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INVERSE SEMIGROUP RINGS

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The rings of the title, semigroup rings formed from inverse semigroups, are a natural generalization of group rings. Inverse semigroups were first considered by V. V. Vagner (1952) and G. B. Preston (1954), in Vagner's case the motivation coming from differential geometry. Such semigroups have been studied intensively – probably more so than any other type apart from groups.

The account that follows is a slightly expanded version of two lectures given at the Stefan Banach International Mathematical Center, Warsaw, in March 1988 during the Semester on Classical Algebraic Structures. For convenience, various key definitions and some basic results on inverse semigroups have been gathered together in an introductory section. The second section is concerned with nil ideals in inverse semigroup rings over a field, while the third deals with the link between an inverse semigroup ring and the group rings of the maximal subgroups of the inverse semigroup in the case where the semilattice (semigroup of idempotents) of the latter satisfies a certain finiteness condition. Virtually all of the results mentioned have been published and a list of references is supplied.

1. Introduction

An *inverse semigroup* is a semigroup S in which, for all $a \in S$, the equations

$$axa = a, \quad xax = x$$

have a unique common solution in S. This solution is usually denoted by a^{-1} and called the "inverse" of a. It should be noted that S need not have an identity element; furthermore, for each $a \in S$, the elements aa^{-1} and $a^{-1}a$ are idempotents, but these may be distinct.

Examples of inverse semigroups include groups, semilattices (by which we mean commutative semigroups consisting of idempotents), the semigroup of $n \times n$ matrix units together with zero, and the semigroup $\mathscr{I}_X$ consisting of all one-to-one partial transformations of a set X under composition of relations.

A useful alternative characterization of inverse semigroups is provided by the following result.

PROPOSITION 1.1. *A semigroup S is inverse if and only if* (i) $\forall a \in S$, $a \in aSa$ *and* (ii) *all idempotents commute.*

Partial one-to-one transformations play the same rôle for inverse semigroups as permutations do for groups. In particular, we have an exact analogue of Cayley's theorem:

PROPOSITION 1.2. *Every inverse semigroup S can be embedded in $\mathscr{I}_S$.*

Now let S be an inverse semigroup. It can be shown that the inverses of elements of S satisfy the conditions below: for all $a, b \in S$,

$$\text{(i) } (a^{-1})^{-1} = a, \quad \text{(ii) } a^2 = a \Rightarrow a^{-1} = a, \quad \text{(iii) } (ab)^{-1} = b^{-1}a^{-1}.$$

Of these, (i) and (ii) are almost immediate from the definition, while (iii) is easily proved by using Proposition 1.1.

We shall also be interested in certain special subsemigroups of S.

1. $E_S := \{e \in S: e^2 = e\}$.

Proposition 1.1 guarantees that E_S is a subsemigroup. It is partially ordered by the rule that

$$e \leqslant f \Leftrightarrow e = ef \; (= fe);$$

moreover, for all e and f in E_S, ef is the greatest lower bound of $\{e, f\}$ under this ordering. We call E_S *the semilattice of S.*

2. $(\forall e \in E_S)$ $P_e := \{x \in S: xx^{-1} = e \text{ and } xe = x\}$.

For each $e \in E_S$, P_e is a right cancellative subsemigroup of S with identity e.

3. $(\forall e \in E_S)$ $H_e := \{x \in S: xx^{-1} = e = x^{-1}x\}$.

For each $e \in E_S$, H_e is a subgroup of S with identity e and it contains every such subgroup. The groups H_e $(e \in E_S)$ are called *the maximal subgroups of S.*

Remark. For all $e \in E_S$ we have $H_e \subseteq P_e$; but, in general, $H_e \neq P_e$. It can also be shown that $S = \bigcup_{e \in E_S} H_e$ if and only if E_S is central in S.

Finally, we consider the relation $\mathscr{D}$ on S defined by the rule

$$x \mathscr{D} y \Leftrightarrow (\exists z \in S) \; xx^{-1} = zz^{-1} \quad \text{and} \quad z^{-1}z = y^{-1}y.$$

PROPOSITION 1.3. *Let S be an inverse semigroup.*

(i) *$\mathscr{D}$ is an equivalence on S.*

(ii) *Each $\mathscr{D}$-class of S contains a maximal subgroup of S and all maximal subgroups in a given $\mathscr{D}$-class are isomorphic.*

We say that S is *bisimple* if and only if $\mathscr{D}$ is the universal relation. Groups are bisimple inverse semigroups. So also is the bicyclic semigroup, which can be defined as the monoid generated by two elements p and q subject to the single defining relation $pq = 1$.

For a fuller discussion of the ideas introduced above, see [4, Chapter V].

To conclude this section, we establish some notation which will be used without further comment in the remainder of the paper.

Notation: R – a ring with a unity (not necessarily commutative),
S – an inverse semigroup,
$R[S]$ – the semigroup ring of S over R,
E – the semilattice of S ($\equiv E_S$),
$\mathscr{M}$ – the set of all maximal subgroups of S,
$J(A)$ – the Jacobson radical of a ring A.

2. Nil ideals in inverse semigroup rings over a field

In this section we restrict ourselves to the case where $R = F$, a field. The characteristic of F will be denoted by $\operatorname{char} F$.

The earliest result on group rings is generally believed to be Maschke's theorem (1899), which, in its modern form, gives necessary and sufficient conditions for the group ring of a *finite* group to be semisimple. The first result on inverse semigroup rings is the analogue of this. It was established independently by Ponizovskii [13], Oganesyan [11] and the author [5] and dates from the mid 1950s.

THEOREM 2.1. *Let S be finite. Then $F[S]$ is semisimple if and only if $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 0$, where p does not divide the order of any subgroup of S.*

This is the starting point for a theory of matrix representations of finite inverse semigroups. The basic problem is to construct the irreducible representations of S from those of the groups in $\mathscr{M}$.

With the development of the theory of group rings of infinite groups from 1950 onwards, it seemed natural to look also at semigroup rings of infinite inverse semigroups. It turns out that many of the group ring results extend; but sometimes the proofs are substantially harder. In this section, I shall focus attention on nil ideals in $F[S]$.

The following concept will be useful in this context. By a *p-element* of S, where p is a prime, we shall mean an element of order a power of p in some subgroup of S.

The main result [8, Theorem A] is

THEOREM 2.2. *Let $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 0$, where S has no p-elements. Then $F[S]$ has no nonzero nil ideals.*

I shall give a brief sketch of the method of proof later. There is, however, a short argument (based on a technique for group rings) that covers an important special case [7] and it is perhaps instructive to consider this first. We require two lemmas.

LEMMA 2.3. *Let T be a finite subset of S and let e be maximal in $\{xx^{-1}\colon x \in T\}$ under the usual partial ordering. Then*

$$(\forall x, y \in T) \quad xy^{-1} = e \Rightarrow x = y.$$

Proof. Suppose that $xy^{-1} = e$ $(x, y \in T)$. Then $xx^{-1}e = xx^{-1}xy^{-1} = xy^{-1} = e$; that is, $e \leqslant xx^{-1}$. Hence, by the maximality of e, $e = xx^{-1}$ and so $\underline{ex = x}$. Also $yx^{-1} = (xy^{-1})^{-1} = e^{-1} = e$. Thus, as above, $e = yy^{-1}$ and so $\underline{y^{-1}e = y^{-1}}$. Hence, from the underlined statements, $xy^{-1}x = x$ and $y^{-1}xy^{-1} = y^{-1}$, from which it follows that $x = (y^{-1})^{-1} = y$.

LEMMA 2.4. *Let F be a subfield of $\mathbf{C}$ that is closed under complex conjugation $z \mapsto \bar{z}$ (e.g. $F = \mathbf{Q}, \mathbf{R}, \mathbf{C}$). The mapping $*\colon F[S] \to F[S]$ defined by*

$$\Big(\sum_{x\in S} a_x x\Big)^* = \sum_{x\in S} \bar{a}_x x^{-1} \qquad (a_x \in F)$$

is an involution and, for all $a \in F[S]$, $aa^* = 0$ implies $a = 0$.

Proof. It is clear that $*$ is an involution. Let $a = \sum_{i=1}^{n} a_i x_i \in F[S]\backslash 0$, where $x_1, \ldots, x_n$ are distinct elements of S and $a_1, \ldots, a_n \in F\backslash 0$. Let e be maximal in $\{x_i x_i^{-1}\colon i = 1, \ldots, n\}$ and suppose, without loss of generality, that, for some $r \geqslant 1$, $e = x_1 x_1^{-1} = \ldots = x_r x_r^{-1}$, while $e \neq x_i x_i^{-1}$ if $i > r$. Now

$$aa^* = \sum_{i,j} a_i \bar{a}_j (x_i x_j^{-1})$$

and so, by Lemma 2.3, the coefficient of e in aa^* is $\sum_{i=1}^{r} |a_i|^2$, which is strictly positive.

Lemma 2.4 was obtained independently by Shehadah [16].

We now establish the following special case of Theorem 2.2.

THEOREM 2.2′. *Let F be a subfield of $\mathbf{C}$ closed under complex conjugation. Then $F[S]$ has no nonzero nil ideals.*

Proof. Suppose that A is a nonzero nil ideal of $F[S]$ and let $a \in A\backslash 0$. By Lemma 2.4, $aa^* \neq 0$. Hence there exists $m > 1$ such that $(aa^*)^m = 0$, while $(aa^*)^{m-1} \neq 0$. Write $b = (aa^*)^{m-1}$. Then $b = b^*$ and so $bb^* = (aa^*)^{2m-2} = 0$, since $2m - 2 \geqslant m$. Thus, by Lemma 2.4, $b = 0$, which is false.

The argument given above does not extend to the case of an arbitrary field F of characteristic zero; nor does it adapt to the case in which F has prime characteristic. However, we can deduce from Theorem 2.2′ that if $\operatorname{char} F = 0$ then $F[S]$ is semiprime.

A completely different method was used to establish Theorem 2.2 in full generality [8]. The main ingredients are the next three lemmas, which we state without proof.

Recall that for $a = \sum_{x \in S} a_x x \in F[S]$ $(a_x \in F)$ the *support* of a, $\operatorname{supp} a$, is defined by $\operatorname{supp} a = \{x \in S\colon\ a_x \neq 0\}$. Thus $|\operatorname{supp} a| < \infty$; and $\operatorname{supp} a = \emptyset$ if and only if $a = 0$.

LEMMA 2.5. *Let A be a nonzero ideal of $F[S]$. Then there exist $e \in E$ and $a \in A$ such that* (i) $e \in \operatorname{supp} a \subseteq eSe$ *and* (ii) $(\operatorname{supp} a) \cap (P_e \backslash H_e) = \emptyset$.

LEMMA 2.6. *Let $e \in E$, let* $\operatorname{char} F = p > 0$ *and let $a \in F[S]$ be such that a is nilpotent and $e \in \operatorname{supp} a \subseteq eSe$. Then either* (i) $(\operatorname{supp} a) \cap H_e$ *contains a p-element or* (ii) $(\operatorname{supp} a) \cap (P_e \backslash H_e) \neq \emptyset$.

From Lemmas 2.5 and 2.6 we see easily that Theorem 2.2 holds for the case where F has prime characteristic. To deal with the zero characteristic case we also need a result depending on the theory of places (see [12, Chapter 2, § 2]).

LEMMA 2.7. *Let* $\operatorname{char} F = 0$ *and let $a \in F[S] \backslash 0$ be nilpotent. Then there is an infinite set $\mathcal{S}_a$ of primes such that, for all p in $\mathcal{S}_a$, there exists a field F_p of characteristic p and a nonzero nilpotent element b in $F_p[S]$ with* $\operatorname{supp} b = \operatorname{supp} a$.

Let $\operatorname{char} F = 0$ and suppose that A is a nonzero nil ideal of $F[S]$. By Lemma 2.5, there exist $e \in E$ and $a \in A$ such that $e \in \operatorname{supp} a \subseteq eSe$ and $(\operatorname{supp} a) \cap (P_e \backslash H_e) = \emptyset$. Let $p \in \mathcal{S}_a$. By Lemma 2.7, there is a field F_p of characteristic p and a nilpotent $b \in F_p[S]$ with $\operatorname{supp} b = \operatorname{supp} a$. Hence $e \in \operatorname{supp} b \subseteq eSe$ and $(\operatorname{supp} b) \cap (P_e \backslash H_e) = \emptyset$. Thus, by Lemma 2.6 (applied to $F_p[S]$), $(\operatorname{supp} b) \cap H_e$ contains a p-element. But this is impossible since $\mathcal{S}_a$ is infinite and $(\operatorname{supp} a) \cap H_e$ is finite. Hence Theorem 2.2 holds for fields of characteristic zero.

The restriction on $\operatorname{char} F$ in Theorem 2.2 is not necessary for the nonexistence of nonzero nil ideals, as is demonstrated by examples due to Teply, Turman and Quesada [17] and Ponizovskii [14].

Finally, we can use Theorem 2.2 to obtain a result on semiprimitivity (that is, semisimplicity in the sense of Jacobson), following the standard route used for group rings.

Let S^1 denote the semigroup obtained from S by adjoining an identity if S has none; otherwise take $S^1 = S$. With this notation we have the following key result essentially due to Amitsur and Passman (see [12, Theorems 7.3.4 and 7.3.6]).

THEOREM 2.8. *Suppose that F is not algebraic over its prime subfield P and that, for all field extensions K of P, $K[S^1]$ has no nonzero nil ideals. Then* $J(F[S]) = 0$.

Remark. This holds for an arbitrary semigroup S and is derived from two general theorems on algebras.

Theorems 2.2 and 2.8 combine to give an important result on inverse semigroup rings due to Domanov [2]:

THEOREM 2.9. *Let F be nonalgebraic over its prime subfield and if* $\operatorname{char} F = p > 0$ *let S have no p-elements. Then* $J(F[S]) = 0$.

COROLLARY 2.10. $J(\mathbf{R}[S]) = 0$ *and* $J(\mathbf{C}[S]) = 0$.

From Theorem 2.9 we can deduce the classical results on the semiprimitivity of group rings.

3. A class of inverse semigroup rings

Throughout this section we allow R to be a nontrivial ring with unity (not necessarily commutative).

Theorem 2.9, at the end of the previous section, was not originally obtained in the manner indicated there. Concerned only with the characteristic zero case (although the prime characteristic case is similar), Domanov deduced Theorem 2.9 by combining Amitsur's theorem [1] on the semiprimitivity of group rings with the following striking result [2, Theorem 1]:

THEOREM 3.1. *If, for all G in $\mathcal{M}$, $J(R[G]) = 0$ then $J(R[S]) = 0$.*

In his proof of Theorem 3.1, Domanov restricts himself to the case where R is a field; but his argument—which consists of constructing a so-called "faithful family" of irreducible $R[S]$-modules from similar families for the group rings $R[G]$ ($G \in \mathcal{M}$)—works equally well for R a general ring with unity.

There is a related result concerning primitivity. This can be derived from the proof of Theorem 3.1, as was pointed out by I. S. Ponizovskii.

THEOREM 3.2. *If S is bisimple and there exists G in $\mathcal{M}$ such that $R[G]$ is primitive then $R[S]$ is primitive.*

It turns out that the converses of Theorems 3.1 and 3.2 are both false. Counterexamples to the converse of Theorem 3.1 appear in [17] and [14], while a counterexample to the converse of Theorem 3.2 was given by the author in [9].

Can we find sufficient conditions on S for the converses of these theorems to hold? The answer is "yes": a fairly natural "finiteness" restriction on E (the semilattice of S) is enough. This condition on E was first considered by Teply, Turman and Quesada [17].

For all $e \in E$, write

$$\hat{e} = \{f \in E\colon e > f \text{ and there is no } g \text{ in } E \text{ with } e > g > f\}.$$

We say that E is *pseudofinite* if and only if

(i) for all $e, g \in E$ with $e > g$ there exists $f \in \hat{e}$ such that $f \geqslant g$;
(ii) for all $e \in E$, $|\hat{e}| < \infty$.

Evidently if E is pseudofinite then $\hat{e} = \emptyset$ if and only if e is the zero of E. Further, we note that E is pseudofinite in each of the following cases:

(a) if $|Ee| < \infty$ for all $e \in E$,

(b) if E is inversely well-ordered,

(c) if S is a free inverse semigroup of finite rank.

Let E be pseudofinite. We now show how to construct a set of pairwise-orthogonal idempotents in $F[S]$ in one-to-one correspondence with the elements of E. For all $e \in E$ define $\sigma(e) \in R[E]$ by the rule

$$\sigma(e) = \begin{cases} \prod_{f \in \hat{e}} (e-f) & \text{if } \hat{e} \neq \emptyset, \\ e & \text{if } \hat{e} = \emptyset. \end{cases}$$

This concept is due to Rukolaine [15], for E finite. Observe that the definition makes sense since (a) the factors $e-f$ commute and (b) $|\hat{e}| < \infty$. If $|\hat{e}| = n > 0$ then $\sigma(e) = e - s_1 + s_2 - s_3 + \ldots + (-1)^n s_n$, where s_r denotes the sum of all products of r distinct elements of $\hat{e}$ $(r = 1, \ldots, n)$.

LEMMA 3.3. *Let E be pseudofinite. Then $\{\sigma(e)\colon e \in E\}$ is a set of pairwise-orthogonal idempotents in $R[S]$.*

These "Rukolaine idempotents" provide a useful tool for analysing the structure of $R[S]$.

We now introduce some further notation.

1. Let A be a ring and let K be a nonempty set. Then A_K denotes the ring of all bounded $K \times K$ matrices over A (that is, $K \times K$ matrices over A with at most finitely many nonzero entries) under the usual matrix operations.

2. Let E be pseudofinite and let D be a $\mathscr{D}$-class of S (see Section 1). Write

$$I(D) := \sum_{e \in E \cap D} R[S]\sigma(e).$$

Proofs of the next four theorems (excluding that of the second part of 3.4(iii)) may be found in [10].

THEOREM 3.4. *Let E be pseudofinite.*

(i) *For all $\mathscr{D}$-classes D of S,*

$$I(D) = \sum_{e \in E \cap D} \sigma(e) R[S]$$

and is a two-sided ideal of $R[S]$.

(ii) *For all $\mathscr{D}$-classes D of S, $I(D) \cong (R[G])_K$, where $K = E \cap D$, $G \in \mathscr{M}$ and $G \subseteq D$. (Recall Proposition 1.3(ii).)*

(iii) *The sum $\sum_D I(D)$ is direct and is an essential ideal of $R[S]$.*

Two natural questions arise when E is pseudofinite. When do the Rukolaine idempotents generate $R[E]$? When does the sum of the ideals $I(D)$ coincide with $R[S]$? These are answered below.

THEOREM 3.5. *Let E be pseudofinite. The following statements are equivalent.*

(i) $|Ee| < \infty$ *for all* $e \in E$.
(ii) $\{\sigma(e): e \in E\}$ *is a basis of* $R[E]$.
(iii) $\sum_D I(D) = R[S]$.

Let us now return to the matter of partial converses for Theorems 3.1 and 3.2. Using Rukolaine idempotents we can prove

THEOREM 3.6. *Let E be pseudofinite.*

(i) $J(R[S]) = 0$ *if and only if, for all G in* $\mathcal{M}$, $J(R[G]) = 0$.
(ii) $R[S]$ *is primitive if and only if S is bisimple and, for some (every) G in* $\mathcal{M}$, $R[G]$ *is primitive.*

For the special case in which E is central in S, (i) was proved by Teply *et al.* in [17].

A similar result holds with "semiprime" and "prime" replacing "semiprimitive" and "primitive" respectively.

THEOREM 3.7. *Let E be pseudofinite.*

(i) $R[S]$ *is semiprime if and only if, for all G in* $\mathcal{M}$, $R[G]$ *is semiprime.*
(ii) $R[S]$ *is prime if and only if S is bisimple and, for some (every) G in* $\mathcal{M}$, $R[G]$ *is prime.*

Necessary and sufficient conditions for a group ring to be (a) semiprime and (b) prime were obtained by I. G. Connell.

We conclude with a brief mention of free inverse semigroups and their semigroup rings.

Inverse semigroups, in common with groups, form a variety of algebras with one binary and one unary operation. It follows that, for a given nonempty set X, the free inverse semigroup FI_X on X exists. It is a somewhat more difficult object to describe than its group-theoretic counterpart. Some of its properties are listed in the lemma below. (For the details, see, for example, [6].)

LEMMA 3.8. *Let X and Y be nonempty sets.*

(i) $FI_X \cong FI_Y$ *if and only if* $|X| = |Y|$.
(ii) *Every subgroup of* FI_X *is trivial.*
(iii) FI_X *has infinitely many* $\mathcal{D}$*-classes.*
(iv) *The semilattice of* FI_X *is pseudofinite if and only if X is finite.*

We call $|X|$ the *rank* of FI_X.

Our final result [10] is easily deduced from Theorems 3.1, 3.6 and 3.7 and Lemma 3.8. Part (i) first appeared in [7].

THEOREM 3.9. *Let S be free.*

(i) *If* $J(R) = 0$ *then* $J(R[S]) = 0$.

(ii) *If S has finite rank and* $J(R[S]) = 0$ *then* $J(R) = 0$.
(iii) *If S has finite rank then* $R[S]$ *is not prime (and so not primitive).*

The third part is somewhat surprising in view of a theorem of Formanek [3] which states that if T is a free group or a free semigroup of rank at least two and F is a field then $F[T]$ is primitive.

The question of whether $R[S]$ can be prime when S is a free inverse semigroup of infinite rank remains open.

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