, set $x = x(\alpha(h))$, $y = x(\phi) = x(\alpha(0))$. Recall (2.3.1) and Corollary 2.2.1(i). Use Proposition 2.2.5 and deduce

$$c(r)\|\alpha(h) - \alpha(0)\| \le \|p\alpha(h) - p\alpha(0)\|.$$

For $0 < |h| < 1$ it follows that

$$\left\|q\left(\frac{1}{h}(\alpha(h) - \alpha(0))\right)\right\| \le \frac{\|q\|}{c(r)}\left\|p\left(\frac{1}{h}(\alpha(h) - \alpha(0))\right)\right\|.$$

Consequently,

$$\|q\psi\| \le \frac{\|q\|}{c(r)}\|p\psi\|,$$

and the assertion becomes obvious. ∎

For $\phi \in A$ define

$$i(T_\phi A) = \sup\{i(\psi) : 0 \ne \psi \in T_\phi A\} \le c_A.$$

PROPOSITION 2. *Suppose $L_0 \subset pA$ is open in L, and the restriction $a|L_0$ is C^1-smooth. Let $\chi \in L_0$. Then*

$$T_{\chi + a(\chi)}A = \{\varrho + Da(\chi)\varrho : \varrho \in L\} \quad \textit{and} \quad \|Da(\chi)\| = i(T_{\chi + a(\chi)}A).$$

The proof is omitted.

For a set $B \subset C$ and an integer $n \ge 0$ define $B_n = F(\{n\} \times B)$. The set B is said to converge to 0 if for every $\varepsilon > 0$ there is an integer $n_\varepsilon \ge 0$ so that

$$B_n \subset \{\phi \in C : \|\phi\| < \varepsilon\} \quad \text{for all integers } n \ge n_\varepsilon.$$

PROPOSITION 3. *Suppose $B \subset A$ converges to 0, and for every integer $n \ge 0$ there is an open subset L_n of $(pA)^\circ$ so that $a|L_n$ is C^1-smooth and $pB_n \subset L_n$. Then*

$$\sup_{\phi \in B_n} i(T_\phi A) \to 0 \quad \textit{as } n \to \infty.$$

P r o o f. 1. The derivative $T(1) = D_2F(1,0)$ defines an isomorphism T_L of L onto L and a continuous linear map of Q into Q with spectra

$$\sigma_L = \exp(\sigma \cap (\mathbb{R} + i[-2\pi, 2\pi])), \quad \sigma_Q = \{0\} \cup \exp(\sigma \setminus (\mathbb{R} + i[-2\pi, 2\pi])),$$

respectively. Choose s, t in $\mathbb{R}$ with

$$\sup_{\lambda \in \sigma_Q} |\lambda| < s < t < \min_{\lambda \in \sigma_L} |\lambda|.$$

Then $\sigma(T_L^{-1}) \subset \{\lambda \in \mathbb{C} : |\lambda| \le 1/t\}$. There exist a norm $\|\cdot\|_0$ on C and constants $c_1 > 0$, $c_2 \ge c_1$ so that

$$\begin{array}{ll} c_1\|\phi\|_0 \le \|\phi\| \le c_2\|\phi\|_0 & \text{for all } \phi \in C, \\ t\|\phi\|_0 \le \|T(1)\phi\|_0 & \text{for all } \phi \in L, \\ \|T(1)\phi\|_0 \le s\|\phi\|_0 & \text{for all } \phi \in Q. \end{array}$$

The remainder map $R = F(1,\cdot) - T(1)$ is C^1-smooth and satisfies $R(0) = 0$, $DR(0) = 0$. Set

$$c = c_A\frac{c_2}{c_1}.$$

For all $\phi \in A$ and all $\psi \in T_\phi A$,

$$\|q\psi\|_0 \le c\|p\psi\|_0,$$

due to Proposition 1, and

$$\begin{aligned}\|qD_2F(1,\phi)\psi\|_0 &\le \|qT(1)\psi\|_0 + \|qDR(\phi)\|_0(1+c)\|p\psi\|_0\\ &= \|T(1)q\psi\|_0 + \|qDR(\phi)\|_0(1+c)\|p\psi\|_0\\ &\le s\|q\psi\|_0 + \|qDR(\phi)\|_0(1+c)\|p\psi\|_0\end{aligned}$$

and

$$\begin{aligned}\|pD_2F(1,\phi)\psi\|_0 &\ge t\|p\psi\|_0 - \|pDR(\phi)\|_0(1+c)\|p\psi\|_0\\ &= (t - \|pDR(\phi)\|_0(1+c))\|p\psi\|_0.\end{aligned}$$

For $\psi \in C \setminus Q$ set

$$\iota_0(\psi) = \|q\psi\|_0/\|p\psi\|_0.$$

Obviously, $\iota_0(\psi) \le c$ for all $\psi \in T_\phi A \setminus \{0\}$, $\phi \in A$. Choose $\beta \in (s/t, 1)$ There exists an integer $n_0 > 0$ such that for all integers $n \ge n_0$ and for all $\phi \in B_n$,

$$0 < \frac{s}{t - \|pDR(\phi)\|_0(1+c)} < \beta,$$

and the sequence

$$\varepsilon_n = \sup_{\phi \in B_n} \frac{\|qDR(\phi)\|_0(1+c)}{t - \|pDR(\phi)\|_0(1+c)}, \quad n \ge n_0,$$

converges to 0 as $n \to \infty$.

2. *Claim*: For every $\varepsilon > 0$ there is an integer $n_\varepsilon \ge n_0$ so that

$$\iota_0(\psi) < \varepsilon \quad \text{for all } \psi \in T_\phi A \setminus \{0\},\ \phi \in B_n,\ n \ge n_\varepsilon.$$

Proof: Let $\varepsilon > 0$. Choose an integer $j \ge n_0$ with

$$\varepsilon_\iota \frac{1}{1-\beta} < \frac{\varepsilon}{2} \quad \text{for all } \iota \ge j,$$

and choose an integer $k \ge 0$ with $\beta^k c < \varepsilon/2$. Set $n_\varepsilon = j + k$. Take an integer $n \ge n_\varepsilon$, $\phi \in B_n$, and $\psi \in T_\phi A \setminus \{0\}$. There exist $\phi_\nu \in B_\nu$, $\nu \in \{0, \dots, n\}$, with $\phi = \phi_n$ and $\phi_{\nu+1} = F(1, \phi_\nu)$ for all $\nu \in \{0, \dots, n-1\}$. Proposition 2 shows that each

$$T_{\phi_\nu} A = \{\chi + Da(\phi_\nu)\chi : \chi \in L\}, \quad \nu \in \{0, \dots, n\},$$

is a 2-dimensional linear space. Recall that all maps $D_2F(1, \phi_\nu)$, $\nu \in \{0, \dots, n\}$, are injective, and

$$D_2F(1,\phi_\nu)T_{\phi_\nu}A \subset T_{\phi_{\nu+1}}A \quad \text{for all } \nu \in \{0, \dots, n-1\}.$$

It follows that there exist $\psi_\nu \in T_{\phi_\nu} A \setminus \{0\}$, $\nu \in \{0, \dots, n\}$, so that $\psi_n = \psi$ and $\psi_{\nu+1} = D_2F(1, \phi_\nu)\psi_\nu$ for all $\nu \in \{0, \dots, n-1\}$. The estimates in part 1 of the proof yield

$$\iota_0(\psi_{\nu+1}) = \iota_0(D_2F(1,\phi_\nu)\psi_\nu) \le \beta\iota_0(\psi_\nu) + \varepsilon_\nu$$

for all $\nu \in \{j, \dots, n-1\}$. Hence

$$\iota_0(\psi) = \iota_0(\psi_n) \le \beta^{n-j}\iota_0(\psi_j) + \sum_{\nu=0}^{n-1-j} \varepsilon_{n-1-\nu}\beta^\nu \le \beta^k c + (\max_{j \le \nu \le n-1} \varepsilon_\nu)\frac{1}{1-\beta} < \varepsilon.$$

3. Use

$$\iota(\psi) \le \frac{c_2}{c_1}\iota_0(\psi) \quad \text{for all } \psi \in C \setminus Q$$

to complete the proof. ■

7.3. The cases of attraction. Recall from Subsection 7.1 that in case

$$0 < I = \inf\{\|py_t\| : t \in \mathbb{R}, \text{ and } y : \mathbb{R} \to \mathbb{R} \text{ is a slowly oscillating periodic solution of equation (1.1.1)}\}$$

there exists a slowly oscillating periodic solution $y^i : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1), with minimal period $\omega > 2$ and orbit parametrization $\eta^i : [0,\omega] \to C$, so that for each $\phi \in A \setminus \{0\}$ with $p\phi \in \mathrm{int}(p \circ \eta^i)$ the solution $x(\phi)$ is not periodic, and either (7.1.3) holds, or (7.1.4) holds.

PROPOSITION 1. *If* $u_0 = 0$, *if* $0 < I$, *and if there exists* $\phi \in A\setminus\{0\}$ *with* $p\phi \in \mathrm{int}(p\circ\eta^i)$ *and property* (7.1.4), *then* (7.1.1) *holds.*

Proof. 1. As in part 1 of the proof of Proposition 7.1.3, there is a translate $y : \mathbb{R} \to \mathbb{R}$ of y^i so that the orbit parametrization $\eta : [0,\omega] \to C$ of y satisfies $\eta(0) = y_0 \in K$. There exist a compact interval $G = [0,g]$, $g > 0$, and a continuous injective map $\gamma : G \to C$ with $\gamma(0) = y_0$, $\gamma(g) = 0$,

$$\gamma(G) = \{\psi \in A \cap K : p\psi \in \mathrm{int}(p \circ \eta)\} \cup \{0, y_0\},$$

and the return map P defines a homeomorphism of the compact set $\gamma(G)$ onto itself.

1.1. *Claim*: There is a trajectory $(s_n)_{n=-\infty}^{\infty}$ of the homeomorphism

$$P_\gamma : G \ni S \mapsto \gamma^{-1}(P(\gamma(s))) \in G$$

in $(0,g)$ with $s_n \to g$ as $n \to \infty$.

Proof: As in part 1.1 of the proof of Proposition 7.1.3 one finds a trajectory $(\phi_n)_{n=-\infty}^{\infty}$ of P in $\gamma((0,g))$ and a sequence $(t_n)_{n=-\infty}^{\infty}$ in $\mathbb{R}$ so that $\phi_n = x_{t_n}$ for all integers n. In particular, $\phi_n \neq 0$ for all n, and therefore $t_{n+1} > t_n + 2$ for all n. Now $\omega(\phi) = \{0\}$ yields $\phi_n \to 0$ as $n \to \infty$, and the sequence $s_n = \gamma^{-1}(\phi_n)$, $n \in \mathbb{Z}$, converges to $g = \gamma^{-1}(0)$ as $n \to \infty$.

1.2 Arguing as in part 1.2 of the proof of Proposition 7.1.3 one obtains $s < P_\gamma(s)$ for every $s \in (0,g)$. In particular, $0 < s_n < s_{n+1} < g$ and $P_\gamma((s_n,g]) = (s_{n+1},g]$ for all integers n.

2. Proposition 2.2.2(v) shows that there exists $b > 1$ with

$$z_2(\psi) + 1 < b \quad \text{for all } \psi \in A \cap (K \cup (-K)).$$

Set $B = F([0,b] \times \gamma([s_0,s_1]))$ and $B_n = F(\{n\} \times B)$ for all integers $n \ge 0$ as in the preceding subsection.

2.1. *Claim*: B converges to 0.

Proof: Let $\varepsilon > 0$. Choose $\delta > 0$ so that $\|F(t,\psi)\| < \varepsilon$ for all $t \in [0,b]$ and all $\psi \in C$ with $\|\psi\| < \delta$. Choose an integer $n_\delta > 0$ with $\|\gamma(s)\| < \delta$ for all $s \in [s_{n_\delta}, g]$. For every $\psi \in B_n = F(\{n\} \times B)$ with $n \ge (n_\delta + 1)b$ there exist $t \in [0,b]$ and $s \in [s_0,s_1]$ so that

$$\psi = F(n, F(t,\gamma(s))) = F(n+t,\gamma(s)),$$

and there is a unique integer $j > 0$ with

$$\sum_{k=0}^{j-1}[z_2(P^k(\gamma(s))) + 1] \leq n + t < \sum_{k=0}^{j}[z_2(P^k(\gamma(s))) + 1].$$

It follows that for some $r \in [0, z_2(P^j(\gamma(s))) + 1) \subset [0, b]$,

$$\begin{aligned}\psi = F(n+t, \gamma(s)) &= F(r, F(\sum_{k=0}^{j-1}[z_2(P^k(\gamma(s))) + 1], \gamma(s)) \\ &= F(r, P^j(\gamma(s))) \in F([0,b] \times \gamma(P_\gamma^j([s_0, s_1]))) = F([0,b] \times \gamma([s_j, s_{j+1}])).\end{aligned}$$

Using $2 \leq z_2(P^k(\gamma(s))) + 1 \leq b$ for all $k \in \{0, \ldots, j\}$ one finds

$$(j+1)b \geq n + t \geq n \geq (n_\delta + 1)b,$$

or $j \geq n_\delta$. Therefore

$$\psi \in F([0,b] \times \gamma([s_j, s_{j+1}])) \subset F([0,b] \times \gamma([s_{n_\delta}, g])),$$

and finally $\|\psi\| < \varepsilon$.

2.2. *Claim*: For every integer $n \geq 0$ there is an open neighbourhood U_n of 0 in C with

$$(A \cap U_n) \setminus \{0\} \subset \bigcup_{k=n}^{\infty} B_k.$$

Proof: Fix an integer $n \geq 0$. There exist an integer $j \geq 1$ with $2j - b - 1 \geq n$, an open neighbourhood U of 0 in C so that $\gamma((s_j, g]) = \gamma(G) \cap U$, and an open neighbourhood U_n of 0 in U so that $F([0,b] \times U_n) \subset U$ and $pU \subset \mathrm{int}(p \circ \eta)$. Let $\psi \in (A \cap U_n) \setminus \{0\}$ be given. Recall that the zeros of $x(\psi)$ are not bounded from below. Use Proposition 2.2.2(i), (v) to obtain $t \in [0,b]$ with $F(t, \psi) \in K$. Proposition 2.3.5 yields $pF(t, \psi) \in \mathrm{int}(p \circ \eta)$ since $p\psi \in pU$. It follows that

$$0 \neq F(t, \psi) \in [A \cap K \cap p^{-1}(\mathrm{int}(p \circ \eta))] \cap U \subset \gamma(G) \cap U = \gamma((s_j, g]),$$

and so $F(t, \psi) \in \gamma((s_j, g))$. There exist an integer $m \geq j$, $r \in (s_m, s_{m+1}]$, and $s \in (s_0, s_1]$ so that

$$F(t, \psi) = \gamma(r) = \gamma(P_\gamma^m(s)) = P^m(\gamma(s)) = F(t_m, \gamma(s))$$

with

$$t_m = \sum_{\kappa=0}^{m-1}[z_2(P^\kappa(\gamma(s))) + 1] \geq 2m \geq 2j > b \geq t.$$

The injectivity of $F(t, \cdot)$ yields $\psi = F(t_m - t, \gamma(s))$. There is an integer $k \geq 0$ with $k \leq t_m - t < k + 1$, and

$$\psi = F(k, F(t_m - t - k, \gamma(s)) \in F(\{k\} \times F([0,1) \times \gamma([s_0, s_1]))) \subset B_k.$$

Observe that $k > t_m - t - 1 \geq 2m - b - 1 \geq 2j - b - 1 \geq n$.

3. Recall $\gamma([s_0, s_1]) \subset A \cap p^{-1}(\mathrm{int}(p \circ \eta))$. Proposition 2.3.5 yields $pB_n \subset \mathrm{int}(p \circ \eta)$ for all integers $n \geq 0$. Moreover,

$$pB_n \subset \mathrm{int}(p \circ \eta) \setminus \{0\} \quad \text{for all } n \geq 0$$

since $0 \notin \gamma([s_0, s_1])$, $B \subset A \setminus \{0\}$. Corollary 4.2.1(i) guarantees that the restriction of a to the open subset $L_0 = \mathrm{int}(p \circ \eta) \setminus \{0\}$ of $(pA)^\circ$ is C^1-smooth. Proposition 7.2.3 applies, and

$$\sup_{\psi \in B_n} i(T_\psi A) \to 0 \quad \text{as } n \to \infty.$$

Using Proposition 7.2.2 one finds

$$\sup_{\chi \in pB_n} \|Da(\chi)\| \to 0 \quad \text{as } n \to \infty.$$

3.1. *Claim*: $Da(\chi) \to 0$ as $0 \neq \chi \to 0$.

Proof: Let $\varepsilon > 0$. There exists an integer n with $\|Da(\chi)\| < \varepsilon$ for all $\chi \in pB_k$, $k \geq n$. Consider an open neighbourhood U_n of 0 in C as in claim 2.2. Choose an open neighbourhood L_ε of 0 in $\mathrm{int}(p \circ \eta)$ with $\{\chi + a(\chi) : \chi \in L_\varepsilon\} \subset U_n$. For every $\chi \in L_\varepsilon \setminus \{0\}$ there exists an integer $k \geq n$ with

$$\chi + a(\chi) \in B_k, \quad \chi = p(\chi + a(\chi)) \in pB_k, \quad \|Da(\chi)\| < \varepsilon.$$

3.2. *Claim*: a is differentiable at $0 \in (pA)^\circ$, and $Da(0) = 0$.

Proof: Let $\varepsilon > 0$. There exists $\delta > 0$ with $\|Da(\chi)\| < \varepsilon$ for $0 < \|\chi\| < \delta$. For such χ and for every integer $n \geq 1$,

$$\left\| a(\chi) - a\left(\frac{1}{n}\chi\right) \right\| \leq \varepsilon \left\| \chi - \frac{1}{n}\chi \right\| = \varepsilon\left(1 - \frac{1}{n}\right)\|\chi\|.$$

As $a\left(\frac{1}{n}\chi\right) \to a(0)$ for $n \to \infty$,

$$\|a(\chi) - a(0) - 0(\chi - 0)\| = \|a(\chi)\| \leq \varepsilon\|\chi\| = \varepsilon\|\chi - 0\|.$$

4. Now it is obvious how to complete the proof that the restriction $a|\mathrm{int}(p \circ \eta)$ is C^1-smooth. ∎

Finally, the case of a stable hyperbolic stationary point remains to be considered.

Proposition 2. *If* $u_0 < 0$ *then* (7.1.1) *holds.*

Proof. 1. According to Proposition 2.5.1(i) there are an open neighbourhood W^s of 0 in C and constants $c \geq 1$, $k \in [0, 1)$ with

$$\|F(n, \phi)\| \leq ck^n\|\phi\| \quad \text{for all } \phi \in W^s \text{ and all integers } n \geq 0.$$

The continuity of a at $0 \in (pA)^\circ$ yields $\varepsilon > 0$ so that $\chi + a(\chi) \in W^s$ and $\|\chi + a(\chi)\| < 1$ for all $\chi \in L$ with $\|\chi\| < \varepsilon$. Then no solution $x(\chi + a(\chi))$, $\|\chi\| < \varepsilon$, is periodic, and

$$\varepsilon \leq \inf\{\|py_t\| : t \in \mathbb{R}, \text{ and } y : \mathbb{R} \to \mathbb{R} \text{ is a slowly oscillating periodic solution of equation (1.1.1)}\}.$$

Proposition 2.3.6 shows that there exists a slowly oscillating periodic solution $y : \mathbb{R} \to \mathbb{R}$ of equation (1.1.1), with minimal period $\omega > 2$ and orbit parametrization $\eta : [0, \omega] \to C$, so that no solution $x(\phi)$ with $\phi \in A \setminus \{0\}$ and $p\phi \in \mathrm{int}(p \circ \eta)$ is periodic. Corollary 4.2.1(i) guarantees that the restriction $a|\mathrm{int}(p \circ \eta) \setminus \{0\}$ is C^1-smooth. The relations $0 \in \mathrm{int}(p \circ \eta)$ and $\varepsilon \leq \|py_t\|$ for $0 \leq t \leq \omega$ yield $\{\chi \in L : \|\chi\| < \varepsilon\} \subset \mathrm{int}(p \circ \eta)$. Set $B = \{\chi + a(\chi) : 0 < \|\chi\| < \varepsilon\}$ and $B_n = F(\{n\} \times B)$ for all integers $n \geq 0$. Then

$B \subset W^s \cap \{\phi \in C : \|\phi\| < 1\}$ converges to 0. Using the inclusion $pB \subset \text{int}(p \circ \eta)$ and Proposition 2.3.5 one finds

$$pB_n \subset \text{int}(p \circ \eta) \quad \text{for all } n \geq 0.$$

As $B \subset A \setminus \{0\}$, we have $B_n \subset A \setminus \{0\}$, and $0 \notin pB_n$ for all $n \geq 0$.

Propositions 7.2.3 and 7.2.2 combined yield

$$\sup_{\chi \in pB_n} \|Da(\chi)\| \to 0 \quad \text{as } n \to \infty.$$

The set

$$pB_n \cup \{0\} = pF_A(\{n\} \times \{\chi + a(\chi) : \|\chi\| < \varepsilon\})$$

is an open neighbourhood of $0 \in (pA)^\circ$ in pA since the map

$$pA \ni \chi \mapsto pF_A(n, \chi + a(\chi)) \in pA$$

is a homeomorphism.

2. *Claim*: $Da(\chi) \to 0$ as $0 \neq \chi \to 0$.

Proof: Let $\varepsilon > 0$. There exists an integer n with $\|Da(\chi)\| < \varepsilon$ for all $\chi \in pB_n = (pB_n \cup \{0\}) \setminus \{0\}$; $pB_n \cup \{0\}$ is a neighbourhood of 0 in pA.

3. Complete the proof that the restriction $a|\text{int}(p \circ \eta)$ is C^1-smooth as in parts 3.1 and 4 of the proof of the preceding proposition. ∎

References

[1] R. Abraham and J. Robbin, *Transversal Mappings and Flows*, Benjamin, New York, 1967.

[2] S. N. Chow and K. Lu, *C^k centre unstable manifolds*, Proc. Roy. Soc. Edinburgh Sect. A 108 (1988), 303–320.

[3] S. N. Chow and H. O. Walther, *Characteristic multipliers and stability of symmetric periodic solutions of $\dot{x}(t) = g(x(t-1))$*, Trans. Amer. Math. Soc. 307 (1988), 127–142.

[4] O. Diekmann, S. van Gils, S. Verduyn Lunel and H. O. Walther, *Delay Equations*: *Complex-*, *Functional- and Nonlinear Analysis*, Springer, New York, 1995.

[5] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, 1988.

[6] J. K. Hale and X. B. Lin, *Symbolic dynamics and nonlinear semiflows*, Ann. Mat. Pura Appl. 144 (1986), 229–259.

[7] G. Iooss, *Bifurcation of Maps and Applications*, North-Holland Math. Stud. 36, North-Holland, Amsterdam, 1979.

[8] B. Lani-Wayda and H. O. Walther, *Chaotic motion generated by delayed negative feedback. Part I*: *A transversality criterion*, Differential Integral Equations 8 (1995), 1407–1452.

[9] B. Lani-Wayda and M. Yebdri, *On local center manifolds*, technical report, in preparation.

[10] J. Mallet-Paret and G. Sell, *Systems of differential delay equations I*: *Floquet multipliers and discrete Lyapunov functions*, J. Differential Equations 125 (1996), 385–440.

[11] —, —, *The Poincaré–Bendixson theorem for monotone cyclic feedback systems with delay*, ibid., 441–489.

[12] J. Mallet-Paret and H. O. Walther, *Rapid oscillations are rare in scalar systems governed by monotone negative feedback with a time lag*, preprint, Universität Gießen, 1994.

[13] A. Neugebauer, *Invariante Mannigfaltigkeiten und Neigungslemmata für Abbildungen in Banachräumen*, diploma thesis, Universität München, 1988.

[14] W. Rinow, *Lehrbuch der Topologie*, Deutscher Verlag der Wiss., Berlin, 1975.

[15] I. Tereščák, *Dynamical systems with discrete Lyapunov functionals*, Ph.D. thesis, Comenius University, Bratislava, 1994.

[16] H. O. Walther, *On instability, ω-limit sets and periodic solutions of nonlinear autonomous differential delay equations*, in: Functional Differential Equations and Approximation of Fixed Points, H. O. Peitgen and H. O. Walther (eds.), Lecture Notes in Math. 730, Springer, Heidelberg, 1979, 489–503.

[17] —, *An invariant manifold of slowly oscillating solutions for $\dot{x}(t) = -\mu x(t) + f(x(t-1))$*, J. Reine Angew. Math. 414 (1991), 67–112.

[18] —, *On Floquet multipliers of periodic solutions of delay equations with monotone nonlinearities*, in: Proc. Internat. Sympos. on Functional Differential Equations (Kyoto, 1990), T. Yoshizawa and J. Kato (eds.), World Sci., Singapore, 1991, 349–356.

[19] —, *Unstable manifolds of periodic orbits of a differential delay equation*, Oscillations and Dynamics in Delay Equations, J. R. Graef and J. K. Hale (eds.), Amer. Math. Soc., Providence, 1992, 177–240.

[20] —, *The 2-dimensional attractor of $x'(t) = -\mu x(t) + f(x(t-1))$*, Mem. Amer. Math. Soc. 544 (1995).