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A WEIGHTED SIEVE OF GREAVES' TYPE I

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1. Introduction

Let $\mathscr{A}$ be a finite integer sequence and $\mathscr{P}$ a set of primes. Following the notation of [2], $|\mathscr{A}|$ denotes the cardinality of $\mathscr{A}$, $\mathscr{A}_d = \{a \in \mathscr{A};\ a \equiv 0 \bmod d\}$ (so that $\mathscr{A}_1 = \mathscr{A}$) and

$$P(z) = \prod_{\substack{p<z\\ p\in\mathscr{P}}} p, \qquad P(z_1, z) = P(z)/P(z_1) = \prod_{\substack{z_1 \leqslant p < z\\ p\in\mathscr{P}}} p \qquad (2 \leqslant z_1 \leqslant z).$$

The classical sifting function

$$S(\mathscr{A}, z) = S(\mathscr{A}, \mathscr{P}, z) = \left|\{a \in \mathscr{A}\colon (a, P(z)) = 1\}\right| = \sum_{d|P(z)} \mu(d)|\mathscr{A}_d|$$

has been intensively studied and much is known about it for a wide range of sieve problems. Later, in Sections 3 and 4, we shall need to recall some of this information and even to amplify it.

In these papers we shall study the weighted sifting function

$$H(\mathscr{A}, z_1, z) = H(\mathscr{A}, \mathscr{P}, z_1, z) = \sum_{a\in\mathscr{A}} \gamma\big((a, P(z))\big) \tag{1.1}$$

where the weight $\gamma(\cdot)$ is defined as follows. Let

$$\{x\}^+ = \max(x, 0)$$

and let $w(p)$ denote an arithmetic function on the primes p of $\mathscr{P}$ that satisfies

$$0 \leqslant w(p) \leqslant 1, \qquad w(p) = 0 \text{ if } p < z_1; \tag{1.2}$$

then

$$\gamma(n) = \{1 - \sum_{\substack{p|n \\ p\in\mathscr{P}}} (1-w(p))\}^{+}. \tag{1.3}$$

It is clear that $\gamma(n) = 0$ if n is divisible by a prime of $\mathscr{P}$ less than z_1, so that H may be written in what will be to some readers the more familiar form

$$(1.4)\quad H(\mathscr{A}, z_1, z) = \sum_{\substack{a\in\mathscr{A} \\ (a,P(z_1))=1}} \gamma\big((a, P(z_1, z))\big) = \sum_{\substack{a\in\mathscr{A} \\ (a,P(z_1))=1}} \{1 - \sum_{\substack{p|a,\, p\in\mathscr{P} \\ z_1 \leqslant p < z}} (1-w(p))\}^{+}.$$

A weighted sifting function of this kind was first introduced by Heath-Brown and the authors in [3]. Inspection of (1.3) suggests that, with a suitable choice of $w(\cdot)$, $\gamma(n) > 0$ only when n has few prime factors in $\mathscr{P}$; that is, $\gamma\big((a, P(z))\big) > 0$ only when a has no prime factor from $\mathscr{P}$ less than z_1 and only a few prime factors from $\mathscr{P}$ between z_1 and z (cf. (1.4)). To show that $\mathscr{A}$ contains such integers a it suffices to show that H is positive. Therefore our objective is to obtain a lower bound for H. In [3] we analysed H in terms of S-functions and obtained a lower bound by bounding each S with a positive sign from below, and each S with a negative sign from above, using the classical upper and lower estimates of S. In his remarkable and important memoir [1], Greaves was the first to show how one might avoid this wasteful process by applying to H (actually Greaves used a somewhat different sifting function) directly the combinatorial method that Rosser and Iwaniec had applied so successfully to S. In these papers we approach H after the manner of Greaves. However, we have learnt from his pioneering work to give a much simplified account of his method (we sketched our new approach in [4]), we have strengthened this method by introducing a more versatile parameter system, and we have extended the scope and generality of the new method to prepare it for applicability. Specifically, in I we establish a general theory for all dimensions $\varkappa$, $1/2 \leqslant \varkappa \leqslant 1$; Lemmas 1, 2, 5, 7 and 9 embody the novelty of our approach. In II we concentrate on the linear sieve, the case $\varkappa = 1$, and show how to incorporate the bilinear form of the remainder sum a la Iwaniec (cf. [8]), and we sketch some applications, notably to the problem of almost-primes in short intervals. There it will be useful to incorporate also the improvement over [3] introduced by Iwaniec and Laborde [7].

2. Description of a class of sieve problems

Let $\mathscr{B} = \mathscr{B}(\mathscr{P})$ denote the set of all positive squarefree integers all of whose prime factors are in $\mathscr{P}$. We assume that $\mathscr{A}$ is well distributed over arithmetic progressions $0 \bmod d$, $d \in \mathscr{B}$, in the following sense: there exists a convenient approximation X to $|\mathscr{A}|$, and there exists also a multiplicative function $\omega(\cdot)$ on $\mathscr{B}$ satisfying

$$(A_0) \qquad 0 \leqslant \omega(p) < p, \quad p \in \mathscr{P},$$

such that

(i) the 'remainders'

$$(2.1) \qquad R_d = |\mathscr{A}_d| - \frac{\omega(d)}{d} X$$

are small on average over the divisors d of $P(z)$ that are less than a certain parameter $y = y(X)$ (with the nature of this average left open for the present but, for example, one might require that

$$\sum_{\substack{d|P(z)\\ d<y}} |R_d| \ll X(\log z)^{-100});$$

(ii) there exist constants $A \geqslant 1$ and $\varkappa$, $1/2 \leqslant \varkappa \leqslant 1$, such that

$$\Omega(\varkappa) \qquad \left|\sum_{\substack{z_1 \leqslant p < z_2\\ p \in \mathscr{P}}} \frac{\omega(p)}{p} \log p - \varkappa \log \frac{z_2}{z_1}\right| \leqslant A, \quad 2 \leqslant z_1 \leqslant z_2;$$

all constants implied by the use of the O- and $\ll$- notations here and later to depend at most on A and $\varkappa$. (There are applications where dependence on A needs to be kept explicit, and this can easily be done where the need arises; for the sake of simplicity we do not do so in this account.)

Condition $\Omega(\varkappa)$ tells us that $\omega(p)$ is about equal to $\varkappa$ on average over the primes of $\mathscr{P}$. In the Iwaniec method for S it suffices to know (in the above sense) that $\omega(p)$ is at most $\varkappa$ on average over $\mathscr{P}$; but here there is one stage where, at present, we require the full force of $\Omega(\varkappa)$.

It is convenient to define $\omega(p) = 0$ when $p \notin \mathscr{P}$. We write

$$V(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right)$$

and quote from [2] several basic results that we shall need about V and ω. We have, by Lemma 5.3 of [2] (because $\Omega(\varkappa)$ together with (A_0) imply condition (Ω_1) in [2]) that

$$(2.2) \qquad \frac{V(z_1)}{V(z_2)} = \left(\frac{\log z_2}{\log z_1}\right)^{\varkappa} \left\{1 + O\left(\frac{1}{\log z_1}\right)\right\} \ll \left(\frac{\log z_2}{\log z_1}\right)^{\varkappa}, \quad 2 \leqslant z_1 \leqslant z_2,$$

and in particular that

$$(2.3) \qquad 1/V(z_2) \ll \log^{\varkappa} z_2, \quad 2 \leqslant z_2.$$

Also, by Lemma 2.3 of [2],

$$\sum_{z_1 \leqslant p < z_2} \frac{\omega(p)}{p} \leqslant \varkappa \log \frac{\log z_2}{\log z_1} + \frac{A}{\log z_1}, \qquad 2 \leqslant z_1 \leqslant z_2, \tag{2.4}$$

whence ([2], (2.3.8))

$$\omega(p)/p \leqslant A/\log p. \tag{2.5}$$

Finally, by (2.2)

$$\sum_{d \mid P(z_1, z_2)} \frac{\omega(d)}{d} \leqslant \frac{V(z_1)}{V(z_2)} \ll \left(\frac{\log z_2}{\log z_1}\right)^{\varkappa}, \qquad 2 \leqslant z_1 \leqslant z_2. \tag{2.6}$$

We end this section with some more notation that we shall require later. Let $\nu(d)$ denote the number of distinct prime divisors of an integer d. Also, if $d > 1$ let $q(d)$ and $p(d)$ denote respectively the largest and least prime factors of d; and, for the sake of completeness, write

$$q(1) = 1, \qquad p(1) = \infty.$$

When $\nu(d) \geqslant 2$, write

$$q_2(d) = q\big(d/q(d)\big), \qquad p_2(d) = p\big(d/p(d)\big)$$

for the second largest and second smallest prime factors of d.

3. Combinatorial sieves

In Lemma 2 below we give a Fundamental Identity, a generalized and weighted form of the Inclusion-Exclusion Principle, that underlies all known 'small' sieves, and to which the simple Lemma 1 gives surprisingly direct access. Lemma 2 occurs for the first time, in relation to $S(\mathscr{A}, \mathscr{P}, z)$, in [2], (2.18), but is not fully exploited there; and it plays a more central rôle, still in relation to $S(\mathscr{A}, \mathscr{P}, z)$, as Lemma 1 of [6] and later as Theorem 6 of [8].

LEMMA 1. *Let $n > 1$ have canonical prime decomposition*

$$n = p_1 \dots p_r, \qquad p_1 > \dots > p_r.$$

Then, for any arithmetic function $g(\cdot)$,

$$\sum_{\substack{1 < d \mid n \\ q(n/d) < p(d)}} g(d) = \sum_{j=1}^{r} g(p_1 \dots p_j).$$

Proof. Either $q(n/d) = 1$, in which case $d = n$ and so contributes the term $g(n) = g(p_1 \dots p_r)$; or $q(n/d) = p_i$ for some i, $1 \leqslant i \leqslant r$. In the latter case we have necessarily that

$$d = p_1 \dots p_{i-1} t, \quad t | p_{i+1} \dots p_r.$$

When $t > 1$, $p(d) = p(t) \leqslant p_{i+1} < p_i \ (= q(n/d))$, and this contradicts the second summation condition on the left. Hence $t = 1$ and, since $d > 1$, $i = 1$ is ruled out. Altogether then, the sum on the left is

$$g(n) + \sum_{i=2}^{r} g(p_1 \dots p_{i-1}) = \sum_{j=1}^{r} g(p_1 \dots p_j).$$

LEMMA 2 (Fundamental Identity). *Let $\chi(\cdot)$ be an arithmetic function satisfying $\chi(1) = 1$, and associate with $\chi(\cdot)$ the 'conjugate' function $\bar{\chi}$ given by*

(3.1) $$\bar{\chi}(1) = 0, \quad \bar{\chi}(d) = \chi(d/p(d)) - \chi(d) \quad \text{when } d > 1.$$

Then, for any arithmetic function $\varphi(\cdot)$,

(3.2) $$\sum_{d|P(z)} \mu(d)\varphi(d) = \sum_{d|P(z)} \mu(d)\chi(d)\varphi(d) + \sum_{d|P(z)} \mu(d)\bar{\chi}(d) \sum_{t|P(p(d))} \mu(t)\varphi(dt).$$

Proof. Apply Lemma 1 with $g = \bar{\chi}$. Then, by (3.1), we have

(3.3) $$\sum_{\substack{d|n \\ q(n/d) < p(d)}} \bar{\chi}(d) = 1 - \chi(n).$$

Let $dt = n$ in the second sum on the right of (3.2), which now takes the form

$$\sum_{n|P(z)} \mu(n)\varphi(n) \sum_{\substack{d|n \\ q(n/d) < p(d)}} \bar{\chi}(d) = \sum_{n|P(z)} \mu(n)\varphi(n)(1 - \chi(n))$$

by (3.3).

When we take $\varphi(d) = |\mathscr{A}_d|$ in (3.2) we obtain at once

(3.4) $$S(\mathscr{A}, z) = \sum_{d|P(z)} \mu(d)\chi(d)|\mathscr{A}_d| + \sum_{d|P(z)} \mu(d)\bar{\chi}(d) S(\mathscr{A}_d, p(d)).$$

Now let χ_y^+ and χ_y^- be the basic upper and lower Buchstab–Rosser–Iwaniec functions for a $\varkappa$-dimensional sieve with parameter y. We recall their definition from [6]. We have $\chi_y^\pm(1) = 1$, and for $1 < d = p_1 \dots p_v$, $p_1 > \dots > p_v$,

(3.5) $$\chi_y^+(d) = 1 \quad \text{if } p_{2i-1}^{\beta+1} p_{2i-2} \dots p_1 < y \quad (1 \leqslant i \leqslant \tfrac{1}{2}(v+1)),$$

(3.6) $$\chi_y^-(d) = 1 \quad \text{if } p_{2i}^{\beta+1} p_{2i-1} \dots p_1 < y \quad (1 \leqslant i \leqslant \tfrac{1}{2}v),$$

and otherwise $\chi_y^\pm(d) = 0$; here $\beta = \beta(\varkappa)$ is a function of $\varkappa$ that satisfies

(3.7) $$\beta(\tfrac{1}{2}) = 1, \quad 1 < \beta(\varkappa) < 2 \quad (\tfrac{1}{2} < \varkappa < 1), \quad \beta(1) = 2.$$

Observe that (3.5) and (3.6) with (3.7) imply $d < y$ whenever $\chi_y^\pm(d) = 1$. With χ_y^+ and χ_y^- thus defined it follows from (3.4) that

(3.8) $$\sum_{d|P(z)} \mu(d)\chi_y^-(d)|\mathscr{A}_d| \leqslant S(\mathscr{A}, z) \leqslant \sum_{d|P(z)} \mu(d)\chi_y^+(d)|\mathscr{A}_d|;$$

replacing $|\mathscr{A}_d|$ in these bounding sums by $X\omega(d)/d+R_d$ and disregarding (here) the remainder sums

(3.9)
$$\sum_{d|P(z)} \mu(d)\chi_y^{\pm}(d)R_d$$

that then occur, the upper and lower estimates of $S(\mathscr{A}, z)$ depend respectively on $XT_\varkappa^+(y, z)$ and $XT_\varkappa^-(y, z)$ where

(3.10)
$$T_\varkappa^{\pm}(\eta, \zeta) = \sum_{d|P(\zeta)} \mu(d)\chi_\eta^{\pm}(d)\frac{\omega(d)}{d}.$$

Here we are not at all interested in bounds for $S(\mathscr{A}, z)$ as such, but the sums $T_\varkappa^{\pm}(\eta, \zeta)$ occur naturally in the Greaves approach and also in ours. We recall from the pioneering work of Iwaniec [6] that, if $\eta \geqslant \zeta \geqslant 2$,

(3.11)
$$T_\varkappa^+(\eta, \zeta) = V(\zeta)\left\{F_\varkappa\left(\frac{\log\eta}{\log\zeta}\right)+O(\log^{-1/3}\eta)\right\} \ll V(\zeta),$$

(3.12)
$$T_\varkappa^-(\eta, \zeta) = V(\zeta)\left\{f_\varkappa\left(\frac{\log\eta}{\log\zeta}\right)+O(\log^{-1/3}\eta)\right\} \ll V(\zeta),$$

where, for some positive constant $A_\varkappa$,

(3.13)
$$F_\varkappa(s) = A_\varkappa s^{-\varkappa}, \quad 0 < s \leqslant \beta+1,$$

(3.14)
$$f_\varkappa(s) = 0, \quad s \leqslant \beta,$$

and otherwise $F_\varkappa, f_\varkappa$ satisfy the simultaneous differential difference equations

(3.15)
$$\begin{aligned} (s^\varkappa F_\varkappa(s))' &= \varkappa s^{\varkappa-1} f_\varkappa(s-1), \quad & s > \beta+1,\\ (s^\varkappa f_\varkappa(s))' &= \varkappa s^{\varkappa-1} F_\varkappa(s-1), \quad & s > \beta, \end{aligned}$$

so that, by (3.13), (3.14) and (3.15),

(3.16)
$$(\beta+2)^\varkappa f_\varkappa(\beta+2) = \varkappa A_\varkappa \int_\beta^{\beta+2} \left(\frac{t}{t-1}\right)^\varkappa \frac{dt}{t}.$$

The O-constants in (3.11) and (3.12) depend at most on $A, \varkappa$ and on

$$s = \log\eta/\log\zeta.$$

Curiously enough, we shall use these formulae only for the 'small' values $s = \beta$, $\beta+1$ and $\beta+2$. We shall develop the properties of $T_\varkappa^{\pm}(\eta, \zeta)$ that we require later, in Section 4.

We return to combinatorial considerations, and apply Lemma 2 – the Fundamental Identity – to H. First define

(3.17)
$$W(d) = \sum_{t|d} \mu(t)\gamma(t), \quad d|P(z),$$

(note that our W differs from Greaves') so that by Möbius inversion,

$$\gamma(n) = \sum_{d|n} \mu(d) W(d), \qquad n | P(z)$$

and, by (1.1),

$$H(\mathscr{A}, z_1, z) = \sum_{d|P(z)} \mu(d) W(d) |\mathscr{A}_d|. \tag{3.18}$$

Also, let

$$\gamma_q(n) = \sum_{d|n} \mu(d) W(qd), \qquad \gamma_1(n) = \gamma(n), \tag{3.19}$$

and

$$H_q(\mathscr{A}, z_1, z) = H_q(\mathscr{A}, \mathscr{P}, z_1, z) = \sum_{d|P(z)} \mu(d) W(qd) |\mathscr{A}_{qd}|, \qquad (q, P(z)) = 1, \tag{3.20}$$

so that $H_1(\mathscr{A}, z_1, z) = H(\mathscr{A}, z_1, z)$. Then, by (3.19) and (3.20),

$$H_q(\mathscr{A}, z_1, z) = \sum_{a \in \mathscr{A}_q} \gamma_q\big((a, P(z))\big), \qquad (q, P(z)) = 1. \tag{3.21}$$

We now apply Lemma 2, with $\varphi(d) = W(d) |\mathscr{A}_d|$ and $\chi = \chi_y^-$, to (3.18) and obtain (cf. (3.4))

$$H(\mathscr{A}, z_1, z) = \sum_{d|P(z)} \mu(d) \chi_y^-(d) W(d) |\mathscr{A}_d| + \sum_{d|P(z)} \mu(d) \bar{\chi}_y^-(d) H_d(\mathscr{A}, z_1, p(d)). \tag{3.22}$$

Comparison with the treatment of $S(\mathscr{A}, z)$ suggests that (cf. (3.8))

$$H(\mathscr{A}, z_1, z) \geqslant \sum_{d|P(z)} \mu(d) \chi_y^-(d) W(d) |\mathscr{A}_d| \tag{3.23}$$

should be true, and we are encouraged toward that belief by the fact that, in the second sum on the right of (3.22), $\mu(d) \bar{\chi}_y^-(d) = 1$ if $\nu(d)$ is even and positive, and is otherwise zero. Thus (3.23) does hold it

$$H_d(\mathscr{A}, z_1, p(d)) \geqslant 0 \qquad \text{whenever } d | P(z),\ 2|\nu(d) > 0,\ \bar{\chi}_y^-(d) = 1\,; \tag{3.24}$$

but in order to prove (3.24) we shall be forced to a choice of $w(\cdot)$ and to place constraints on various parameters (yet to be introduced) that are less advantageous than one might have hoped for. Moreover, even with (3.24) established and (3.23) true, we are not sure that retention of some of the terms in the second sum on the right of (3.22) might not lead to better results than (3.23) yields. Indeed, it could well be that $\chi = \chi_y^-$ is not the right choice in Lemma 2 when applied to H. We shall set these doubts aside in the present study, and deal with (3.24) in Section 5.

4. The sums $T_\varkappa^\pm(\eta, \zeta)$

We define

(4.1) $$\sigma_\varkappa^+(\eta) = T_\varkappa^+(\eta, \eta^{1/(\beta+1)}), \quad \sigma_\varkappa^-(\eta) = T_\varkappa^-(\eta, \eta^{1/\beta}) \quad (\eta > 1)$$

and summarize the only results we require from formulae (3.11) to (3.16) in

LEMMA 3. *For* $\eta > 1$ *we have*

(4.2) $$T_\varkappa^-(\eta, \eta^{1/(\beta+2)}) \geqslant V(\eta)\{(\beta+2)^\varkappa f_\varkappa(\beta+2) + O(\log^{-1/3}\eta)\},$$

(4.3) $$\sigma_k^+(\eta) \ll 1, \quad \sigma_k^-(\eta) \ll V(\eta)\log^{-1/3}\eta,$$

and

(4.4) $$\sigma_\varkappa^+(\eta/\zeta) = V(\eta) A_\varkappa \frac{\log^\varkappa \eta}{\log^\varkappa(\eta/\zeta)}\{1 + O(\log^{-1/3}(\eta/\zeta))\}, \quad 2 \leqslant \zeta \leqslant \tfrac{1}{4}\eta.$$

Proof. Of these, (4.2) follows from (3.12) and (2.2) (even with asymptotic equality), as also does, after use of (3.13),

(4.5) $$\sigma_\varkappa^+(\eta) = V(\eta) A_\varkappa\{1 + O(\log^{-1/3}\eta)\}$$

and hence the first part of (4.3), and (4.4). The second part of (4.3) follows from (3.12), (3.14) and (2.2).

(*Note.* It would be of interest to give an independent 'ab initio' proof of (4.5).)

Our main tool in the treatment of the sum $T_\varkappa^\pm(\eta, \zeta)$ leading to Lemma 5 below – the Reduction Lemma – is a form of the ever useful Buchstab identity.

LEMMA 4 (Buchstab). *For* $2 \leqslant \zeta_1 < \zeta_2$ *we have*

(4.6) $$T_\varkappa^+(\eta, \zeta_2) = T_\varkappa^+(\eta, \zeta_1) - \sum_{\substack{\zeta_1 \leqslant p < \zeta_2 \\ p^{\beta+1} < \eta}} \frac{\omega(p)}{p} T_\varkappa^-\left(\frac{\eta}{p}, p\right)$$

and

(4.7) $$T_\varkappa^-(\eta, \zeta_2) = T_\varkappa^-(\eta, \zeta_1) - \sum_{\zeta_1 \leqslant p < \zeta_2} \frac{\omega(p)}{p} T_\varkappa^+\left(\frac{\eta}{p}, p\right).$$

Proof. By (3.10)

$$T_\varkappa^\pm(\eta, \zeta_2) - T_\varkappa^\pm(\eta, \zeta_1) = \sum_{\zeta_1 \leqslant q(d) < \zeta_2} \mu(d)\chi_\eta^\pm(d)\frac{\omega(d)}{d}.$$

Put $q(d) = p$ and $d = pt$ and observe that, from the definitions of $\chi_\eta^\pm$,

(4.8) $$\chi_\eta^{(-)^\nu}(d_1 d_2) = \chi_\eta^{(-)^\nu}(d_1)\chi_{\eta/d_1}^{(-)^{\nu(d_1)+\nu}}(d_2) \quad \text{if} \quad q(d_2) < p(d_1)$$

$(\nu = 0, 1),$

so that, in particular (taking $d_1 = p$, $d_2 = t$), $\chi_\eta^{\pm}(pt) = \chi_\eta^{\pm}(p)\chi_{\eta/p}^{\mp}(t)$. Hence

$$T_\varkappa^{\pm}(\eta, \zeta_2) - T_\varkappa^{\pm}(\eta, \zeta_1) = -\sum_{\zeta_1 \leqslant p < \zeta_2} \frac{\omega(p)}{p} \chi_\eta^{\pm}(p) \sum_{q(t) < p} \mu(t) \chi_{\eta/p}^{\mp}(t) \frac{\omega(t)}{t}$$
$$= -\sum_{\zeta_1 \leqslant p < \zeta_2} \frac{\omega(p)}{p} \chi_\eta^{\pm}(p)\, T_\varkappa^{\mp}\left(\frac{\eta}{p}, p\right).$$

Since $\chi_\eta^-(p) = 1$, and $\chi_\eta^+(p) = 1$ if and only if $p^{\beta+1} < \eta$, both results follow.

Let us record here also the Buchstab formula ([2], Lemma 7.1)

$$\sum_{\zeta_1 \leqslant p < \zeta_2} \frac{\omega(p)}{p} V(p) = V(\zeta_1) - V(\zeta_2), \quad 2 \leqslant \zeta_1 < \zeta_2. \tag{4.9}$$

We come now to the principal result of this section, the Reduction Lemma. This result does not in itself represent any significant improvement over (3.11) and (3.12), but its form proves singularly convenient when, later on, the sums $T_\varkappa^{\pm}(\eta, \zeta)$ occur inside some dauntingly complicated expressions (see (6.5) below). The Reduction Lemma in continuous form, and with $\varkappa = 1$, was first known to us from a paper of Siebert [9] and occurs also in Greaves [1].

LEMMA 5 (Reduction Lemma). *Let*

$$\log \eta / \log \zeta \ll 1. \tag{4.10}$$

Then

$$T_\varkappa^{(-)^\nu}(\eta, \zeta) = \sum_{\substack{p(t) > \zeta \\ q^{\beta-1}(t) < \eta/t \\ \nu(t) \equiv \nu \bmod 2}} \mu^2(t) \frac{\omega(t)}{t} \sigma_\varkappa^+\left(\frac{\eta}{t}\right) + O\left(V(\zeta) \log^{-1/3} \zeta\right),$$

where

$$2 \leqslant \zeta,\ 1 < \eta \quad \textit{if} \quad \nu = 0, \quad \textit{and} \quad 2 \leqslant \zeta < \eta^{1/\beta} \quad \textit{if} \quad \nu = 1. \tag{4.11}$$

Proof. We begin with Lemma 4. By (4.6) with $\zeta_1 = \zeta$ and $\zeta_2 = \eta^{1/(\beta+1)}$ (so that the conditions $p < \zeta_2$ and $p^{\beta+1} < \eta$ in (4.6) coincide) and by (4.7) with $\zeta_1 = \zeta$ and $\zeta_2 = \eta^{1/\beta}$ respectively, we have

$$T_\varkappa^+(\eta, \zeta) = \sigma_\varkappa^+(\eta) + \sum_{\zeta \leqslant p < \eta^{1/(\beta+1)}} \frac{\omega(p)}{p} T_\varkappa^-\left(\frac{\eta}{p}, p\right), \quad 2 \leqslant \zeta < \eta^{1/(\beta+1)}, \tag{4.12}$$

and

$$T_\varkappa^-(\eta, \zeta) = \sigma_\varkappa^-(\eta) + \sum_{\zeta \leqslant p < \eta^{1/\beta}} \frac{\omega(p)}{p} T_\varkappa^+\left(\frac{\eta}{p}, p\right), \quad 2 \leqslant \zeta < \eta^{1/\beta}. \tag{4.13}$$

Actually, the condition $\zeta < \eta^{1/(\beta+1)}$ in (4.12) may be dropped and replaced by the minimal requirement $\eta > 1$. For if $\zeta \geqslant \eta^{1/(\beta+1)}$ the sum in (4.12) is empty and (4.12) reads

$$(4.14) \qquad T_\varkappa^+(\eta, \zeta) = \sigma_\varkappa^+(\eta), \quad \zeta \geqslant \eta^{1/(\beta+1)},$$

which is correct. To see this note, in the definition (3.10) of $T_\varkappa^+(\eta, \zeta)$, that $\chi_\eta^+(d) = 1$ implies (cf. (3.5)) that the prime factors of d satisfy certain inequalities of which the first ($i = 1$ in (3.5) with η in place of y) reads $q(d)^{\beta+1} < \eta$. This means that $d|P(\eta^{1/(\beta+1)})$, and this renders the summation condition $d|P(\zeta)$ in (3.10) redundant if $\zeta \geqslant \eta^{1/(\beta+1)}$. Thus

$$(4.12') \quad T_\varkappa^+(\eta, \zeta) = \sigma_\varkappa^+(\eta) + \sum_{\zeta \leqslant p < \eta^{1/(\beta+1)}} \frac{\omega(p)}{p} T_k^-\left(\frac{\eta}{p}, p\right), \quad 2 \leqslant \zeta,\ 1 < \eta.$$

We now modify (4.12′) and (4.13) in one respect – in the sum on the right of each we prefer to have the summation condition $\zeta < p$ and we merely estimate the contribution from the possible term $T_\varkappa^\pm\left(\frac{\eta}{\zeta}, \zeta\right)\frac{\omega(\zeta)}{\zeta}$. In (4.12′) – (4.12) really, with $\zeta^{\beta+1} < \eta$, otherwise the sum is empty – we have $\eta/\zeta > \zeta^\beta \geqslant \zeta \geqslant 2$, so that (3.12) and (2.5) together give

$$T_\varkappa^-\left(\frac{\eta}{\zeta}, \zeta\right)\frac{\omega(\zeta)}{\zeta} \ll V(\zeta)\frac{1}{\log \zeta}.$$

The estimation is a little more complicated in the case of (4.13), although (2.5) applies again and we have to deal with $T_\varkappa^+(\eta/\zeta, \zeta)/\log\zeta$. If $\eta/\zeta \geqslant \zeta \geqslant 2$ we apply (3.11) and obtain at once the estimate $\ll V(\zeta)/\log\zeta$ as before. But if $\eta/\zeta < \zeta$, or $\eta^{1/2} < \zeta$, then $(\eta/\zeta)^{1/(\beta+1)} < \eta^{1/(\beta+1)} \leqslant \eta^{1/2} < \zeta$ and (4.14) applies; we have therefore to estimate

$$\sigma_\varkappa^+(\eta/\zeta)/\log\zeta.$$

Suppose first that $\varkappa > 1/2$. Here $\beta = \beta(\varkappa) > 1$ and therefore $\eta/\zeta > \zeta^{\beta-1}$ together with $\eta < \zeta^2$ allows us to deduce from (4.4) that $\sigma_\varkappa^+(\eta/\zeta) \ll V(\eta) \leqslant V(\zeta)$ and we obtain once more the estimate $V(\zeta)/\log\zeta$. Now take the case $\varkappa = 1/2$, when $\beta = 1$. Here $\eta > \zeta$ and therefore, by (4.3), $\sigma_{1/2}^+(\eta/\zeta) \ll 1$. Also, (2.3) with $z_2 = \zeta$ and $\varkappa = 1/2$ tells us that

$$1 \ll V(\zeta)\log^{1/2}\zeta,$$

so that we obtain this time the estimate $V(\zeta)/\log^{1/2}\zeta$. To summarize, we have proved that

$$(4.15) \quad T_\varkappa^+(\eta, \zeta) = \sigma_\varkappa^+(\eta) + \sum_{\zeta < p < \eta^{1/(\beta+1)}} \frac{\omega(p)}{p} T_\varkappa^-\left(\frac{\eta}{p}, p\right) + O\left(V(\zeta)\log^{-1}\zeta\right),$$
$$2 \leqslant \zeta,\ 1 < \eta,$$

and

$$(4.16) \qquad T_\varkappa^-(\eta, \zeta) = \sigma_\varkappa^-(\eta) + \sum_{\zeta < p < \eta^{1/\beta}} \frac{\omega(p)}{p} T_\varkappa^+\left(\frac{\eta}{p}, p\right) + O\left(V(\zeta)\log^{-1/2}\zeta\right), \quad 2 \leqslant \zeta < \eta^{1/\beta}.$$

We proceed, on the basis of these formulae, to a proof by induction on $k \geqslant 1$ of the two statements

$$(4.17)_k \qquad T_\varkappa^+(\eta, \zeta) = \sum_{\substack{p(t) > \zeta \\ q^{\beta-1}(t) < \eta/t \\ 2 \mid \nu(t)}} \mu^2(t) \frac{\omega(t)}{t} \sigma_\varkappa^+\left(\frac{\eta}{t}\right) + O\left(V(\zeta)\log^{-1/3}\zeta\right), \quad \zeta \geqslant \eta^{1/(\beta+2k-1)},\ \eta > 1,$$

and

$$(4.18)_k \qquad T_\varkappa^-(\eta, \zeta) = \sum_{\substack{p(t) > \zeta \\ q^{\beta-1}(t) < \eta/t \\ 2 \nmid \nu(t)}} \mu^2(t) \frac{\omega(t)}{t} \sigma_\varkappa^+\left(\frac{\eta}{t}\right) + O\left(V(\zeta)\log^{-1/3}\zeta\right), \quad \eta^{1/(\beta+2k)} \leqslant \zeta < \eta^{1/\beta}.$$

The summation conditions $\zeta < p(t)$, $tq^{\beta-1}(t) < \eta$ together imply $\zeta^{\nu(t)+\beta-1} < \eta$. Hence $\zeta \geqslant \eta^{1/(\beta+2k-1)}$ and $\nu(t)$ *even* give

$$\nu(t) \leqslant 2k-2 \quad \text{in } (4.17)_k,$$

while $\zeta \geqslant \eta^{1/(\beta+2k)}$ and $\nu(t)$ *odd* give

$$\nu(t) \leqslant 2k-1 \quad \text{in } (4.18)_k.$$

Hence $(4.17)_1$ is true by (4.14). To see that $(4.18)_1$ is true (note that, from above, the sum in $(4.18)_1$ extends over primes p only with $\zeta < p < \eta^{1/\beta}$) we invoke (4.16); since $\zeta \geqslant \eta^{1/(\beta+2)}$, we have for each term in the sum on the right of (4.16) that $p \geqslant (\eta/p)^{1/(\beta+1)}$ and hence $T_\varkappa^+(\eta/p, p) = \sigma_\varkappa^+(\eta/p)$ by (4.14). Also, $\sigma_\varkappa^-(\eta)$ in (4.16) can be estimated by (4.3). This proves $(4.18)_1$.

Suppose now that $(4.17)_k$ and $(4.18)_k$ have been proved for some $k \geqslant 1$, and consider the case $k+1$. Take $(4.17)_{k+1}$ first. Here it is legitimate to apply $(4.18)_k$ to each term in the sum on the right of (4.15) because $\zeta \geqslant \eta^{1/(\beta+2k+1)}$ implies that $p \geqslant (\eta/p)^{1/(\beta+2k)}$ whenever $p > \zeta$ and $p < (\eta/p)^{1/\beta}$ follows from $p < \eta^{1/(\beta+1)}$. Hence by (4.15) and $(4.18)_k$,

$$T_\varkappa^+(\eta, \zeta) = \sigma_\varkappa^+(\eta) + \sum_{\zeta < p < \eta^{1/(\beta+1)}} \frac{\omega(p)}{p} \left\{ \sum_{\substack{p(t) > p \\ q^{\beta-1}(t) < \eta/(pt) \\ 2 \nmid \nu(t)}} \mu^2(t) \frac{\omega(t)}{t} \sigma_\varkappa^+\left(\frac{\eta}{pt}\right) + O\left(V(p)\log^{-1/3}p\right) \right\} + O\left(V(\zeta)\log^{-1}\zeta\right)$$

and, writing $n = pt$ on the right,

$$T_\varkappa^+(\eta, \zeta) = \sigma_\varkappa^+(\eta) + \sum_{\substack{p(n) > \zeta \\ q^{\beta-1}(n) < \eta/n \\ 2|\nu(n) > 0}} \mu^2(n) \frac{\omega(n)}{n} \sigma_\varkappa^+\left(\frac{\eta}{n}\right) + O\left(V(\zeta) \log^{-1/3} \zeta\right)$$

after an application of (4.9) (with $\zeta_1 = \zeta$) to handle the accumulation of error terms $O\left(V(p) \log^{-1/3} p\right)$. This proves $(4.17)_{k+1}$, since $\sigma_\varkappa^+(\eta)$ can be put into the sum over n as corresponding to $\nu(n) = 0$.

We deal with $(4.18)_{k+1}$ in similar fashion. Apply $(4.17)_{k+1}$ to each term in the sum on the right of (4.16), as is permitted since $p \geqslant (\eta/p)^{1/(\beta+2k+1)}$ follows from $p > \zeta \geqslant \eta^{1/(\beta+2k+2)}$. Then, by the same procedure,

$$\begin{aligned} T_\varkappa^-(\eta, \zeta) &= \sigma_\varkappa^-(\eta) + \sum_{\zeta < p < \eta^{1/\beta}} \frac{\omega(p)}{p} \Bigg\{ \sum_{\substack{p(t) > p \\ q^{\beta-1}(t) < \eta/(pt) \\ 2|\nu(t)}} \mu^2(t) \frac{\omega(t)}{t} \sigma_\varkappa^+\left(\frac{\eta}{pt}\right) + \\ &\qquad + O\left(V(p) \log^{-1/3} p\right) \Bigg\} + O\left(V(\zeta) \log^{-1/2} \zeta\right) \\ &= \sum_{\substack{p(n) > \zeta \\ q^{\beta-1}(n) < \eta/n \\ 2\nmid\nu(n)}} \mu^2(n) \frac{\omega(n)}{n} \sigma_\varkappa^+\left(\frac{\eta}{n}\right) + O\left(V(\zeta) \log^{-1/3} \zeta\right) \end{aligned}$$

after disposing of $\sigma_\varkappa^-(\eta)$ by means of (4.3). This completes the proof of the inductive step, and hence of Lemma 5, since the number of steps in our induction argument is bounded, by (4.10).

We conclude this section with a, by now standard, result for transforming sums over primes to integrals.

LEMMA 6 (cf. Iwaniec [5], Lemma 8). *Let $B(\tau)$ be a positive, continuous and monotonic function in the range $(2 \leqslant) z_1 \leqslant \tau \leqslant z_2$. Then*

$$\left| \sum_{z_1 \leqslant p < z_2} \frac{\omega(p)}{p} B(p) - \varkappa \int_{z_1}^{z_2} \frac{B(\tau)}{\tau \log \tau} d\tau \right| \leqslant AB \log^{-1} z_1$$

where

$$B = \max\left(B(z_1), B(z_2)\right).$$

5. The inequalities (3.23) and (3.24)

As we mentioned earlier, towards the end of Section 3, we cannot prove these inequalities without defining $w(\cdot)$ suitably and being more specific about the parameters in use. We have already indicated that y is our basic parameter. We shall assume y to be sufficiently large and indicate this formally by

$$(5.1) \qquad y \geqslant y_0.$$

We write

$$(5.2) \qquad z_1 = y^V, \quad z_2 = y^U \quad (0 < V_1 \leqslant V < U)$$

in (3.22) ($0 < V_1 \leqslant V$ says merely that we require V to be bounded away from 0) and introduce two new parameters E and T, with $E \leqslant V$ and $U \leqslant T < 1$; also, we write

$$(5.3) \qquad E_0 = \max\left(E, \frac{1}{3}\left(\frac{4}{2+\beta} - T\right)\right).$$

We impose on E, V, U and T the conditions

$$(5.4) \qquad E_0 \leqslant V \leqslant 1/(\beta+2), \quad 1/2 \leqslant U \leqslant T < 1,$$

and

$$(5.5) \qquad U + (\beta+1)\,V \geqslant 1,$$

and we then define

$$(5.6) \qquad w(p) = \begin{cases} \dfrac{1}{T-E}\left(\dfrac{\log p}{\log y} - E\right), & y^{1/(\beta+2)} \leqslant p < y^U,\ p \in \mathscr{P}, \\[2ex] \dfrac{1}{T-E}\left(\dfrac{\log p}{\log y} - E_0\right), & y^V \leqslant p < y^{1/(\beta+2)},\ p \in \mathscr{P}, \\[2ex] 0, & p < y^V,\ p \in \mathscr{P}. \end{cases}$$

Note that (1.2) with $z_1 = y^V$ is satisfied, and that $w(p)$ is defined only for $p < y^U$, $p \in \mathscr{P}$. When $\varkappa = 1$, this function w is close to the corresponding function in Greaves [1], but we have introduced two new parameters which will turn out, eventually (in paper II), to make our method more flexible in applications. For convenience write

$$(5.7) \qquad W_0(d) = 1 - \sum_{p|d} w(p), \quad d \mid P(y^U).$$

We are now in a position to prove (3.24) (with $z_1 = y^V$ and $z = y^U$) and hence to derive (3.23) from (3.22). Indeed, it will transpire in the process that (3.23) takes the simpler form

$$(5.8) \qquad H(\mathscr{A}, y^V, y^U) \geqslant \sum_{d|P(y^U)} \mu(d)\,\chi_y^-(d)\,W_0(d)\,|\mathscr{A}_d|.$$

The key result is Lemma 7 below which, together with Lemmas 5, 8, and 9, characterize our approach.

LEMMA 7. *Suppose that* $d|P(y^U)$ *and that* $n|P(p(d))$. *Then*

$$\gamma_d(n) \geqslant 0 \quad \text{if} \quad \nu(d) = 2. \tag{5.9}$$

On the other hand, if $\nu(d) > 2$ *and if also*

$$q_2^{\beta+1}(d)\,q(d) < y, \tag{5.10}$$

then

$$\gamma_d(n) = \begin{cases} W_0(d), & n = 1, \\ w(p), & n = p, \\ 0, & \nu(n) \geqslant 2. \end{cases} \tag{5.11}$$

Finally, we have

$$W_0(d) \geqslant 0 \quad \text{when } \bar{\chi}_y^-(d) = 1 \text{ and } \nu(d) > 2. \tag{5.12}$$

Proof. The proof of (5.9) does not depend on the precise choice of w. Let $d = p_1 \dots p_s$ with $p_1 > \dots > p_s$ and $s \geqslant 2$, so that $n|P(p_s)$. By (3.19) and (3.17) we obtain easily that

$$\gamma_d(n) = \sum_{m|d} \mu(m)\gamma(mn). \tag{5.13}$$

Begin with $s = 2$, so that $\gamma_d(n) = \gamma(n) - \gamma(p_1 n) - \gamma(p_2 n) + \gamma(p_1 p_2 n)$. By (1.3) and (1.2) it is clear that $\gamma(mn) \leqslant \gamma(n)$ for each divisor m of d. Hence $\gamma_d(n) = 0$ if $\gamma(n) = 0$; $\gamma_d(n) = \gamma(n) - \gamma(p_1 n) \geqslant 0$ if $\gamma(n) > 0$ but $\gamma(p_2 n) = 0$ and $\gamma_d(n) = \gamma(n) - \gamma(p_2 n) \geqslant 0$ if $\gamma(n) > 0$ but $\gamma(p_1 n) = 0$; and if $\gamma(n)$, $\gamma(p_1 n)$, $\gamma(p_2 n)$ are all three positive, then

$$\gamma_d(n) = \gamma(p_1 p_2 n) - \{1 - \sum_{p|p_1 p_2 n} (1 - w(p))\} \geqslant 0$$

by (1.3), since $\{x\}^+ - x \geqslant 0$. This proves (5.9).

Now suppose that $\nu(d) > 2$ and that

$$d = p_1 \dots p_s \quad (p_1 > \dots > p_s), \quad s \geqslant 3, \quad p_2^{\beta+1} p_1 < y, \quad n|P(p_s) \tag{5.14}$$

in accordance with (5.10). Since $d|P(y^U)$ we have $p_1 < y^U$. It is evident from (1.3) that

$$\gamma(1) = 1 \quad \text{and} \quad \gamma(p) = w(p) \quad (p \in \mathcal{P},\ p < y^U). \tag{5.15}$$

We begin by proving that

$$\gamma(p_1 p_2) = 0 \tag{5.16}$$

in the equivalent form $w(p_1) + w(p_2) \leqslant 1$. Since $p_2^{\beta+1} p_1 < y$ we have $p_s < \dots < p_2 < y^{1/(\beta+2)}$, and only p_1 may exceed $y^{1/(\beta+2)}$; also we have $p_1 p_2 < y^{\frac{1+\beta U}{1+\beta}}$. Hence, by (5.6) and (5.3),

$$w(p_1)+w(p_2) \leqslant \frac{1}{T-E}\left(\frac{\log p_1 p_2}{\log y} - E - E_0\right) \leqslant \frac{1}{T-E}\left(\frac{1-\beta U}{1+\beta} - E_0 - E\right)$$
$$\leqslant \frac{1}{T-E}\left(\frac{\beta U + T}{\beta+1} - E\right) \leqslant 1$$

since $U \leqslant T$ by (5.4). We have assumed that $p_2 \geqslant y^V$ in this argument when quoting (5.6); otherwise $w(p_2) = 0$ and the result is trivial simply by virtue of (1.2).

The result (5.16) seems rather modest, but (1.3) tells us at once not only that then $\gamma(d) = 0$, but even that $\gamma(dn) = 0$ $(n|P(p_s))$. More is true. The argument used to prove (5.16) shows that $\gamma(p'p'') = 0$ for any two distinct prime divisors p', p'' of dn, so that our preceding remarks show that

$$\gamma(mn) = 0 \quad \text{whenever} \quad m|d,\ \nu(mn) \geqslant 2. \tag{5.17}$$

Hence, by (5.13),

$$\gamma_d(n) = \sum_{\substack{m|d \\ \nu(mn) \leqslant 1}} \mu(m)\gamma(mn). \tag{5.18}$$

When $n = 1$ we get immediately the first part of (5.11), using (5.7). When $n = p$, only $m = 1$ makes a contribution and $\gamma_d(p) = \gamma(p) = w(p)$ by (5.15). This is the second part of (5.11). As for the third part, this follows at once from (5.17) since the sum on the right of (5.18) is now empty.

It remains to prove (5.12). With the notation (5.14),

$$W_0(d) = 1 - \sum_{i=1}^{s} w(p_i) = 1 - \sum_{\substack{i=1 \\ p_i \geqslant y^V}}^{s_1} w(p_i)$$

where $0 \leqslant s_1 \leqslant s$ and $p_i < y^V (i = s_1+1, \ldots, s)$. The result is obvious when $s_1 = 0$ and 1. In proving (5.16) we have already shown that the result is true also when $s_1 = 2$. Consider $s_1 = 3$ next. Here

$$w(p_1)+w(p_2)+w(p_3) \leqslant \frac{1}{T-E}\left(\frac{\log p_1 p_2 p_3}{\log y} - 2E_0 - E\right),$$

and $p_2^{\beta+1} p_1 < y$ implies that

$$p_1 p_2 p_3 < p_1 p_2^2 < p_1 (y/p_1)^{2/(\beta+1)} = y^{2/(\beta+1)} p_1^{(\beta-1)/(\beta+1)} < y^{\frac{(\beta-1)U+2}{\beta+1}}.$$

Hence, by (5.3) and (5.4),

$$w(p_1)+w(p_2)+w(p_3) < \frac{1}{T-E}\left(\frac{(\beta-1)U+2}{\beta+1} - 2E_0 - E\right)$$
$$\leqslant 1 - 2\left(E_0 - \frac{1-T}{\beta+1}\right) \leqslant 1.$$

Hence $W_0(d) \geqslant 0$ when $s_1 = 3$.

Suppose finally that $s_1 \geqslant 4$. Then $\bar{\chi}_y^-(d) = 1$ tells us that $p_{2j}^{\beta+1} \dots p_1 < y$ ($j = 1, \dots, \frac{1}{2}s-1$) and

$$p_{s-2}^{\beta+1} \dots p_1 < y \leqslant p_s^{\beta+1} \dots p_1.$$

We have to consider

$$\sum_{i=1}^{s_1} w(p_i) \leqslant \frac{1}{T-E}\left(\frac{\log p_1 \dots p_{s_1}}{\log y} - (s_1-1)E_0 - E\right).$$

If $s_1 < s$ the foregoing inequalities imply that $p_1 \dots p_{s_1} < y$ and therefore – remember $s_1 \geqslant 4$ –

$$\sum_{i=1}^{s_1} w(p_i) \leqslant \frac{1}{T-E}(1-3E_0-E) \leqslant 1.$$

The last stem is valid by virtue of (5.3) and because $\beta \leqslant 2$ by (3.7). This leaves the case $s_1 = s$. Here $p_s^{\beta-2}(p_s \dots p_1) < y$, implying in particular that $p_s^{\beta-2+s} < y$, and therefore

$$\sum_{i=1}^{s} w(p_i) < \frac{1}{T-E}\left(1+(2-\beta)\frac{\log p_s}{\log y} - (s-1)E_0 - E\right)$$
$$< \frac{1}{T-E}\left(1+\frac{2-\beta}{\beta-2+s} - (s-1)E_0 - E\right) \leqslant \frac{1}{T-E}\left(\frac{4}{\beta+2} - 3E_0 - E\right) \leqslant 1$$

by (5.3). This completes the proof of (5.12) and hence of Lemma 7.

We are now in a position to prove (3.24); specifically, to show that

$$H_d(\mathscr{A}, y^V, p(d)) = \sum_{a \in \mathscr{A}_d} \gamma_d\big((a, P(p(d)))\big) \geqslant 0 \tag{5.19}$$
$$\text{if} \quad d \mid P(y^U),\ 2 \mid \nu(d) > 0 \text{ and } \bar{\chi}_y^-(d) = 1.$$

We write $n = (a, P(p(d)))$, so that $n \mid P(p(d))$. Then (5.19) follows at once from (5.9) when $\nu(d) = 2$. When $\nu(d) \geqslant 4$, $\bar{\chi}_y^-(d) = 1$ implies that $\chi_y^-(d/p(d)) = 1$ (cf. (3.1)) and hence that (5.10) is true (cf. (3.6), the inequality corresponding to $i = 1$). We may therefore invoke (5.11) and obtain

$$H_d(\mathscr{A}, y^V, p(d))$$
$$= \sum_{\substack{a \in \mathscr{A}_d \\ (a, P(p(d))) = 1}} W_0(d) + \sum_{\substack{p < p(d) \\ p \in \mathscr{P}}} \sum_{\substack{a \in \mathscr{A}_d \\ (a, P(p(d))) = p}} w(p)$$
$$= W_0(d) S(\mathscr{A}_d, \mathscr{P}, p(d)) + \sum_{\substack{p < p(d) \\ p \in \mathscr{P}}} w(p) \left|\{a \in \mathscr{A}_d : (a, P(p(d))) = p\}\right|.$$

The second sum evidently contains only non-negative terms because $w(p) \geqslant 0$ when $p \in \mathscr{P}$ and the first expression is non-negative by (5.12). We have now proved (5.19). It follows by (3.22) and (3.23) that

$$H(\mathscr{A}, y^V, y^U) \geqslant \sum_{d \mid P(y^U)} \mu(d) \chi_y^-(d) W(d) |\mathscr{A}_d|. \tag{5.20}$$

We complete the combinatorial preparation for a lower bound of H with

LEMMA 8. *We have, subject to* (5.3) *and* (5.4) *and with the choice* (5.6) *of* $w(\cdot)$, *that*

$$H(\mathscr{A}, y^V, y^U) \geqslant \sum_{d \mid P(y^U)} \mu(d)\chi_y^-(d) W_0(d) |\mathscr{A}_d|.$$

Proof. The result follows at once from (5.20) if we can show that $W(d) = W_0(d)$ when $d \mid P(y^U)$, and $\chi_y^-(d) = 1$. This is in any case obvious when $\nu(d) = 0$ and 1, so we may suppose that $\nu(d) \geqslant 2$. Here, by (3.17) and (5.15),

$$W(d) = W_0(d) + \sum_{\substack{t \mid d \\ \nu(t) \geqslant 2}} \mu(t)\gamma(t),$$

and it suffices to prove that

$$\gamma(t) = 0 \quad \text{whenever} \quad \nu(t) \geqslant 2,\ t \mid d \mid P(y^U) \text{ and } \chi_y^-(d) = 1.$$

But by (1.3) this amounts to no more than showing that, in these circumstances,

$$\sum_{p \mid t} w(p) \leqslant \nu(t) - 1,$$

and this has already been proved in the course of demonstrating the truth of (5.12). Indeed, this inequality is trivial if $p(t) < y^V$ and otherwise follows at once from $w(q(t)) + w(q_2(t)) \leqslant 1$ (cf. the proof of (5.16)).

We shall analyse the expression on the right in Lemma 8 in the next section; but to make the exposition there flow smoothly we establish here one more combinatorial identity that will be required and that is, perhaps, the most distinctive feature of our method.

LEMMA 9. *Let* $n > 1$ *be a squarefree integer with* $\nu(n)$ *even and* $p(n) \geqslant p$. *Then*

$$\sum_{\substack{d \mid n \\ q^{\beta-1}(n/d) < y/(pn)}} \mu(d)\chi_y^-(d) = \begin{cases} -\bar{\chi}_y^-(n) & \text{if} \quad np^{\beta-1}(n) < y/p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof resembles the argument of Lemma 1. Let

$$n = p_1 \ldots p_r, \qquad p_1 > \ldots > p_r \geqslant p, \qquad 2 \mid r,$$

and define

$$\delta(x) = \begin{cases} 1, & x > 1, \\ 0, & x \leqslant 1. \end{cases}$$

In our sum, $q(n/d) = 1$ when $d = n$ and the contribution at $d = n$ to the sum is

$$\delta\left(\frac{y}{pn}\right)\chi_y^-(n);$$

otherwise $q(n/d) = p_j$ for some j, $1 \leqslant j \leqslant r$, and those d's in the sum having $q(n/d) = p_j$ contribute to it

$$\Sigma_j = \delta\left(\frac{y}{pnp_j^{\beta-1}}\right) \sum_{\substack{d|n \\ q(n/d)=p_j}} \mu(d)\chi_y^-(d).$$

But if $d|n$ and $q(n/d) = p_j$, then necessarily

$$d = p_1 \dots p_{j-1} t, \quad t | p_{j+1} \dots p_r,$$

and therefore

$$\Sigma_j = \delta\left(\frac{y}{pnp_j^{\beta-1}}\right)(-1)^{j-1} \sum_{t|p_{j+1}\dots p_r} \mu(t)\chi_y^-(p_1 \dots p_{j-1} t).$$

By (4.8)

$$\chi_y^-(p_1 \dots p_{j-1} t) = \chi_y^-(p_1 \dots p_{j-1})\chi_{y/(p_1\dots p_{j-1})}^{(-)^j}(t)$$

and so

$$\Sigma_j = \delta\left(\frac{y}{pnp_j^{\beta-1}}\right)(-1)^{j-1}\chi_y^-(p_1 \dots p_{j-1}) \sum_{t|p_{j+1}\dots p_r} \mu(t)\chi_{y/(p_1\dots p_{j-1})}^{(-)^j}(t).$$

If $\delta(y/(pnp_j^{\beta-1})) = 0$, $\Sigma_j = 0$. Suppose $\delta(y/(pnp_j^{\beta-1})) = 1$. Then

$$pp_r \dots p_{j+1} p_j^\beta p_{j-1} \dots p_1 < y,$$

or

$$pp_r \dots p_{j+1} p_j^\beta < y/(p_1 \dots p_{j-1}),$$

and this implies

$$p_s^{\beta+1} p_{s+1} \dots p_{j+1} < y/(p_1 \dots p_{j-1}) \qquad \text{for each } s = j+1, \dots, r;$$

hence $\chi_{y/(p_1\dots p_{j-1})}^{(-)^j}(t) = 1$ for every $t|p_{j+1}\dots p_r$, so that

$$\begin{aligned}\Sigma_j &= \delta\left(\frac{y}{pnp_j^{\beta-1}}\right)(-1)^{j-1}\chi_y^-(p_1 \dots p_{j-1}) \sum_{t|p_{j+1}\dots p_r} \mu(t) \\ &= \begin{cases} -\delta\left(\dfrac{y}{pnp_r^{\beta-1}}\right)\chi_y^-\left(\dfrac{n}{p_r}\right), & j = r, \\ 0, & j = 1, \dots, r-1. \end{cases}\end{aligned}$$

The sum in the lemma therefore equals

$$\delta\left(\frac{y}{pn}\right)\chi_y^-(n) - \delta\left(\frac{y}{pnp(n)^{\beta-1}}\right)\chi_y^-\left(\frac{n}{p(n)}\right).$$

If $np(n)^{\beta-1} < y/p$, both δ's here equal 1 and we obtain the first result of the lemma. Suppose then that $pnp(n)^{\beta-1} \geqslant y$, when the sum equals

$$\delta\left(\frac{y}{pn}\right)\chi_y^-(n).$$

But since $p \leqslant p(n)$ we now have $p(n)^\beta n \geqslant y$, whence, by (3.6), $\chi_y^-(n) = 0$ (remember that $\nu(n)$ is even). The proof of the lemma is now complete.

6. A lower bound for H: preparation

By Lemma 8 and (2.1),

$$H(\mathscr{A}, y^V, y^U) \geqslant XG + R \tag{6.1}$$

where

$$\begin{aligned} G = G(\mathscr{A}, y^V, y^U) &= \sum_{d \mid P(y^U)} \mu(d)\chi_y^-(d)\, W_0(d)\frac{\omega(d)}{d} \\ &= \sum_{d \mid P(y^U)} \mu(d)\chi_y^-(d)\frac{\omega(d)}{d}\Big(1 - \sum_{p \mid d} w(p)\Big) \end{aligned} \tag{6.2}$$

by (5.7), and

$$R = R(\mathscr{A}, y^V, y^U) = \sum_{\substack{d \mid P(y^U) \\ d < y}} \mu(d)\chi_y^-(d)\, W_0(d)\, R_d; \tag{6.3}$$

the summation condition $d < y$ here is implicit in $\chi_y^-(d) = 1$, but is stressed here to clarify the nature of R. For example, by the proof of Lemma 7, (5.12),

$$|R| \leqslant \sum_{\substack{d \mid P(y^U) \\ d < y}} |R_d|. \tag{6.4}$$

Our main business is with G. By (6.2) and (3.10)

$$G = T_\kappa^-(y, y^U) - \sum_{\substack{p < y^U \\ p \in \mathscr{P}}} w(p) \sum_{\substack{d \mid P(y^U) \\ p \mid d}} \mu(d)\chi_y^-(d)\frac{\omega(d)}{d}.$$

The inner sums over d, on the right, look beguilingly like T-functions but the connection is actually rather complicated. We write d uniquely in the form $d = d_2 p d_1$ where $d_2 \mid P(p)$ and $d_1 \mid P(p^+, y^U)$, here p^+ denotes the successor of p in $\mathscr{P}$, so that $p(d_1) > p$. By two successive applications of (4.8) we have that

$$\chi_y^-(d) = \chi_y^-(d_1)\,\chi_{y/d_1}^{(-)^{\nu(d_1)+1}}(pd_2) = \chi_y^-(d_1)\,\chi_{y/d_1}^{(-)^{\nu(d_1)+1}}(p)\,\chi_{y/(pd_1)}^{(-)^{\nu(d_1)}}(d_2),$$

and accordingly we obtain with the help of (3.10) (replacing notation d_1 by d at the end)

(6.5) $$G = T_x^-(y, y^U) + \sum_{p<y^U} w(p)\frac{\omega(p)}{p} \sum_{d|P(p^+, y^U)} \mu(d)\chi_y^-(d)\chi_{y/d}^{(-)^{\nu(d)+1}}(p)\frac{\omega(d)}{d} T_x^{(-)^{\nu(d)}}\left(\frac{y}{pd}, p\right).$$

We apply (4.7) to the first term on the right, with $\eta = y$, $\zeta_2 = y^U$ and $\zeta_1 = y^{1/(\beta+2)}$ (remember that $U > 1/(\beta+2)$), and obtain

(6.6) $$T_x^-(y, y^U) = T_x^-(y, y^{1/(\beta+2)}) - \sum_{y^{1/(\beta+2)} \leqslant p < y^U} \frac{\omega(p)}{p} T_x^+\left(\frac{y}{p}, p\right).$$

In the double sum on the right of (6.5) consider first the terms with $p \geqslant y^{1/(\beta+2)}$. We claim for these that only $d = 1$ can occur. For suppose there does occur a d with $\nu(d) \geqslant 1$; if $\nu(d)$ is even $\chi_y^-(d) = 1$ implies that $p(d)^\beta d < y$, by (the last inequality of) (3.6), so that $p^{\beta+\nu(d)} < y$ and therefore $p^{\beta+2} < y$ contrary to hypothesis, and if $\nu(d)$ is odd $\chi_{y/d}^{(-)^{\nu(d)+1}}(p) = \chi_{y/d}^+(p) = 1$ implies $p^{\beta+1} < y/d$ and therefore $p^{\beta+2} < y$ again, the same contradiction. Hence, by (6.5) and (6.6)

(6.7) $$G = T_x^-(y, y^{1/(\beta+2)}) - \sum_{y^{1/(\beta+2)} \leqslant p < y^U} (1 - w(p))\frac{\omega(p)}{p} T_x^+\left(\frac{y}{p}, p\right) + G_0,$$

where

(6.8) $$G_0 = \sum_{y^V \leqslant p < y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p} \times \sum_{d|P(p^+, y^U)} \mu(d)\chi_y^-(d)\chi_{y/d}^{(-)^{\nu(d)+1}}(p)\frac{\omega(d)}{d} T_x^{(-)^{\nu(d)}}\left(\frac{y}{pd}, p\right).$$

We attack this complicated expression by means of Lemma 5 – the Reduction Lemma – with $\nu = \nu(d)$, $\eta = y/(pd)$ and $\zeta = p$. Requirement (4.10) of that lemma is satisfied because here $\log \eta/\log \zeta \leqslant \log y/\log p - 1 \leqslant 1/V - 1$ and V is bounded away from the origin (see (5.2)). Also (4.11) is satisfied. For $\zeta = p \geqslant 2$ always; when $\nu(d)$ is even and positive, $\eta = y/(pd) > p(d)^{\beta-1} \geqslant 1$ from $\chi_y^-(d) = 1$, when $\nu(d) = 0$ then $\eta = y/p > p^{\beta+1} > 1$, and when $\nu(d)$ is odd $\chi_{y/d}^+(p) = 1$ implies that $y/(pd) > p^\beta$, which translates into $\eta > \zeta^\beta$, as required. Thus Lemma 5 may indeed be applied to each term on the right in (6.8). We may as well dispose right away of the error terms arising from the term $O(V(p)\log^{-1/3} p)$ in the lemma. The total contribution is

$$\ll \sum_{y^V \leqslant p < y^{1/(\beta+2)}} \frac{\omega(p)}{p} V(p)\log^{-1/3} p \sum_{d|P(p^+, y^U)} \frac{\omega(d)}{d}$$

$$\ll V(y)(\log y^V)^{-1/3} \sum_{y^V \leqslant p < y^{1/3}} \frac{\omega(p)}{p}\left(\frac{V(p)}{V(y)}\right)^2$$

$$\ll V(y)\log^{-1/3} y$$

by (2.6), (2.4) and (5.2) ($V \geqslant V_1 > 0$). Hence

$$(6.9)\quad G_0 = \sum_{y^V \leqslant p < y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p} \sum_{d|P(p^+, y^U)} \mu(d)\chi_y^-(d)\chi_{y/d}^{(-)^{\nu(d)+1}}(p) \times$$
$$\times \frac{\omega(d)}{d} \sum_{\substack{p(t)>p \\ q(t)^{\beta-1}<y/(pdt) \\ \nu(t)\equiv\nu(d) \bmod 2}} \mu^2(t)\frac{\omega(t)}{t}\sigma_\varkappa^+\left(\frac{y}{pdt}\right) + O\left(V(y)\log^{-1/3} y\right).$$

This expression for G_0 looks more complicated than ever, so let us say right away that the main contribution to G_0 comes from $d = t = 1$ and from the terms with $(d, t) = 1$, and that, under the circumstances, it is surprisingly simple. All the other terms will be absorbed into the error term.

Let us prepare the ground. First, we may add the summation condition $t|P(p^+, y^U)$ since $p(t) > p$ and, for $t > 1$, $q(t)^\beta < (yq(t))/(pdt) < y/p^2 < y^{1-2V}$ implies $q(t) < y^U$ by (5.4) and (5.5). Next, we may omit the factor $\chi_{y/d}^{(-)^{\nu(d)+1}}(p)$; for this is 1 if $\nu(d)$ is even, or if $\nu(d)$ is odd and $p^{\beta+1}d < y$. But if $\nu(d)$ is odd so is $\nu(t)$ and then the summation condition $q(t)^{\beta-1} < y/(pdt)$ implies that $p^{\beta+1} < pq(t)^{\beta-1}t < y/d$. Now separate the terms in G_0 into the singleton corresponding to $d = t = 1$, the group of terms with $(d, t) = 1$ and the group with $(d, t) > 1$; and in the group with $(d, t) = 1$ write $dt = n$ and note that then $\nu(n)$ is even since $\nu(d) \equiv \nu(t) \bmod 2$. Then, by (6.9),

$$G_0 = \sum_{y^V \leqslant p < y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p} \times$$
$$\times\left\{\sigma_\varkappa^+\left(\frac{y}{p}\right) + \sum_{\substack{1<n|P(p^+, y^U) \\ 2|\nu(n)}} \frac{\omega(n)}{n}\sigma_\varkappa^+\left(\frac{y}{pn}\right) \sum_{\substack{d|n \\ q(n/d)^{\beta-1}<y/(pn)}} \mu(d)\chi_y^-(d) + G_0^{(p)}\right\} +$$
$$+ O\left(V(y)\log^{-1/3} y\right)$$

where, for each p,

$$(6.10)\quad G_0^{(p)} = \sum_{d|P(p^+, y^U)} \mu(d)\chi_y^-(d)\frac{\omega(d)}{d} \sum_{\substack{t|P(p^+, y^U) \\ q(t)^{\beta-1}<y/(pdt) \\ \nu(t)\equiv\nu(d) \bmod 2 \\ (d,t)>1}} \frac{\omega(t)}{t}\sigma_\varkappa^+\left(\frac{y}{pdt}\right).$$

Before estimating $G_0^{(p)}$ we record at once that, by Lemma 9,

$$(6.11)\quad G_0 = \sum_{y^V \leqslant p < y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p} \times$$
$$\times\left\{\sigma_\varkappa^+\left(\frac{y}{p}\right) - \sum_{\substack{n|P(p^+, y^U) \\ p(n)^{\beta-1}<y/(pn)}} \bar{\chi}_y^-(n)\frac{\omega(n)}{n}\sigma_\varkappa^+\left(\frac{y}{pn}\right) + G_0^{(p)}\right\} + O\left(V(y)\log^{-1/3} y\right);$$

the conditions $n > 1$ and $\nu(n)$ even are implicit in $\bar{\chi}_y^-(n) = 1$.

We turn to (6.10). Here let $(d, t) = r$ and write $d = d_1 r$, $t = t_1 r$ – so that $(d_1, t_1) = 1$ – and $n = d_1 t_1$. Since $\nu(d) \equiv \nu(t) \bmod 2$ we have $\nu(n)$ even again. Thus

$$G_0^{(p)} = \sum_{1 < r | P(p^+, y^U)} \mu(r) \frac{\omega^2(r)}{r^2} \sum_{\substack{n | P(p^+, y^U)/r \\ 2 | \nu(n)}} \frac{\omega(n)}{n} \sigma_\varkappa^+ \left(\frac{y}{pr^2 n}\right) \times$$

$$\times \sum_{\substack{d_1 | n \\ q(nr/d_1)^{\beta - 1} < y/(pr^2 n)}} \mu(d_1) \chi_y^- (rd_1).$$

Let $\tau(\cdot)$ denote the divisor function. Then, since $q(nr/d_1) \geqslant q(r)$ $(d_1 | n)$,

$$|G_0^{(p)}| \leqslant \sum_{1 < r | P(p^+, y^U)} \frac{\omega^2(r)}{r^2} \sum_{\substack{n | P(p^+, y^U)/r \\ q(r)^{\beta - 1} pr^2 n < y \\ 2 | \nu(n)}} \frac{\omega(n)}{n} \tau(n) \sigma_\varkappa^+ \left(\frac{y}{pr^2 n}\right). \tag{6.12}$$

We prove next that in each term of this double sum

$$\sigma_\varkappa^+ \left(\frac{y}{pr^2 n}\right) \ll V(y) \log^{1/2} y. \tag{6.13}$$

For $\varkappa = 1/2$, so that $\beta = 1$ and $pr^2 n < y$, this follows at once from (4.3) – $\sigma_\varkappa^+ (y/(pr^2 n)) \ll 1$ – and (2.3). When $\varkappa > 1/2$ and therefore $\beta > 1$ we have $q(r) > p$ and therefore $y/(pr^2 n) > p^{\beta - 1}$. Hence (4.4) (with $\eta = y$, $\zeta = pr^2 n$) applies and $\sigma_\varkappa^+ (y/(pr^2 n)) \ll V(y)$. It follows that (6.13) is true in any case and so, by (6.12), and then by (2.5),

$$G_0^{(p)} = V(y) \log^{1/2} y \sum_{1 < r | P(p^+, y^U)} \frac{\omega^2(r)}{r^2} \sum_{n | P(p^+, y^U)} \frac{\omega(n)}{n} \tau(n)$$

$$\ll V(y) \log^{-1/2} y \sum_{r | P(p, y)} \frac{\omega(r)}{r} \sum_{n | P(p, y)} \frac{\omega(n)}{n} \tau(n)$$

$$\ll V(y) \log^{-1/2} y \left(\sum_{r | P(p, y)} \frac{\omega(r)}{r}\right)^3 \ll V(y) \log^{-1/2} y$$

after invoking (2.6) at the last step. When we substitute this estimate of $G_0^{(p)}$ in (6.11) and then apply (1.2) and (2.4) we see that the total contribution arising from all the sums $G_0^{(p)}$ in (6.11) is still $\ll V(y) \log^{-1/2} y$ so that

$$G_0 = \sum_{y^V \leqslant p < y^{1/(\beta + 2)}} w(p) \frac{\omega(p)}{p} \left\{ \sigma_\varkappa^+ \left(\frac{y}{p}\right) - \right.$$

$$\left. - \sum_{\substack{n | P(p^+, y^U) \\ p(n)^{\beta - 1} < y/(pn)}} \bar{\chi}_y^- (n) \frac{\omega(n)}{n} \sigma_\varkappa^+ \left(\frac{y}{pn}\right) \right\} + O\left(\frac{V(y)}{\log^{1/3} y}\right). \tag{6.14}$$

We may now substitute this formula in (6.7). Before we do so we observe that if $y^{1/(\beta+2)} \leqslant p$, $T_\varkappa^+(y/p, p) = \sigma_\varkappa^+(y/p)$ by (4.14). Hence, by (6.7) and (6.14),

$$(6.15)\quad G = T_\varkappa^-(y, y^{1/(\beta+2)}) - \sum_{y^{1/(\beta+2)} \leqslant p < y^U} (1-w(p))\frac{\omega(p)}{p}\sigma_\varkappa^+\left(\frac{y}{p}\right) +$$
$$+ \sum_{y^V \leqslant p < y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p}\left\{\sigma_\varkappa^+\left(\frac{y}{p}\right) - \sum_{\substack{n | P(p^+, y^U) \\ p(n)^{\beta-1} < y/(pn)}} \bar{\chi}_y^-(n)\frac{\omega(n)}{n}\sigma_\varkappa^+\left(\frac{y}{pn}\right)\right\} +$$
$$+ O\left(V(y)\log^{-1/3} y\right).$$

7. A lower bound for H: completion

It remains for us to transform (6.15) to an applicable condition. We begin with the first expression on the right of (6.15). By (4.2) of Lemma 3, and by (3.16),

$$(7.1)\qquad T_\varkappa^-(y, y^{1/(\beta+2)}) \geqslant V(y)\varkappa A_\varkappa\left\{\int_\beta^{\beta+2}\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t} + O(\log^{-1/3} y)\right\}.$$

Next, by (4.4) of Lemma 3 and by (5.6),

$$\sum_{y^{1/(\beta+2)} \leqslant p < y^U} (1-w(p))\frac{\omega(p)}{p}\sigma_\varkappa^+\left(\frac{y}{p}\right)$$
$$= \frac{V(y)A_\varkappa \log^\varkappa y}{T-E} \cdot \sum_{y^{1/(\beta+2)} \leqslant p < y^U} \frac{\omega(p)}{p}\left(T - \frac{\log p}{\log y}\right)\log^{-\varkappa}\frac{y}{p} + O\left(V(y)\log^{-1/3} y\right)$$

where, as usual, we have used (1.2) and (2.4) to estimate the contribution from the error terms([1]) in (4.4). We apply Lemma 6 to the sum on the right, with $B \ll \log^{-\varkappa} y$, so that the error introduced by moving from a sum to an integral is $\ll \log^{-1-\varkappa} y$.

The integral itself is

$$\varkappa \int_{y^{1/(\beta+2)}}^{y^U} \frac{T - \dfrac{\log\tau}{\log y}}{\log^\varkappa(y/\tau)}\frac{d\tau}{\tau\log\tau} = \frac{\varkappa}{\log^\varkappa y}\int_{1/U}^{\beta+2}\left(T - \frac{1}{t}\right)\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t},$$

([1]) From now on our O- and $\ll$-constants may depend also on U.

so that

$$(7.2)\qquad \sum_{y^{1/(\beta+2)}\leqslant p<y^U} (1-w(p))\frac{\omega(p)}{p}\sigma_\varkappa^+\left(\frac{y}{p}\right)$$
$$= V(y)\frac{\varkappa A_\varkappa}{T-E}\left\{\int_{1/U}^{\beta+2}\left(T-\frac{1}{t}\right)\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t}+O(\log^{-1/3}y)\right\}.$$

Exactly the same procedure applies to the first (the positive) part of the third expression on the right of (6.15) and we obtain

$$(7.3)\qquad \sum_{y^V\leqslant p<y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p}\sigma_\varkappa^+\left(\frac{y}{p}\right)$$
$$= V(y)\frac{\varkappa A_\varkappa}{T-E}\left\{\int_{\beta+2}^{1/V}\left(\frac{1}{t}-E_0\right)\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t}+O(\log^{-1/3}y)\right\}.$$

This brings us to the second (negative) part of that same expression. In the end it makes the smallest (but negative) contribution, but it gives the most technical trouble. In the sum over n we have $\nu(n)$ positive and even, $p(n) > p$ and $p(n)^{\beta-1} < y/(pn)$; the condition $q(n) < y^U$ may be omitted since $\nu(n) \geqslant 2$ and $np(n)^{\beta-1}p < y$ together imply $q(n)p(n)^\beta p < y$ whence $q(n) < y/p^{\beta+1} \leqslant y^{1-(\beta+1)V} < y^U$ by (5.5). The condition $p(n)^{\beta-1} < y/(pn)$ implies also that $y > p^{\nu(n)+\beta} \geqslant y^{(\nu(n)+\beta)V}$, so that

$$(7.4)\qquad \nu(n) < 1/V-\beta,$$

where by (5.4), (5.3) and (5.5)

$$(7.5)\qquad \frac{1}{V}\leqslant \min\left(\frac{1}{E},\ 3\left(\frac{4}{2+\beta}-T\right)^{-1},\ \frac{\beta+1}{1-U}\right).$$

Since $T < 1$ we have, in particular, that

$$(7.6)\qquad \frac{1}{V}<3\frac{2+\beta}{2-\beta},$$

which is no restriction at all when $\beta = 2$ ($\varkappa = 1$) but when $\beta = 1$ ($\varkappa = 1/2$) gives $1/V < 9$ and hence, by (7.4), $\nu(n) < 8$ or $\nu(n) = 2$, 4 or 6. Thus, in the half-dimensional sieve the summation over n has at most three terms.

Next, we certainly want to apply (4.4) from Lemma 3, with $\eta = y$ and $\zeta = pn$, so as to substitute for $\sigma_\varkappa^+(y/pn)$. The condition $\zeta \geqslant 2$ in (4.4) is obviously fulfilled; and the condition $\eta/\zeta \geqslant 4$ (requiring η/ζ to be bounded away from 1) translates into $y/(pn) \geqslant 4$ and is certainly satisfied when $\beta > 1$ (i.e. $\varkappa > 1/2$) in view of the summation condition $y/(pn) > p(n)^{\beta-1}$. When β

$= 1$ the terms with $y/4 < pn < y$ require special attention. For each of these terms we invoke (4.3) and (2.3) to yield

$$\sigma_\varkappa^+\left(\frac{y}{pn}\right) \ll 1 \ll V(y)\log^{1/2} y,$$

and together their contribution after summation over p is

$$\ll V(y)\log^{1/2} y \sum_{n|P(y^V, y^U)} \frac{\omega(n)}{n} \sum_{\max(y^V, y/4n) \leqslant p < y/n} \frac{\omega(p)}{p}.$$

The inner sum is empty unless $y/n > y^V$; and if $y/n > y^V$ it is, by (2.4) (with $\varkappa = 1/2$),

$$\ll \log\left(1+\frac{\log 4}{\log(y/4n)}\right)+\frac{1}{\log(y/4n)} \ll \frac{1}{\log y}.$$

Hence when $\beta = 1$ the terms with $y/4 < pn < y$ make altogether a contribution, by (2.6), $\ll V(y)\log^{-1/2} y$. It follows that we may substitute from (4.4) in the sum over n on the right of (6.15), where the $p(n)$-condition should now be altered to $\max(p(n)^{\beta-1}, 4) < y/(pn)$. Of course, (4.4) brings with it an error term, involving a negative power of $\log(y/pn)$. For $\beta > 1$ we have at once $\log(y/pn) \gg \log y$, but when $\beta = 1$, we have to interchange summation over p and n as above and then use Lemma 6 with $B(\tau) = (\log(y/n\tau))^{-5/6} \ll 1$ to obtain again for the error term the estimate $\ll V(y)\log^{-1/3} y$. To sum up, we have by (6.15), (7.1), (7.2), (7.3) and the preceding discussion that

$$G \geqslant V(y)\varkappa A_\varkappa \Bigg\{ \int_\beta^{\beta+2} \left(\frac{t}{t-1}\right)^\varkappa \frac{dt}{t} - \frac{1}{T-E}\int_{1/U}^{\beta+2} \left(T-\frac{1}{t}\right)\left(\frac{t}{t-1}\right)^\varkappa \frac{dt}{t} + \tag{7.7}$$
$$+\frac{1}{T-E}\int_{\beta+2}^{1/V} \left(\frac{1}{t}-E_0\right)\left(\frac{t}{t-1}\right)^\varkappa \frac{dt}{t} - \frac{1}{\varkappa}\sum_{y^V \leqslant p < y^{1/(\beta+2)}} w(p)\frac{\omega(p)}{p} \times$$
$$\times \sum_{\substack{p(n) > p \\ \max(p(n)^{\beta-1}, 4) < y/(pn)}} \bar{\chi}_y^-(n)\frac{\omega(n)}{n}\frac{\log^\varkappa y}{\log^\varkappa(y/(pn))} + O(\log^{-1/3} y)\Bigg\}.$$

Remember that in the summation over n, $\nu(n)$ is even and positive and satisfies (7.4).

We simplify the exposition from here on by imposing the requirement

$$V \geqslant 1/(\beta+4). \tag{7.8}$$

Then (7.4) implies $\nu(n) < 4$ so that the summation over n has just the one term corresponding to $\nu(n)=2$, namely

$$\sum_{\substack{p<p_2<p_1 \\ pp_2^\beta p_1<y\leqslant p_2^{\beta+1}p_1}} \frac{\omega(p_1)}{p_1}\frac{\omega(p_2)}{p_2}\frac{\log^\varkappa y}{\log^\varkappa(y/pp_2p_1)} \qquad (1/2<\varkappa\leqslant 1) \tag{7.9}$$

and

$$\sum_{\substack{p<p_2<p_1 \\ 4pp_2p_1<y\leqslant p_2^2p_1}} \frac{\omega(p_1)}{p_1}\frac{\omega(p_2)}{p_2}\frac{\log^{1/2} y}{\log^{1/2}(y/pp_2p_1)}, \qquad \varkappa=1/2. \tag{7.9'}$$

A two-fold application of Lemma 6 is indicated, with $B\ll\log^{-\varkappa}y$ when $1/2<\varkappa\leqslant 1$ and $B\ll 1$ when $\varkappa=1/2$. After some straightforward calculation, and writing

$$h_2(\varkappa,x)=\iint\limits_{\substack{x<x_2<x_1 \\ \beta x_2+x_1<1-x \\ (\beta+1)x_2+x_1\geqslant 1}} \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{1}{(1-x-x_1-x_2)^\varkappa}, \qquad x=\frac{\log p}{\log y}, \tag{7.10}$$

we find for $1/2\leqslant\varkappa\leqslant 1$ that either of the sums (7.9), (7.9′) is equal to

$$\varkappa^2 h_2(\varkappa,x)+O(\log^{-1/3}y).$$

It follows from (7.7) that

$$G\geqslant V(y)\frac{\varkappa A_\varkappa}{T-E}\Bigg\{(T-E)\int\limits_\beta^{\beta+2}\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t}-\int\limits_{1/U}^{\beta+2}\left(T-\frac{1}{t}\right)\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t}+ \tag{7.11}$$

$$+\int\limits_{\beta+2}^{1/V}\left(\frac{1}{t}-E_0\right)\left(\frac{t}{t-1}\right)^\varkappa\frac{dt}{t}-\varkappa\sum_{y^V\leqslant p<y^{1/(\beta+2)}}\left(\frac{\log p}{\log y}-E_0\right)h_2\left(\varkappa,\frac{\log p}{\log y}\right)\frac{\omega(p)}{p}+$$

$$+O(\log^{-1/3}y)\Bigg\}.$$

The function $\left(\frac{\log p}{\log y}-E_0\right)h_2\left(\varkappa,\frac{\log p}{\log y}\right)$ of p is piecewise monotonic for $y^V\leqslant p<y^{1/(\beta+2)}$ (actually, one can show that $xh_2(\varkappa,x)$ has a unique maximum in the interval $V\leqslant x<1/(\beta+2)$ and also that $-h_2(\varkappa,x)$ is increasing) so that Lemma 6 applies to the sum over p on the right of (7.11), with $B\ll 1$, and gives for it (with the factor $\varkappa$ included)

$$\varkappa^2\int\limits_{y^V}^{y^{1/(\beta+2)}}\left(\frac{\log\tau}{\log y}-E_0\right)h_2\left(\varkappa,\frac{\log\tau}{\log y}\right)\frac{d\tau}{\tau\log\tau}+O(\log^{-1}y)$$

$$=\varkappa^2\int\limits_{\beta+2}^{1/V}\left(\frac{1}{t}-E_0\right)h_2\left(\varkappa,\frac{1}{t}\right)\frac{dt}{t}+O(\log^{-1}y)$$

by means of the substitution $\tau = y^{1/t}$. Now the double integral $h_2(\varkappa, 1/t)$ can be integrated once, giving

$$(7.12) \qquad h_2\left(\varkappa, \frac{1}{t}\right) = \left(\frac{t}{t-1}\right)^{\varkappa} k_2(\varkappa, t)$$

where

$$(7.13) \qquad k_2(\varkappa, t) = \int_2^{t-\beta} \left\{\frac{\beta(t-1)^{\varkappa}}{(w+\beta)^{1-\varkappa}\big((t-1)\beta-w\big)^{\varkappa}} - \frac{(\beta-1)^{1-\varkappa}}{(\beta-1+w)^{1-\varkappa}}\right\} \frac{\log(w-1)}{w} dw.$$

With these remarks we finally derive from (7.11) the inequality

$$(7.14) \qquad G \geqslant V(y)\frac{\varkappa A_{\varkappa}}{T-E}\left\{(T-E)\int_{\beta}^{\beta+2}\left(\frac{t}{t-1}\right)^{\varkappa}\frac{dt}{t} - \int_{1/U}^{\beta+2}\left(T-\frac{1}{t}\right)\left(\frac{t}{t-1}\right)^{\varkappa}\frac{dt}{t} + \int_{\beta+2}^{1/V}\left(\frac{1}{t}-E_0\right)\left(\frac{t}{t-1}\right)^{\varkappa}\left(1-\varkappa^2 k_2(\varkappa, t)\right)\frac{dt}{t} + O(\log^{-1/3} y)\right\}.$$

A combination of (6.1), (7.14) and (6.4) yields the following

THEOREM. *Given a $\varkappa$-dimensional sieve problem $(1/2 \leqslant \varkappa \leqslant 1)$ that involves the integer sequence $\mathscr{A}$ and the sifting set $\mathscr{P}$ of primes, we assume that $\mathscr{A}$ and $\mathscr{P}$ satisfy the conditions* (A_0) *and* $(\Omega(\varkappa))$ *and that the numerical constants T, U, V and E satisfy* (5.3), (5.4) *and* (5.5). *Then the weighted sifting-function H defined in* (1.1), (1.3) *and* (5.6) *is estimated from below by*

$$H(\mathscr{A}, \mathscr{P}, y^V, y^U) \geqslant XV(y)\{g_{\varkappa}(T, U, V, E) + O(\log^{-1/3} y)\} - \sum_{\substack{d|P(y^U)\\ d<y}} |R_d|,$$

where for $V \geqslant 1/(\beta+4)$

$$(7.15) \qquad g_{\varkappa}(T, U, V, E) = \frac{\varkappa A_{\varkappa}}{T-E}\left\{(T-E)\int_{\beta}^{\beta+2}\left(\frac{t}{t-1}\right)^{\varkappa}\frac{dt}{t} - \int_{1/U}^{\beta+2}\left(T-\frac{1}{t}\right)\left(\frac{t}{t-1}\right)^{\varkappa}\frac{dt}{t} + \int_{\beta+2}^{1/V}\left(\frac{1}{t}-E_0\right)\left(\frac{t}{t-1}\right)^{\varkappa}\left(1-\varkappa^2 k_2(\varkappa, t)\right)\frac{dt}{t}\right\}.$$

In this formula k_2 is as given by (7.13) *and $\beta = \beta(\varkappa)$ is the function associated with $\varkappa$* (cf. (3.7)). *The O-constant depends at most on $\varkappa$ and A* (*from condition* $(\Omega(\varkappa))$).

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