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Direct summands of systems of continuous linear transformations

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Introduction

In his study of the reduction problem of Hermitian quadratic forms Aronszajn [1] introduced a theory of ***systems*** $(V, W; A_1, A_2)$, where V and W are complex locally convex topological vector spaces and A_1, A_2: $V \to W$ are continuous linear transformations. A homomorphism from $(V, W; A_1, A_2)$ to a system $(X, Y; B_1, B_2)$ is a pair (φ, ψ) of continuous linear transformations φ: $V \to X$, ψ: $W \to Y$ such that $B_j \varphi = \psi A_j$, $j = 1, 2$. Composing homomorphisms componentwise, we obtain a category in which two systems are isomorphic if and only if the corresponding pairs of linear transformations are equivalent in the classical sense. In the special case that $V = W$, $X = Y$ and A_1, B_1 are identity operators the systems above are isomorphic precisely when the operators A_2 and B_2 are similar.

In this paper we consider systems $(V, W; A_1, \ldots, A_N)$ involving $N \geqslant 2$ continuous linear transformations. A classification of such systems up to isomorphism is of course out of the question in view of the fact that the problem poses insurmountable difficulties for very special subcategories already in the case $N = 2$. Consider for example the fact that for systems with zero second spaces isomorphism amounts to topological linear isomorphism of their first spaces, or take the similarity question for single operators in Hilbert space (Ernest [4] has recently classified bounded linear operators on a separable Hilbert space up to unitary equivalence. However, of necessity, the invariants are of a very complicated nature). The detailed investigation of special classes of linear transformations, deep as it may be, anyway seldom approaches the goal of a complete classification. Hence there is place for a more *qualitative* study of *general* systems which may still yield useful information on their structure.

We outline some of the background. An algebraic system $(V, W; A_1, \ldots, A_N)$ is one in which V and W are complex vector spaces in the algebraic sense and $A_1, \ldots, A_N$: $V \to W$ are arbitrary linear transformations. As pointed out in [1], such a system can be viewed as a topological system with V and W carrying their finest weak topologies. The category of algebraic systems with N linear transformations is equivalent to the category of right modules over a certain subring of the matrix ring $M_{N+1}(C)$ (details in [6]). Hence general concepts of module theory are applicable to algebraic systems. E.g., subspaces G of V, H of W determine a subsystem $(G, H;\ A_1|_G, \ldots, A_N|_G)$ of $(V, W;\ A_1, \ldots, A_N)$ provided $A_1G + \ldots + A_NG \subset H$. In the rest of the introduction we write (V, W), (G, H) for a system and its subsystem when no ambiguity arises. (The concept of a system, and therefore the meaning of these symbols, is modified in the paper itself.) A subsystem (G, H) of (V, W) is a direct summand [1] of (V, W) in case there exists a subsystem (K, L) of (V, W) such that $V = G + K$, $W = H + L$ with the sums direct. Note the restriction $\sum A_j K \subset L$. (For topological systems one requires in addition that the associated projections of V onto G and of W onto H be continuous.) The subsystem (G, H) is *pure* in (V, W) iff it is a direct summand in every intermediate subsystem the spaces of which are finite-dimensional extensions of G and H.

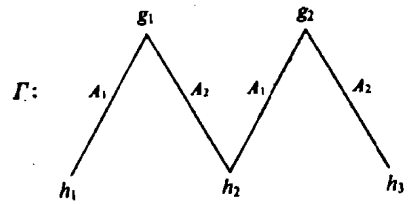
The structure of algebraic systems $(V, W; A_1, A_2)$ (with 2 transformations!) was investigated by Aronszajn and one of the authors ([1], [2] and [5]). The category of such systems contains subcategories equivalent to the category of modules over the principal ideal domain of complex polynomials in one variable. Thus concepts and results from the theory of modules over integral domains and in particular abelian groups (such as torsion-submodule, torsion-free rank, Prüfer groups) could be generalized to systems, though often in a more complicated form requiring new methods. One of the main lines of attack was to first describe completely the structure of purely simple (in particular indecomposable) systems of increasingly complex isomorphism types Ω. Then one gives necessary and sufficient conditions on subsystems of type Ω within a given system (V, W) to form a direct sum which is pure in (V, W). For certain types Ω the cardinal number of summands in a maximal sum of this sort is the dimension of a certain vector space attached to (V, W). It is thus an isomorphism invariant of (V, W) – the multiplicity of Ω in (V, W). This is combined with an analysis of increasingly involved canonically defined subsystems of a given system (e.g., its maximal divisible subsystem or its torsion part).

For topological systems such an approach would require substantial modifications. However, the first steps can be carried out. In this paper we give necessary and sufficient conditions for any finite-dimensional subsystem to be a direct summand in the topological sense. The result can be put in its most explicit form for subsystems whose structure is known; in particular, for all finite-dimensional subsystems in case $N = 2$.

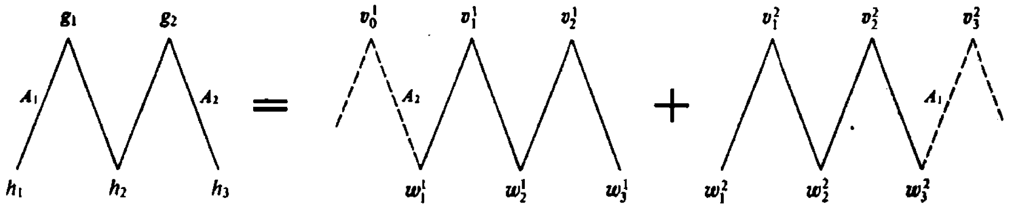
Let us describe the nature of the conditions obtained by means of a simple example. Consider sequences of vectors (g_1, g_2) in V and

[1] The terms "spectral", "quasi-spectral", "quasi-spectrally irreducible" and "eigenvalue part" were used in some of our references instead of "direct summand", "pure", "purely simple" and "torsion part" respectively.

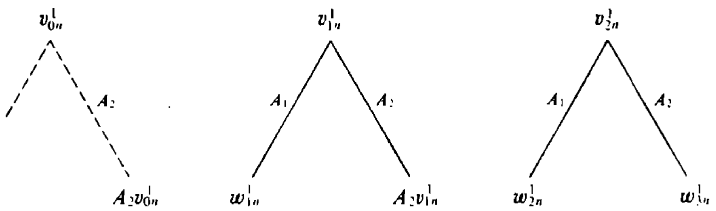
(h_1, h_2, h_3) in W such that $A_1 g_j = h_j$, $A_2 g_j = h_{j+1}$ for $j = 1, 2$. Such a pair of sequences is an example of what we call a *chain* Γ in (V, W). We represent it by the diagram:



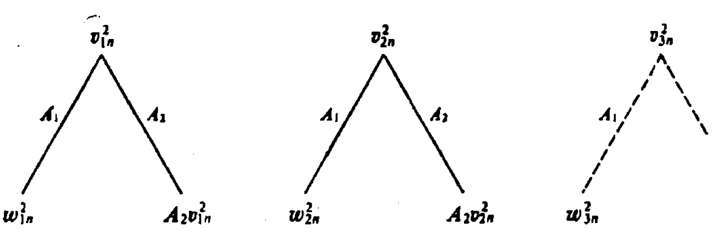
In case (g_1, g_2) and (h_1, h_2, h_3) are *linearly independent* sequences, the subspaces G and H spanned by them determine a subsystem (G, H) of a well defined isomorphism type which we denote by III^3. It was shown in [2] that to have these sequences independent and (G, H) a direct summand of (V, W) in the *algebraic* sense it is necessary and sufficient that the chain not be the sum of restrictions of two longer chains in (V, W):



(termwise addition). Assume for simplicity that the topologies of V and W are metrizable. We show here that (G, H) is a *topological* direct summand of type III^3 if and only if there do not exist two sequences of "broken chains":



and



(i.e., $\lim_{n\to\infty} (A_2 v^1_{jn} - w^1_{j+1,n}) = 0$, $j = 0, 1$; $\lim_{n\to\infty} (A_2 v^2_{jn} - w^2_{j+1,n}) = 0$, $j = 1, 2$) such that the given chain is the sum of their restrictions (i.e., $g_j = v^1_{jn} + v^2_{jn}$, $j = 1, 2$). In this condition the sequences (v^1_{jn}) etc. themselves need not converge. In other words, the topological condition is the algebraic condition obtained by passing to sequences of vectors modulo null sequences.

After getting the initial definitions out of the way, we reformulate our problem in Section 2 in terms of split monomorphisms and outline the method followed in characterizing them. In Section 3 we introduce the tools of internal hom and tensor products for topological systems (these are discussed in [6] for algebraic systems). Criteria for split monomorphisms from systems whose spaces carry their weak topologies and from finite-dimensional systems are given in Theorems 4.4 and 4.5 respectively. These pivotal results of the paper can be obtained without the machinery of Section 3 (as in [10]). However, we want to emphasize the functorial aspect of our procedure.

Our insistence on a general concept of a system which includes algebraic systems as a special case now pays off in several ways. We define functors from topological to algebraic systems which enable us to reduce the question of when a finite-dimensional subsystem is a topological direct summand to the algebraic case (Theorem 5.2). A further reduction, to finite-dimensional indecomposable subsystems, is made in Section 6. In Section 7 we use these reductions to transform the chain conditions of [2] to broken chain conditions. The chain conditions in the algebraic case themselves can also be deduced from Theorem 4.5. Rather than repeat the results of [2] for pairs of linear transformations, we treat in the concluding section an example of one isomorphism type of systems involving three linear transformations.

Theorem 4.5, dealing with topological systems, is used in [11] to prove the algebraic result that a copure subsystem is also pure (copurity being defined in a dual fashion to purity). This in turn yields one possible way to obtain a purity criterion for pairs of linear transformations [7].

1. The category of C^N-systems

We wish to consider finitely many continuous linear transformations $A_1, \ldots, A_N$ from a topological vector space V to a topological vector space W. *All the topological vector spaces considered in this work are assumed to be complex, locally convex and separated.* It is helpful to view the data $(V, W; A_1, \ldots, A_N)$ as an object of a category with suitable morphisms. Just as a set of endomorphisms of an abelian group M determines a module over the ring R generated by the endomorphisms, the linear transformations $A_1, \ldots, A_N$ give rise to a module-like object. To each N-tuple $(a_1, \ldots, a_N) \in C^N$ a linear transformation $\sum_{i=1}^{N} a_i A_i$: $V \to W$ is attached, and one gets a vector space of transformations from V to W. In the case of the module M it is convenient to replace R by an abstract ring, because the action of R on submodules and quotient modules of M may not be faithful. So in the case of $(V, W; A_1, \ldots, A_N)$ it is for similar reasons convenient to have the space C^N of N-tuples $(a_1, \ldots, a_N)$ acting as linear transformations from V to W, rather than the space of combinations $\sum a_i A_i$.

We therefore define a *topological C^N-system*, or briefly a *C^N-system*, to be a pair (V, W) of complex, separated, locally convex topological vector spaces along with a *system operation* assigning to every N-tuple $e \in C^N$ and $v \in V$ a vector $ev \in W$ such that:

(i) for each $e \in C^N$ the map $v \mapsto ev$ is a continuous linear transformation of V to W;

(ii) $(ae_1 + \beta e_2)v = a(e_1 v) + \beta(e_2 v)$ for all $v \in V$, $e_1, e_2 \in C^N$ and $a, \beta \in C$.

For any pair of topological vector spaces V and W we shall henceforth denote by $\mathrm{Hom}(V, W)$ the space of continuous linear maps from V to W. Essentially, a C^N-system (V, W) is given by a linear representation T: $C^N \to \mathrm{Hom}(V, W)$ such that for each $e \in C^N$ and $v \in V$ $T_e(v) = ev$. Any base $e^1, \ldots, e^N$ of C^N determines N continuous linear maps $T_{e^1}, \ldots, T_{e^N}$: $V \to W$. Conversely, having N continuous linear maps $A_1, \ldots, A_N$: $V \to W$ and selecting a base $e^1, \ldots, e^N$ of C^N, a representation T: $C^N \to \mathrm{Hom}(V, W)$ is given by sending $e = \sum a_i e^i \in C^N$ to $T_e = \sum a_i A_i$. When there is no confusion we denote the transformation T_e simply by e.

Let (X, Y), (V, W) be two topological C^N-systems. A *homomorphism* (φ, ψ): $(X, Y) \to (V, W)$ is a pair of continuous linear transformations φ: $X \to V$, ψ: $Y \to W$ such that

$$e\varphi(x) = \psi(ex)$$

for every $e \in C^N$ and $x \in X$. In fact it suffices to test this commutativity condition only on a basis of C^N and a set of generators of X. We *compose* two homomorphisms (φ, ψ): $(X, Y) \to (V, W)$, (σ, τ): $(V, W) \to (U, Z)$ by the rule $(\sigma, \tau)(\varphi, \psi) = (\sigma\varphi, \tau\psi)$: $(X, Y) \to (U, Z)$. The *identity* homomorphism $1_{(V,W)}$ on (V, W) is the pair of identity maps $(1_V, 1_W)$. The notions of *endomorphism*, *isomorphism* and *isomorphic systems* now become apparent. The set of homomorphisms from (X, Y) to (V, W) forms a subspace of $\mathrm{Hom}(X, V) \times \mathrm{Hom}(Y, W)$ in the obvious manner, and we shall denote it by $\mathrm{Hom}\big((X, Y), (V, W)\big)$.

A *subsystem* (G, H) of a C^N-system (V, W) is determined by two linear subspaces G of V and H of W such that $e(G) \subset H$ for all $e \in C^N$.

With the induced topologies, G and H are again separated and locally convex. The action of C^N from V to W restricts to a continuous action from G to H. Hence (G, H) has the structure of a C^N-system. Moreover, the pair $(\iota_{V,G}, \iota_{W,H})$ of inclusions of G to V and of H to W is a homomorphism of C^N-systems. If $(\varphi, \psi) \in \mathrm{Hom}((V, W), (U, Z))$, then $(\varphi, \psi)(\iota_{V,G}, \iota_{W,H})$ is called the *restriction* of (φ, ψ) to (G, H).

A *closed subsystem* (G, H) of (V, W) is a subsystem such that G is closed in V and H is closed in W.

If $(\varphi, \psi) \in \mathrm{Hom}((X, Y), (V, W))$, then $(\varphi, \psi)(X, Y) = (\varphi X, \psi Y)$ is a subsystem of (V, W) and $\mathrm{Ker}(\varphi, \psi) = (\mathrm{Ker}\varphi, \mathrm{Ker}\psi)$ is a closed subsystem of (X, Y).

The *sum* $(G^1, H^1)+(G^2, H^2)$ and *intersection* $(G^1, H^1)\cap(G^2, H^2)$ of two subsystems (G^1, H^1), (G^2, H^2) of (V, W) are respectively the subsystems (G^1+G^2, H^1+H^2) and $(G^1\cap G^2, H^1\cap H^2)$.

In the sequel we shall denote the dual space of a locally convex space V, consisting of continuous linear forms on V, by V'. We put $\langle v, v'\rangle$ for the value of a linear form $v' \in V'$ on an element $v \in V$. Any topological C^N-system (V, W) has a *dual* C^N-system $(V, W)' = (W', V')$ defined as follows. We let each $e \in C^N$ act from W' to V' via the transpose rule

$$\langle v, ew'\rangle = \langle ev, w'\rangle .$$

We assume that W', V' carry their weak topologies (i.e., the $\sigma(W', W)$, $\sigma(V', V)$ topologies), in which case $e \in C^N$ acts continuously from W' to V'.

An *algebraic C^N-system* (V, W) is initially defined in the same way as a topological C^N-system except that V and W are simple linear spaces without topologies (and consequently each $e \in C^N$ acts as a simple linear map from V to W). Every algebraic C^N-system (V, W) can be viewed as a topological C^N-system by putting on V and W their finest weak topologies (i.e., $\sigma(V, V^*)$, $\sigma(W, W^*)$, where V^*, W^* are the algebraic duals). This is because every linear map between such spaces is continuous (and every linear subspace of such a space is closed). When the term "algebraic C^N-system" is used it will be clear from the context whether the initial algebraic concept or its topological equivalent is meant.

To every topological C^N-system (V, W) one can attach its *underlying algebraic C^N-system* obtained by forgetting the topologies on V and W (or replacing them by the finest weak topologies).

2. The problem of split monomorphisms

A C^N-system (V, W) is said to be the *direct sum* of a family of subsystems $((G_j, H_j))_{j\in J}$ in case V is the topological direct sum of $(G_j)_{j\in J}$ and W is the topological direct sum of $(H_j)_{j\in J}$. If this is the case, we write

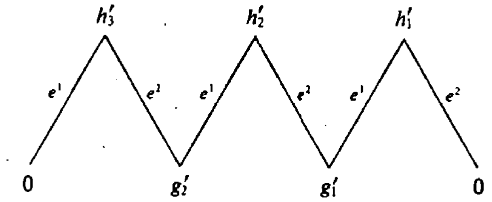
$$(V, W) = \sum_{j \in J}{}^{\cdot} (G_j, H_j)$$

(or $(V, W) = (G_1, H_1) \dot{+} \ldots \dot{+} (G_n, H_n)$ for a finite number of summands). A subsystem (G, H) of (V, W) is said to be a *direct summand* of (V, W) in case it has a *supplement*, namely a subsystem (K, L) such that $(V, W) = (G, H) \dot{+} (K, L)$. As V and W are separated, a direct summand and any of its supplements must be closed subsystems of (V, W). A subsystem (G, H) of (V, W) is a direct summand of (V, W) if and only if there is an endomorphism (π, ϱ) of (V, W) such that $(\pi, \varrho)^2 = (\pi, \varrho)$ and $(\pi, \varrho)(V, W) = (G, H)$ (in which case $\mathrm{Ker}(\pi, \varrho)$ is a supplement to (G, H) in (V, W)). We call such a (π, ϱ) a *projection*. Note that the existence of a closed subsystem (K, L) of (V, W) such that $(V, W) = (G, H) + (K, L)$ and $(G, H) \cap (K, L) = (0, 0)$ does not by itself imply that (G, H) is a direct summand of (V, W) (except in situations where the closed graph theorem can be invoked to show that the projection in the algebraic sense is continuous).

Let us return for a moment to the example of the subsystem (G, H) spanned by the chain Γ of the introduction; where now A_1, A_2 are replaced by the basis e^1, e^2 of C^2. The two components of a projection onto (G, H) must be of the form

$$\pi(v) = \sum_{j=1}^{2} \langle v, g_j' \rangle g_j, \qquad \varrho(w) = \sum_{j=1}^{3} \langle w, h_j' \rangle h_j, \qquad g_j' \in V',\ h_j' \in W'.$$

From the requirements that (π, ϱ) be a projection and (h_1, h_2, h_3) be linearly independent one deduces that the functionals must form a chain



in (W', V') which is dual to Γ in the sense that $\langle g_j, g_k' \rangle = \delta_{jk}$, $\langle h_j, h_k' \rangle = \delta_{jk}$. Conversely, the existence of a dual chain implies that (G, H) is a direct summand of type III^3. This is a special case of Aronszajn's characterization of finite-dimensional direct summands of C^2-systems [1], Section 6, Theorem 1. Our task is to eliminate the dual system from this characterization.

Chains may be regarded as descriptions of homomorphisms (see Proposition 7.2 below). In general, to describe direct summands of a given isomorphism type Ω in a C^N-system (V, W) we consider homomorphisms (φ, ψ) from a model system (X, Y) of type Ω into (V, W). Such a homomorphism is said to be a *split monomorphism* whenever it has a left inverse $(\sigma,\tau)\colon (V,W)\to(X,Y)$ (i.e., $(\sigma,\tau)(\varphi,\psi) = 1_{(X,Y)}$). If $(\varphi,\psi)\in \mathrm{Hom}\big((X, Y),(V,W)\big)$ is a split monomorphism with left inverse (σ,τ), then $(\varphi,\psi)\times\times(\sigma,\tau)$ is a projection of (V,W) onto $(\varphi,\psi)(X,Y)$, and the homomorphism of (X,Y) onto $(\varphi,\psi)(X,Y)$ induced by (φ,ψ) has the restriction $(\sigma,\tau)|_{(\varphi,\psi)(X,Y)}$ as a two-sided inverse. Thus $(\varphi,\psi)(X,Y)$ is a direct summand of (V,W) which is isomorphic to (X,Y). Conversely, given a projection (π,ϱ) of (V,W) onto a subsystem (G,H) and an isomorphism $(\varphi,\psi)\colon (X,Y)\approx(G,H)$, we have (but for change of targets) that $(\varphi,\psi)\in\mathrm{Hom}\big((X,Y),(V,W)\big)$ and $(\sigma,\tau) = (\varphi^{-1},\psi^{-1})(\pi,\varrho)$ is a well defined left inverse of it. We therefore develop conditions for a given homomorphism to be a split monomorphism. We shall study in particular the case that (X,Y) is finite-dimensional, that is both X and Y are of finite dimension.

We observe that for isomorphism types Ω of infinite-dimensional systems there is some advantage in studying a given system in terms of its epimorphisms onto a system of type Ω (rather than in terms of subsystems of type Ω). This is because the kernel of a homomorphism is always closed. However, as far as single direct summands are concerned the two approaches are equivalent.

As a first step in eliminating linear functionals, we characterize split monomorphisms $(\varphi,\psi)\colon (X,Y)\to(V,W)$ in terms involving (Y',X') rather than (W',V'). Our technique is to generalize the following considerations to C^N-systems. Let X and V be vector spaces, which for simplicity we take to be finite-dimensional. Let $(x_i)_{i=1}^m$ be a basis of X and $(x_j')_{j=1}^m$ its dual in X'. A linear transformation $\varphi\colon X\to V$ is left invertible if and only if φ is monic or, equivalently, $\big(\varphi(x_i)\big)_{i=1}^m$ is linearly independent. However, we shall imitate the following transforms of these conditions. Look at the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(V,X) & \xrightarrow{\ \mathrm{Hom}(\varphi,X)\ } & \mathrm{Hom}(X,X)\\
\downarrow\wr & & \downarrow\wr\\
\mathrm{Hom}\big(V,\mathrm{Hom}(X',C)\big) & \xrightarrow{\ \mathrm{Hom}(\varphi,\mathrm{Hom}(X',C))\ } & \mathrm{Hom}\big(X,\mathrm{Hom}(X',C)\big)\\
\downarrow\wr & & \downarrow\wr\\
\mathrm{Hom}(V\otimes X',C) & \xrightarrow{\ \mathrm{Hom}(\varphi\oplus 1_{X'},C)\ } & \mathrm{Hom}(X\otimes X',C)
\end{array}$$

Clearly, φ has a left inverse if and only if 1_X is in the image of $\mathrm{Hom}(\varphi,X)$. The vertical isomorphisms to the second row stem from the canonical isomorphism $\hat{}\ \colon X\approx X'' = \mathrm{Hom}(X',C)$, while those to the third come from the adjointness relation of Hom and $\otimes$. 1_X is carried first to $\hat{}$ and then to the trace form $\varkappa$ (a particular contraction of tensors) defined by the requirements $\varkappa\colon x \otimes x' \mapsto \langle x', \hat{x} \rangle = \langle x, x' \rangle$. Thus φ has a left inverse if and only if there exists a linear functional F on $V \otimes X'$ such that $F(\varphi \otimes 1_{X'}) = \varkappa$. This is the condition generalized in 4.4. Since any linear functional on the image of $\varphi \otimes 1_{X'}$ is extendible to $V \otimes X'$, this is equivalent to the requirement that

$$\sum_{i,j=1}^{m} c_{ij} \varphi(x_i) \otimes x_j' = 0, \qquad c_{ij} \quad \text{scalars},$$

implies

$$\sum_{i=1}^{m} c_{ii} = \varkappa\Big(\sum_{i,j=1}^{m} c_{ij} x_i \otimes x_j'\Big) = 0.$$

This is generalized in 4.5.

3. Internal hom and tensor product

The desired functors, which will enable us to carry through the argument of the last section in the category of C^N-systems, can be defined with various choices of topologies. We shall consider here only the simplest definitions adequate for application to finite-dimensional direct summands.

We first choose internal hom and tensor products in the category of separated locally convex spaces. If X and V are such spaces, we endow the space $\mathrm{Hom}(X, V)$ of continuous linear maps from X to V with the topology of pointwise (or simple) convergence. This space with the assigned separated locally convex topology will be denoted by $\mathrm{Hom}_s(X, V)$. It is clear that defining the action of Hom_s on continuous linear maps in the usual way we obtain a bifunctor. Note that with the topology we have already chosen for X' we have $\mathrm{Hom}_s(X, C) = X'$.

We endow the algebraic tensor product $X \otimes Y$ with the inductive topology and denote the resulting separated locally convex space by $X \otimes_\iota V$ (see [8], Definition 3). This inductive topology is the finest locally convex topology making the tensor map $\otimes\colon X \times V \to X \otimes V$ separately continuous. For any separated locally convex space U the map $\varphi \mapsto \varphi \circ \otimes$ is a bijection of $\mathrm{Hom}(X \otimes_\iota V, U)$ onto the space $\mathscr{B}(X, V;\ U)$ of separately continuous bilinear maps of $X \times V$ into U (see [8], Proposition 13). This universal property enables us to define the bifunctor $\otimes_\iota$ as usual.

It is clear from the definitions that the map $\omega \mapsto \big(x \mapsto (v \mapsto \omega(x, v))\big)$ is a bijection of $\mathscr{B}(X, V;\ U)$ onto $\mathrm{Hom}\big(X, \mathrm{Hom}_s(V, U)\big)$. Thus we get a bijection

$$\text{(1)} \qquad \mathrm{Hom}(X \otimes_\iota V, U) \approx \mathrm{Hom}\big(X, \mathrm{Hom}_s(V, U)\big),$$

where a continuous linear map $\varphi: X \otimes_\iota V \to U$ corresponds to a continuous linear map $\psi: X \to \mathrm{Hom}_s(V, U)$ if and only if

$$(2) \qquad \varphi(x \otimes v) = \psi(x)(v) \quad \text{for all } x \in X, v \in V.$$

The isomorphisms (1) are evidently linear and constitute a natural isomorphism or an *adjointness relation.*

We define an *internal homfunctor* hom in the category of C^N-systems as follows. We put $\mathrm{Hom}_s\big((V, W), (U, Z)\big)$ for the space $\mathrm{Hom}\big((V, W), (U, Z)\big)$ of system homomorphisms with the topology induced on it as a subspace of the topological product $\mathrm{Hom}_s(V, U) \times \mathrm{Hom}_s(W, Z)$. We define the continuous action of any $e \in C^N$ from $\mathrm{Hom}_s\big((V, W), (U, Z)\big)$ to $\mathrm{Hom}_s(V, Z)$ by

$$(3) \qquad e(\varphi, \psi) = e \circ \varphi = \psi \circ e, \qquad (\varphi, \psi) \in \mathrm{Hom}\big((V, W), (U, Z)\big).$$

(Here we have written e for T_e.) As the spaces involved are separated locally convex and $\big(e, (\varphi, \psi)\big) \mapsto e(\varphi, \psi)$ is bilinear, we obtain a C^N-system

$$\mathrm{hom}\big((V, W), (U, Z)\big) = \big(\mathrm{Hom}_s\big((V, W), (U, Z)\big), \mathrm{Hom}_s(V, Z)\big)$$

with the system operation specified by (3).

Given homomorphisms $(\mu, \nu): (V^1, W^1) \to (V, W)$ and $(\sigma, \tau): (U, Z) \to (U^1, Z^1)$, we set

$$\mathrm{hom}\big((\mu, \nu), (\sigma, \tau)\big) = \big(\mathrm{Hom}\big((\mu, \nu), (\sigma, \tau)\big), \mathrm{Hom}(\mu, \tau)\big),$$

where

$$\mathrm{Hom}\big((\mu, \nu), (\sigma, \tau)\big)\big((\varphi, \psi)\big) = (\sigma, \tau)(\varphi, \psi)(\mu, \nu),$$
$$(\varphi, \psi) \in \mathrm{Hom}\big((V, W), (U, Z)\big).$$

This makes hom a bifunctor with the usual variances.

The dual of a C^N-system, defined in Section 1, can be constructed (up to isomorphism) as a value of hom. To do this, we consider the C^N-system $(C^{N'}, C)$, where $C^{N'}$ is the dual of C^N and $e \in C^N$ acts on $f \in C^{N'}$ via the rule $ef = \langle e, f \rangle$.

LEMMA 3.1. *If (X, Y) is a C^N-system, then the map $(\varphi, \psi) \mapsto \psi$ is an isomorphism of* $\mathrm{Hom}_s\big((X, Y), (C^{N'}, C)\big)$ *onto the dual space Y' of Y.*

Proof. If $(\varphi, \psi): (X, Y) \to (C^{N'}, C)$ is a homomorphism of C^N-systems, we must have $e\varphi(x) = \psi(ex)$ for all $e \in C^N$ and $x \in X$. By the action of e defined for $(C^{N'}, C)$ we have $e\varphi(x) = \langle e, \varphi(x) \rangle$. Thus $\langle e, \varphi(x) \rangle = \langle ex, \psi \rangle$ and φ is determined by ψ. In other words, the map of the lemma is injective. It is also surjective because if $\varphi(x)$ is defined for every x in X by means of the last equation, then φ is a continuous linear map of X into $C^{N'}$ and (φ, ψ) is a homomorphism. It is easy to see that $(\varphi, \psi) \mapsto \psi$ is a homeomorphism.

PROPOSITION 3.2. *For any C^N-system (X, Y) the pair of maps $(\varphi, \psi) \mapsto \psi$ and $1_{X'}$ is an isomorphism*

$$\hom((X, Y), (C^{N'}, C)) \approx (Y', X').$$

These isomorphisms constitute a natural isomorphism.

Proof. By Lemma 3.1 and the fact that $\mathrm{Hom}_s(X, C) = X'$, we have a pair of topological linear isomorphisms. That this pair is a homomorphism of systems is immediate from the definitions.

The assignment $(X, Y) \mapsto (Y', X')$ is completed to a contravariant functor by sending $(\varphi, \psi) \in \mathrm{Hom}((X, Y), (U, Z))$ to $(\psi', \varphi') \in \mathrm{Hom}((U, Z)', (X, Y)')$, where φ', ψ' are the transposes of φ, ψ. It is then easy to verify that the isomorphisms of the proposition are natural, namely constitute a corepresentation of this *duality functor.*

We now construct a left adjoint to hom. We consider the space $(X \otimes_\iota W) \oplus (Y \otimes_\iota V)$, with the direct sum (same as product) topology, and the (not necessarily closed) subspace $R((X, Y), (V, W))$ algebraically generated by all terms of the form

$$(x \otimes ev, -ex \otimes v) \quad \text{for} \quad x \in X, v \in V, e \in C^N.$$

This subspace, which we shall denote simply by R when there is no confusion, is called the *space of tensor relations* in $X \otimes_\iota W \oplus Y \otimes_\iota V$. Taking the closure $\bar{R}$ and considering the quotient space $(X \otimes_\iota W \oplus Y \otimes_\iota V)/\bar{R}$ (which is still locally convex and separated), we determine the action of any $e \in C^N$ from $X \otimes_\iota V$ to $(X \otimes_\iota W \oplus Y \otimes_\iota V)/\bar{R}$ by the requirement

$$e(x \otimes v) = (x \otimes ev, 0) + \bar{R} = (0, ex \otimes v) + \bar{R} \tag{4}$$

for $x \in X$, $v \in V$. This action is linear and continuous because it is $1_X \otimes T_e$: $X \otimes_\iota V \to X \otimes_\iota W$ followed by the canonical injection and quotient maps. The linearity in e is evident. The *internal tensor product* $(X, Y) \otimes (V, W)$ is then the C^N-system $(X \otimes_\iota V, (X \otimes_\iota W \oplus Y \otimes_\iota V)/\bar{R})$ with the system operation given by (4).

The *tensor product of two homomorphisms* (φ, ψ): $(X, Y) \to (X^1, Y^1)$, (σ, τ): $(V, W) \to (V^1, W^1)$ is the homomorphism

$$(\varphi, \psi) \otimes (\sigma, \tau): (X, Y) \otimes (V, W) \to (X^1, Y^1) \otimes (V^1, W^1)$$

defined as the pair

$$(\varphi \otimes \sigma, \overline{\varphi \otimes \tau \oplus \psi \otimes \sigma}).$$

Here $\varphi \otimes \sigma$: $X \otimes_\iota V \to X^1 \otimes_\iota V^1$ is the usual tensor product of linear transformations. As to the second component, we note that due to the commutativity condition satisfied by (φ, ψ) and (σ, τ) the continuous linear map $\varphi \otimes \tau \oplus \psi \otimes \sigma$ maps $R = R((X, Y), (V, W))$ into $R^1 = R((X^1, Y^1), (V^1, W^1))$, hence $\bar{R}$ into $(R^1)^-$. It thus induces a continuous linear map $\overline{\varphi \otimes \tau \oplus \psi \otimes \sigma}$ on passing to the quotients. This map is determined by the requirement that

$$\overline{\varphi \otimes \tau \oplus \psi \otimes \sigma}\big((x \otimes w, y \otimes v) + \bar{R}\big) = \big(\varphi(x) \otimes \tau(w), \psi(y) \otimes \sigma(v)\big) + (R^1)^-$$

for all $x \in X$, etc. It is easy to see that $(\varphi, \psi) \otimes (\sigma, \tau)$ is a homomorphism of C^N-systems and that our assignments constitute a bifunctor — the *internal tensor product functor of* C^N-systems.

Proposition 3.3. *The internal tensor product functor of the category of C^N-systems is a left adjoint of* hom. *In other words, we have natural bijections*

$$(5) \qquad \xi\colon \operatorname{Hom}\big((X, Y) \otimes (V, W), (U, Z)\big) \approx \operatorname{Hom}\Big((X, Y), \operatorname{hom}\big((V, W), (U, Z)\big)\Big).$$

(In the notation for natural transformations we often suppress indices indicating dependence on the objects involved.)

Proof. For given $(X, Y), (U, Z), (V, W)$ we define the required bijection ξ as the composite of three bijections.

(a) *A bijection of* $\operatorname{Hom}\Big((X, Y), \operatorname{hom}\big((V, W), (U, Z)\big)\Big)$ *onto the set A of all triples of continuous linear maps*

$$a_1\colon X \to \operatorname{Hom}_s(V, U), \qquad a_2\colon X \to \operatorname{Hom}_s(W, Z), \qquad a_3\colon Y \to \operatorname{Hom}_s(V, Z)$$

satisfying the equations

$$(6) \qquad e a_1(x)(v) = a_2(x)(ev) = a_3(ex)(v)$$

for all $x \in X$, $v \in V$, $e \in C^N$.

A homomorphism $(\sigma, \tau)\colon (X, Y) \to \operatorname{hom}\big((V, W), (U, Z)\big)$ is a pair of continuous linear maps

$$\sigma\colon X \to \operatorname{Hom}_s\big((V, W), (U, Z)\big)$$

and

$$\tau\colon Y \to \operatorname{Hom}_s(V, Z)$$

satisfying

$$(*) \qquad \big(e\sigma(x)\big)(v) = \tau(ex)(v).$$

Now $\sigma(x)$, being in $\operatorname{Hom}_s\big((V, W), (U, Z)\big)$, is a pair of continuous linear maps $\sigma_1(x)\colon V \to U$ and $\sigma_2(x)\colon W \to Z$ which satisfy the commutativity condition for homomorphisms of C^N-systems. Define $\sigma_1\colon X \to \operatorname{Hom}_s(V, U)$ and $\sigma_2\colon X \to \operatorname{Hom}_s(W, Z)$ by $\sigma_1\colon x \mapsto \sigma_1(x)$, $\sigma_2\colon x \mapsto \sigma_2(x)$. The fact that σ is linear is equivalent to the linearity of σ_1 and σ_2. Since the topology of $\operatorname{Hom}_s\big((V, W), (U, Z)\big)$ is that induced from the product topology of

$\mathrm{Hom}_s(V, U) \times \mathrm{Hom}_s(W, Z)$, the continuity of σ is equivalent to that of σ_1 and σ_2. Equation $(*)$ can be rewritten

$$\big(e(\sigma_1(x), \sigma_2(x))\big)(v) = \tau(ex)(v).$$

By the definition (3) of the system operation of $\mathrm{hom}\big((V, W), (U, Z)\big)$, this can be expressed in the form

$$e\sigma_1(x)(v) = \sigma_2(x)(ev) = \tau(ex)(v),$$

the first equality of which is the commutativity condition for $\sigma(x)$. Thus, if *we attach to each* (σ, τ) *the triple* $(\sigma_1, \sigma_2, \tau)$, we obtain a map into A. This map is the desired first bijection. Indeed it is clear from the above considerations that it has as inverse the map sending $(a_1, a_2, a_3) \in A$ to $\big((a_1, a_2), a_3\big)$.

(b) *A bijection of A onto the set B of all triples of continuous linear maps*

$$\beta_1\colon X \otimes_\iota V \to U, \quad \beta_2\colon X \otimes_\iota W \to Z, \quad \beta_3\colon Y \otimes_\iota V \to Z$$

satisfying the equations

$$e\beta_1(x \otimes v) = \beta_2(x \otimes ev) = \beta_3(ex \otimes v) \tag{7}$$

for all $x \in X$, $v \in V$, $e \in C^N$.

For $i = 1, 2, 3$ we take β_i to be the map corresponding to a_i in the adjointness relation (1). The formulas of type (2) show that equations (7) are equivalent to the equations (6).

(c) *A bijection of B onto* $\mathrm{Hom}\big((X, Y) \otimes (V, W), (U, Z)\big)$.

Let $(\beta_1, \beta_2, \beta_3) \in B$. Since $X \otimes_\iota W \oplus Y \otimes_\iota V$ is a topological direct sum, the linearity and continuity of the map $[\beta_2, \beta_3]$ sending $(t_1, t_2) \in X \otimes_\iota W \oplus \oplus Y \otimes_\iota V$ to $\beta_2(t_1) + \beta_3(t_2) \in Z$ is equivalent to the linearity and continuity of β_2 and β_3. The second equality of (7) expresses the fact that $[\beta_2, \beta_3]$ vanishes on generators of $R\big((X, Y), (V, W)\big)$, hence on $\bar{R}$. Therefore $[\beta_2, \beta_3]$ induces a continuous linear map

$$\overline{[\beta_2, \beta_3]}\colon (X \otimes_\iota W \oplus Y \otimes_\iota V)/\bar{R} \to Z$$

(and conversely, every continuous linear map ϱ between these spaces is of this form with $\beta_2(t_1) = \varrho\big((t_1, 0) + \bar{R}\big)$, $\beta_3(t_2) = \varrho\big((0, t_2) + \bar{R}\big)$). Referring to the definition (4) of the system operation of $(X, Y) \otimes (V, W)$, we see that equations (7) state that $(\beta_1, \overline{[\beta_2, \beta_3]})$ satisfies the commutativity condition of system homomorphisms. Thus, *our third bijection is the map* $(\beta_1, \beta_2, \beta_3) \mapsto (\beta_1, \overline{[\beta_2, \beta_3]})$. Its inverse attaches to $(\pi, \varrho) \in \mathrm{Hom}\big((X, Y) \otimes \otimes (V, W), (U, Z)\big)$ the triple (π, β_2, β_3), where β_2, β_3 are defined in terms of ϱ as in the parenthetical remark above.

Of the statement that (5) is an adjointness relation we shall verify only the part actually used here; namely, the naturality in the first variable. (In fact, for algebraic systems at least we have a natural isomorphism even between the internal functors:

$$\hom\big((X,\ Y)\otimes(V,\ W),\ (U,Z)\big)\approx\hom\Big((X,\ Y),\ \hom\big((V,\ W),\ (U,Z)\big)\Big),$$

cf. [6]. Here it is clear that if we endow A and B with the structures of complex vector spaces by defining the operations componentwise, our three bijections become linear isomorphisms. Hence so is their composite ξ.)

Let $(\varphi,\psi)\colon(X^1,\ Y^1)\to(X,\ Y)$ be a homomorphism of C^N-systems. We distinguish maps belonging to $(X^1,\ Y^1)$, (V,W), (U,Z) from the former ones by adding the superscript 1, and show that the following square commutes.

$$(8)\qquad \begin{array}{ccc} \begin{array}{l}\operatorname{Hom}\big((X,\ Y),\\ \hom\big((V,\ W),\ (U,Z)\big)\big)\end{array} & \xrightarrow{\operatorname{Hom}((\varphi,\psi),\hom((V,W),(U,Z)))} & \begin{array}{l}\operatorname{Hom}\big((X^1,\ Y^1),\\ \hom\big((V,\ W),\ (U,Z)\big)\big)\end{array} \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \xi^1} \\ \begin{array}{l}\operatorname{Hom}\big((X,\ Y)\otimes\\ \otimes(V,\ W),\ (U,Z)\big)\end{array} & \xrightarrow{\operatorname{Hom}((\varphi,\psi)\otimes 1_{(V,W)},(U,Z))} & \begin{array}{l}\operatorname{Hom}\big((X^1,\ Y^1)\otimes\\ \otimes(V,\ W),\ (U,Z)\big)\end{array} \end{array}$$

Let $(\sigma,\ \tau)$ belong to the upper left corner. It is sent by the top map to $(\sigma^1,\ \tau^1)=(\sigma\varphi,\ \tau\psi)$. We have $\sigma_1^1=\sigma_1\varphi$, $\sigma_2^1=\sigma_2\varphi$. Hence, the first bijection defining ξ^1 sends $(\sigma^1,\ \tau^1)$ to $(\sigma_1\varphi,\ \sigma_2\varphi,\ \tau\psi)$. The second bijection carries this to $(\beta_1^1,\ \beta_2^1,\ \beta_3^1)$, where $\beta_1^1(x^1\otimes v)=\sigma_1\big(\varphi(x^1)\big)(v)$, etc. Thus

$$(\beta_1^1,\ \beta_2^1,\ \beta_3^1)=\big(\beta_1(\varphi\otimes 1_V),\ \beta_2(\varphi\otimes 1_W),\ \beta_3(\psi\otimes 1_V)\big).$$

This goes by the third bijection to

$$\big(\beta_1(\varphi\otimes 1_V),\ \overline{[\beta_2(\varphi\otimes 1_W),\ \beta_3(\psi\otimes 1_V)]^1}\big).$$

Evaluating the second component on a generator $\overline{(x^1\otimes w,\ y^1\otimes v)+(R^1)^-}$ of the second space of $(X^1,\ Y^1)\otimes(V,\ W)$, we see that it equals $\overline{[\beta_2,\ \beta_3]}\times\times(\varphi\otimes 1_W\oplus\psi\otimes 1_V)$. Thus the image of $(\sigma,\ \tau)$ we just computed is the same as its image along the other path; namely,

$$\big(\beta_1,\ \overline{[\beta_2,\ \beta_3]}\big)\big((\varphi,\ \psi)\otimes(1_W,\ 1_V)\big).$$

4. Characterizations of split monomorphisms

Let X be a separated locally convex space. As in the definition of the dual system, its topological dual X' is endowed with the topology $\sigma(X', X)$. Hence, deviating from standard notation, we shall denote here by X'' the topological dual of the above X' endowed with the topology $\sigma(X'', X')$.

A C^N-system (X, Y) is said to be *weak* in case X and Y have the weak topologies $\sigma(X, X')$ and $\sigma(Y, Y')$. Algebraic C^N-systems and in particular finite-dimensional C^N-systems are weak. For a weak system (X, Y) the canonical map $\hat{}^X: X \to X''$, sending x to $\hat{x}$ with $\langle x', \hat{x} \rangle = \langle x, x' \rangle$ for all $x' \in X'$, is a topological linear isomorphism; similarly for Y. It is easy to see that we have an isomorphism

$$(\hat{}^X, \hat{}^Y): (X, Y) \approx (X'', Y'').$$

Combining this isomorphism with the inverse of the isomorphism of Proposition 3.2 for $(X, Y)'$ in place of (X, Y), we obtain

LEMMA 4.1. *For a weak C^N-system (X, Y) we have an isomorphism*

$$(\delta, \varepsilon): (X, Y) \approx \hom\big((X, Y)', (C^{N'}, C)\big).$$

Here $\varepsilon(y) = \hat{y}$ and $\delta(x)$ is of the form $\big(\delta_1(x), \hat{x}\big)$, where $\delta_1(x) \in \operatorname{Hom}(Y', C^{N'})$.

The determination of $\delta_1(x)$, which is given by $\langle e, \delta_1(x)(y')\rangle = \langle e(x), y'\rangle$ for all $y' \in Y'$, $e \in C^N$, is immaterial for our purpose. So also is the fact that the isomorphisms of 4.1 are natural.

LEMMA 4.2. *Let (X, Y) be a weak C^N-system. Then the image of the isomorphism (δ, ε) of Lemma 4.1 under the bijection*

$$\operatorname{Hom}\Big((X, Y), \hom\big((X, Y)', (C^{N'}, C)\big)\Big) \approx \operatorname{Hom}\big((X, Y) \otimes (X, Y)', (C^{N'}, C)\big)$$

described in the proof of Proposition 3.3 is the trace homomorphism

$$(\lambda, \varkappa): (X, Y) \otimes (X, Y)' \to (C^{N'}, C),$$

where the trace form $\varkappa$ is defined by the requirements

$$\varkappa\Big((x \otimes x', y \otimes y') + R\big((X, Y), (X, Y)'\big)^-\Big) = \langle x, x'\rangle + \langle y, y'\rangle$$

for all $x \in X, y \in Y, x' \in X', y' \in Y'$ (and λ satisfies $\langle e, \lambda(x \otimes y')\rangle = \langle ex, y'\rangle$).

Proof. The first bijection of 3.3 sends (δ, ε) to $(\delta_1, \hat{}^X, \hat{}^Y)$, where δ_1 maps x to $\delta_1(x)$ as in 4.1. This is sent by the second bijection to $(\lambda, \beta_2, \beta_3)$, where $\lambda(x \otimes y') = \delta_1(x)(y')$ (and thus λ is as defined in the lemma), $\beta_2(x \otimes x') = \hat{x}(x') = \langle x, x'\rangle$ and similarly $\beta_3(y \otimes y') = \langle y, y'\rangle$. The last bijection leads to $(\lambda, \overline{[\beta_2, \beta_3]})$, where clearly $\overline{[\beta_2, \beta_3]} = \varkappa$.

PROPOSITION 4.3. *Let (X, Y) be a weak C^N-system. A homomorphism $(\varphi, \psi)\colon (X, Y)\to(V, W)$ is a split monomorphism if and only if there exists a homomorphism*

$$(\sigma, \tau)\colon (V, W)\otimes(X, Y)'\to(C^{N'}, C)$$

such that

$$(\sigma, \tau)\big((\varphi, \psi)\otimes 1_{(X,Y)'}\big) = (\lambda, \varkappa), \tag{1}$$

where $(\lambda, \varkappa)$ is the trace homomorphism on $(X, Y)\otimes(X, Y)'$.

Proof. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}\big((V, W), (X, Y)\big) & \xrightarrow{\mathrm{Hom}((\varphi,\psi),(X,Y))} & \mathrm{Hom}\big((X, Y), (X, Y)\big) \\
\big\downarrow \wr\, \mathrm{Hom}((V,W),(\delta,\varepsilon)) & & \big\downarrow \wr\, \mathrm{Hom}((X,Y),(\delta,\varepsilon)) \\
\mathrm{Hom}\big((V, W), \mathrm{hom}((X, Y)', (C^{N'}, C))\big) & \longrightarrow & \mathrm{Hom}\big((X, Y), \mathrm{hom}((X, Y)', (C^{N'}, C))\big) \\
\big\downarrow \wr & & \big\downarrow \wr\, \xi^1 \\
\mathrm{Hom}\big((V, W)\otimes (X, Y)', (C^{N'}, C)\big) & \xrightarrow[\mathrm{Hom}((\varphi,\psi)\otimes 1_{(X,Y)'},(C^{N'},C))]{} & \mathrm{Hom}\big((X, Y)\otimes (X, Y)', (C^{N'}, C)\big)
\end{array}$$

The top square is commutative and its vertical maps are isomorphisms because Hom is a bifunctor and (δ, ε) (of Lemma 4.1) is an isomorphism. The bottom square is a case of square (8) of Section 3. The homomorphism (φ, ψ) is left invertible if and only if $1_{(X,Y)}$ is in the image of $\mathrm{Hom}\big((\varphi, \psi), (X, Y)\big)$. We have $\mathrm{Hom}\big((X, Y), (\delta, \varepsilon)\big)(1_{(X,Y)}) = (\delta, \varepsilon)$ and by Lemma 4.2 $\xi^1\big((\delta, \varepsilon)\big) = (\lambda, \varkappa)$. Hence (φ, ψ) is left invertible if and only if $(\lambda, \varkappa)$ is in the image of the bottom horizontal map; which is what the proposition states.

THEOREM 4.4. *Let (X, Y) be a weak C^N-system, and let the linear trace form $\tilde{\varkappa}\colon X\otimes_{\iota}X'\oplus Y\otimes_{\iota}Y'\to C$ be defined by the requirements*

$$\tilde{\varkappa}\big((x\otimes x', y\otimes y')\big) = \langle x, x'\rangle + \langle y, y'\rangle.$$

A homomorphism $(\varphi, \psi)\colon (X, Y)\to(V, W)$ is a split monomorphism if and only if for some continuous linear functional $F\colon V\otimes_{\iota}X'\oplus W\otimes_{\iota}Y'\to C$, vanishing on the space of tensor relations $R\big((V, W), (X, Y)'\big)$, we have

$$F(\varphi\otimes 1_{X'}\oplus\psi\otimes 1_{Y'}) = \tilde{\varkappa}. \tag{2}$$

Proof. Due to the naturality of the isomorphisms of Lemma 3.1

(see 3.2), we have a commutative square

$$\begin{array}{ccc}
\operatorname{Hom}\big((V,W)\otimes (X,Y)',(C^{N'},C)\big) & \xrightarrow{\operatorname{Hom}((\varphi,\psi)\otimes 1_{(X,Y)'},(C^{N'},C))} & \operatorname{Hom}\big((X,Y)\otimes (X,Y'),(C^{N'},C)\big) \\
\downarrow \wr & & \downarrow \wr \\
\big((V\otimes_\iota X'\oplus W\otimes_\iota Y')/\bar{R}\big)' & \xrightarrow{\operatorname{Hom}(\varphi\otimes 1_{X'}\oplus\psi\otimes 1_{Y'},C)} & \big((X\otimes_\iota X'\oplus Y\otimes_\iota Y')/(R^1)^-\big)'
\end{array}$$

where the vertical isomorphisms take a homomorphism to its second component. Thus there exists a (σ,τ) satisfying (1) of Proposition 4.3 if and only if there exists a linear functional τ satisfying

$$\tau(\overline{\varphi\otimes 1_{X'}\oplus\psi\otimes 1_{Y'}}) = \varkappa.$$

This last condition is clearly tantamount to the condition stated in the theorem.

We see that the consideration of the systems $\hom\big((V,W),(U,Z)\big)$ and $(X,Y)\otimes(V,W)$, rather than just their second spaces, has been extra baggage; carried along to motivate the procedure.

We now specialize to the problem of splitting $(\varphi,\psi)\colon (X,Y)\to(V,W)$, *under the standing assumption that* (X,Y) *is finite-dimensional.* Let $(x_1,\dots,x_m)$, $(y_1,\dots,y_n)$ be bases of X and Y respectively, and let $(x_1',\dots,x_m')$, $(y_1',\dots,y_n')$ be their dual bases in X', Y' respectively. That is, $\langle x_i,x_j'\rangle=\delta_{ij}$ and $\langle y_k,y_l'\rangle=\delta_{kl}$. We say that the elements

$$\{(\varphi x_i\otimes x_j',0),\ (0,\psi y_k\otimes y_l');\ i,j=1,\dots,m;\ k,\ l=1,\dots,n\}$$

in $V\otimes X'\oplus W\otimes Y'$ are *independent of* $R\big((V,W),(X,Y)'\big)^-$ *up to a zero trace* whenever any pair of linear combinations

$$\Big(\sum_{i,j=1}^{m} c_{ij}x_i\otimes x_j',\ \sum_{k,l=1}^{n} d_{kl}\psi y_k\otimes y_l'\Big),$$

which belongs to $R\big((V,W),(X,Y)'\big)^-$ must satisfy

$$\sum_{i=1}^{m} c_{ii}+\sum_{k=1}^{n} d_{kk}=0.$$

THEOREM 4.5. *The homomorphism* $(\varphi,\psi)\colon (X,Y)\to(V,W)$ ***is a split*** *monomorphism if and only if the elements* $\{(\varphi x_i\otimes x_j',0),\ (0,y_k\otimes y_l')\}$ ***are*** *independent of* $R\big((V,W),(X,Y)'\big)^-$ *up to a zero trace.*

Proof. Supposing (φ,ψ) is split, we get a continuous functional $F\colon V\otimes_\iota X'\oplus W\otimes_\iota Y'\to C$, vanishing on the closure of the space $R=R\big((V,W),(X,Y)'\big)$ of tensor relations, and satisfying condition (2) of Theorem 4.4. In particular

$$F\big((\varphi x_i\otimes x_j',0)\big)=\langle x_i,x_j'\rangle=\delta_{ij}\quad\text{for}\quad i,j=1,\dots,m,$$

and

$$F((0,\psi y_k\otimes y_l')) = \langle y_k, y_l'\rangle = \delta_{kl} \quad \text{for} \quad k,l = 1,\dots,n.$$

If some pair of combinations $(\sum c_{ij}\varphi x_i\otimes x_j',\ \sum d_{kl}\psi y_k\otimes y_l')$ belongs to $\bar{R}$, then

$$0 = F((\sum c_{ij}\varphi x_i\otimes x_j',\ \sum d_{kl}\psi y_k\otimes y_l'))$$
$$= \sum c_{ij}F((\varphi x_i\otimes x_j',\ 0)) + \sum d_{kl}F((0,\ \psi y_k\otimes y_l')) = \sum c_{ii}+\sum d_{kk}.$$

On the other hand, if the elements in question are independent of $\bar{R}$ up to a zero trace, one defines a linear functional F_1 on the finite-dimensional subspace of $(V\otimes_\iota X'\oplus W\otimes_\iota Y')/\bar{R}$ generated by the terms

$$\{(\varphi x_i\otimes x_j',\ 0)+\bar{R};\ (0,\ \psi y_k\otimes y_l')+\bar{R}\}$$

by

$$F_1((\sum c_{ij}\varphi x_i\otimes x_j',\ \sum d_{kl}\psi y_k\otimes y_l')+\bar{R}) = \sum c_{ii}+\sum d_{kk}.$$

This F_1 is well defined because of the hypothesis, and continuous because it is given on a finite-dimensional separated vector space. We take a continuous extension of F_1 to all of $(V\otimes_\iota X'\oplus W\otimes_\iota Y')/\bar{R}$ and let F be the functional defined on $V\otimes_\iota X'\oplus W\otimes_\iota Y'$ as the composite of that extension with the canonical map

$$V\otimes_\iota X'\oplus W\otimes_\iota Y'\to(V\otimes_\iota X'\oplus W\otimes_\iota Y')/\bar{R}.$$

This F will satisfy condition (2) of 4.4. Indeed, it suffices to compare the values of the members of (2) only on the basis $\{(x_i\otimes x_j',0),\ (0,y_k\otimes y_l')\}$ of $X\otimes_\iota X'\oplus Y\otimes_\iota Y'$. In that case $F((\varphi x_i\otimes x_j',0)) = \delta_{ij} = \langle x_i, x_j'\rangle$ and $F((0,\psi y_k\otimes y_l')) = \delta_{kl} = \langle y_k, y_l'\rangle$, and the proof is ended.

Remark. Without employing bases our theorem states that (φ,ψ) is a split monomorphism if and only if the preimage of $R((V,W),(X,Y)')^-$ under $\varphi\otimes 1_{X'}\oplus\psi\otimes 1_{Y'}$, is contained in $\mathrm{Ker}(\tilde{\varkappa})$.

In order to pursue the splitting question further, we require a more explicit description of the space R of tensor relations in $V\otimes_\iota X'\oplus W\otimes_\iota Y'$ and of the topology of the latter, under which the closure of R is taken (or more simply, of the topologies of $V\otimes_\iota X'$ and $W\otimes_\iota Y'$). This is the purpose of the following two propositions.

Keeping the above notations for bases of X, Y and their duals, let $(e^1,\dots,e^N)$ be a basis of C^N. For each $k\in\{1,\dots,N\}$ let (e_{ij}^k) be the $n\times m$ matrix of the linear transformation $x\mapsto e^k x$ relative to the bases $(x_1,\dots$ $\dots,x_m)$, $(y_1,\dots,y_n)$ of X and Y; so that $e^k x_i = \sum_{j=1}^{n} e_{ji}^k y_j$.

PROPOSITION 4.6. *An element z in $V\otimes X'\oplus W\otimes Y'$ belongs to R if* *and only if it is of the form*

$$\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{k=1}^{N} e^k_{ji} v_{jk}\right)\otimes x'_i,\ -\sum_{j=1}^{n}\left(\sum_{k=1}^{N} e^k v_{jk}\right)\otimes y'_j\right), \tag{3}$$

where the $v_{jk} \in V$.

Proof. If z is of the above form we may rewrite it as

$$z = \sum_{j=1}^{n}\sum_{k=1}^{N}\left(v_{jk}\otimes \sum_{i=1}^{m} e^k_{ji} x'_i,\ -e^k v_{jk}\otimes y'_j\right).$$

Since the matrix associated to the action of e^k from Y' to X' is the transpose of (e^k_{ij}), we have $\sum_{i=1}^{m} e^k_{ji} x'_i = e^k y'_j$. Thus

$$z = \sum_{j=1}^{n}\sum_{k=1}^{N}(v_{jk}\otimes e^k y'_j,\ -e^k v_{jk}\otimes y'_j). \tag{4}$$

Each summand above is in R, by definition of the space of tensor relations, and hence $z \in R$.

For the converse it suffices to show that any generator $(v\otimes ey',\ -ev\otimes y')$ of R, with $v \in V$, $e \in C^N$, $y' \in Y'$, is expressible as a term of the form (3). Letting $e = \sum_{k=1}^{N} c_k e^k$, $y' = \sum_{j=1}^{n} d_j y'_j$ we get

$$(v\otimes ey',\ -ev\otimes y') = \left(\sum_{j=1}^{n}\sum_{k=1}^{N} d_j c_k v\otimes e^k y'_j,\ -\sum_{j=1}^{n}\left(\sum_{k=1}^{N} e^k d_j c_k v\right)\otimes y'_j\right).$$

Upon noting that $e^k y'_j = \sum_{i=1}^{m} e^k_{ji} x'_i$, this equals

$$\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{k=1}^{N} e^k_{ji} d_j c_k v\right)\otimes x'_i,\ -\sum_{j=1}^{n}\left(\sum_{k=1}^{N} e^k d_j c_k v\right)\otimes y'_j\right),$$

an expression of the desired form (3).

PROPOSITION 4.7. *The mapping* $V^m \to V\otimes_t X'$ *given by* $(v_1, \ldots, v_m) \mapsto \sum_{j=1}^{m} v_j\otimes x'_j$ *is an algebraic and topological isomorphism of* $V\otimes_t X'$ *with the m-fold cartesian product* V^m. *Thus a net* $\sum v_j(a)\otimes x'_j$ *in* $V\otimes_t X'$, *where* a *runs over a directed set, tends to an element* $\sum v_j\otimes x'_j$ *if and only if* $v_j(a)\to v_j$ *for each* $j = 1, \ldots, m$ *in* V. *Similarly for* $W\otimes_t Y'$.

Proof. That we have an algebraic isomorphism is obvious since the x'_j's form a basis of X'. The topological part falls out from the fact that the topology on $V\otimes X'$, carried over from the product space V^m, makes the tensor product map $V \times X \to V \otimes X'$ separately continuous (in fact continuous) and is finer than the topology of $V \otimes_\varepsilon X'$, hence coincides with it.

5. Reduction to algebraic systems

In [2] Theorem 6.6 explicit "chain" conditions are given so that a homomorphism $(S, T) \to (U, Z)$ of algebraic C^2-systems, with (S, T) finite-dimensional, will be a split monomorphism. We now intend to reduce the question of splitting a homomorphism $(\varphi, \psi): (X, Y) \to (V, W)$ between *topological* C^N-systems, with (X, Y) finite-dimensional, to the same question concerning the simpler *algebraic* C^N-systems. This will put us in a position in Section 7 to apply the chain conditions of [2], Theorem 6.6, to (φ, ψ).

For any separated locally convex space V and directed set D let V^D be the vector space of all nets over D in V and, denoting by V_0^D the subspace of nets tending to zero, consider $Q^D(V) = V^D/V_0^D$. For any continuous linear map $A: V \to W$ between our spaces, there is a functorially induced linear map $Q^D(A): Q^D(V) \to Q^D(W)$ such that $Q^D(A)\big((v_a)_{a\in D} + V_0^D\big) = (Av_a)_{a\in D} + W_0^D$. The continuity of A guarantees that $Q^D(A)$ is well defined. For any separated locally convex space V there is a natural embedding $\Delta_V: V \to Q^D(V)$ given by $\Delta_V(v) = (v_a)_{a\in D} + V_0^D$, where $v_a = v$ for all $a \in D$. (Although Δ_V depends also on D, we leave this out of its notation, as the symbol $Q^D(V)$ for its target will be a sufficient reminder of this fact.) The naturality of Δ is expressed by $Q^D(A)\Delta_V = \Delta_W A$.

Using this construction on a topological C^N-system (V, W), we derive the *algebraic* C^N-system $Q^D(V, W) = \big(Q^D(V), Q^D(W)\big)$, where the action of $e \in C^N$ from $Q^D(V)$ to $Q^D(W)$ is given by $e\big((v_a)_{a\in D} + V_0^D\big) = (ev_a)_{a\in D} + W_0^D$. To a homomorphism $(\sigma, \tau): (V, W) \to (U, Z)$ we associate functorially the homomorphism $Q^D(\sigma, \tau) = \big(Q^D(\sigma), Q^D(\tau)\big): Q^D(V, W) \to Q^D(U, Z)$ of algebraic C^N-systems. For any C^N-system (V, W) there is a natural monomorphism of algebraic C^N-systems $\Delta_{(V,W)} = (\Delta_V, \Delta_W): (V, W) \to Q^D(V, W)$, where the source (V, W) stands for the underlying algebraic system of the given system. Thus, for any $(\sigma, \tau): (V, W) \to (U, Z)$ the equation, $Q^D(\sigma, \tau)\Delta_{(V,W)} = \Delta_{(U,Z)}(\sigma, \tau)$ holds.

LEMMA 5.1. *For any finite-dimensional C^N-system (X, Y) and every directed set D the homomorphism $\Delta_{(X,Y)}: (X, Y) \to Q^D(X, Y)$ is a split monomorphism of algebraic C^N-systems.*

Proof. We pick any linear map $\pi_C: Q^D(C) \to C$ which is a left inverse of the injection $\Delta_C: C \to Q^D(C)$. Being finite-dimensional, the space

X is reflexive, and thus for each $u \in Q^D(X)$ there is a uniquely determined element $\pi_X(u) \in X$ satisfying

$$\langle \pi_X(u), x' \rangle = \pi_C\big(Q^D(x')(u)\big)$$

for every $x' \in X'$. Here $Q^D(x')$ is the map of $Q^D(X)$ into $Q^D(C)$ attached to x': $X \to C$. The ensuing map π_X: $Q^D(X) \to X$ is linear. Likewise for Y we get a linear map π_Y: $Q^D(Y) \to Y$ satisfying

$$\langle \pi_Y(z), y' \rangle = \pi_C\big(Q^D(y')(z)\big)$$

for all $z \in Q^D(Y)$, $y' \in Y'$. The equations

$$\begin{aligned}\langle \pi_X(\Delta_X(x)), x' \rangle &= \pi_C\big(Q^D(x')(\Delta_X(x))\big) = \pi_C(\Delta_C(\langle x, x' \rangle)) \\ &= \langle x, x' \rangle\end{aligned}$$

establish that π_X is a left inverse of Δ_X. Similarly, π_Y is a left inverse of Δ_Y.

Thus the pair (π_X, π_Y) will be a left inverse of $\Delta_{(X,Y)}$, provided it satisfies the commutativity condition required of homomorphisms. We note that

$$Q^D(ey')(u) = Q^D(y')(eu)$$

for $e \in C^N$, $y' \in Y'$ and $u \in Q^D(X)$. Indeed, if T_e is the linear transformation corresponding to e in (X, Y), we have $ey' = y' \circ T_e$. Hence by functoriality $Q^D(ey') = Q^D(y') \circ Q^D(T_e)$. Testing (π_X, π_Y) for the commutativity requirement yields

$$\begin{aligned}\langle e\pi_X(u), y' \rangle &= \langle \pi_X(u), ey' \rangle = \pi_C\big(Q^D(ey')(u)\big) \\ &= \pi_C\big(Q^D(y')(eu)\big) = \langle \pi_Y(eu), y' \rangle.\end{aligned}$$

Hence $\Delta_{(X,Y)}$ is a split monomorphism.

THEOREM 5.2. *Let (X, Y), (V, W) be C^N-systems with (X, Y) finite-dimensional. Let D be a directed set which is order isomorphic to a base of neighbourhoods of zero in $V \otimes_t X' \oplus W \otimes_t Y'$* (right directed with respect to the relation $\supset$). *Then a homomorphism (φ, ψ): $(X, Y) \to (V, W)$ is a split monomorphism if and only if*

$$(\Delta_V, \Delta_W)(\varphi, \psi)\colon (X, Y) \to Q^D(V, W)$$

is a split monomorphism of algebraic C^N-systems.

Proof. Suppose (σ, τ): $(V, W) \to (X, Y)$ is a left inverse for (φ, ψ), so that $(\sigma, \tau)(\varphi, \psi) = (1_X, 1_Y)$. By functoriality, $Q^D(\varphi, \psi)$ is then split by the left inverse $Q^D(\sigma, \tau)$ for any directed set D. By 5.1, $\Delta_{(X,Y)}$ has a left inverse (π_X, π_Y). Then $(\pi_X, \pi_Y)Q^D(\sigma, \tau)$ serves as a left inverse of $Q^D(\varphi, \psi)\Delta_{(X,Y)} = \Delta_{(V,W)}(\varphi, \psi)$, and one half of the proof is done.

On the other hand, if (φ, ψ) is *not* a split monomorphism, then choosing bases $\{x_1, \dots, x_m\}$ of X, $\{y_1, \dots, y_n\}$ of Y and dual bases $\{x_1', \dots, x_m'\}$ of X', $\{y_1', \dots, y_n'\}$ of Y', we infer from 4.5 that the terms $\{(\varphi x_i \otimes x_j', 0), (0, \psi y_k \otimes y_l')\}$ in $V \otimes_\iota X' \oplus W \otimes_\iota Y'$ are not independent of $\bar{R} = R\big((V, W), (X, Y)'\big)^-$ up to a zero trace. Thus there exists a pair of linear combinations

$$z = \Big(\sum_{p,q=1}^{m} c_{pq}\varphi x_p \otimes x_q', \sum_{r,s=1}^{n} d_{rs}\psi y_r \otimes y_s'\Big) \in \bar{R}$$

such that $\sum_{p=1}^{m} c_{pp} + \sum_{r=1}^{n} d_{rr} \neq 0$. Let $(z_a)_{a\in D}$ be a net in R converging to z in its closure. Due to 4.6 each z_a looks like

$$z_a = \Big(\sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\sum_{k=1}^{N} e_{ji}^k v_{jk}(a)\Big)\otimes x_i', -\sum_{j=1}^{n}\Big(\sum_{k=1}^{N} e^k v_{jk}(a)\Big)\otimes y_j'\Big),$$

where $\{e^1, \dots, e^N\}$ forms a base of C^N, (e_{ij}^k) is the matrix of e^k as it acts from X to Y and $v_{jk}(a) \in V$. To say $z_a \to z$ as a runs through D means, according to 4.7, that

$$\sum_{j=1}^{n}\sum_{k=1}^{N} e_{ji}^k v_{jk}(a) \to \sum_{p=1}^{m} c_{pi}\varphi x_p, \qquad i = 1, \dots, m,$$
$$\sum_{k=1}^{N} -e^k v_{jk}(a) \to \sum_{r=1}^{n} d_{rj}\psi y_r, \qquad j = 1, \dots, n,$$

as a runs through D. These convergence relations may be respectively interpreted as the following equations inside $Q^D(V)$ and $Q^D(W)$:

$$\sum_{j=1}^{n}\sum_{k=1}^{N} e_{ji}^k\big((v_{jk}(a))_{a\in D} + V_0^D\big) = \sum_{p=1}^{m} c_{pi}\Delta_V(\varphi x_p), \qquad i = 1, \dots, m,$$
$$\sum_{k=1}^{N} -e^k\big((v_{jk}(a))_{a\in D} + V_0^D\big) = \sum_{r=1}^{n} d_{rj}\Delta_W(\psi y_r), \qquad j = 1, \dots, n.$$

Regrouping these equations into one inside $Q^D(V)\otimes_\iota X' \oplus Q^D(W)\otimes_\iota Y'$ gives

$$\Big(\sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\sum_{k=1}^{N} e_{ji}^k\big((v_{jk}(a))_{a\in D} + V_0^D\big)\otimes x_i', -\sum_{j=1}^{n}\Big(\sum_{k=1}^{N} e^k\big((v_{jk}(a))_{a\in D} + V_0^D\big)\otimes y_j'\Big)$$
$$= \Big(\sum_{i=1}^{m}\sum_{p=1}^{m} c_{pi}\Delta_V(\varphi x_p)\otimes x_i', \sum_{j=1}^{n}\sum_{r=1}^{n} d_{rj}\Delta_W(\psi y_r)\otimes y_j'\Big).$$

But the left-hand side is now seen, because of 4.6, to belong to $R\big(Q^D(V, W), (X, Y)'\big)$, the tensor relations space contained in $Q^D(V)\otimes_\iota X' \oplus Q^D(W)\otimes_\iota Y'$, and hence to $R(Q^D(V, W), (X, Y)')^-$. (We note in passing that, since $Q^D(V, W)$ is algebraic, $Q^D(V)$ and $Q^D(W)$ have their finest weak topologies. It turns out thereby that so does $Q^D(V) \otimes_t X' \oplus Q^D(W) \otimes_t Y'$, and hence $R(Q^D(V, W), (X, Y)')$ is closed.) Recalling that $\sum_{p=1}^{m} c_{pp} + \sum_{r=1}^{n} d_{rr} \neq 0$ the equation shows that the elements $\{(\Delta_V \varphi x_i \otimes x_j', 0), (0, \Delta_W \psi y_k \otimes y_l')\}$ are not independent of $R(Q^D(V, W), (X, Y)')$ up to a zero trace. Because of 4.5, $\Delta_{(V,W)}(\varphi, \psi)$ is not a split monomorphism, which ends the proof.

Remarks. (a) In proving sufficiency of our condition we cannot argue simply that if (μ, ν) is a left inverse of $\Delta_{(V,W)}(\varphi, \psi)$, then $(\mu, \nu)\Delta_{(V,W)}$ is a left inverse of (φ, ψ). This is because, as can be shown by simple examples, $(\mu, \nu)\Delta_{(V,W)}$ is not always continuous from the given topological system (V, W) to (X, Y). (In particular $\Delta_{(V,W)}$ is not generally continuous from (V, W) to the algebraic system $Q^D(V, W)$.) The fact that a left inverse (μ, ν) of $\Delta_{(V,W)}(\varphi, \psi)$ making $(\mu, \nu)\Delta_{(V,W)}$ continuous does indeed exist is a consequence of both parts of the above proof.

(b) When V and W are metrizable, so is $V \otimes_t X' \oplus W \otimes_t Y'$, and one prefers to take for D the natural numbers.

COROLLARY 5.3. *Given a homomorphism* (φ, ψ) *as in* 5.2, *the following statements are equivalent:*

(a) (φ, ψ) *is a split monomorphism.*

(b) $\Delta_{(V,W)}(\varphi, \psi)$: $(X, Y) \to Q^D(V, W)$ *is a split monomorphism for every directed set* D.

(c) $Q^D(\varphi, \psi)$: $Q^D(X, Y) \to Q^D(V, W)$ *is a split monomorphism of algebraic* C^N*-systems for every directed set* D.

Proof. (a)$\Leftrightarrow$(b). In the necessity proof of 5.2 no use was made of any particular property of the directed set.

(a)$\Rightarrow$(c), since Q^D is a functor on systems.

(c)$\Rightarrow$(a). With the particular D of 5.2, the assumption and the fact that by 5.1 $\Delta_{(X,Y)}$ is a split monomorphism imply that $Q^D(\varphi, \psi)\Delta_{(X,Y)} = \Delta_{(V,W)}(\varphi, \psi)$ is also a split monomorphism. Hence (a) follows from Theorem 5.2.

6. Reduction to finite-dimensional indecomposable sources

Given C^N-systems (X, Y) and (V, W), we denote the set of homomorphisms (φ, ψ): $(X, Y) \to (V, W)$ which are *not* split monomorphisms by $\mathrm{Nli}((X, Y), (V, W))$ (for "*Not left-invertible*"). A C^N-system is said to be *indecomposable* in case it is not a zero system and it has no direct summands except the trivial ones — itself and the zero subsystem. Our aim is to reduce the determination of $\mathrm{Nli}((X, Y), (V, W))$, where (X, Y) is finite-dimensional, to the case that (X, Y) is finite-dimensional and indecomposable. Of the various ways to achieve this reduction we prefer to use the last section because the special case that (V, W) is algebraic follows easily from former papers.

THEOREM 6.1 (Krull–Schmidt, Fitting).

(a) *Let (X, Y) be a finite-dimensional indecomposable C^N-system. Then every endomorphism of (X, Y) is either nilpotent or an automorphism. The algebra* $\mathrm{End}(X, Y)$ *of endomorphisms of (X, Y) is local with the unique maximal ideal* $\mathrm{Nli}((X, Y), (X, Y))$ *consisting of the nilpotent endomorphisms. We have a decomposition*

$$\mathrm{End}(X, Y) = C\cdot 1_{(X,Y)} \dot{+} \mathrm{Nli}((X, Y), (X, Y))$$

(*as C-subspaces*).

(b) *Let (X, Y) be a finite-dimensional C^N-system. Then (X, Y) is a finite direct sum $\sum_{j=1}^{n}\cdot(X_j, Y_j)$, where the subsystems (X_j, Y_j) are indecomposable. If $(X, Y) = \sum_{k=1}^{p}\cdot(U_k, Z_k)$ is any other decomposition with the (U_k, Z_k) indecomposable, then $p = n$ and there exists a permutation π of $\{1, \ldots, n\}$ such that for all $j = 1, \ldots, n$, $(U_j, Z_j) \approx (X_{\pi(j)}, Y_{\pi(j)})$ and*

$$(X, Y) = \sum_{l=1}^{j}\cdot(X_{\pi(l)}, Y_{\pi(l)}) \dot{+} \sum_{l=j+1}^{n}\cdot(U_l, Z_l).$$

Proof. Since the category of algebraic C^N-systems is equivalent to a category of modules and since a finite-dimensional C^N-system (X, Y) is of finite length $\dim X + \dim Y$, Fitting's lemma and the Krull–Schmidt theorem are applicable to finite-dimensional C^N-systems. Alternatively, one may easily imitate the classical proof (see e.g. [9], Chapter 5, and [3], Chapter 8, 2.1). This yields the theorem except for the last statement of (a). However, denoting for the moment $\mathrm{Nli}((X, Y), (X, Y))$ by M, it is evident that $C\cdot 1_{(X,Y)} \cap M = 0$. Since $\mathrm{End}(X, Y)/M$ is a finite-dimensional division algebra over the subfield $(C\cdot 1_{(X,Y)} + M)/M \approx C$ and the latter is algebraically closed, $\mathrm{End}(X, Y)/M$ coincides with this subfield, i.e., $\mathrm{End}(X, Y) = C\cdot 1_{(X,Y)} + M$.

PROPOSITION 6.2. *Let (X, Y) and (V, W) be C^N-systems with (X, Y) finite-dimensional and indecomposable. Then* $\mathrm{Nli}((X, Y), (V, W))$ *is a subspace of* $\mathrm{Hom}((X, Y), (V, W))$.

Proof. It is clear that $\mathrm{Nli}((X, Y), (V, W))$ is stable under multiplication by complex numbers. Suppose that $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathrm{Nli}((X, Y),$

$(V, W))$ while $(\varphi, \psi) = (\varphi_1, \psi_1) + (\varphi_2, \psi_2)$ is a split monomorphism. Let (σ, τ) be a left inverse of (φ, ψ). Then

$$(\sigma, \tau)(\varphi_1, \psi_1) + (\sigma, \tau)(\varphi_2, \psi_2) = (\sigma, \tau)(\varphi, \psi) = 1_{(X, Y)}.$$

Since $\operatorname{End}(X, Y)$ is a local ring, at least one of the endomorphisms $(\sigma, \tau)(\varphi_j, \psi_j)$ has an inverse (μ, ν). But then $(\mu, \nu)(\sigma, \tau)$ is a left inverse of (φ_j, ψ_j); against assumption.

LEMMA 6.3. *Let (X_1, Y_1), (X_2, Y_2) and (V, W) be algebraic C^N-systems, where (X_1, Y_1) and (X_2, Y_2) are finite-dimensional, indecomposable and not isomorphic. Let $(\varphi_j, \psi_j) \in \operatorname{Hom}((X_j, Y_j), (V, W))$, $j = 1, 2$, be split monomorphisms. Then $(\varphi_1, \psi_1)(X_1, Y_1) + (\varphi_2, \psi_2)(X_2, Y_2)$ is a direct sum which is a direct summand in (V, W).*

Proof. Let (U, Z) be any finite-dimensional subsystem of (V, W) such that

$$(\varphi_1, \psi_1)(X_1, Y_1) + (\varphi_2, \psi_2)(X_2, Y_2) \subset (U, Z).$$

Since $(\varphi_j, \psi_j)(X_j, Y_j)$ are direct summands of (V, W), they are direct summands of (U, Z) with finite-dimensional supplements. Decomposing such supplements, we obtain representations of (U, Z) as direct sums of indecomposable subsystems:

$$(U, Z) = \sum_{j=1}^{n} \cdot (G_j, H_j) = \sum_{k=1}^{p} \cdot (U_k, Z_k),$$

where $(G_1, H_1) = (\varphi_1, \psi_1)(X_1, Y_1)$ and $(U_p, Z_p) = (\varphi_2, \psi_2)(X_2, Y_2)$. By 6.1 (b), $p = n$ and there exists a permutation π such that $(U_j, Z_j) \approx (G_{\pi(j)}, H_{\pi(j)})$, $j = 1, \ldots, n$, and

$$(U, Z) = \sum_{j=1}^{n-1} \cdot (G_{\pi(j)}, H_{\pi(j)}) \dot{+} (U_n, Z_n). \tag{1}$$

Since (φ_j, ψ_j), $j = 1, 2$ are monomorphisms, the assumption implies that (G_1, H_1) is not isomorphic to (U_n, Z_n). Hence $\pi(n) \neq 1$, and there exists $j_0 \leqslant n-1$ with $\pi(j_0) = 1$. Thus (1) shows that the sum $(\varphi_1, \psi_1)(X_1, Y_1) + (\varphi_2, \psi_2)(X_2, Y_2) = (G_1, H_1) + (U_n, Z_n)$ is direct and a direct summand of (U, Z). That it is a direct summand of (V, W) follows from Theorem 5.5 of [2], which states that a pure finite-dimensional subsystem of an algebraic system is a direct summand. (In [2] this is formulated for C^2-systems. However, the proof is valid for C^N-systems.)

Let (X, Y) and (V, W) be C^N-systems with (X, Y) finite-dimensional and indecomposable. A family $((\varphi_j, \psi_j))_{j \in J}$ of homomorphisms of (X, Y) into (V, W) is said to be *linearly independent modulo* $\operatorname{Nli}((X, Y), (V, W))$ in case no finite non-trivial C-linear combination of the (φ_j, ψ_j)'s belongs to $\operatorname{Nli}((X, Y), (V, W))$; i.e., $((\varphi_j, \psi_j) + \operatorname{Nli}((X, Y), (V, W)))_{j \in J}$ is linearly independent in the complex vector space $\mathrm{Hom}((X, Y), (V, W))/\mathrm{Nli}((X, Y), (V, W))$.

LEMMA 6.4. *Let $((X_k, Y_k))_{k\in K}$ be a finite family of finite-dimensional indecomposable C^N-systems of distinct isomorphism types, and let (V, W) be an algebraic C^N-system. For each $k \in K$, let $((\varphi_{jk}, \psi_{jk}))_{j\in J(k)}$ be a finite family of homomorphisms of (X_k, Y_k) into (V, W). Then in order that each $(\varphi_{jk}, \psi_{jk})$ be a monomorphism and the sum $\sum_{k\in K}\sum_{j\in J(k)}(\varphi_{jk}, \psi_{jk})(X_k, Y_k)$ be a direct sum which is a direct summand of (V, W) it is necessary and sufficient that for every $k \in K$ the family $((\varphi_{jk}, \psi_{jk}))_{j\in J(k)}$ be linearly independent modulo* $\mathrm{Nli}((X_k, Y_k), (V, W))$.

Proof. This follows from Section 4 of [5] and in particular Theorem 4.2 there. Although [5] discusses only C^2-systems, no changes are required to make this section valid for C^N-systems (indeed for modules). Whenever Theorem 4.2 mentions a pure subsystem (see footnote (1), p. 6), under the conditions of the present lemma this subsystem is finite-dimensional. Hence, by [2], 5.5, it is a direct summand. Conversely, it is evident that any direct summand is pure. In particular, the sets $D(X_k, Y_k; V, W)$ of Theorem 4.2, consisting of homomorphisms of (X_k, Y_k) into (V, W) which are not pure monomorphisms, coincide for finite-dimensiona, (X_k, Y_k) with our $\mathrm{Nli}((X_k, Y_k), (V, W))$.

In [5] a family $((\varphi_i, \psi_j))_{j\in J}$ of homomorphisms from an algebraic system (X, Y) to an algebraic system (V, W) is said to be *linearly independent modulo* $D(X, Y; V, W)$ in case

$$(2)\qquad \sum_{j\in I}(\varphi_j, \psi_j)(\mu_j, \nu_j) \in D(X, Y; V, W), \quad J \supset I \text{ finite}, (\mu_j, \nu_j) \in \mathrm{End}(X, Y),$$

implies that none of the (μ_j, ν_j) is an automorphism. If (X, Y) is finite-dimensional, indecomposable, then by 6.1 (a) we have for $j \in I$

$$(\mu_j, \nu_j) = a_j 1_{(X,Y)} + (\sigma_j, \tau_j), \qquad a_j \in C,\ (\sigma_j, \tau_j) \in \mathrm{Nli}((X, Y), (X, Y)).$$

Since the (σ_j, τ_j) are not monomorphisms, $(\varphi_j, \psi_j)(\sigma_j, \tau_j)$ belong to $\mathrm{Nli}((X, Y), (V, W))$. Thus it follows from 6.2 that (2) is equivalent to

$$\sum_{j\in I} a_j(\varphi_j, \psi_j) \in \mathrm{Nli}((X, Y), (V, W)).$$

Since (μ_j, ν_j) is an automorphism if and only if $a_j \neq 0$, we see that for a finite-dimensional indecomposable source the linear independence concept of [5] coincides with the one introduced here.

As to the hypothesis of 4.2, our Lemma 6.3 shows that it is satisfied for homomorphisms with non-isomorphic sources; while by [5], Theorem 4.3, and [2], Theorem 5.5, it is satisfied also for homomorphisms with a common source.

Our remarks show that indeed the present lemma is a special case of [5], 4.2.

Remark. By Fitting's lemma, if (X, Y) is finite-dimensional and indecomposable, then $\mathrm{Nli}((X, Y), (X, Y))$ is the radical of $\mathrm{End}(X, Y)$; i.e., of $\mathrm{Hom}((X, Y), (X, Y))$ as a right module over $\mathrm{End}(X, Y)$. However, even for (V, W) of finite dimension $\mathrm{Nli}((X, Y), (V, W))$ need not equal the radical of $\mathrm{Hom}((X, Y), (V, W))$ as a right module over $\mathrm{End}(X, Y)$. For example, if (X, Y) is of type III^3 (see introduction or Section 7), there clearly exists a non-zero epimorphism (φ, ψ): $(X, Y)\to(V, W)$ which is not monomorphic; so that $\mathrm{Nli}((X, Y), (V, W)) \neq 0$. Yet the radical in question vanishes, because $\mathrm{End}(X, Y) \approx \boldsymbol{C}$.

THEOREM 6.5. *Lemma* 6.4 *stays valid if the condition that the* $\boldsymbol{C}^N$*-system* (V, W) *be algebraic is removed.*

Proof. Let (X, Y) be a $\boldsymbol{C}^N$-system of the form

$$(X, Y) = \sum_{k\in K}{}^{\cdot} \sum_{j\in J(k)}{}^{\cdot} (X_{jk}, Y_{jk}), \tag{3}$$

where there exist isomorphisms $(\varepsilon_{jk}, \zeta_{jk})$: $(X_{jk}, Y_{jk}) \approx (X_k, Y_k)$. Let $(\varphi, \psi) \in \mathrm{Hom}((X, Y), (V, W))$ be defined by the requirement that for all $k \in K$, $j \in J(k)$ the restriction of (φ, ψ) to (X_{jk}, Y_{jk}) is $(\varphi_{jk}, \psi_{jk})(\varepsilon_{jk}, \zeta_{jk})$.

Consider the following statements:

(a) At least one of the $(\varphi_{jk}, \psi_{jk})$ is not a monomorphism or the sum $\sum\sum(\varphi_{jk}, \psi_{jk})(X_k, Y_k)$ is not direct or it is not a direct summand of (V, W).

(b) $(\varphi, \psi) \in \mathrm{Nli}((X, Y), (V, W))$.

(c) There exists a directed set D such that

$$\Delta_{(V,W)}(\varphi, \psi) \in \mathrm{Nli}((X, Y), Q^D(V, W)).$$

(d) There exists a directed set D such that at least one of the $\Delta_{(V,W)}(\varphi_{jk}, \psi_{jk})$ is not a monomorphism or the sum $\sum\sum\Delta_{(V,W)}(\varphi_{jk}, \psi_{jk}) \times$ $\times (X_k, Y_k)$ is not direct or it is not a direct summand of $Q^D(V, W)$.

(e) There exist a directed set D and $k \in K$ for which we have a non-trivial linear dependence relation

$$\sum_{j\in J(k)} a_j \Delta_{(V,W)}(\varphi_{jk}, \psi_{jk}) \in \mathrm{Nli}((X_k, Y_k), Q^D(V, W)).$$

(f) There exists a k in K for which we have a non-trivial dependence relation

$$\sum a_j(\varphi_{jk}, \psi_{jk}) \in \mathrm{Nli}((X_k, Y_k), (V, W)).$$

It is clear that (a)$\Leftrightarrow$(b) and (c)$\Leftrightarrow$(d); the latter because the restriction of $\Delta_{(V,W)}(\varphi, \psi)$ to (X_{jk}, Y_{jk}) is $\Delta_{(V,W)}(\varphi_{jk}, \psi_{jk})(\varepsilon_{jk}, \zeta_{jk})$. Corollary 5.3 yields (b)$\Leftrightarrow$(c) and, upon factoring $\Delta_{(V,W)}$ out of the dependence relation of (e), also (e)$\Leftrightarrow$(f). By Lemma 6.4, (d)$\Leftrightarrow$(e). Hence the negations of (a) and (f) are equivalent, as we wanted to show.

Since every finite-dimensional C^N-system (X, Y) is a finite direct sum of indecomposable subsystems we can represent it in the form (3), where $\big((X_k, Y_k)\big)_{k\in K}$ is a finite family of pairwise non-isomorphic C^N-systems. Given $(\varphi, \psi) \in \mathrm{Hom}\big((X, Y), (V, W)\big)$, let $(\varphi_{jk}, \psi_{jk})$ be $(\varepsilon_{jk}, \zeta_{jk})^{-1}$ followed by the restriction of (φ, ψ) to (X_{jk}, Y_{jk}). Then (φ, ψ) and the $(\varphi_{jk}, \psi_{jk})$ are related as in the proof of Theorem 6.5, and the equivalence of (b) and (f) shows how to determine $\mathrm{Nli}\big((X, Y), (V, W)\big)$ in terms of $\mathrm{Nli}\big((X_k, Y_k), (V, W)\big)$, $k \in K$. Hence, we are interested now in finding $\mathrm{Nli}\big((X, Y), (V, W)\big)$ when (X, Y) is finite-dimensional and indecomposable. The following two sections treat special cases of this problem.

7. The broken chain condition for C^2-systems

In [2] all the isomorphism types of finite-dimensional indecomposable C^2-systems are characterized in terms of chains. Conditions expressed in terms of chains are given for a homomorphism from an indecomposable finite-dimensional C^2-system (X, Y) to an *algebraic* C^N-system (V, W) to be a split monomorphism. Equivalently, $\mathrm{Nli}\big((X, Y), (V, W)\big)$ is determined. The purpose of this section is to determine $\mathrm{Nli}\big((X, Y), (V, W)\big)$ for (V, W) a *general topological* C^2-system.

We first review some of the terminology and results of [2]. Henceforth, (a, b) will be a fixed basis of C^2. Let (V, W) be a C^2-system, and let n denote a positive integer.

(i) A pair of tuples $\big((x_1, \ldots, x_n), (y_1, \ldots, y_{n-1})\big)$ in $V^n \times W^{n-1}$ is a *chain of type I^n in (V, W)* whenever the equations

$$0 = ax_1, \quad bx_i = y_i = ax_{i+1} \quad \text{for} \quad i = 1, \ldots, n-1, \quad bx_n = 0$$

hold true.

(ii) A pair of tuples $\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big)$ in $V^n \times W^n$ is a *chain of type II^n_θ* for some $\theta \in C \cup \{\infty\}$ whenever the equations

$$y_1 = ax_1, \quad (b - \theta a)x_i = y_{i+1} = ax_{i+1} \quad \text{for} \quad i = 1, \ldots, n-1,$$
$$(b - \theta a)x_n = 0$$

in case $\theta \in C$; or

$$y_1 = bx_1, \quad ax_i = y_{i+1} = bx_{i+1} \quad \text{for} \quad i = 1, \ldots, n-1, \quad ax_n = 0$$

in case $\theta = \infty$, hold true.

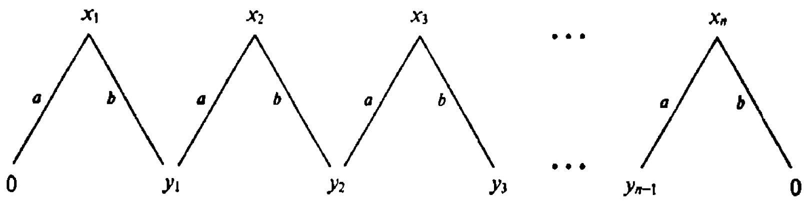
(iii) A pair of tuples $((x_1, \ldots, x_{n-1}), (y_1, \ldots, y_n))$ in $V^{n-1} \times W^n$ is called a *chain of type* III^n whenever the equations

$$y_1 = ax_1, \quad bx_i = y_{i+1} = ax_{i+1} \quad \text{for} \quad i = 1, \ldots, n-2, \quad bx_{n-1} = y_n$$

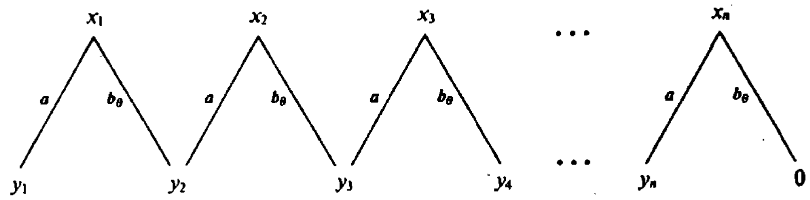
hold true. In case $n = 1$ $(x_1, \ldots, x_{n-1})$ denotes the empty sequence.

For a chain $((x_i), (y_j))$ of type I^n, II^n_θ or III^n in (V, W) let X and Y be the respective spans of the x_i's and y_j's. The subsystem (X, Y) of (V, W) is then called the *subsystem spanned by the chain* $((x_i), (y_j))$. In case the x_i's and y_j's form bases of X and Y respectively, (X, Y) is itself called a *C^2-system of type I^n, II^n_θ or III^n*, depending on the type of chain which spans it. Chains can be represented by the following kind of diagrams.

Type I^n:

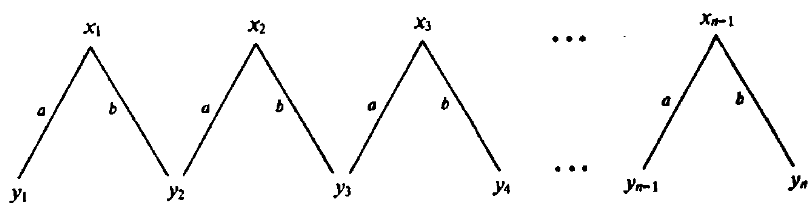


Type II^n_θ:



(Here $b_\theta = b - \theta a$, and if $\theta = \infty$, the understanding is that b replaces a and a replaces b_θ.)

Type III^n:



In [2], Proposition 2.6, Theorem 4.3, one can find a proof of the following theorem of Kronecker.

THEOREM 7.1. *A finite-dimensional C^2-system is indecomposable if and only if it is of one of the types I^n, II^n_θ or III^n.*

Let Ω denote any one of the types I^n, II_θ^n or III^n. Chains of type Ω in a C^2-system (V, W) may be added and multiplied by scalars componentwise (e.g., $\big((x_i^1), (y_j^1)\big)+\big((x_i^2), (y_j^2)\big) = \big((x_i^1+x_i^2), (y_j^1+y_j^2)\big)$) to form the vector space of chains of type Ω in (V, W), which we denote by $C\Omega(a, b;\ V, W)$.

PROPOSITION 7.2. *If (X, Y), (V, W) are C^2-systems and (X, Y) is of type Ω, the space* $\operatorname{Hom}\big((X, Y), (V, W)\big)$ *is isomorphic to $C\Omega(a, b;\ V, W)$.*

Those $(\varphi, \psi) \in \operatorname{Hom}\big((X, Y), (V, W)\big)$, which are split monomorphisms, correspond exactly to those chains in $C\Omega(a, b;\ V, W)$ spanning direct summands of type Ω.

Proof. If $\big((x_i), (y_j)\big)$ is a chain of type Ω spanning (X, Y) and $(\varphi, \psi) \in \operatorname{Hom}\big((X, Y), (V, W)\big)$, then $\big((\varphi x_i), (\psi y_j)\big)$ is a chain of type Ω in (V, W). The association $(\varphi, \psi) \mapsto \big((\varphi x_i), (\psi y_j)\big)$ is linear. Conversely, a chain $\big((v_i), (w_j)\big) \in C\Omega(a, b;\ V, W)$ determines a homomorphism (φ, ψ): $(X, Y) \to (V, W)$ given by extending the assignments $\varphi(x_i) = v_i$, $\psi(y_j) = w_j$ linearly. Here, since (X, Y) is finite-dimensional, φ and ψ are continuous, and we have our isomorphism.

Now suppose (φ, ψ) is a split monomorphism. As we noted in section 2, the subsystem $(\varphi X, \psi Y)$, spanned by the chain $\big((\varphi x_i), (\psi y_j)\big)$ in (V, W), is a direct summand of (V, W) isomorphic to (X, Y), and hence of type Ω. Conversely, suppose the chain $\big((\varphi x_i), (\psi y_j)\big)$ of type Ω spans $(\varphi X, \psi Y)$ as a system of type Ω (i.e., the φx_i's form a base of φX, the ψy_j's form a base of ψY), which is a direct summand of (V, W). Then $(\varphi X, \psi Y) \approx (X, Y)$, and a projection of (V, W) onto $(\varphi X, \psi Y)$ followed by this last isomorphism is a left inverse of (φ, ψ).

For each of the types $\Omega = I^n$, II_θ^n, III^n and a C^2-system (V, W), we now describe a subspace $\hat{C}\Omega(a, b;\ V, W)$ of $C\Omega(a, b;\ V, W)$, as in [2], Definition 6.1.

(i) Let $\hat{C}I^n(a, b;\ V, W)$ consist of the chains in $CI^n(a, b;\ V, W)$ which decompose into a sum of two chains of type I^n in (V, W), one of the form $\big((0, v_2^1, \dots, v_n^1), (0, w_2^1, \dots, w_{n-1}^1)\big)$ and the other of the form $\big((v_1^2, \dots, v_{n-1}^2, 0), (w_1^2, \dots, w_{n-2}^2, 0)\big)$.

(ii) Let $\hat{C}II_\theta^n(a, b;\ V, W)$ consist of the chains in $CII_\theta^n(a, b;\ V, W)$, which are a sum of two chains of type II_θ^n in (V, W), one of the form $\big((v_1^1, \dots, v_{n-1}^1, 0), (w_1^1, \dots, w_{n-1}^1, 0)\big)$ and the other of the form $\big((v_1^2, \dots, v_n^2), (w_1^2, \dots, w_n^2)\big)$, where $w_1^2 = (b-\theta a)v_0^2$ for some v_0^2 in V. (In case $\theta = \infty$, $w_1^2 = av_0^2$.)

(iii) Let $\hat{C}III^n(a, b;\ V, W)$ consist of the chains in $CIII^n(a, b;\ V, W)$, which are a sum of two chains of type III^n in (V, W), one of the form $\big((v_1^1, \dots, v_{n-1}^1), (w_1^1, \dots, w_n^1)\big)$, where $w_1^1 = bv_0^1$ for some $v_0^1 \in V$, and the other of the form $\big((v_1^2, \dots, v_{n-1}^2), (w_1^2, \dots, w_n^2)\big)$, where $w_n^2 = av_n^2$ for some $v_n^2 \in V$.

The following result is a special case of [2], Theorem 6.6 (see also Theorem 5.5 there).

THEOREM 7.3. *A subsystem* (X, Y) *spanned by a chain* Γ *of type* $\Omega = I^n$, II_θ^n *or* III^n *inside an algebraic* C^2*-system* (V, W) *fails to be a direct summand of type* Ω *inside* (V, W) *if and only if* $\Gamma \in \hat{C}\Omega(a, b;\ V, W)$.

We are now in a position to give the *broken chain condition.*

THEOREM 7.4. *Let* $(\varphi, \psi)\colon (X, Y) \to (V, W)$ *be a homomorphism of topological* C^2*-systems, where* (X, Y) *is of type* $\Omega = I^n$, II_θ^n *or* III^n. *Let* $\Gamma = \big((x_i), (y_j)\big)$ *be a chain of type* Ω *spanning* (X, Y). *Then* $(\varphi, \psi) \in \mathrm{Nli}\big((X, Y), (V, W)\big)$ *if and only if for some directed set* D *(alternatively, for* D *as in Theorem* 5.2) *the chain*

$$\Delta_{(V,W)}\Gamma = \big((\Delta_V \varphi x_i), (\Delta_W \psi y_j)\big)$$

belongs to $\hat{C}\Omega\big(a, b;\ Q^D(V, W)\big)$.

Proof. By 5.3, $(\varphi, \psi) \in \mathrm{Nli}\big((X, Y), (V, W)\big)$ if and only if there is a directed set D such that

$$\Delta_{(V,W)}(\varphi, \psi) \in \mathrm{Nli}\big((X, Y), Q^D(V, W)\big).$$

(By 5.2 it suffices to consider one directed set D of an order type specified there.) According to 7.2, this is the case if and only if $\Delta_{(V,W)}\Gamma$ fails to span a direct summand of type Ω in the algebraic system $Q^D(V, W)$. By 7.3, this in turn happens if and only if $\Delta_{(V,W)}\Gamma \in \hat{C}\Omega\big(a, b; Q^D(V, W)\big)$.

We now interpret Theorem 7.4 in the case $\Omega = II_\theta^n$, $\theta \in C$, in a manner which explains the choice of the terminology "broken chains". Analogously, one may expound the other cases $\Omega = I^n$, III^n or II_∞^n. The result obtained for (X, Y) of type III^3 (and V, W metrizable) was already given in the introduction. For simplicity, we formulate the next proposition only in terms of arbitrary directed sets.

PROPOSITION 7.5. *Let* $(\varphi, \psi)\colon (X, Y) \to (V, W)$ *be a homomorphism of* C^2*-systems, where* (X, Y) *is of type* II_θ^n, $\theta \in C$, *and is spanned by a chain* $\Gamma = \big((x_i)_1^n, (y_j)_1^n\big)$ *of type* II_θ^n. *Then* $(\varphi, \psi) \in \mathrm{Nli}\big((X, Y), (V, W)\big)$ *if and only if for some directed set* D *there exist* n *nets* $\{(v_{0a})_{a\in D}, \ldots, (v_{n-1,a})_{a\in D}\}$ *satisfying*

$$(*)\qquad (b-\theta a)v_{0a} + av_{1a} \to \psi y_1,\quad av_{j+1,a} - (b-\theta a)v_{ja} \to 0$$
$$\textit{for } j = 1, \ldots, n-2,\quad (b-\theta a)v_{n-1,a} \to 0$$

as a *runs over* D.

Proof. By 7.4, all we have to verify is that the existence of n nets satisfying $(*)$ is equivalent to $\Delta_{(V,W)}\Gamma \in \hat{C}II_\theta^n\big(a, b;\ Q^D(V, W)\big)$.

Suppose such nets exist. Then $\Delta_{(V,W)}\Gamma$ is the sum of the following two chains of type II_θ^n:

$$\Big(\big((v_{1a})_{a\in D} + V_0^D, \ldots, (v_{n-1,a})_{a\in D} + V_0^D, 0\big), \big((av_{1a})_{a\in D} + W_0^D, \ldots, (av_{n-1,a})_{a\in D} + W_0^D, 0\big)\Big)$$

and

$$\Big(\big((\varphi x_1 - v_{1a})_{a\in D} + V_0^D, \ldots, (\varphi x_{n-1} - v_{n-1,a})_{a\in D} + V_0^D, \Delta_V \varphi x_n\big),$$
$$\big((\psi y_1 - a v_{1a})_{a\in D} + W_0^D, \ldots, (\psi y_n - a v_{n-1,a})_{a\in D} + W_0^D, \Delta_W \psi y_n\big)\Big)$$

with

$$(\psi y_1 - a v_{1a})_{a\in D} + W_0^D = (b - \theta a)\big((v_{0a})_{a\in D} + V_0^D\big).$$

Thus $\Delta_{(V,W)}\Gamma$ belongs to the required subspace.

Conversely, suppose that $\Delta_{(V,W)}\Gamma$ is the sum of two chains of type II_θ^n in $Q^D(V, W)$, one of the form

$$\Big(\big((v_{1a}^1)_{a\in D} + V_0^D, \ldots, (v_{n-1,a}^1)_{a\in D} + V_0^D, 0\big),$$
$$\big((w_{1a}^1)_{a\in D} + W_0^D, \ldots, (w_{n-1,a}^1)_{a\in D} + W_0^D, 0\big)\Big),$$

and the other of the form

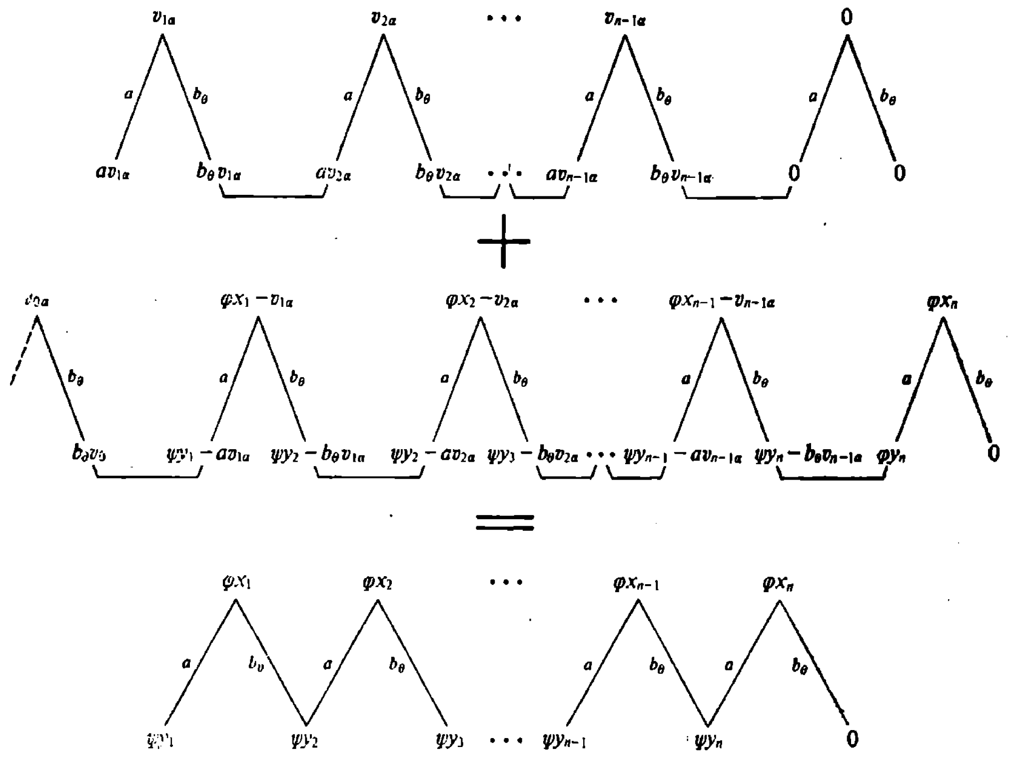
$$\Big(\big((v_{1a}^2)_{a\in D} + V_0^D, \ldots, (v_{na}^2)_{a\in D} + V_0^D\big), \big((w_{1a}^2)_{a\in D} + W_0^D, \ldots, (w_{na}^2)_{a\in D} + W_0^D\big)\Big),$$

where $(w_{1a}^2)_{a\in D} = (b - \theta a)\big((v_{0a}^2)_{a\in D} + V_0^D\big)$ for some $(v_{0a}^2)_{a\in D} + V_0^D \in Q^D(V)$. Then the n nets

$$\{(v_{0a}^2)_{a\in D}, (v_{1a}^1)_{a\in D}, \ldots, (v_{n-1,a}^1)_{a\in D}\}$$

in the given order satisfy relations (*), as desired.

The conditions of 7.5 can be expressed by the following "broken chain" diagram. In this diagram it is to be understood that $b_\theta = b - \theta a$, sum-



mation is taken term by term, and the nets of differences between terms joined by the mark $\diagdown\_\_\_\diagup$ always tend to zero.

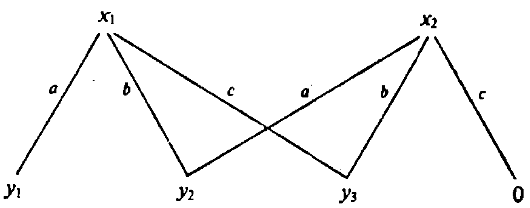
Theorem 6.5 and the remarks following it show how to formulate in terms of broken chains the condition that a homomorphism of C^2-systems (φ, ψ): $(X, Y) \to (V, W)$ belong to $\mathrm{Nli}((X, Y), (V, W))$ for the case in which (X, Y) is finite-dimensional but not necessarily indecomposable.

8. An example for computation of $\mathrm{Nli}((X, Y), (V, W))$

This section deals with a specific example of a finite-dimensional C^3-system (X, Y). The intention is to illustrate the use of Theorem 4.5 in testing for the left-invertibility of homomorphisms (φ, ψ): $(X, Y) \to (V, W)$, in case (V, W) is algebraic. The conditions obtained are comparable to the result of [2], Theorem 6.6, which has been quoted earlier as Theorem 7.3, for C^2-systems. The generalization to the case that (V, W) is topological can then be done as in the last section.

Let X be 2-dimensional with a basis (x_1, x_2), and let Y be 3-dimensional with a basis (y_1, y_2, y_3). Let a, b, c form a basis of C^3. Because of linearity, the action of C^3 from X to Y is fully described by specifying the values of ax_1, ax_2, bx_1, bx_2, cx_1, cx_2 in Y. Take these values as specified by the following chain diagram:

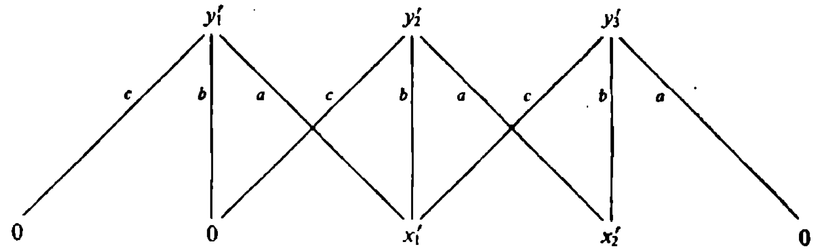
(1)



The resulting C^3-system (X, Y) is indecomposable.

The dual $(X, Y)' = (Y', X')$ of (X, Y) can be described by the chain diagram:

(2)



where (x_1', x_2'), (y_1', y_2', y_3') are the dual bases.

According to Theorem 4.5, a homomorphism (φ, ψ): $(X, Y) \to (V, W)$, where (V, W) is algebraic, is a split monomorphism if and only if the terms $(\varphi x_i \otimes x_j', 0)$, $(0, \psi y_k \otimes y_l')$ in $V \otimes X' \oplus W \otimes Y'$ are independent of the space R of tensor relations up to a zero trace. From the definition of R it follows that this subspace is spanned by elements of the form

$$(3) \qquad (v \otimes ay', \; -av \otimes y'), \; (v \otimes by', \; -bv \otimes y'), \; (v \otimes cy', \; -cv \otimes y'),$$

where $v \in V$ and $y' \in Y'$. If $z_1, z_2 \in V \otimes X' \oplus W \otimes Y'$ are congruent modulo R, i.e., $z_1 - z_2 \in R$, we write $z_1 \equiv z_2$.

LEMMA 8.1. *With the notations as above, and for* $i, j = 1, 2$, $k, l = 1, 2, 3$, *the following hold*:

$$\text{(i)} \qquad (\varphi x_i \otimes x_j', 0) \equiv (0, \psi y_i \otimes y_j'),$$

$$\text{(ii)} \qquad (0, \psi y_k \otimes y_l') \equiv 0 \quad \textit{when } k \neq l,$$

$$\text{(iii)} \qquad (0, \psi y_k \otimes y_k') \equiv (0, \psi y_1 \otimes y_1').$$

Proof. Only one case from each of (i), (ii) and (iii) is presented. The other cases all follow the same pattern. The form (3) of the elements spanning R and the equalities inherent in the chain diagrams (1), (2) yield:

$$\begin{aligned}
\text{(i)} \quad (\varphi x_2 \otimes x_1', 0) &= (\varphi x_2 \otimes a y_1', 0) \equiv (0, a\varphi x_2 \otimes y_1') = (0, \psi a x_2 \otimes y_1') \\
&= (0, \psi y_2 \otimes y_1'), \\
\text{(ii)} \quad (0, \psi y_1 \otimes y_2') &= (0, \psi a x_1 \otimes y_2') = (0, a\varphi x_1 \otimes y_2') \equiv (\varphi x_1 \otimes a y_2', 0) \\
&= (\varphi x_1 \otimes b y_3', 0) \equiv (0, \psi b x_1 \otimes y_3') = (0, \psi a x_2 \otimes y_3') \\
&\equiv (\varphi x_2 \otimes a y_3', 0) = (\varphi x_2 \otimes 0, 0) = 0, \\
\text{(iii)} \quad (0, \psi y_3 \otimes y_3') &= (0, \psi c x_1 \otimes y_3') = (0, c\varphi x_1 \otimes y_3') \equiv (\varphi x_1 \otimes c y_3', 0) \\
&= (\varphi x_1 \otimes x_1', 0) \equiv (0, \psi y_1 \otimes y_1').
\end{aligned}$$

PROPOSITION 8.2. *With notations as above, the homomorphism* (φ, ψ): $(X, Y) \to (V, W)$ *fails to be a split monomorphism if and only if* $(0, \psi y_1 \otimes y_1') \in R$.

Proof. If $(0, \psi y_1 \otimes y_1') \in R$, then the zero trace condition of Theorem 4.5 is clearly violated. Therefore (φ, ψ) cannot be a split monomorphism.

If $(\varphi, \psi) \in \mathrm{Nli}((X, Y), (V, W))$, then by Theorem 4.5 there exist scalars c_{ij}, d_{kl}, with $i, j = 1, 2$, $k, l = 1, 2, 3$, such that

$$(4) \qquad \sum c_{ii} + \sum d_{kk} \neq 0$$

while

$$(5) \qquad \sum c_{ij} (\varphi x_i \otimes x_j', 0) + \sum d_{kl} (0, \psi y_k \otimes y_l') \in R.$$

Due to Lemma 8.1, the sum in (5) is congruent modulo R to

$$\sum c_{ii} (0, \psi y_i \otimes y_i') + \sum d_{kk} (0, \psi y_k \otimes y_k').$$

This in turn is congruent modulo R to

$$\Big(\sum c_{ii} + \sum d_{kk}\Big) (0, \psi y_1 \otimes y_1'),$$

which then must be in R. From (4), it follows that $(0, \psi y_1 \otimes y_1') \in R$, as desired.

From Proposition 6.2 it is known that, since (X, Y) is indecomposable, $\mathrm{Nli}((X, Y), (V, W))$ is a subspace of $\mathrm{Hom}((X, Y), (V, W))$. The result just proved makes this fact transparent.

One can carry Proposition 8.2 one step further by using the fact that according to formula (4) of Section 4, R is generated by elements of the form

$$(v \otimes ay_k', -av \otimes y_k'), (v \otimes by_k', -bv \otimes y_k'), (v \otimes cy_k', -cv \otimes y_k'),$$

where $v \in V$ and $k = 1, 2, 3$. Using diagram (2) to evaluate ay_k', etc., we see that R is generated by the collection of the following 9 types of vectors of $V \otimes X' \oplus W \otimes Y'$:

$$\begin{gathered}(v \otimes x_1', -av \otimes y_1'),\ (v \otimes x_2', -av \otimes y_2'),\ (0, -av \otimes y_3'),\\ (0, -bv \otimes y_1'),\ (v \otimes x_1', -bv \otimes y_2'),\ (v \otimes x_2', -bv \otimes y_3'),\\ (0, -cv \otimes y_1'),\ (0, -cv \otimes y_2'),\ (v \otimes x_1', -cv \otimes y_3').\end{gathered}$$

These generators can be replaced by

$$\begin{gathered}(v \otimes x_1', -av \otimes y_1'),\ (v \otimes x_2', -av \otimes y_2'),\ (0, av \otimes y_3'),\\ (0, bv \otimes y_1'),\ (0, bv \otimes y_2' - av \otimes y_1'),\ (0, bv \otimes y_3' - av \otimes y_2'),\\ (0, cv \otimes y_1'),\ (0, cv \otimes y_2'),\ (0, cv \otimes y_3' - av \otimes y_1').\end{gathered}$$

For each of the last 9 types of generators of R, v can vary freely through V. Thus the set of generators of a fixed type is a subspace of R; and R is the sum of these 9 subspaces.

PROPOSITION 8.3. *The homomorphism $(\varphi, \psi)\colon (X, Y) \to (V, W)$ is not a split monomorphism if and only if there exist seven vectors $v_1, v_2, \ldots, v_7$ in V satisfying*

$$\psi y_1 = bv_2 - a(v_3 + v_7) + cv_5, \qquad 0 = bv_3 - av_4 + cv_6, \qquad 0 = av_1 + bv_4 + cv_7.$$

Proof. By the last remark we have $(0, \psi y_1 \otimes y_1') \in R$ if and only if there exist $v_1, v_2, \ldots, v_9$ in V such that

$$\begin{aligned}\psi y_1 \otimes y_1' = {} & av_1 \otimes y_3' + bv_2 \otimes y_1' + (bv_3 \otimes y_2' - av_3 \otimes y_1') + (bv_4 \otimes y_3' - av_4 \otimes y_2') + \\ & + cv_5 \otimes y_1' + cv_6 \otimes y_2' + (cv_7 \otimes y_3' - av_7 \otimes y_1') - av_8 \otimes y_1' - av_9 \otimes y_2',\end{aligned}$$

and

$$0 = v_8 \otimes x_1' + v_9 \otimes x_2'.$$

The last equation implies $v_8 = v_9 = 0$. Equating coefficients of y_1', y_2' and y_3' on both sides of the first equation then provides the answer.

The above technique can be carried out for several types of finite-dimensional indecomposable C^N-systems described by chain diagrams. For example, we state the result of applying the above process to a C^2-system (X, Y) of type II_0^n (see Section 7). Let (a, b) be a basis of C^2, and let $\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big)$ be a chain of type II_0^n spanning (X, Y). The following can be deduced from the zero trace condition of Theorem 4.5.

PROPOSITION 8.4. *For (X, Y) as above, a homomorphism (φ, ψ): $(X, Y) \to (V, W)$ of algebraic C^2-systems is not a split monomorphism if and only if there exist $v_0, v_1, \dots, v_{n-1} \in V$ satisfying*

$$\begin{aligned} \psi y_1 &= bv_0 + av_1, \\ 0 &= bv_1 - av_2, \\ 0 &= bv_2 - av_3, \\ &\dots\dots\dots\dots \\ 0 &= bv_{n-2} - av_{n-1}, \\ 0 &= bv_{n-1}. \end{aligned}$$

This result is exactly the statement quoted in Theorem 7.3 that the chain $\Gamma = \big((\varphi x_1, \dots, \varphi x_n), (\psi y_1, \dots, \psi y_n)\big)$ inside (V, W) fails to span a direct summand of (V, W) if and only if $\Gamma \in \hat{C}II_0^n(a, b; V, W)$. Indeed, the conditions of Proposition 8.4 state that the chain Γ is the sum of the type II_0^n chains

$$\big((v_1, v_2, \dots, v_{n-1}, 0), (av_1, av_2, \dots, av_{n-1}, 0)\big)$$

and

$$\big((\varphi x_1 - v_1, \varphi x_2 - v_2, \dots, \varphi x_{n-1} - v_{n-1}, \varphi x_n), \\ (\psi y_1 - av_1, \psi y_2 - av_2, \dots, \psi y_{n-1} - av_{n-1}, \psi y_n)\big),$$

where $\psi y_1 - av_1 = bv_0$ for some v_0 in V.

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References

[1] N. Aronszajn, *Quadratic forms on vector spaces*, Proc. Intern. Symposium on Linear Spaces 1960, p. 29–87, Jerusalem 1961.

[2] –, and U. Fixman, *Algebraic spectral problems*, Studia Math. 30 (1968), p. 273–338.

[3] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton 1956.

[4] J. Ernest, *A classification, decomposition and spectral multiplicity theory for bounded operators on a separable Hilbert space*, Preprint, Univ. of California Santa Barbara.

[5] U. Fixman, *On algebraic equivalence between pairs of linear transformations*, Trans. Amer. Math. Soc. 113 (1964), p. 424–453.

[6] –, and N. Sankaran, *The fundamental functors for pairs of linear transformations* (in preparation).

[7] –, and F. A. Zorzitto, *A purity criterion of pairs of linear transformations*, Canad. J. Math. 26 (1974), p. 734–745.

[8] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[9] N. Jacobson, *Lectures in abstract algebra*, Vol. 1, D. Van Nostrand, New York 1951.

[10] F. A. Zorzitto, *Topological decompositions in systems of linear transformations*, Doctoral Thesis (1972), Queen's University, Kingston.

[11] –, *Purity and copurity in systems of linear transformations*, Canad. J. Math. 28 (1976), p. 889–896.

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