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VECTOR BUNDLES AND LOW-CODIMENSIONAL SUBMANIFOLDS OF PROJECTIVE SPACE: A PROBLEM LIST

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The interplay between submanifolds of complex projective space $\mathbf{P}_n$ and vector bundles on $\mathbf{P}_n$ has been very fruitful (cf. [OSS], [H4], [V]). The purpose of this paper is to make accessible to the non-expert some open problems and questions. This list consists of 37 problems (and one theorem) and extends largely the one given in [S3]. In particular, we included a fair number of "easier" problems. Some of them can also be found in the collection [OV3]. In the whole paper it is assumed that submanifolds of $\mathbf{P}_n$ are nondegenerate.

1. Surfaces in $\mathbf{P}_4$

Surfaces in $\mathbf{P}_4$ of low degree have been classified by various authors [A1], [A2], [I1], [I2], [L2], [O2], [O3]. The classification is complete up to degree 8 and at present there are efforts to go up to degree 10 [R1], [R2]. Ellingsrud and Peskine [EP] have shown that surfaces in $\mathbf{P}_4$ of large degree are necessarily of general type. All the known surfaces X in $\mathbf{P}_4$ satisfy the estimate

$$q(X) = h^1(X, \mathcal{O}_X) \leqslant 2.$$

$q(X) = 2$ is attained by the Horrocks–Mumford torus. The value $q(X) = 1$ is achieved by the elliptic quintic scroll in $\mathbf{P}_4$. The following problem seems to go back to Van de Ven.

PROBLEM 1. Is the irregularity of all smooth surfaces $X \subset \mathbf{P}_4$ bounded by 2?

Another general problem for surfaces $X \subset \mathbf{P}_4$ is also due to Van de Ven, cf. [O4].

This paper is in final form and no version of it will be submitted for publication elsewhere.

PROBLEM 2. Let $X \subset \mathbf{P}_4$ be a smooth surface such that the normal bundle $N_{X/\mathbf{P}_4} \cong L_1 \oplus L_2$ splits into a direct sum of line bundles. Is X a complete intersection?

By a result of Faltings [F2] the answer is yes if $L_i = \mathcal{O}_X(a_i)$, $i = 1, 2$.

The normal bundle of some surfaces $X \subset \mathbf{P}_4$ of low degree has been studied by Braun [Br].

A surface $X \subset \mathbf{P}_4$ is called subcanonical if $K_X = \mathcal{O}_X(a)$. For $a < 0$ X is necessarily a complete intersection. For $a = 0$ one gets a Horrocks–Mumford torus [HM]. For $a = 1$ one gets only complete intersections by [BC]. The case $a = 2$ is partly treated in [DPPS]. Subcanonical surfaces X in $\mathbf{P}_4$ give rise to rank 2-bundles E on $\mathbf{P}_4$ fitting into an exact sequence

$$0 \to \mathcal{O} \to E \to J_X(a+5) \to 0$$

where $K_X = \mathcal{O}_X(a)$.

This makes a classification of subcanonical surfaces in $\mathbf{P}_4$ particularly interesting and gives candidates for new rank 2-bundles on $\mathbf{P}_4$.

PROBLEM 3. Classify all smooth surfaces $X \subset \mathbf{P}_4$ with $K_X = \mathcal{O}_X(a)$ for small values of a.

Here *small* means for instance $a = 2$ or $a = 3$.

Finally, the classification of smooth surfaces in $\mathbf{P}_4$ of small degree should be pushed to a reasonable point.

PROBLEM 4. Give a complete classification of smooth surfaces $X \subset \mathbf{P}_4$ of degree 9 and 10.

2. Threefolds in $\mathbf{P}_5$

Threefolds in $\mathbf{P}_5$ are even more "rigid" than surfaces in $\mathbf{P}_4$. By Barth's Lefschetz theorem [B1] their irregularity is zero. By a result of Severi all surfaces in $\mathbf{P}_4$ except the Veronese surface are linearly normal. This implies that all 3-folds in $\mathbf{P}_5$ are linearly normal. The Veronese surface can be described as the dependency locus of a map

$$\mathcal{O}_{\mathbf{P}_4}^{\oplus 3} \to \Omega^1_{\mathbf{P}_4}(2).$$

Okonek [O2] gave a similar example of a 3-fold X in $\mathbf{P}_5$ as dependency locus of a map

$$\mathcal{O}_{\mathbf{P}_5}^{\oplus 4} \to \Omega^1_{\mathbf{P}_5}(2).$$

This 3-fold X is not quadratically normal. Peskine has posed the problem if this is the only 3-fold in $\mathbf{P}_5$ which is not quadratically normal, resp. if there is a finite list of those 3-folds.

PROBLEM 5. Classify all 3-folds $X \subset \mathbf{P}_5$ which are not quadratically normal.

In analogy with the result of Ellingsrud and Peskine [EP] for surfaces in $\mathbf{P}_4$ the following problem is natural.

PROBLEM 6. Does there exist $d \in \mathbf{N}$ such that all 3-folds $X \subset \mathbf{P}_5$ with $\deg X \geqslant d$ are of general type?

The problem for 3-folds in $\mathbf{P}_5$ corresponding to Problem 2 has an affirmative answer due to Peskine [P], i.e. a smooth 3-fold $X \subset \mathbf{P}_5$ with $N_{X/\mathbf{P}_5} \cong L_1 \oplus L_2$ is necessarily a complete intersection. We will see later on a generalisation to higher dimension. As a testing ground for conjectures in higher dimensions it seems worthwhile to investigate the normal bundle of 3-folds in $\mathbf{P}_5$ of low degree analogously to the surface case [Br].

PROBLEM 7. Study the normal bundle of 3-folds in $\mathbf{P}_5$.

In particular, decide if $N_{X/\mathbf{P}_5}$ is stable, semi-stable, indecomposable and compute $H^0(X, N(-2))$.

There are only two surfaces in $\mathbf{P}_4$ which are scrolls [L1]. The corresponding classification of 3-folds in $\mathbf{P}_5$ should be done.

PROBLEM 8. Make a complete list of 3-folds in $\mathbf{P}_5$ which are scrolls.

As with surfaces in $\mathbf{P}_4$ the classification of 3-folds in $\mathbf{P}_5$ of low degree should be pushed to a reasonable limit. Okonek [O2] has done the classification up to degree 7. In [I2] one can find the case of degree 8.

PROBLEM 9. Give a complete classification of 3-folds in $\mathbf{P}_5$ of degree 9 and 10.

In [BSS] one may eventually find a solution.

3. Around the Hartshorne conjecture

The guiding conjecture for submanifolds of projective space of low codimension is Hartshorne's conjecture [H1].

CONJECTURE 10. *Let $X \subset \mathbf{P}_n$ be a smooth subvariety satisfying* $\dim X > \frac{2}{3}n$. *Then X is a complete intersection.*

There are some results supporting this conjecture. If the degree of X is small compared to n then the conjecture is true [B2]. Moreover, Zak [Z] has proved a first order version of this conjecture, namely, that X is linearly normal for $\dim X \geqslant \frac{2}{3}(n-1)$. Note that for $\operatorname{codim} X = 2$ the subvariety $X \subset \mathbf{P}_n$ is a complete intersection, precisely if X is projectively normal i.e.

$$H^0(\mathbf{P}_n, \mathcal{O}_{\mathbf{P}_n}(k)) \to H^0(X, \mathcal{O}_X(k))$$

is surjective for all $k > 0$.

Ran [R] has given quadratic estimates in codimension 2 for k-normality. These have been improved for $k = 2$ in [PPS] and [E2]. At least a linear estimate would be desirable for k-normality in codimension 2.

Peskine has an approach to the following

PROBLEM 11. Let $X \subset \mathbf{P}_n$ be a smooth submanifold of codimension 2, $n \geqslant 5$. Then X is $(n-4)$-normal.

As a general test for the Hartshorne conjecture it would be interesting to establish properties weaker than "complete intersection" in the "Hartshorne range" $\dim X > \frac{2}{3}n$. We propose three problems in this spirit.

PROBLEM 12. Let $X \subset \mathbf{P}_n$ be a smooth subvariety with $\dim X > \frac{2}{3}n$. Show that X is quadratically normal.

Note that for $\operatorname{codim} X = 2$ the previous problem includes the present one.

PROBLEM 13. Let $X \subset \mathbf{P}_n$ be a smooth subvariety with $\dim X > \frac{2}{3}n$. Show that $N_{X/\mathbf{P}_n}(-2)$ is generated by sections.

Note that the condition $\dim X > \frac{2}{3}n$ cannot be improved as the example $X = \operatorname{Gr}(2, 5) \subset \mathbf{P}_9$ shows. Here $H^0(X, N_{X/\mathbf{P}_9}(-2)) = 0$.

A weaker problem concerns the dual variety $X^\vee$ of X. The dual variety can be described either as the closure of all hyperplanes $H \in \mathbf{P}_n^\vee$ tangent to X or as the image of $\mathbf{P}(N_{X/\mathbf{P}_n}(-1))$ in $\mathbf{P}_n^\vee$. From this description it can easily be seen that the ampleness of $N_{X/\mathbf{P}_n}(-1)$ implies that $X^\vee \subset \mathbf{P}_n^\vee$ is a hypersurface. Hence the preceding problem is stronger than

PROBLEM 14. Let $X \subset \mathbf{P}_n$ be a smooth subvariety with $\dim X > \frac{2}{3}n$. Then $X^\vee \subset \mathbf{P}_n^\vee$ is a hypersurface.

For $\operatorname{codim} X = 2$ this is true [HS], [E1].

4. Vector bundles and subvarieties

There is a relation between vector bundles on $\mathbf{P}_n$ and subvarieties of $\mathbf{P}_n$ which is well understood only in codimension 2 (Serre construction cf. [OSS]). The following question goes back to Hartshorne [H1].

PROBLEM 15. Let $X \subset \mathbf{P}_n$ be a smooth subvariety of codimension r.

(i) When does there exist a rank-r vector bundle E on $\mathbf{P}_n$ extending $N_{X/\mathbf{P}_n}$?

(ii) When does there exist a rank-r vector bundle E on $\mathbf{P}_n$ and a section $\sigma \in H^0(\mathbf{P}_n, E)$ such that $X = V(\sigma)$ as schemes?

Note that in case (ii) one has automatically $E|X \cong N_{X/\mathbf{P}_n}$. For $r = 2$ and $n \geqslant 3$ by Serre's construction (ii)) is true precisely if X is subcanonical. By [B1] X is subcanonical for $n \geqslant 6$. A way of constructing higher rank bundles from codimension 2 submanifolds with some extra structure has been given by Vogelaar [Vo].

Sometimes it is possible to put a double structure on a subvariety $X \subset \mathbf{P}_n$ which is subcanonical, hence giving rise (in codimension 2) to a vector bundle on $\mathbf{P}_n$. For surfaces in $\mathbf{P}_4$ this has been used by Hulek and Van de Ven [HV] to construct the Horrocks–Mumford bundle from an elliptic quintic scroll. In [HOV] the Castelnuovo surface was used in the same spirit. But here the resulting double structure was a complete intersection. For smaller codimension this should be impossible.

PROBLEM 16. Let $X \subset \mathbf{P}_n$ be a smooth submanifold with $\dim(X) \geqslant (n+1)/2$ admitting a double structure which is a complete intersection. Is then X a complete intersection?

The answer is yes for $n = 5$ as has been shown by Peskine [P]. Problem 2 is interesting in any dimension.

PROBLEM 17. Let $X \subset \mathbf{P}_n$ be a smooth subvariety with splitting normal bundle $N_{X/\mathbf{P}_n} \cong L_1 \oplus \ldots \oplus L_r$. Under what conditions is X a complete intersection?

Faltings [F2] has shown that X is a complete intersection if $\dim X \geqslant n/2$ and $N_{X/\mathbf{P}_n} \cong \mathcal{O}_X(a_1) \oplus \ldots \oplus \mathcal{O}_X(a_r)$. Peskine [P] has shown that the answer is yes for 3-folds in $\mathbf{P}_5$. I will sketch a different proof of a more general result.

THEOREM. *Let $X \subset \mathbf{P}_n$ be a smooth subvariety with $\dim X \geqslant (n+1)/2$ and $N_{X/\mathbf{P}_n} \cong L_1 \oplus \ldots \oplus L_r$, $L_i \in \operatorname{Pic}(X)$. If $n \geqslant 5$, X is a complete intersection.*

Proof. N being ample implies the ampleness of all L_i. Hence we get by Kodaira's vanishing theorem

$$H^q(X, S^k(N^\vee)) = 0, \qquad k \geqslant 1,\ 1 \leqslant q \leqslant 2. \tag{1}$$

Denote by $\hat{X}$ the formal completion of $\mathbf{P}_n$ along X. The obstructions to extend a line bundle on X to $\hat{X}$ lie in $H^2(X, S^k(N^\vee))$ and the obstructions for uniqueness of the extension lie in $H^1(X, S^k(N^\vee))$. Hence we get by (1) an isomorphism

$$\operatorname{Pic}(\hat{X}) \xrightarrow{\sim} \operatorname{Pic}(X).$$

By a theorem of Faltings [F1] for $d = \dim X \geqslant (n+1)/2$ there is an isomorphism

$$\operatorname{Pic}(\mathbf{P}_n) \xrightarrow{\sim} \operatorname{Pic}(\hat{X}).$$

Hence we obtain an isomorphism

$$\mathrm{Pic}(\mathbf{P}_n) \xrightarrow{\sim} \mathrm{Pic}(X).$$

This implies the existence of $a_i \in \mathbf{N}$ with

$$L_i = \mathcal{O}_X(a_i), \quad 1 \leqslant i \leqslant n-d.$$

But in the case

$$N \cong \mathcal{O}_X(a_1) \oplus \ldots \oplus \mathcal{O}_X(a_{n-d})$$

Faltings [F2] has shown that under the assumption $d = \dim X \geqslant n/2$ the submanifold X has to be a complete intersection.

The case of curves in $\mathbf{P}_3$ is not really fitting very well into our considerations but the following problem due to K. Hulek belongs to this circle of questions.

PROBLEM 18. Let $C \subset \mathbf{P}_3$ be a smooth curve whose normal bundle splits in the form $N_{C/\mathbf{P}_3} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$. Is C a complete intersection?

5. Vector bundles and moduli spaces

The analogue of the Hartshorne conjecture for vector bundles on $\mathbf{P}_n$ is the following problem.

PROBLEM 19. Let E be a rank r-bundle on $\mathbf{P}_n$. Does E split if $n \gg r$?

For $r = 2$ the Hartshorne conjecture is in fact equivalent to the splitting of all 2-bundles on $\mathbf{P}_n$, $n \geqslant 7$. Remember that over $\mathbf{C}$ the only known 2-bundles on $\mathbf{P}_n$, $n \geqslant 5$, are of the form $\mathcal{O}(a) \oplus \mathcal{O}(b)$. For unstable rank-2 bundles there is the following conjecture cf. [GS].

CONJECTURE 20. *Every unstable rank-2 bundle on $\mathbf{P}_n$, $n \geqslant 5$, splits.*

Here the unstability of the 2-bundle E can be defined by $H^0(\mathbf{P}_n, E_{\mathrm{norm}}) \neq 0$ where $E_{\mathrm{norm}} = E(k)$ and k is such that $c_1(E(k)) \in \{0, -1\}$. Note that even on $\mathbf{P}_4$ there is no example known of an unstable indecomposable 2-bundle.

More generally, a vector bundle E on a projective variety X is called to be *Gieseker-(semi)-stable* with respect to some ample line bundle H if for $\nu \gg 0$

$$\frac{\chi(F(\nu))}{\mathrm{rk}(F)} \underset{(-)}{<} \frac{\chi(E(\nu))}{\mathrm{rk}(E)}$$

for all coherent subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$.

If this condition is modified to

$$\mu(F) \underset{(-)}{<} \mu(E),$$

where $\mu(F) = ((\det F)\cdot H^{n-1})/\mathrm{rk}(F)$, $n = \dim X$, one gets the notion of μ-(semi)-stability in the sense of Mumford.

These two notions are connected by

$$\begin{array}{ccc} \mu\text{-stable} & \Rightarrow & \text{Gieseker-stable} \\ \Downarrow & & \Downarrow \\ \mu\text{-semi-stable} & \Leftarrow & \text{Gieseker-semi-stable.} \end{array}$$

Maruyama [M1], [M2] has shown that equivalence classes of semi-stable torsion-free sheaves with fixed rank and fixed Hilbert polynomial p have a coarse moduli scheme $\bar{M}_X(p)$. $\bar{M}_X(p)$ contains the stable sheaves $M_X(p)$ as an open part. These moduli-spaces have been studied by many authors for various special base manifolds X, cf. [OSS]. There is at present no counterexample known to the question whether the moduli spaces are always rational provided X is rational. In the special case $X = \mathbf{P}_2$ this should be true.

PROBLEM 21. Show that $M_{\mathbf{P}_2}(r, c_1, c_2)$ is rational.

Here, as on all projective spaces, the Hilbert polynomial of a bundle is determined by the rank r and the Chern classes c_i. For $r = 2$ this is known to be true [B4], [Hu1], [Ma]. For all r the spaces $M_{\mathbf{P}_2}(r, c_1, c_2)$ are unirational [El].

The properties of the moduli spaces $M_{\mathbf{P}_2}$ have been proved important in the study of the solutions of the Yang–Mills equations on S^4. For instance Donaldson [Do1] related the space of real k-instantons on S^4 via the moment map to $M_{\mathbf{P}_2}(2, 0, k)$. Using the irreducibility of $M_{\mathbf{P}_2}$, he obtained the connectedness of the space of k-instantons. In fact, one of the driving forces in vector bundle theory has been the relation to real instantons. For the case of "complex instantons" the outstanding problem is the following.

PROBLEM 22. Are the spaces $M^{\mathrm{inst}}_{\mathbf{P}_3}(k)$ nonsingular, irreducible, rational?

Here $M^{\mathrm{inst}}_{\mathbf{P}_3}(k)$ denotes the moduli space of stable rank-2 bundles E on $\mathbf{P}_3$ with $c_1(E) = 0$, $c_2(E) = k$, and the instanton condition $H^1(\mathbf{P}_3, E(-2)) = 0$. The problem has a positive answer for $k \leqslant 4$ (cf. [B5], [ES], [H2]; rationality is known only for $k \leqslant 3$).

Returning to $M_{\mathbf{P}_2}(2, c_1, c_2)$ there is an interesting problem which has been posed by Hirschowitz and Hulek [HH] which is an analogue of the corresponding problem for the moduli spaces of curves of genus g.

PROBLEM 23. Determine the maximal dimension of a complete subvariety of $M^{\mathrm{stab}}_{\mathbf{P}_2}(2, c_1, c_2)$.

Here $M^{\mathrm{stab}}_{\mathbf{P}_2} \subset M_{\mathbf{P}_2}$ is the open part of stable bundles. Hirschowitz and Hulek have shown [HH] that except in some obvious cases there always exist complete curves. What one would expect is a function of c_2 (once c_1 normalized to 0 or -1). A stronger form would be the

PROBLEM 24. Determine the q-completeness in the sense of Andreotti–Grauert [AG] of $M^{\text{stab}}_{\mathbf{P}_2}(2, c_1, c_2)$.

Note that the q-completeness of a space implies that it does not contain complete subvarieties of dimension $\geqslant q$.

A solution of this problem might have implications on the existence of 2-bundles on higher dimensional projective space.

Another hard problem in vector bundle theory is the analogue of the Hodge conjecture.

PROBLEM 25. Let E be a topological complex vector bundle on a projective manifold such that all Chern classes $c_i(E)$ are of type (i, i). Under what conditions does E admit an algebraic structure?

For a surface the answer is yes [Sch1]. For $X = \mathbf{P}_n$ the conditions on the Chern classes are automatic. For $n \leqslant 3$ all topological vector bundles on $\mathbf{P}_n$ are algebraic [AR], [Sch1], [Vo], [Ho]. Topologists [Sm], [Sw] have constructed many nontrivial topological 2-bundles on $\mathbf{P}_n$ satisfying $c_1 = c_2 = 0$. By Barth's inequality [B3] $c_1^2 < 4c_2$ these can never have a stable algebraic structure. Therefore they cannot admit any algebraic structure if conjecture 20 is true.

Another potential topological 2-bundle without algebraic structure is the one on $\mathbf{P}_4$ with $c_1 = 0$, $c_2 = 3$. Barth and Elencwajg [BE] and Ballico, Chiantini [BC] have shown that it does not admit a (semi-) stable structure.

PROBLEM 26. Is there an algebraic 2-bundle on $\mathbf{P}_4$ with $c_1 = 0$, $c_2 = 3$?

In the same spirit we pose

PROBLEM 27. Does the topological rank-2 bundle on $\mathbf{P}_4$ with $c_1 = -1$, $c_2 = 4$ admit an unstable algebraic structure?

This topological bundle admits a stable algebraic structure — this is the famous Horrocks–Mumford bundle. If the answer to Problem 27 is no, then one has found three topological 2-bundles on $\mathbf{P}_5$ without algebraic structure. This follows from the fact that there are precisely three topological extensions to $\mathbf{P}_5$ [Sw] and the work of Decker [D] that there is no algebraic 2-bundle on $\mathbf{P}_5$ which restricts to the Horrocks–Mumford bundle.

All the problems in this section are certainly very hard. In order not to depress the reader too much let's conclude with a more accessible problem posed by L. Ein.

PROBLEM 28. Classify all n-bundles E on $\mathbf{P}_n$, generated by sections and $c_n(E) = 1$.

If there will be more than the obvious ones ($\mathcal{O}(1)^{\oplus n}$; $T_{\mathbf{P}_n}(-1)$, $\Omega_{\mathbf{P}_n}(1)$ for n even) this would lead to new Cremona transformations cf. [ESB].

6. Chern classes

Bogomolov [Bo] proved an important inequality for the Chern classes of semi-stable rank-r bundles E on surfaces:

$$(r-1)c_1(E)^2-2rc_2(E)\leqslant 0.$$

For bundles on n-dimensional compact Kähler manifolds (X, ω_X) admitting a Hermite–Einstein metric this inequality is still true [L2], [K].

$$((r-1)c_1(E)^2-2rc_2(E))\omega_X^{n-2}\leqslant 0.$$

Since stable bundles admit such metrics [Do3], [UY] this inequality is also true for stable bundles on higher dimensional base manifolds.

A long time ago Barth has asked the following question:

PROBLEM 29. Does there exist an analogous homogeneous inequality involving higher Chern classes?

It follows from the estimates given in [S2] that the answer is no for rank $\leqslant 3$ or dim $\leqslant 3$. Hence the first case to consider would be 4-bundles on 4-folds.

A relevant homogeneous expression would be

$$12c_4-3c_1c_3+c_2^2.$$

Bogomolov [Bo] has shown that the cotangent bundle Ω_X^1 of a surface of general type is semi-stable (with respect to K_X). Hence $c_1(X)^2-4c_2(X)\leqslant 0$. Miyaoka [Mi] improved this to $c_1(X)^2\leqslant 3c_2(X)$ and Yau [Y1], [Y2] has generalized this inequality to n-dimensional projective manifolds with ample canonical bundle K_X

$$(nc_1(X)^2-2(n+1)c_2(X))\cdot K_X^{n-2}\leqslant 0.$$

Observe that this is exactly Bogomolov's inequality for a rank-$(n+1)$-bundle E having Chern classes $c_1(E)=c_1(X)$, $c_2(E)=c_2(X)$. Hence it seems natural to ask if the Miyaoka–Yau inequality can be deduced from Bogomolov's inequality.

PROBLEM 30. Let X be a n-dimensional compact manifold with K_X ample. Does there exist a semi-stable bundle E of rank $(n+1)$ such that $c_1(E)=c_1(X)$ and $c_2(E)=c_2(X)$?

Tai [T] has proved inequalities for the Chern classes of smooth complete intersections in $\mathbf{P}_n$. As a further test towards the Hartshorne conjecture I propose the following

PROBLEM 31. Let $X\subset \mathbf{P}_n$ be a smooth subvariety with $\dim X>\frac{2}{3}n$. Are the inequalities of Tai for the Chern classes of X satisfied?

7. Metrics

Vector bundles admitting a Hermite–Einstein metric [K] are always μ-semi-stable [K], [Lü3]. The converse — known as the Hitchin–Kobayashi conjecture — has been solved recently by Donaldson [Do3] for projective manifolds and by Uhlenbeck–Yau [UY] for compact Kähler manifolds.

The Hermite–Einstein metric on a stable bundle is essentially unique. On $\mathbf{P}_n$ stable bundles E are often given by monads, i.e. there is a complex

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of vector bundles and vector bundle maps such that β is an epimorphism and α a monomorphism of bundles such that $E \cong \ker \beta / \operatorname{im} \alpha$. In many cases B has an obvious Hermite–Einstein metric. This metric induces one on E and it is natural to ask if this metric is Hermite–Einstein. Lübke [Lü1] has shown this to be true for the Null-correlation bundles on $\mathbf{P}_3$.

PROBLEM 32. Is the Hermite–Einstein metric on the Horrocks–Mumford bundle the one induced by this procedure?

Note that the Horrocks–Mumford bundle belongs to a monad

$$0 \to \mathcal{O}_{\mathbf{P}_4}(-1)^{\oplus 5} \to \Omega^2_{\mathbf{P}_4}(2)^{\oplus 2} \to \mathcal{O}_{\mathbf{P}_4}^{\oplus 5} \to 0.$$

Let us conclude this section by asking a rather vague question due to K. Hulek.

PROBLEM 33. Do the moduli spaces $M_{\mathbf{P}_2}(r, c_1, c_2)$ admit a natural metric?

Such a metric — if it exists — should serve to compute the q-convexity of $M_{\mathbf{P}_2}$, cf. Problem 23.

8. Miscellaneous problems

In this section we collect some problems which do not fit exactly into one of the preceding sections but are either closely related to the above mentioned problems or are so interesting that I could not refrain from mentioning them.

PROBLEM 34. Classify smooth 3-folds in $\mathbf{P}_6$ and 4-folds in $\mathbf{P}_7$ up to a reasonable degree.

The classification of 3-folds in $\mathbf{P}_6$ is known for degree $\leqslant 6$ [I1], [Fu]. It is expected that one can get to higher degrees for 4-folds in $\mathbf{P}_7$. In analogy with the Horrocks–Mumford torus in $\mathbf{P}_4$, Hulek [H2] has posed the following

PROBLEM 35. Are there any abelian surfaces in $\mathbf{P}_1 \times \mathbf{P}_3$?

Mori [Mo] has verified Hartshorne's conjecture on characterizing $\mathbf{P}_n$ by the ampleness of the tangent bundle. The following characterization problem of the quadric has been brought to my attention by L. Ein.

PROBLEM 36. Let X be a n-dimensional compact manifold, $n \geqslant 3$, such that $\bigwedge^2 T_X$ is ample. Show that either $X \cong \mathbf{P}_n$ or $X \cong Q_n$, $Q_n \subset \mathbf{P}_{n+1}$ being a smooth hyperquadric.

Note the wonderful differential geometric characterization of compact Kähler manifolds of nonnegative holomorphic bisectional curvature due to Mok [Mk]. As a last problem we recall one form Hirzebruch's famous problem list [Hi].

PROBLEM 37. Let X be a compact complex surface diffeomorphic to $S^2 \times S^2$. Is X then necessarily a Hirzebruch surface?

One would have to show that there is no projective surface of general type diffeomorphic to $S^2 \times S^2$. Hence this problem is a special case of a question due to Van de Ven, if the Kodaira-dimension is a diffeomorphism invariant. In a few special cases this problem has been affirmatively answered [B0], [Do4], [FM1], [FM2], [L0], [OA1], [OV2], [O5], [Y1]. It is a pleasure to see that moduli spaces of rank 2-bundles yield diffeomorphism-invariants.

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Added in proof (April 1990)

1) F. Serrano (*Divisors of bielliptic surfaces and embeddings in* $\mathbf{P}_4$, preprint) has constructed another class of surfaces in $\mathbf{P}_4$ with irregularity $q = 1$ (certain bielliptic surfaces).

2) Problem 2 has been solved in the meantime by Basili and C. Peskine (*Décomposition du fibré normal des surfaces lisses de* $\mathbf{P}_4$ *et structures doubles sur les solides de* $\mathbf{P}_5$, preprint).

3) Problem 4 is essentially solved now due to the efforts of Alexander, Aure, Ranestad and Decker, Schreyer who showed existence of some "potential" surfaces. For degree 9 see: A. Aure, K. Ranestad, *The smooth surfaces of degree* 9 *in* $\mathbf{P}^4$, preprint.

4) Problem 9 is completely solved by M. Beltrametti, A. Sommese and M. Schneider, *Threefolds of degree* 9 *and* 10 *in* $\mathbf{P}^5$, preprint.

Moreover the classification in degree 11 is almost complete (*Threefolds of degree* 11 *in* $\mathbf{P}^5$, preprint).

5) A. Alzati and G. Ottaviani have a contribution to Problem 11, *A linear bound on the t-normality of codimension two subvarieties of* $\mathbf{P}^n$, to appear in Crelle's Journal, and *A vanishing theorem for the ideal sheaf of codimension two subvarieties of* $\mathbf{P}^n$, preprint.

6) Problem 36 has been solved in dimension 3 by J. Wiśniewski by using classification of Fano 3-folds (*Length of extremal rays and generalized adjunction*, Math. Z. 200 (1989). 409–427.