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ON HALLIAN DIGRAPHS AND THEIR BINDING NUMBER

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I. Definitions and notation

Throughout the paper, $D = (V, A)$ denotes a *digraph*, i.e., a finite directed graph without multiple arcs and loops, where $V = V(D)$ is the set of vertices and $A = A(D)$ is the set of arcs.

A *circuit* of D is a sequence of different vertices $v_1, \ldots, v_n$ such that (v_i, v_{i+1}) for $i = 1, \ldots, n-1$ and (v_n, v_1) are arcs of D.

For $v \in V(D)$, let us denote by $\Gamma_D^+(v)$ (shortly $\Gamma^+(v)$) the set $\{w\colon (v, w) \in A(D)\}$ and $d^+(v) = |\Gamma^+(v)|$, $\delta^+(D) = \min_{v \in V(D)} d^+(v)$. For $X \subseteq V(D)$, $\Gamma^+(X) = \bigcup_{v \in X} \Gamma^+(v)$. We define $\Gamma^-(X)$, $d^-(v)$, $\delta^-(D)$ similarly.

If each vertex v of D has $d^+(v) = d^-(v) = 1$, then D is a *permutation digraph.*

A digraph D is called *hallian* if D contains a partial permutation digraph [1].

Equivalently, D is hallian if and only of D has a (1, 1)-*factor*, i.e., if its vertices can be covered by a set of vertex disjoint circuits.

(1.1) [1]. *A digraph $D = (V, A)$ is hallian if and only if $|\Gamma^+(X)| \geqslant |X|$ for every nonempty set $X \subseteq V$.* ■

A hallian diagraph D is *k-hallian* if for any set $X \subseteq V(D)$ of cardinality at most k a subgraph of D induced by the set $V(D) \setminus X$ is hallian. The largest k such that D is k-hallian is called the *hallian index* of D and is denoted by $h(D)$. It is clear that $h(D) \leqslant \min\{\delta^+(D)-1, \delta^-(D)-1\}$.

A digraph D is *k-strongly connected* if for any two distinct vertices u and v there exist k internally vertex disjoint paths from u to v in D. The largest

k such that D is k-strongly connected is called the *strong connectivity* of D and is denoted by $\varkappa(D)$. Note that $\varkappa(D) \leqslant \min\{\delta^+(D), \delta^-(D)\}$.

For concepts not defined here, see [4].

II. Some properties of hallian digraphs

Now we will show relations between cardinalities of $\Gamma^+(X)$ and X, $X \subseteq V(D)$, for k-hallian and k-strongly connected digraphs. Let D be a given digraph and let $\mathscr{F}_D = \{X \subseteq V(D)\colon X \neq \emptyset \text{ and } \Gamma^+(X) \neq V(D)\}$.

(2.1). *Let D be a digraph on n vertices. If D is l-strongly connected, then $|\Gamma^+(X)\backslash X| \geqslant l$ for any set $X \in \mathscr{F}_D$.*

Proof. If $l = 0$, then this is obvious. Let $l \geqslant 1$ and $X \in \mathscr{F}_D$. There exists a vertex $v \notin \Gamma^+(X)$. If $v \in V\backslash(X \cup \Gamma^+(X))$, then $\Gamma^+(X)\backslash X$ is a cutset of D. Hence $|\Gamma^+(X)\backslash X| \geqslant l$. If $V\backslash(X \cup \Gamma^+(X)) = \emptyset$, then $v \in X\backslash\Gamma^+(X)$. This implies that $\Gamma^-(v) \cap X = \emptyset$, thus $|\Gamma^+(X)\backslash X| \geqslant |\Gamma^-(v)| \geqslant \delta^-(D) \geqslant l$. ∎

(2.2). *Let D be a digraph on n vertices. Then D is k-strongly connected and k-hallian if and only if for any set $X \in \mathscr{F}_D$, $|\Gamma^+(X)| \geqslant |X|+k$.*

Proof. Assume $|\Gamma^+(X)| \geqslant |X|+k$ for any $X \in \mathscr{F}_D$. Suppose there exists a set $A \subseteq V(D)$, $|A| \leqslant k-1$, such that $D' = \langle V(D)\backslash A\rangle$ is not strongly connected. Let $Q_1, \ldots, Q_r$, $r \geqslant 2$, be the strongly connected components of D'. Hence for some i, $1 \leqslant i \leqslant r$,

$$|\Gamma_D^+(Q_i)| \leqslant |V(Q_i)|+|A| \leqslant |V(Q_i)|+k-1 < |V(Q_i)|+k,$$

a contradiction. This implies D is k-strongly connected.

Suppose that D is not k-hallian, i.e., there exists a set $A \subseteq V(D)$, $|A| \leqslant k$, such that $D' = \langle V(D)\backslash A\rangle$ is not hallian. Hence, there exists a set $N \subseteq V(D)\backslash A$ such that $|\Gamma_{D'}^+(N)| < |N|$. So, $\Gamma_{D'}^+(N) \neq V(D')$ and therefore $\Gamma_D^+(N) \neq V(D)$. By the assumption $|\Gamma_D^+(N)| \geqslant |N|+k$ and by the fact that $|\Gamma_{D'}^+(N)| < |N|$ we obtain $|\Gamma_D^+(N)| > |\Gamma_{D'}^+(N)|+k$, a contradiction. Hence, D is k-hallian.

Conversely, assume D is k-hallian and k-strongly connected and suppose there exists a set $A \in \mathscr{F}_D$ such that $|\Gamma^+(A)| < |A|+k$. By (2.1), $|\Gamma^+(A)\backslash A| \geqslant k$. Hence, there is a set $B \subseteq \Gamma^+(A)\backslash A$, $|B| = k$. So, $D' = \langle V(D)\backslash B\rangle$ is hallian. Thus for any $X \in \mathscr{F}_{D'}$, $|\Gamma_{D'}^+(X)| \geqslant |X|$. This implies that $|\Gamma_{D'}^+(A)| \geqslant |A|$.

On the other hand, $|\Gamma_{D'}^+(A)| = |\Gamma_D^+(A)|-|B| < |A|+k-|B| = |A|$, a contradiction. ∎

By (2.2) we easily get the following two theorems.

(2.3). *If D is an l-connected k-hallian digraph on n vertices, then $|\Gamma^+(X)| \geqslant |X|+r$ for any $X \in \mathscr{F}_D$, where $r = \min\{k, l\}$.* ∎

(2.4). *Let D be a hallian digraph on n vertices and let $r = \min\{h(D), \varkappa(D)\}$. Then there exists a set $X \in \mathscr{F}_D$ such that $|\Gamma^+(X)| = |X|+r$.*

Proof. We consider two cases:

Case 1: $r = h(D)$. Then there exists a set $A \subseteq V(D)$, $|A| = r+1$, such that $D' = \langle V(D)\backslash A\rangle$ is not hallian. Hence there exists $X \subseteq V(D')$ such that $|\Gamma^+_{D'}(X)| < |X|$ and $|\Gamma^+_D(X)| \leqslant |\Gamma^+_{D'}(X)| + |A| < |X|+r+1$. So, $|\Gamma^+_D(X)| \leqslant |X|+r$. Since $\Gamma^+_{D'}(X) \neq V(D')$, we have $\Gamma^+_D(X) \neq V(D)$ and the reverse inequality follows from (2.3), proving the result in this case.

Case 2: $r = \varkappa(D)$. Let $Y \subseteq V(D)$ be a cutset of cardinality r and let $Q_1, \ldots, Q_t$, $t \geqslant 2$, be the strongly connected components of $\langle V(D)\backslash Y\rangle = D'$. For some i, $1 \leqslant i \leqslant t$, $\Gamma^+_{D'}(Q_i) = Q_i$. Then $|\Gamma^+_D(Q_i)| \leqslant |Q_i|+r$. Since $Q_i \in \mathscr{F}_D$, we also obtain the reverse inequality. ∎

The Chvátal–Erdős Theorem [2] states that an undirected graph G is hamiltonian if its connectivity is at least as large as its independence number, $\alpha(G)$. Häggkvist and Thomassen [3] have generalized this result by proving the following conjecture of Berge: if G is $(\alpha(G)+p)$-connected, then any set of vertex disjoint paths of G of total length at most p is contained in a hamiltonian cycle of G.

B. Jackson and O. Ordaz in [5] have proved the corresponding theorems for digraphs.

(2.5) [5]. *If $\alpha(D) \leqslant \varkappa(D)$, then D is hallian.* ∎

(2.6) [5]. *Let D be a k-strongly connected digraph on at least two vertices such that $k \geqslant \alpha(D)+p$. Then any set of vertex disjoint paths of D of total length at most p is contained in a $(1, 1)$-factor of D.* ∎

(2.7) [5]. *If D is a digraph such that $|\Gamma^+(S)| \geqslant \min\{|V(D)|, |S|+p\}$ for all $S \subseteq V(D)$, then any set of vertex disjoint paths of D of total length at most p is contained in a $(1, 1)$-factor of D.* ∎

In view of the proof of (2.6) it is easy to prove that

(2.8). *Let D be a k-strongly connected digraph such that $k \geqslant \alpha(D)+p$. Then D is p-hallian.*

Proof. For a digraph on one or two vertices, the theorem is obvious. Let D be a digraph on at least three vertices. Suppose that D is not p-hallian. According to (2.3) there exists a set $X \in \mathscr{F}_D$ such that

$$(*) \qquad |\Gamma^+(X)| < |X|+p.$$

Since $d^+(v) \geqslant k \geqslant \alpha(D)+p$ for all $v \in V(D)$ we have $|X| \geqslant 2$. It is easy to see that $|\Gamma^+(X)| = |\Gamma^+(X)\backslash X| + |X \cap \Gamma^+(X)|$ and $|X| = |X\backslash\Gamma^+(X)| + |X \cap \Gamma^+(X)|$. Thus $(*)$ is equivalent to $|\Gamma^+(X)\backslash X| < |X\backslash\Gamma^+(X)|+p$. Moreover, $X\backslash\Gamma^+(X)$ is an independent set of vertices of D, hence $|\Gamma^+(X)\backslash X| < \alpha(D)+p \leqslant k$. From (2.1), $|\Gamma^+(X)\backslash X| \geqslant k$ for any $X \in \mathscr{F}_D$, a contradiction. ∎

III. On the binding number of hallian digraphs

The *binding number* of a digraph D, denoted by $\mathrm{bind}^+(D)$, is defined by

$$\mathrm{bind}^+(D) = \min_{X\in\mathscr{F}_D} \frac{|\Gamma^+(X)|}{|X|}.$$

For any hallian digraph D, $\mathrm{bind}^+(D) \geqslant 1$.

(3.1). *Let D be a digraph on n vertices. If $X \in \mathscr{F}_D$, then $|X| \leqslant n-\delta^-(D)$.*

Proof. There exists $v \in V(D)\backslash\Gamma^+(X)$. This implies $\Gamma^-(v)\cap X = \emptyset$ and $|\Gamma^-(v)| \geqslant \delta^-(D)$, thus $|X| \leqslant n-\delta^-(D)$. ∎

(3.2). *For any digraph D,* $\mathrm{bind}^+(D) \leqslant \dfrac{n-1}{n-\delta^-(D)}$.

Proof. Let $X \subseteq V(D)$ and $X = V(D)\backslash\{v\}$, where $d^-(v) = \delta^-(D)$. Hence $\Gamma^+(X) = V\backslash\{v\}$. Thus $\dfrac{|\Gamma^+(X)|}{|X|} = \dfrac{n-1}{n-\delta^-(D)}$ and $\mathrm{bind}^+(D) \leqslant \dfrac{n-1}{n-\delta^-(D)}$. ∎

(3.3). *If D is an l-strongly connected k-hallian digraph on n vertices and $r = \min\{k, l\}$, then* $\mathrm{bind}^+(D) \geqslant 1+\dfrac{r}{n-\delta^-(D)}$.

Proof. By (2.3), for any $X \in \mathscr{F}_D$, $|\Gamma^+(X)| \geqslant |X|+r$. So, $|\Gamma^+(X)|/|X| \geqslant 1 + r/|X|$ and by (3.1) the theorem is proved. ∎

(3.4). *If a digraph D on n vertices has $h(D) = \delta^-(D)-1$ and $\varkappa(D) \geqslant h(D)$, then* $\mathrm{bind}^+(D) = \dfrac{n-1}{n-\delta^-(D)}$.

Proof. According to (3.3) we have $\mathrm{bind}^+(D) \geqslant \dfrac{n-1}{n-\delta^-(D)}$. By (3.2) the theorem follows. ∎

(3.5). *Let C_n be a circuit on n vertices. Then* $\mathrm{bind}^+(C_n) = 1$, $n \geqslant 2$.

Proof. $\varkappa(C_n) = 1$, $h(C_n) = 0$ and $\delta^-(C_n) = 1$. By (3.4) the result follows. ∎

Let D_1, D_2 be digraphs with vertex sets $V(D_1) = \{v_1, \ldots, v_n\}$, $V(D_2) = \{w_1, \ldots, w_m\}$.

The *Cartesian product* of the digraphs D_1 and D_2 is a digraph $D_1 \times D_2$ with vertex set $V(D_1)\times V(D_2)$ and with $\{(v_i, w_j), (v_k, w_l)\} \in A(D_1 \times D_2)$ if and only if either $i = k$ and $(w_j, w_l) \in A(D_2)$, or $j = l$ and $(v_i, v_k) \in A(D_1)$.

It is not difficult to see that $\varkappa(C_n \times C_m) = 2$ and $h(C_n \times C_m) = 0$. Moreover, $\delta^-(C_n \times C_m) = 2$.

(3.6). $\mathrm{bind}^+(C_n \times C_m) = 1$, $n \geqslant 2$, $m \geqslant 2$.

Proof. $C_n \times C_m$ is a hallian digraph, hence $\operatorname{bind}^+(C_n \times C_m) \geqslant 1$. By (2.4) there exists a set $X \in \mathscr{F}_{C_n \times C_m}$ such that $|\Gamma^+(X)| = |X|$. The result is proved. ■

If G is an undirected graph, then G^* will denote the digraph obtained from G by replacing each edge $\{u, v\}$ of G by the arcs (u, v) and (v, u) in G^*. It is easy to see that $\operatorname{bind}(G) = \operatorname{bind}^+(G^*)$. Woodall's [6] main result is the following.

(3.7) [6]. *Let G be a graph. If* $\operatorname{bind}(G) \geqslant \frac{3}{2}$, *then G is hamiltonian.* ■

For digraphs the similar result is not true. Let D_7 be the digraph shown in Fig. 1. It is not hamiltonian.

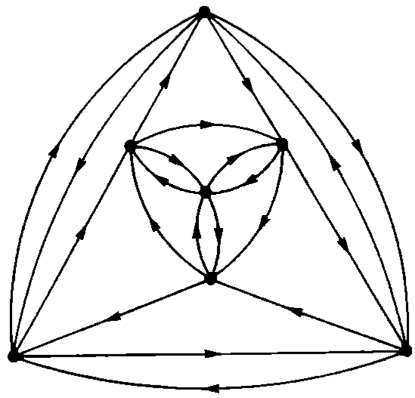

Fig. 1

(3.8). $\operatorname{bind}^+(D_7) = \frac{3}{2}$.

Proof. $\varkappa(D_7) = 3$, $h(D_7) = 2$ and $\delta^-(D_7) = 3$. Thus by (3.4) the result follows. ■

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