GEOMETRIC AND ALGEBRAIC TOPOLOGY
BANACH CENTER PUBLICATIONS, VOLUME 18
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1986

ON INVERSE LIMITS OF $(2k+m+1)$-DISK BUNDLE MAPS*

T. B. RUSHING

Salt Lake City, UT, U.S.A.

0. Introduction

Consider the question, "What maps can arise as inverse limits of n-disk bundle maps?" After a moment's reflection (see Section 1) one concludes that all such maps must be cell-like maps, and so the question then becomes, "Just which cell-like maps can arise as the limit of n-disk bundle maps?" It turns out that every cell-like map between finite dimensional metric compacta is the limit of n-disk bundle maps for some n. In fact in [8] (Theorem 3) it is shown that if $f\colon X \to Y$ is any such map with $\dim X = k$, then f is the inverse limit of $(2k+1)$-disk bundle maps.

At this point one might conjecture that if $\dim f = k$ (i.e., $\sup\{\dim f^{-1}(y) \mid y \in Y\} = k$), then $n = 2k+1$ still suffices. This is false. This leads us to the following conjecture (first given as Conjecture 2 of [8]):

CONJECTURE 1. *A map $f\colon X^n \to Y$ between finite dimensional metric compacta with $\dim f^{-1}(y) \leqslant k$ for all $y \in Y$ is a cell-like map if and only if it is the inverse limit of $(n+k+1)$-disk bundle maps.*

The techniques discussed in this note do not allow us to establish the conjecture, however, they do permit us to establish a result in that direction. In particular, we observe that the following theorem results by combining techniques of [8] along with [7] (or alternately [11]).

THEOREM 1. *A map $f\colon X \to A^m$ from a finite dimensional metric compactum onto an m-dimensional* ANR, *with $\dim f^{-1}(y) \leqslant k$ for all $y \in Y$, is cell-like if and only if it is the inverse limit of $(2k+m+1)$-disk bundle maps.*

The proof of the above theorem uses as a fundamental ingredient the notion of pseudoimmersion (first defined in Section 11 of [8]). We indicate in this note why relative embeddings and pseudoimmersions are of interest and why further study of them is justified.

* This paper is in final form and no version of it will be submitted for publication elsewhere.

The author is very indebted to H. Toruńczyk for bringing to his attention the announced theorem of W. A. Pasynkov discussed in Section 4 and for giving him a detailed presentation of a proof of theorem 2 based on work by S. Eilenberg in [5] and H. Toruńczyk in [11]. In general, Henryk Toruńczyk contributed greatly toward this paper.

1. Discussion of limits of disk bundles

We consider inverse sequences $\underline{X} = (X_i, q_{ij})$, $\underline{Y} = (Y_i, r_{ij})$ of compact, metric ANR's and maps $\underline{f} = (f_i)$: $\underline{X} \to \underline{Y}$ where the f_i: $X_i \to Y_i$ commute with the bonding maps. A map f: $X \to Y$ is the *inverse limit* of $\underline{f}$ if $\lim\limits_{\leftarrow} \underline{X} = X$, $\lim\limits_{\leftarrow} \underline{Y} = Y$ and for every i we have $f_i q_i = r_i f$ where q_i: $X \to X_i$ and r_i: $Y \to Y_i$ are the natural projections. (For the conclusion of Theorem 1 of this paper, one may further assume that all the X_i and Y_i are manifolds and all the bonding maps are inclusions.) In the case that each f_i is an n-disk bundle map, we say that f is the *inverse limit of n-disk bundle maps*. F is *cell-like* if $F = \lim\limits_{\leftarrow} \underline{F}$ where $\underline{F} = (F_i, g_{ij})$ and each g_{ij} is null-homotopic. A proper map f: $X \to Y$ is a *cell-like map* if $f^{-1}(y)$ is cell-like for every $y \in Y$.

A priori, perhaps it is not clear that the characterization of cell-like maps given by Theorem 1 (as a limit of disk bundle maps in the sense just defined) is the "right" way to view cell-like maps. We shall now give observations and examples to indicate that this is indeed the case.

First note that any map which is a limit of disk bundle maps is cell-like since any point inverse is the inverse limit of cells. The fact that a limit of disk bundle maps can so trivially be seen to have the "defining property" of cell-like maps is a pretty good reason in itself to just forget the old definition and simply study limits of disk bundle maps. (Of course, this reasoning is sound only because we know that all cell-like maps between metric compacta are such limits.) One is further inclined to change the defining point-of-view since certain important properties of cell-like maps become immediately transparent from the formulation of them as limits of disk bundle maps. In particular, the result proved independently by R. D. Anderson and R. B. Sher [9] that a cell-like map between finite dimensional metric compacta is a shape equivalence is immediate since a limit of disk bundle maps is a limit of homotopy equivalences. Also the result of Sibe Mardešić and the author [6] that a cell-like map between finite dimensional metric compacta is a shape fibration is immediate since a limit of disk bundle maps is a limit of fibrations. Thus, these stronger properties are at your fingertips when studying limits of disk bundle maps.

Even in the case of maps between manifolds, it is appropriate to *characterize* cell-like maps in terms of inverse limits rather than, say, in terms of approximations. To illustrate this, consider the next three examples.

EXAMPLE 1 (given in [8]). One might conjecture that every cell-like map from an n-manifold onto an m-manifold can be approximated by a $(n-m)$-disk bundle map between the same two manifolds. This is false as can be seen by the following example. Let cH^3 denote the cone over a nonsimply connected homology sphere H^3. Then it is not difficult to see that the projection map $f\colon (cH^3)\times S^1 \to S^1$ cannot be approximated by a disk bundle map (with the same domain and range). Clearly f is a cell-like map and it follows from the double suspension theorem that $(cH^3)\times S^1$ is a manifold.

EXAMPLE 2. Bruce Hughes simplified an example of the author to produce the following elementary example of a cell-like map of a 4-manifold onto S^2 which cannot be approximated by a disk bundle map. (Analogs of this example exist in higher dimensions.) Let B^4 represent the unit closed 4-ball and attach a handle of index 2 to B^4 along a tubular neighborhood of a trefoil knot in its boundary. Call the resulting 4-manifold M^4. Represent the handle by $B^2\times B^2$ with "core" $B^2\times 0$. Let $p\colon B^2\times B^2 \to B^2\times 0$ be the "projection". Now there is a map $g\colon B^2\times 0 \to S^2$ such that $g|\mathrm{Int}(B^2\times 0)$ is a homeomorphism and $g|(\partial B^2\times 0)$ has image a fixed point $x_0\in S^2$. We can extend the map $gp\colon B^2\times B^2\to S^2$ to all of M^4 by mapping B^4 to x_0. Let $f\colon M^4\to S^2$ denote this extension. We claim that f cannot be approximated by a disk bundle map. If f could be so approximated then $f|\partial M^4$ would be an approximate fibration [2]. Thus, the fibers of $f|\partial M^4$ would have the same homotopy type. This is a contradiction since $\pi_1((f|\partial M^4)^{-1}(x_0)) \not\approx \pi_1((f|\partial M^4)^{-1}(y))$ for any $y\in S^2$ such that $y\neq x_0$.

Notice that Example 2 shows that a cell-like map between manifolds when restricted to the boundary of the domain manifold, need not be an approximate fibration.

EXAMPLE 3 (given in [8]). Now it is true that if f is a cell-like map between closed manifolds of the *same dimension* then p can be approximated by a homeomorphism (Siebenmann [10]). In fact, it is true that if f is a cell-like map from a manifold onto an ANR with the disjoint disk property, then it can be approximated by a homeomorphism (Edwards [4]). (Of course such an ANR will consequently have the same dimension as the manifold.) However, in general, cell-like maps between metric spaces of the same dimension cannot be approximated by homeomorphisms ($=$ 0-disk bundle maps). This applies even for the domain a closed manifold and the range an ANR. For instance let $f\colon S^n\to S^n/\alpha$ where $\alpha\subset S^n$ is a noncellular arc. More interesting such examples abound in the literature. For instance, let $f\colon S^n \to S^n/G$ be the projection where S^n/G is the Daverman–Walsh ghastly generalized n-manifold [3]. Thus, even in codimension zero one cannot *characterize* cell-like maps by using "strict" approximations even if the metric spaces are quite nice (i.e., ANR's).

Certainly, in general, it is desirable to minimize dimension hypotheses in topological results. For example, in the past great expenditures of energy have gone into minimizing dimension restrictions on various embedding approximation and immersion theorems. In our case, we are interested in minimizing the fiber dimension of the "approximating" bundle maps. Thus, one wishes to impose hypotheses on a cell-like map $f: X \to Y$ which will reduce this fiber dimension. In Theorem 1, we impose restrictions on the dimension of point-inverses. Siebenmann [10] imposed the restriction that X and Y be closed n-manifolds, $n \geqslant 5$, and showed the minimum obtainable fiber dimension to be zero. Thus, Siebenmann's CE-approximation theorem as well as Edwards' CE-approximation theorem [4] may be viewed as special results within the general program of limiting fiber dimension on "approximating" disk bundle maps to cell-like maps. Of course, one would have expected that strenuous restrictions might be necessary in order to insure that the approximating disk bundles have fiber dimension zero, i.e., that they be homeomorphisms.

2. Discussion of relative embeddings and pseudoimmersions

The notion of relative embeddings (defined below) is intimately related to the main point of concern in this note, i.e., that of minimizing the fiber dimension of the "approximating" bundle maps. Indeed, in the next section we shall present the particular relative embedding theorem which leads to the proof of Theorem 1. On the other hand, relative embeddings are of independent interest and are worthy of study for their own sake.

Consider the following general question.

QUESTION 1 (given in [8]). *If $f: X^n \to Y^m$ is a map between finite dimensional compacta such that* $\dim f^{-1}(y) \leqslant k$ *for all $y \in Y$, then for how small a q does there exist a map $r: X \to \boldsymbol{R}^q$ such that $r|p^{-1}(y)$ is an embedding for each $y \in Y$?*

One might expect that since classically each $f^{-1}(y)$ can be embedded in $\boldsymbol{R}^{2k+1}$, then all the $f^{-1}(y)$'s can be "continuously" embedded in $\boldsymbol{R}^{2k+1}$ as required. This certainly is the case if f is a constant map or the identity map. It is easy to see that this is not in general the case as is pointed out by John Bryant and David Wilson with the following interesting example. (Note that f is a cell-like map in this example.)

EXAMPLE 4 (given in [8]). Let $\pi: S^3 \to RP(3)$ be the natural projection. Let M_π be the mapping cylinder of π and let $f: M_\pi \to RP(3)$ be the natural projection. Then the fibre dimension of f is 1, however, the Borsuk–Ulam theorem assures that there cannot be $r: M_\pi \to \boldsymbol{R}^3$ which embeds all the point-inverses of f.

We now give the formal definitions of relative embedding and pseudoimmersion. Pseudoimmersion was originally defined in [8] and the notion of relative embedding was implicit there.

DEFINITION. Let $f: X \to Y$ be a map. Then a map $r: X \to Z$ such that $r|f^{-1}(y)$ is an embedding for each $y \in Y$ is called an *embedding of X into Z relative to f.*

DEFINITION. If $r: X \to Z$ is an embedding rel a cell-like map $f: X \to Y$, then r is called a *pseudoimmersion of X into Z relative to f.*

In terms of the above terminology, we may rephrase Question 1 as follows.

QUESTION 1'. *If $f: X^n \to Y^m$ is a map between finite dimensional compacta such that* $\dim f \leq k$, *then for how small a q does there exist an embedding $r: X \to R^q$ relative to f?*

CONJECTURE 2. $q = n+k+1$ *suffices for Question* 1.

The following are a weaker question and conjecture.

QUESTION 2. *If $f: X^n \to Y^m$ is a cell-like map between finite dimensional compacta such that* $\dim f \leq k$, *then for how small a q does there exists a pseudoimmersion $r: X \to R^q$ relative to f?*

CONJECTURE 3 (Conjecture 1 of [8]). $q = n+k+1$ *suffices for Question* 2.

As observed in [8], an affirmative answer to Conjecture 3 combined with techniques of [8] would establish Conjecture 1.

EXAMPLE 5. To see that $q = m+k+1$ will not suffice for Question 2, let f be the map of the cone over the 1-skeleton of a three-simplex to a point.

QUESTION 3. *Restrict Question 1 to the case that f is a* PL *map between polyhedra.*

QUESTION 4. *Restrict Question 1 to the case that f is a locally trivial bundle map.* (Question 4 is answered in this paper by the Corollary to Theorem 2.)

3. A proposed program for attacking Conjectures 1–3 and Questions 1–4

All of the conjectures and questions raised heretofore are related. We shall suggest a general program for attacking Conjecture 2 and in the process will adress the other conjectures and questions.

Sibe Mardešić and the author made the first progress. The simple idea is to adapt the classical Menger–Nöbeling proof that a k-dimensional compactum X embeds into R^{2k+1} to this situation. Recall the steps are as follows:

I. X compactum, Y metric space. The set of all ε-maps $f: X \to Y$ is open in Y^X.

II. The set of ε-maps $X^k \to R^{2k+1}$, denoted $C_\varepsilon(X, R^{2k+1})$, is dense in $C(X, R^{2k+1})$.

III. X compactum, Y complete metric space, then $C(X, Y)$ is complete.

IV. Baire Category Theorem.

To adapt the proof one replaces $C_\varepsilon(X, Y)$ by $C_\varepsilon^p(E, Y)$ where $p\colon E \to B$ is a map between metric spaces and $C_\varepsilon^p(E, Y)$ represents all maps $f\colon E \to Y$ such that $f|F_b = p^{-1}(b)$ is an ε-map of F_b into Y.

Everything goes through pretty smoothly except Step II. Recall there X is mapped to nerve P which, in turn, is mapped into R^{2k+1}. Hence, basically the problem is reduced to a somewhat similar problem for PL maps.

H. C. Hsiang has suggested a program for handling the PL case. Briefly, the rough idea is to first do the locally trivial fiber bundle case over a manifold, then use it to do the PL case. The locally trivial fiber bundle case reduces to finding a cross section in the associated bundle to a bundle of embeddings. To solve this problem one applies obstruction theory. (The locally trivial fiber bundle case is established in this paper by a different method. It appears in the next section as the Corollary to Theorem 2.)

The PL case is then attacked by applying the above case inductively to an open simplex at a time and using the mapping cylinder structure on a simplicial complex due to M. Cohen [1]. This again leads to an obstruction. For the cell-like mapping case the obstruction is easier to analyze. Obviously, quite a bit of work will be required to accomplish the details of this program.

4. A relative embedding theorem

The following theorem is a consequence of a result announced in 1975 by W. A. Pasynkov [7], however, so far as the author is aware, the proof was not published and no manuscript of the proof is available. The author is indebted to H. Toruńczyk for providing the proof of Theorem 2 given below. The author is also indebted to Vo Thanh Liem for noticing that the Corollary follows from Theorem 2.

THEOREM 2. *If $f\colon X \to Y^m$ is a map of compacta and* $\dim f \leqslant k$, *then there exists a map $r\colon X \to I^{2k+m+1}$ such that $r|f^{-1}(y)$ is an embedding for each $y \in Y$ (i.e., X embeds in R^{2k+m+1} relative to f).*

If f is a map as in Theorem 2, then a result of Hurewicz assures that $\dim X \leqslant m+k$. (This result is given in Hurewicz–Wallman, 91–92.) Consequently, X must be finite dimensional and Theorem 2 is weaker than Conjecture 2 since $n+k+1 \leqslant m+2k+1$. Certainly $2k+m+1$ in Theorem 2 cannot be replaced by $k+m+1$ (see Example 5).

COROLLARY. *If $f\colon X^n \to Y^m$ is a locally trivial bundle map between finite dimensional compacta and* $\dim f \leqslant k$, *then there exists a map $r\colon X \to I^{n+k+1}$ such that $r|f^{-1}(y)$ is an embedding for each $y \in Y$ (i.e., X embeds in R^{n+k+1} relative to f).*

Proof of Theorem 2. There are two cases:

(a) $k = 0$. This case has been considered in [5].

(b) The general case. By a result announced in [7] (see [11], Corollary 1 for a proof) there is a map $g\colon X \to R^k$ such that $h = f \times g$ satisfies $\dim h = 0$. Applying case (a) we get a map $u\colon X \to R^{k+m+1}$ such that $h \times u$ is one-to-one. We take $r = g \times u\colon X \to R^{2k+m+1}$; then $f \times r$ is one-to-one and so r is an embedding rel f.

Proof of Corollary. Since f is a locally trivial bundle map, $m+k = n$. Therefore, $2k+m+1 = (m+k)+k+1 = n+k+1$.

5. The proof of Theorem 1

As is observed on page 18 of [8], the techniques of [8] show that pseudoimmersion theorems yield theorems on limits of disk bundles. Theorem 1 concerns maps between finite dimensional compacta and its proof is along the lines of the proof of Theorem 3 of [8]. In particular, the hypothesis of Theorem 1 gives us a cell-like map $f\colon X \to A^m$ between finite dimensional metric compacta with $\dim f^{-1}(y) \leqslant k$ for all $y \in A$. Thus, Theorem 2 applies to give an embedding $r\colon X \to R^{2k+m+1}$ relative to f. In fact, r is a pseudoimmersion of X into R^{2k+m+1} and it is not difficult to conclude that each $f^{-1}(y)$, $y \in Y$, is embedded cellularly under r.

At this point, one may replace A by a polyhedron by taking a pull-back via the procedure used to show that Lemma 1′ implies fundamental lemma on page 9 of [8]. (The domain of the pull-back is easily seen to pseudoimmerse in R^{2k+m+1}.) Now one observes that the hypothesis of Lemma 1′ of [8] is satisfied. The key to the fact that a pseudoimmersion of X into R^{2k+m+1} suffices and that an embedding is not required, lies in consideration of the space $H_{\check{p}}(\check{E})$ of Lemma 1′. In particular, the fact that $\check{r}$ is a pseudoimmersion is enough to insure that this space is a copy of $\check{E}$. Furthermore, the bulk of paper [8] is concerned with showing that this set-up is enough to yield a "close" $(2k+m+1)$-disk bundle map.

It seems that some technique along the lines of the proof of Theorem 3 (modulo Fundamental Lemma) on page 7 of [8] should suffice to allow us to replace the ANR A in Theorem 1 by a general finite dimensional metric compactum, however, we haven't quite seen how to accomplish this. This leads to the following question.

QUESTION 5. *Can the* ANR *A of Theorem* 1 *be replaced by a general finite dimensional metric compactum?*

References

[1] M. Cohen, *Simplicial structure and transverse cellularity*, Ann. of Math., 85 (1967), 218–245.

[2] D. Coram and P. Duvall, *Approximate fibrations*, Rocky Mountain J. Math., 7 (1977), 275–288.

[3] R. J. Daverman and J. J. Walsh, *A ghastly generalized n-manifold*, Illinois J. Math. 25 (1981), 555–576.

[4] R. D. Edwards, *Approximating certain cell-like maps by homeomorphisms*, preprint. See Notices Amer. Math. Soc. 24 (1977), A-649, # 751–G5 and *Proc. Int. Congr. Math., Helsinki* 1978, Academia Sci. Fennica, Helsinki 1980, 111–127.

[5] S. Eilenberg, *Remarque sur un théorème de M. Hurewicz*, Fund. Math., 24 (1935), 156–159.

[6] S. Mardešić and T. B. Rushing, *Shape fibrations*, I, General Topology and Appl., 9 (1978), 193–215.

[7] W. A. Pasynkov, *On dimension and geometry of mappings*, Dokl. AN SSSR 221 (1975), 543–546 (in Russian).

[8] T. B. Rushing, *Approximating cell-like maps by disk-bundle projections*, Amer. J. Math., 106, 1 (1984), 1–20.

[9] R. B. Sher, *Realizing cell-like maps in Euclidean space*, General Topology and Appl., 2 (1972), 75–89.

[10] L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), 271–294.

[11] H. Toruńczyk, *Finite-to-one restrictions of continuous functions*, Fund. Math. 125 (1985), 253–265.

Presented to the Topology Semester
April 3 – June 29, 1984