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COMPLETE CUBICS IN ENUMERATIVE GEOMETRY

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The construction of the variety of complete plane cubics is described by a sequence of five blow-ups. A coordinate description is given of each of the varieties obtained by the blow-ups. Homology bases and homology relations in codimension 1 for these varieties are given.

1. Introduction

In the enumerative geometry of conics problems concerning contact conditions were the occasion of a completion of point conics to the variety M_5 of complete conics or point-line conics [9, 11]. A point conic in the plane P_2 is given by $\sum p^{ik} x_i x_k = 0$ with symmetric coefficients p^{ik}, $i, k = 1, 2, 3$, and (p^{ik}) is regarded as a point in P_5. The minors L_{ik} of the symmetric matrix $((p^{ik}))$ determine the equation $\sum L_{ik} w^i w^k = 0$ (in coordinates (w^i) of a line in P_2) of the set of tangents of a reduced conic (p^{ik}), so we have in P_5 the condition $(P_5|B_4)$ of simple contact for conics.

The regular projective transformations

$$x_i = \sum t_i^k \bar{x}_k, \qquad w^i = \sum \tau_k^i \bar{w}^k \tag{1}$$

of P_2 induce the group $T(P_5)$ of transformations by

$$p^{ik} = \sum \tau_m^i \tau_n^k \bar{p}^{mn}. \tag{2}$$

For $\operatorname{codim} B_4 = 1$ a variety A in P_5 is a degeneration with respect to B_4 or with respect to simple contact if $A \subseteq t(B_4)$ for all $t \in T(P_5)$ [5], that is, if all minors L_{ik} vanish on A. In P_5 the Veronese surface V_2 is a degeneration with respect to simple contact, and φ: $M_5 \to P_5$ is a blow-up of P_5 with center V_2. A generic

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point of M_5 is $(p, l) = (p^{ik}, l_{mn})$ with $l_{mn} = \varrho \cdot L_{mn}$ [11], and the transformations of $T(M_5)$ are given by (2) and $l_{ik} = \sum t_i^m t_k^n \bar{l}_{mn}$.

On M_5 there is no degeneration with respect to simple contact, that is with respect to the proper transform $(M_5|B_4)$ of $(P_5|B_4)$; M_5 is *complete* with respect to simple contact.

A cubic in P_2 over the complex numbers with equation $\sum p^{ijk} x_i x_j x_k = 0$ is regarded as a point (p^{ijk}) in P_9, and the condition $(P_9|B_8)$ of simple contact is given by $\sum L_{ijklmn}(p) w^i w^j w^k w^l w^m w^n = 0$. The $L(p)$ are forms of degree 4 in p^{ijk}, explicitly, given in [8], I, p. 50. By (1) the trnasformations of the group $T(P_9)$ are

$$p^{ijk} = \sum \tau_l^i \tau_m^j \tau_n^k \bar{p}^{lmn}, \tag{3}$$

and, moreover, we have

$$L_{ijklmn} = \sum t_i^\alpha t_j^\beta t_k^\gamma t_l^\delta t_m^\varepsilon t_n^\varphi \bar{L}_{\alpha\beta\gamma\delta\varepsilon\varphi}.$$

The degeneration on P_9 with respect to simple contact is the variety $(P_9|D_4)$ of nonreduced cubics; $(P_9|D_4) \subseteq t(P_9|B_8)$ for all $t \in T(P_9)$. On $(P_9|D_4)$ all the forms $L(p)$ vanish.

Analogously to conics we can regard the variety of point-line cubics, given by the generic point (p, l), $(l) = \varrho \cdot (L(p))$ [8], I, [13], but this variety is not smooth, on $(P_9|D_4)$ the variety $(P_9|A_2)$ of triple lines is singular.

2. The sequence of five blow-ups

In [8], II, we constructed, by means of five blow-ups

$$P_9 \leftarrow M_9 \leftarrow N_9 \leftarrow R_9 \leftarrow S_9 \leftarrow \check{Q}_9, \tag{4}$$

a smooth variety which is complete, that is, without degenerations with respect to simple contact; we will describe this construction now.

If $V_9^{(i)}$ is a variety in (4), then a degeneration on $V_9^{(i)}$ is a variety which is in $t(V_9^{(i)}|B_8)$ for all $t \in T(V_9^{(i)})$, where $T(V_9^{(i)})$ is induced by (1) and $(V_9^{(i)}|B_8)$ is the proper transform of $(P_9|B_8)$. We choose a nonsingular center W for the next blow-up $\varphi_{i+1}\colon V_9^{(i+1)} \to V_9^{(i)}$ among the degenerations on $V_9^{(i)}$, and we write down forms which define W; this gives the coordinates for $V_9^{(i+1)}$. In [8], II, it is demonstrated that the local parameters for the blow-up are found among these forms. The same blow-ups were obtained by P. Aluffi [1] from a different point of view.

We choose the variety $(P_9|A_2)$ of triple lines for the center of the first blow-up $\varphi_1\colon M_9 \to P_9$. The variety $(P_9|A_2)$ is defined by the forms B_{ij}^{rs}, where

$$\begin{aligned} B_{ii}^{rs} &= \tfrac{1}{2}(p^{rjj} p^{skk} - 2p^{rjk} p^{sjk} + p^{rkk} p^{sjj}), \\ B_{ij}^{rs} &= \tfrac{1}{2}(p^{rik} p^{sjk} - p^{rik} p^{skk} + p^{rjk} p^{sik} - p^{rkk} p^{sij}), \end{aligned} \tag{5}$$

for $i, j, k = 1, 2, 3$; $i \neq j \neq k \neq i$; $r, s = 1, 2, 3$.

The B_{ij}^{rs} are the coefficients of the first covariant and contravariant form $\sum B_{ij}^{rs} x_r x_s w^i w^j$ [2, 4], which describes for the cubic (p) and for a point (x) the tangents (w) to the polar conic from (x) with respect to (p) [12].

M_9 is given in $P_9 \times P_{35}$ by the generic point (p^{ijk}, b_{lm}^{rs}), $(b_{lm}^{rs}) = \varrho \cdot (B_{lm}^{rs})$. Let $(M_9|A_8)$ be the exceptional divisor, and let $(M_9|D_4)$ and $(M_9|B_8)$ be the proper transforms of $(P_9|D_4)$ and $(P_9|B_8)$ respectively.

From (1) we get the transformations (3) and $b_{kl}^{ij} = \sum t_k^r t_l^s \tau_m^i \tau_n^j \bar{b}_{rs}^{mn}$ in $T(M_9)$. Analogously we can get the transformations for the varieties described below.

The forms $L_{ijklmn}(p)$ of degree 4 in p^{ijk}, vanishing on $(P_9|D_4)$, can be written using (5) as forms L_{ijklmn} of degree 2 in b_{kl}^{ij} given (in [8], II, p. 124). These forms in b_{kl}^{ij} can also be got as follows. For given (x) or (w) the two pairs (x, w), described by

$$\sum b_{kl}^{ij} x_i x_j w^k w^l = 0, \qquad \sum w^i x_i = 0,$$

coincide on $(M_9|D_4)$. This coincidence occurs for some (w) if $\sum L_{ijklmn} w^i w^j w^k w^l w^m w^n = 0$ with the forms L_{ijklmn} of degree 2 in b_{kl}^{ij}, and it occurs for some (x) if $\sum L^{ijklmn} x_i x_j x_k x_l x_m x_n = 0$ and the L^{ijklmn} are of degree 2 in b_{kl}^{ij} too (see [8], II, p. 125). Therefore on $(M_9|D_4)$ we have $L_{ijklmn} = 0$ and $L^{ijklmn} = 0$.

There are two degenerations on M_9 with respect to simple contact, that is, with respect to $(M_9|B_8)$, namely the variety $(M_9|D_4)$ and a variety $(M_9|C_4) \subset (M_9|A_8)$ of dimension 4 too, such that on $(M_9|C_4)$ we have $p^{ijk} b_{rs}^{lm} = p^{ilm} b_{rs}^{jk}$. We choose the variety $(M_9|C_4)$ for the center of the second blow-up φ_2: $N_9 \to M_9$. $(M_9|C_4)$ is defined by the forms $G^{ijklm}_{rs} = p^{ijk} b_{rs}^{lm} - p^{ilm} b_{rs}^{ik}$ and the L_{ijklmn}, or in homogeneous form by

$$^{(1)}M^{ijklm}_{rs}{}^{\mu\nu}_{\sigma\tau} = G^{ijklm}_{rs} \cdot b^{\mu\nu}_{\sigma\tau},$$

$$^{(2)}M_{ijklmn}{}^{\alpha\beta\gamma} = L_{ijklmn} \cdot p^{\alpha\beta\gamma}.$$

The variety N_9 is given by the generic point (p, b, m), $(m) = ({}^{(1)}m, {}^{(2)}m) = \varrho \cdot ({}^{(1)}M, {}^{(2)}M) = \varrho \cdot (M)$. Let $(N_9|C_8)$ be the exceptional divisor, $(N_9|D_4)$, $(N_9|A_8)$, $(N_9|B_8)$ be the proper transforms of $(M_9|D_4)$, $(M_9|A_8)$, $(M_9|B_8)$ respectively. Now there are two degenerations on N_9 with respect to $(N_9|B_8)$, the variety $(N_9|D_4)$ and the 7-dimensional variety $(N_9|E_7) = (N_9|A_8) \cap (N_9|C_8)$.

We choose the variety $(N_9|E_7)$ for the center of the third blow-up φ_3: $R_9 \to N_9$. $(N_9|E_7)$ is defined by the forms G_{rs}^{ijklm} and ${}^{(2)}m_{ijklmn}{}^{\alpha\beta\gamma}$, or in homogeneous form by

$$^{(1)}N^{ijklm}_{rs}{}^{abcde}_{fg}{}^{\mu\nu}_{\sigma\tau} = G^{ijklm}_{rs} \cdot {}^{(1)}m^{abcde}_{fg}{}^{\mu\nu}_{\sigma\tau},$$

$$^{(2)}N_{ijklmn}{}^{\alpha\beta\gamma}{}^{\mu\sigma\tau}{}^{ab}_{cd} = {}^{(2)}m_{ijklmn}{}^{\alpha\beta\gamma} \cdot p^{\mu\sigma\tau} \cdot b^{ab}_{cd},$$

and R_9 is given by the generic point (p, b, m, n) with $({}^{(1)}n, {}^{(2)}n) = \varrho \cdot ({}^{(1)}N, {}^{(2)}N)$. Let $(R_9|E_8)$ be the exceptional divisor, and $(R_9|D_4)$, $(R_9|C_8)$, $(R_9|A_8)$, $(R_9|B_8)$

the proper transforms of $(N_9|D_4)$, $(N_9|C_8)$, $(N_9|A_8)$, $(N_9|B_8)$ respectively. We find one degeneration on R_9 with respect to simple contact, the variety $(R_9|D_4)$, which will be the center of the fourth blow-up $\varphi_4\colon S_9 \to R_9$ and which is defined by the forms ${}^{(2)}n_{ijklmn}{}^{\alpha\beta\gamma\,\mu\sigma\tau\,ab}_{\qquad\quad cd}$ and by the forms L^{ijklmn} of degree 2 in b^{ij}_{kl}, or in homogeneous form by

$$
\begin{aligned}
{}^{(1)}R^{\alpha\beta\gamma\delta\varepsilon\varphi\,ijklm\,abcde\,\mu\nu}_{\qquad\quad rs\quad fg\,\sigma\tau} &= L^{\alpha\beta\gamma\varepsilon\varphi}\cdot{}^{(1)}n^{ijklm\,abcde\,\mu\nu}_{\quad rs\quad fg\,\sigma\tau},\\
{}^{(2)}R_{ijklmn}{}^{\alpha\beta\gamma\,\mu\sigma\tau\,ab\,ef\,qr}_{\qquad\quad cd\,gh\,st} &= {}^{(2)}n_{ijklmn}{}^{\alpha\beta\gamma\,\mu\sigma\tau\,ab}_{\qquad\quad cd}\cdot b^{ef}_{gh}\cdot b^{qr}_{st}.
\end{aligned}
$$

The variety S_9 is given by the generic point (p, b, m, n, r) with $({}^{(1)}r, {}^{(2)}r) = \varrho\cdot({}^{(1)}R, {}^{(2)}R)$, and we have the exceptional divisor $(S_9|D_8)$ and proper transforms $(S_9|E_8)$, $(S_9|C_8)$, $(S_9|A_8)$, $(S_9|B_8)$.

Finally, there is one degeneration on S_9 with respect to simple contact, the variety $(S_9|Z_4) \subset (S_9|D_8)$, which will be the center of the fifth blow-up φ_5: $\check{Q}_9 \to S_9$. This variety is defined by the ${}^{(2)}r$, which we will be denoted by $S_{ijklmn}{}^{\alpha\beta\gamma\,\mu\sigma\tau\,ab\,ef\,qr}_{\qquad\quad cd\,gh\,st}$.

$\check{Q}_9$ is given by the generic point (p, b, m, n, r, s), $(s) = \varrho\cdot(S)$, and we have the exceptional divisor $(\check{Q}_9|Z_8)$ and proper transforms $(\check{Q}_9|D_8)$, $(\check{Q}_9|E_8)$, $(\check{Q}_9|C_8)$, $(\check{Q}_9|A_8)$, $(\check{Q}_9|B_8)$. There are no degenerations on $\check{Q}_9$ with respect to $(\check{Q}_9|B_8)$, that is, $\check{Q}_9$ is complete with respect to simple contact.

$\check{Q}_9$ is biregularly equivalent to the smooth variety Q_9 with the generic point (p, b, m, n, r, s, l), $(l) = \varrho\cdot(L_{ijklmn})$; Q_9 is complete with respect to simple contact too, and we call Q_9 the variety of complete cubics.

3. Relations for simple conditions

The relations

(6) $$(M_5|L) \sim 2(M_5|P) - (M_5|V_4)$$

and

(7) $$\big(M_5|B(c, 1)\big) \sim 2(M_5|P) + 2(M_5|L)$$

for complete conics are among the first relations of numerical equivalence in enumerative geometry [3, 6, 7]. $(M_5|V_4)$ is the exceptional divisor on M_5; $(M_5|P)$ is the condition to contain a point, in other words $(M_5|P)$ is the subvariety given by one linear equation in the coordinates p^{ik}; $(M_5|L)$ is the condition to contact a line, that is, $(M_5|L)$ is the subvariety given by one linear equation in the coordinates l_{ik}; and $\big(M_5|B(c, 1)\big)$ is the condition to contact a given conic.

Following van der Waerden [10], for the condition $(Q_9|Y_8) = (Q_9|L)$ and for $(Q_9|Y_8) = \big(Q_9|B(c, 1)\big)$ we will exhibit on Q_9 a homology relation

(8) $$(Q_9|Y_8) \sim \alpha_1(Q_9|X_8^{(1)}) + \ldots + \alpha_m(Q_9|X_8^{(m)})$$

in codimension 1 with $(Q_9|X_8^{(i)})$ basic elements and α_i integers. According to

Schubert [7] the integers α_i are called the characteristic numbers of the simple condition $(Q_9 | Y_8)$. We compute them by intersecting (8) with suitable one-dimensional subvarieties. Analogously we can get relations on the other varieties $V_9^{(i)}$ in (4) (see [8], III, p. 171).

Let $(P_9|P)$, $(M_9|B)$, $(N_9|M)$, $(R_9|N)$, $(S_9|R)$, $(\check{Q}_9|S)$ be the 8-dimensional varieties given by one linear equation in the coordinates p^{ijk}; b_{ij}^{rs}; ${}^{(1)}m$, ${}^{(2)}m$; ${}^{(1)}n$, ${}^{(2)}n$; ${}^{(1)}r$, ${}^{(2)}r$; and s, respectively. If we already have $(V_9^{(i)} | Y_8)$, then $(V_9^{(i+1)} | Y_8)$ is the proper transform on $V_9^{(i+1)}$ in (4). Also we get $(Q_9 | Y_8)$ from $(\check{Q}_9 | Y_8)$, and $(Q_9|L)$ is the variety on Q_9 given by one linear equation in the coordinates l_{ijklmn} on Q_9.

Starting with the homology group $H_{16}(P_9)$, we compute in [8], III, step by step the homology groups $H_{16}(M_9)$, $H_{16}(N_9)$, $H_{16}(R_9)$, $H_{16}(S_9)$, $H_{16}(Q_9) \cong H_{16}(Q_9)$ to be free abelian groups of rank 2, 3, 4, 5 and 6 respectively.

Basic elements on M_9 (that is, elements whose classes form a basis of $H_{16}(M_9)$) are $(M_9|P)$, $(M_9|A_8)$ or $(M_9|P)$, $(M_9|B)$, and there is a relation $(M_9|B) \sim 2(M_9|P) - (M_9|A_8)$.

Basic elements on N_9 are $(N_9|P)$, $(N_9|B)$, $(N_9|M)$ or $(N_9|P)$, $(N_9|A_8)$, $(N_9|C_8)$, and there is a relation $(N_9|M) \sim (N_9|P) + 2(N_9|B) - (N_9|C_8)$.

On R_9 we have basic elements $(R_9|P)$, $(R_9|B)$, $(R_9|M)$, $(R_9|N)$ or $(R_9|P)$, $(R_9|A_8)$, $(R_9|C_8)$, $(R_9|E_8)$, and $(R_9|N) \sim (R_9|P) + (R_9|B) + (R_9|M) - (R_9|E_8)$.

On S_9 we have basic elements $(S_9|P)$, $(S_9|B)$, $(S_9|M)$, $(S_9|N)$, $(S_9|R)$, or $(S_9|P)$, $(S_9|A_8)$, $(S_9|C_8)$, $(S_9|E_8)$, $(S_9|D_8)$, and $(S_9|R) \sim 2(S_9|B) + (S_9|N) - (S_9|D_8)$.

Finally, basic elements on Q_9 are $(Q_9|P)$, $(Q_9|B)$, $(Q_9|M)$, $(Q_9|N)$, $(Q_9|R)$, $(Q_9|L)$ or $(Q_9|P)$, $(Q_9|A_8)$, $(Q_9|C_8)$, $(Q_9|E_8)$, $(Q_9|D_8)$, $(Q_9|Z_8)$, and we have the relations $(Q_9|S) \sim (Q_9|R) - (Q_9|Z_8)$, $(Q_9|L) \sim (Q_9|S) - 3(Q_9|B) - 2(Q_9|P)$ and

$$(Q_9|L) \sim 4(Q_9|P) - 2(Q_9|A_8) - 3(Q_9|C_8) - 6(Q_9|E_8) - (Q_9|D_8) - 2(Q_9|Z_8). \tag{9}$$

(9) corresponds to (6) and differs from Zeuthen's relation [13] (see [8], IV, p. 102). In [8], IV, we have shown that the relation

$$(Q_9|B(c, 1)) \sim 2(Q_9|P) + 2(Q_9|L),$$

corresponding to (7) for complete conics, is valid for complete cubics too.

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