POLSKA AKADEMIA NAUK • INSTYTUT MATEMATYCZNY

S.5133
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The elementary theory of distributions (II)

WARSZAWA 1961
PAŃSTWOWE WYDAWNICTWO NAUKOWE

S. 7133

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1816

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

Introduction

This paper (called here ETD II) contains an introduction to the theory of distributions of several variables. The case of a single variable was the subject of "The elementary theory of distributions, I" (called here ETD I) (*). For teaching reasons, particularly as regards beginners, it is advisible to read ETD I before ETD II. However, the exposition in ETD II is complete and does not presuppose any knowledge of ETD I.

The main idea is the same in the two papers, but some modifications introduced in the case of several variables are of striking advantage. In ETD I, distributions are defined, roughly speaking, as limits of sequences of continuous functions. The same could be done for several variables, but this would cause complications due to the necessity of introducing a superflous auxiliary notion of generalized derivatives of continuous functions. That notion can be avoided by using infinitely derivable functions or polynomials. But polynomials have no local properties, which — as we have verified experimentally — spoils the elegance of the theory. Thus we have decided on infinitely derivable functions as the starting point ot the theory.

Another modification is that in ETD I (except for the last Section) we dealt with distributions of finite order, while in ETD II this restriction is dropped.

ETD II contains a theory of elementary operations on distributions, such as addition, multiplication, derivation, substitution. It is worth remarking that substitution is presented here more generally than in earlier papers. Other operations, as the integral, convolution, the Fourier transform, will be introduced in ETD III.

The abbreviation "iff" is used instead of "if and only if".

(*) Rozprawy Matematyczne XII, Warszawa 1957.

§ 1. Terminology and notation

Given two systems of finite or infinite numbers

$$a = (\alpha_1, \ldots, \alpha_q), \quad b = (\beta_1, \ldots, \beta_q),$$

we write

$$a < b,$$

iff

$$\alpha_j < \beta_j \quad \text{for} \quad j = 1, \ldots, q.$$

Similarly we write

$$a \leqslant b,$$

iff

$$\alpha_j \leqslant \beta_j \quad \text{for} \quad j = 1, \ldots, q.$$

If real numbers $\xi_1, \ldots, \xi_q$ are finite, then

$$x = (\xi_1, \ldots, \xi_q)$$

can be interpreted as a point of the q-dimensional Euclidean space.

The above convention enables us to denote the q-dimensional open intervals

$$\alpha_j < \xi_j < \beta_j \quad (j = 1, \ldots, q)$$

by

$$a < x < b,$$

just as in the one-dimensional case. Similarly, if α_j and β_j are finite, then the q-dimensional closed interval

$$\alpha_j \leqslant \xi_j \leqslant \beta_j \quad (j = 1, \ldots, q)$$

will be denoted by

$$a \leqslant x \leqslant b.$$

Unless the contrary is explicitly stated, by an interval we always understand a bounded interval. Usually the word "interval" means "open interval". An interval $a < x < b$ is said to be *inside* an open set O iff the closed interval $a \leqslant x \leqslant b$ is contained in O.

We adopt the usual notation

$$x+y = (\xi_1+\eta_1, \ldots, \xi_q+\eta_q), \quad x-y = (\xi_1-\eta_1, \ldots, \xi_q-\eta_q),$$

$$\lambda x = (\lambda\xi_1, \ldots, \lambda\xi_q), \quad |x| = \sqrt{\xi_1^2 + \ldots + \xi_q^2},$$

where $y = (\eta_1, \ldots, \eta_q)$ and λ is a number.

Functions defined on subsets of a q-dimensional space will usually be denoted by symbols $\varphi(x)$, $f(x)$, $F(x)$, ... instead of $\varphi(\xi_1, \ldots, \xi_q)$, $f(\xi_1, \ldots, \xi_q)$, $F(\xi_1, \ldots, \xi_q)$, ... All functions under consideration are defined on open subsets of the q-dimensional Euclidean space, unless the contrary is explicitely stated.

Let $F(x)$ be a continuous function in an interval I, let $x_0 = (\xi_{01}, \ldots, \xi_{0q})$ be a fixed point in this interval, and let $k = (\varkappa_1, \ldots, \varkappa_q)$ be a system of non-negative integers. Integrating $F(x)$ first $\varkappa_1$ times in ξ_1, then $\varkappa_2$ times in ξ_2, and so on, we obtain the iterated integral of order k

$$\int_{\xi_{0q}}^{\xi_q} d\tau_{q\varkappa_q} \ldots \int_{\xi_{0q}}^{\tau_{q2}} d\tau_{q1} \ldots \int_{\xi_{01}}^{\xi_1} d\tau_{1\varkappa_1} \ldots \int_{\xi_{01}}^{\tau_{11}} d\tau_{11} F(\tau_{11}, \ldots, \tau_{q1}).$$

This integral will be denoted shortly by

$$\int_{x_0}^{x} F(t)\, dt^k,$$

or, in the particular case $k = (1, \ldots, 1)$, by

$$\int_{x_0}^{x} F(t)\, dt.$$

Observe the obvious formulae

$$\int_{x_0}^{x} \lambda f(t)\, dt^k = \lambda \int_{x_0}^{x} f(t)\, dt^k \qquad (\lambda \text{ number}),$$

$$\int_{x_0}^{x} \big(f(t) + g(t)\big)\, dt^k = \int_{x_0}^{x} f(t)\, dt^k + \int_{x_0}^{x} g(t)\, dt^k,$$

$$\frac{\partial^{\varkappa_1 + \ldots + \varkappa_q}}{\partial \xi_1^{\varkappa_1} \ldots \partial \xi_q^{\varkappa_q}} \int_{x_0}^{x} f(t)\, dt^k = f(x).$$

Infinitely derivable functions will be called *smooth functions*. If $\varphi(x)$ is a smooth function and $k = (\varkappa_1, \ldots, \varkappa_q)$ is a system of non-negative integers, then by its *derivative of order* k we understand the function

$$\varphi^{(k)}(x) = \frac{\partial^{\varkappa_1 + \ldots + \varkappa_q}}{\partial \xi_1^{\varkappa_1} \ldots \partial \xi_q^{\varkappa_q}} \varphi(\xi_1, \ldots, \xi_q).$$

Generally we understand by *order* any sequence $k = (\varkappa_1, \ldots, \varkappa_q)$ of non-negative integers. It will be convenient to use also the notation

$$\begin{array}{ll} e_1 = (1, 0, \ldots, 0), & \mathit{0} = (0, 0, \ldots, 0), \\ e_2 = (0, 1, \ldots, 0), & \mathit{1} = (1, 1, \ldots, 1), \\ \ldots\ldots\ldots\ldots & \mathit{2} = (2, 2, \ldots, 2), \\ e_q = (0, 0, \ldots, 1), & \ldots\ldots\ldots\ldots \end{array}$$

Instead of $\mathit{0}$ one may also write 0, which does not lead to a misunderstanding.

§ 2. Uniform and almost uniform convergence

Given any set I, we say that a sequence of functions $f_n(x)$ *converges uniformly* in I to $f(x)$ and we write

$$f_n(x) \rightrightarrows f(x) \quad \text{in } I,$$

iff the function $f(x)$ is defined on I and, for any given number $\varepsilon > 0$, there is an index n_0 such that for every $n > n_0$ the function $f_n(x)$ is defined on the whole set I and satisfies there the inequality $|f_n(x) - f(x)| < \varepsilon$. Thus for initial indexes n the functions $f_n(x)$ need not be defined in I.

We write

$$f_n(x) \rightrightarrows \quad \text{in } I,$$

iff there exists a function $f(x)$ such that $f_n(x) \rightrightarrows f(x)$ in I. We shall use this symbol when it is not necessary to evidence the limit function.

We write

$$f_n(x) \rightrightarrows\leftleftarrows g_n(x) \quad \text{in } I,$$

iff both sequences $f_n(x)$ and $g_n(x)$ converge uniformly on I to the same limit.

A sequence $f_n(x)$ is said to converge to $f(x)$ *almost uniformly* in an open set O iff $f_n(x) \rightrightarrows f(x)$ on every interval I inside O. The limit function is defined in the whole set O, but according to the definition none of the functions $f_n(x)$ need be defined in the whole set O. If O_n is the open set where $f_n(x)$ is defined, then for every interval I inside O there exists an integer n_0 such that I is inside O_n for $n > n_0$.

§ 3. Fundamental sequences of smooth functions

Let O be an open set in the q-dimensional space.

A sequence $\varphi_n(x)$ of smooth functions is said to be *fundamental* in O iff for every interval I inside O there exists an order k and smooth functions $\Phi_n(x)$ such that

(F_1) $\quad \Phi_n^{(k)}(x) = \varphi_n(x)$,

(F_2) $\quad \Phi_n(x) \rightrightarrows$ in I.

The order k and the sequence $\Phi_n(x)$ depend, in general, on I. According to the definition, none of the function $\varphi_n(x)$ need be defined in the whole set O. If O_n is the open set where $\varphi_n(x)$ is defined, then for every interval I inside O there exists an index n_0 such that I is inside O_n for $n > n_0$. The functions $\Phi_n(x)$ are defined in I for $n > n_0$ and satisfy there (F_1) and (F_2).

It follows immediately from the definition (for $k = 0$) that:

3.1. *Every sequence of smooth functions convergent almost uniformly in O is fundamental.*

Differentiating (F_1) m times, we find that:

3.2. *If $\varphi_n(x)$ is a fundamental sequence, so is $\varphi_n^{(m)}(x)$.*

It is useful to observe that the order k which occurs in the condition (F_1) can, if necessary, be replaced by any greater order. This results from the following statement:

3.3. *If $\Phi_n(x)$ satisfies (F_1) and (F_2) and if $l \geqslant k$, then the sequence of smooth functions*

$$\tilde{\Phi}_n(x) = \int_{x_0}^{x} \Phi_n(t)\,dt^{l-k} \qquad (x_0 \text{ in } I)$$

satisfies also (F_1) and (F_2), where k is replaced by l.

Observe also that:

3.4. *If a sequence $\varphi_n(x)$ is fundamental in every interval I inside O, it is fundamental in O.*

For, if I is any interval inside O, there is an interval I' inside O such that I is inside I'. Since the sequence $\varphi_n(x)$ is fundamental in I', there are smooth functions $\Phi_n(x)$ and an order k such that (F_1) and (F_2) hold in I.

§ 4. The definition of distributions

We say that two sequences $\varphi_n(x)$ and $\psi_n(x)$ fundamental in O are *equivalent* in O and we write

$$\varphi_n(x) \sim \psi_n(x),$$

iff the interlaced sequence

$$\varphi_1(x),\ \psi_1(x),\ \varphi_2(x),\ \psi_2(x),\ \ldots$$

is fundamental.

Evidently, the following condition is necessary and sufficient that $\varphi_n(x)$ and $\psi_n(x)$ be equivalent: For every interval I inside O there exist sequences of smooth functions $\Phi_n(x)$ and $\Psi_n(x)$ and an order k such that

(E_1) $\quad \Phi_n^{(k)}(x) = \varphi_n(x) \quad$ and $\quad \Psi_n^{(k)}(x) = \psi_n(x)$,

(E_2) $\quad \Phi_n(x) \rightrightarrows\!\leftleftarrows \Psi_n(x)$ in I.

The sequences $\Phi_n(x)$ and $\Psi_n(x)$ and the order k depend, in general, on I.

From 3.3 it follows that

4.1. *The order k in the condition (E_1) can, if necessary, be replaced by any greater order l.*

It is easy to see that the relation $\sim$ is reflexive and symmetric, i.e. that

1° $\varphi_n(x) \sim \varphi_n(x)$,

2° $\varphi_n(x) \sim \psi_n(x)$ implies $\psi_n(x) \sim \varphi_n(x)$.

It is also transitive, i.e.

3° $\varphi_n(x) \sim \psi_n(x)$ and $\psi_n(x) \sim \vartheta_n(x)$ imply $\varphi_n(x) \sim \vartheta_n(x)$.

In fact, the supposition in 3° means that, for every interval I inside O, there exist an order k and smooth functions $\Phi_n(x)$ and $\Psi_n(x)$ satisfying (E_1) and (E_2), and there exist an order l and smooth functions $\tilde{\Psi}_n(x)$, $\Theta_n(x)$ such that

$$\tilde{\Psi}_n^{(l)}(x) = \psi_n(x), \quad \Theta_n^{(l)}(x) = \vartheta_n(x), \quad \tilde{\Psi}_n(x) \rightrightarrows\leftleftarrows \Theta_n(x).$$

By 4.1 we may assume that $k = l$. The sequences $\Phi_n(x)$ and $\tilde{\Theta}_n(x) = \Psi_n(x) - \tilde{\Psi}_n(x) + \Theta_n(x)$ converge uniformly in I to the same limit, and $\Phi_n^{(k)}(x) = \varphi_n(x)$, $\tilde{\Theta}_n^{(k)}(x) = \vartheta_n(x)$, which proves that $\varphi_n(x) \sim \vartheta_n(x)$.

The relation $\sim$ being reflexive, symmetric and transitive, the set of all sequences fundamental in O gets decomposed into disjoint classes (equivalence classes of the relation $\sim$) such that two fundamental sequences are in the same class iff they are equivalent. These equivalence classes are called *distributions* (defined in O). Thus the notion of distribution is obtained by identification of equivalent fundamental sequences.

The distribution determined by a fundamental sequence $\varphi_n(x)$, i.e. the class of all fundamental sequences equivalent to $\varphi_n(x)$ will be denoted by the symbol $[\varphi_n(x)]$. Two sequences $\varphi_n(x)$ and $\psi_n(x)$ fundamental in O determine the same distribution iff they are equivalent. Thus

$$[\varphi_n(x)] = [\psi_n(x)] \quad \text{iff} \quad \varphi_n(x) \sim \psi_n(x).$$

Distributions will be denoted by $f(x)$, $g(x)$, etc., as functions. It should be emphasized that this notation is purely symbolic and, in general, it does not allow us to substitute points for the variable x.

§ 5. Multiplication by a number

The operation $\lambda\varphi(x)$ of multiplication of a function $\varphi(x)$ by a number λ has the following property:

1° If $\varphi_n(x)$ is a fundamental sequence, so is $\lambda\varphi_n(x)$.

This property enables us to extend the operation onto arbitrary distributions $f(x) = [\varphi_n(x)]$ by assuming

$$\lambda f(x) = [\lambda\varphi_n(x)].$$

In order to verify the uniqueness of the product $\lambda\varphi(x)$ we have to show that the product does not depend on the choice of the fundamental sequence $\varphi_n(x)$. In other words:

2° If $\varphi_n(x) \sim \bar{\varphi}_n(x)$, then $\lambda\varphi_n(x) \sim \lambda\bar{\varphi}_n(x)$.

In fact, the sequence

$$\varphi_1(x),\ \bar{\varphi}_1(x),\ \varphi_2(x),\ \bar{\varphi}_2(x),\ \ldots$$

is fundamental. Thus by 1° so is the sequence

$$\lambda\varphi_1(x),\ \lambda\bar{\varphi}_1(x),\ \lambda\varphi_2(x),\ \lambda\bar{\varphi}_2(x),\ \ldots,$$

which implies the assertion.

§ 6. Addition

The operation $\varphi(x)+\psi(x)$ of addition of two functions $\varphi(x)$ and $\psi(x)$ has the following property:

1° If $\varphi_n(x)$ and $\psi_n(x)$ are fundamental sequences, so is $\varphi_n(x)+\psi_n(x)$.

In order to prove 1°, suppose that, for any interval I inside O, there are orders k and l and functions $\Phi_n(x)$, $\Psi_n(x)$ such that

$$\Phi^{(k)}(x) = \varphi_n(x), \qquad \Phi_n(x) \rightrightarrows,$$
$$\Psi^{(l)}(x) = \psi_n(x), \qquad \Psi_n(x) \rightrightarrows.$$

We can assume that $k = l$ since each of the orders k and l can be arbitrarily enlarged (see 3.3). Since

$$\big(\Phi_n(x)+\Psi_n(x)\big)^{(k)} = \varphi_n(x)+\psi_n(x), \qquad \Phi_n(x)+\Psi_n(x) \rightrightarrows,$$

the sequence $\varphi_n(x)+\psi_n(x)$ is fundamental.

Property 1° enables us to extend the addition onto arbitrary distributions $f(x) = [\varphi_n(x)]$ and $g(x) = [\psi_n(x)]$ by assuming

$$f(x)+g(x) = [\varphi_n(x)+\psi_n(x)].$$

The sum so defined is unique, for it does not depend on the choice of fundamental sequences $\varphi_n(x)$ and $\psi_n(x)$. In other words:

2° If $\varphi_n(x) \sim \bar{\varphi}_n(x)$ and $\psi_n(x) \sim \bar{\psi}_n(x)$, then

$$\varphi_n(x)+\psi_n(x) \sim \bar{\varphi}_n(x)+\bar{\psi}_n(x).$$

In fact, the sequences

$$\varphi_1(x),\ \bar{\varphi}_1(x),\ \varphi_2(x),\ \bar{\varphi}_2(x),\ \ldots,$$
$$\psi_1(x),\ \bar{\psi}_1(x),\ \psi_2(x),\ \bar{\psi}_2(x),\ \ldots$$

are fundamental. By 1° so is the sequence

$$\varphi_1(x)+\psi_1(x),\ \bar{\varphi}_1(x)+\bar{\psi}_1(x),\ \varphi_2(x)+\psi_2(x),\ \bar{\varphi}_2(x)+\bar{\psi}_2(x),\ \ldots,$$

which implies the assertion.

§ 7. Regular operations

Multiplication by a given number λ is an operation on a single function (or distribution). Addition is an operation on two functions (or distributions). Generally, we may consider operations on an arbitrary number of functions and extend them onto distributions. The method of extension is similar. It would be tedious and unnecessary to repeat the argument in every particular case. In this Section it will be shown generally that the extension is feasible for a large class of operations.

Denote by

$$A\big(\varphi(x), \psi(x), \ldots\big)$$

an operation on a finite number of functions $\varphi(x)$, $\psi(x)$, ... Suppose that this operation has the following property:

1° If $\varphi_n(x), \psi_n(x), \ldots$ are fundamental sequences, so is

$$A\big(\varphi_n(x), \psi_n(x), \ldots\big).$$

Such an operation is extended onto distributions $f(x) = [\varphi_n(x)]$, $g(x) = [\varphi_n(x)], \ldots$, by assuming

$$A\big(f(x), g(x), \ldots\big) = \big[A\big(\varphi_n(x), \psi_n(x), \ldots\big)\big].$$

The extension is unique, i.e. it does not depend on the choice of the fundamental sequences $\varphi_n(x)$, $\psi_n(x)$, ... In other words:

2° If

(1) $$\varphi_n(x) \sim \bar{\varphi}_n(x), \qquad \psi_n(x) \sim \bar{\psi}_n(x), \qquad \ldots,$$

then

(2) $$A\big(\varphi_n(x), \psi_n(x), \ldots\big) \sim A\big(\bar{\varphi}_n(x), \bar{\psi}_n(x), \ldots\big).$$

In fact, by supposition, the sequences

$$\varphi_1(x),\ \bar{\varphi}_1(x),\ \varphi_2(x),\ \bar{\varphi}_2(x), \ldots,$$

$$\psi_1(x),\ \bar{\psi}_1(x),\ \psi_2(x),\ \bar{\psi}_2(x), \ldots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

are fundamental. By 1° so is the sequence

$$A\big(\varphi_1(x), \psi_1(x), \ldots\big),\ A\big(\bar{\varphi}_1(x), \bar{\psi}_1(x), \ldots\big),\ A\big(\varphi_2(x), \psi_2(x), \ldots\big), \quad \ldots,$$

which proves the assertion.

All operations $A\big(\varphi(x), \psi(x), \ldots\big)$ with property 1° will be called *regular operations*. Every regular operation defined on smooth functions is extended automatically onto distributions. This extension is always unique.

Multiplication by a number and addition are regular operations, as we have seen in Sections 5 and 6.

§ 8. Subtraction, translation, derivation

We are now going to give further examples of regular operations.

Subtraction. The subtraction $\varphi(x)-\psi(x)$ is a regular operation. In fact, if $\varphi_n(x)$ and $\psi_n(x)$ are fundamental sequences, so is $\varphi_n(x)-\psi_n(x)$. The proof is analogous to that in Section 6. Thus we define the difference of two distributions $f(x) = [\varphi_n(x)]$ and $g(x) = [\psi_n(x)]$ by the formula

$$f(x)-g(x) = [\varphi_n(x)-\psi_n(x)].$$

Translation. The translation $\varphi(x+h)$ is a regular operation. More exactly, if $\varphi_n(x)$ is a fundamental sequence in the open set O, then $\varphi_n(x+h)$ is a fundamental sequence in the translated set O_h, which consists of all points x such that $x+h$ is in O. Thus, if $f(x) = [\varphi_n(x)]$ is a distribution defined in O, then

$$f(x+h) = [\varphi_n(x+h)]$$

is a distribution defined in O_h.

Derivation. The derivation $\varphi^{(m)}(x)$ of an arbitrary order m is a regular operation. In fact, by 3.2, if $\varphi_n(x)$ is a fundamental sequence, so is $\varphi^{(m)}(x)$. Thus we define the derivative of order m of any distribution $f(x) = [\varphi_n(x)]$ by assuming

$$f^{(m)}(x) = [\varphi_n^{(m)}(x)].$$

Evidently:

8.1. *Each distribution has derivatives of all orders.*

Property 8.1 is of striking advantage in calculations with distributions, and makes them easier and more elegant than the calculations in Classical Analysis.

§ 9. Multiplication of a distribution by a smooth function

The multiplication $\varphi(x)\psi(x)$, when considered as an operation on two functions $\varphi(x)$ and $\psi(x)$, is not regular, for if the sequences $\varphi_n(x)$ and $\psi_n(x)$ are fundamental, their product $\varphi_n(x)\psi_n(x)$ need not be fundamental.

However, multiplication may also be considered as an operation on a single function, the other factor being kept fixed. Denote by $\omega(x)$ that fixed factor. We shall prove that, if $\omega(x)$ is a smooth function, multiplication $\omega(x)\varphi(x)$ is a regular operation on $\varphi(x)$. In other words, if the sequence $\varphi_n(x)$ is fundamental, so is $\omega(x)\varphi_n(x)$.

In fact, since $\varphi_n(x)$ is fundamental, for every *interval I inside O* there exist an order k and smooth functions $\Phi_n(x)$ such that

$$\Phi_n^{(k)}(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \rightrightarrows \quad \text{in } I.$$

For every order m and every smooth function $\omega(x)$ the sequence $\omega(x)\Phi_n^{(m)}(x)$ is fundamental in I. The proof is by induction. The case $m = 0$ follows from 3.1. If the sequence is fundamental for some m, then the sequence is also fundamental for $m+e_j$, since

$$\omega(x)\Phi_n^{(m+e_j)}(x) = \big(\omega(x)\Phi_n^{(m)}(x)\big)^{(e_j)} - \omega^{(e_j)}(x)\Phi_n^{(m)}(x)$$

and the right side is the difference of two sequences which are fundamental by 3.2 and the induction hypothesis. For $m = k$ we find that $\omega(x)\varphi_n(x)$ is fundamental in I. The interval I being arbitrary, the sequence $\omega(x)\varphi_n(x)$ is fundamental in the whole set O, on account of 3.4. Thus we have proved that multiplication by a smooth function $\omega(x)$ is a regular operation.

According to the general method, we define the product of an arbitrary distribution $f(x) = [\varphi_n(x)]$ by a smooth function $\omega(x)$ by means of the formula

$$\omega(x)f(x) = [\omega(x)\varphi_n(x)].$$

Occasionally we shall also write $f(x)\omega(x)$ instead of $\omega(x)f(x)$.

Observe that if $\omega(x)$ is a constant function, then the multiplication just defined coincides with that from Section 5.

§ 10. Substitution

Let $\sigma(x)$ be a fixed smooth function in an open q-dimensional set O such that

$$(1) \qquad \left(\frac{\partial\sigma(x)}{\partial\xi_1}\right)^2 + \ldots + \left(\frac{\partial\sigma(x)}{\partial\xi_q}\right)^2 \neq 0 \quad \text{in } O;$$

suppose that the values of $\sigma(x)$ are in an open set O' of real numbers y. The substitution

$$\varphi\big(\sigma(x)\big)$$

is a regular operation on $\varphi(y)$ ($\sigma(x)$ being fixed). More precisely, we shall show that if $\varphi_n(y)$ is fundamental in O', so is $\varphi_n\big(\sigma(x)\big)$ in O.

Let I be any interval inside O. The function $\sigma(x)$ maps I onto an interval I' inside O'.

Observe first that if, for some smooth functions $\Phi_n(y)$, the sequence $\Phi_n\big(\sigma(x)\big)$ is fundamental in I, so is the sequence $\Phi_n'\big(\sigma(x)\big)$. In fact, from

$$\frac{\partial}{\partial\xi_j}\Phi_n\big(\sigma(x)\big) = \Phi_n'\big(\sigma(x)\big)\frac{\partial}{\partial\xi_j}\sigma(x) \qquad (j = 1, \ldots, q)$$

we find by algebraic calculations

$$(2)\quad \Phi_n'(\sigma(x)) = \frac{\frac{\partial}{\partial \xi_1}\Phi_n(\sigma(x))\cdot\frac{\partial}{\partial \xi_1}\sigma(x) + \ldots + \frac{\partial}{\partial \xi_q}\Phi_n(\sigma(x))\cdot\frac{\partial}{\partial \xi_q}\sigma(x)}{\left(\frac{\partial\sigma(x)}{\partial\xi_1}\right)^2 + \ldots + \left(\frac{\partial\sigma(x)}{\partial\xi_q}\right)^2}.$$

Here the derivatives $\frac{\partial}{\partial \xi_i}\Phi_n(\sigma(x))$ form fundamental sequences, by 3.2. Also the products of those derivatives by smooth functions $\frac{\partial}{\partial \xi_j}\sigma(x)$ are fundamental sequences, since multiplication by smooth functions is a regular operation. Thus the numerator in (2) is a fundamental sequence, as the sum of fundamental sequences. Finally, the whole fraction on the right side represents a fundamental sequence, for it can be represented as the product of the numerator by the inverse of the denominator.

By induction, if $\Phi_n(\sigma(x))$ is fundamental, so is $\Phi_n^{(k)}(\sigma(x))$ for every non-negative integer k.

Now, let $\Phi_n(y)$ be a sequence of smooth functions such that, for an integer $k \geqslant 0$,

$$\Phi_n^{(k)}(y) = \varphi_n(y) \quad \text{and} \quad \Phi_n(y) \rightrightarrows \quad \text{in } I'.$$

Then $\Phi_n(\sigma(x)) \rightrightarrows$ in I. Thus $\Phi_n(\sigma(x))$ is a fundamental sequence and so is $\Phi_n^{(k)}(\sigma(x))$, i.e. $\varphi_n(\sigma(x))$. Since the interval I is arbitrary, the sequence $\varphi_n(\sigma(x))$ is fundamental in the whole set O, on account of 3.4.

Thus we have proved that the substitution of a given smooth function $\sigma(x)$ satisfying (1) is a regular operation. According to the general method we define the substitution of $\sigma(x)$ into an arbitrary distribution $f(y) = [\varphi_n(y)]$ in O' by the formula

$$f(\sigma(x)) = [\varphi_n(\sigma(x))].$$

The distribution $f(y)$ is one-dimensional, i.e. is defined in a one-dimensional set, the distribution $f(\sigma(x))$ is q-dimensional, i.e. is defined in a q-dimensional set.

In Section 25 we shall also consider the more general case where the outer distribution $f(y)$ is p-dimensional, $1 \leqslant p \leqslant q$.

§ 11. Product of distributions with separated variables

The product of two smooth functions $\varphi(\xi_1, \ldots, \xi_q)$, $\psi(\eta_1, \ldots, \eta_r)$ can be written in the form $\varphi(x)\psi(y)$, where $x = (\xi_1, \ldots, \xi_q)$, $y = (\eta_1, \ldots, \eta_r)$. If $\varphi(x)$ is defined in an open subset O' of the q-dimensional space, and $\psi(y)$ is defined in an open subset O'' of the r-dimensional space, then the product $(x)\varphi(y)$ is defined in the open set O of all points $(\xi_1, \ldots, \xi_q, \eta_1, \ldots, \eta_r)$ such that $(\xi_1, \ldots, \xi_q)$ is in O' and $(\eta_1, \ldots, \eta_r)$ is in O''.

It is evident that the product $\varphi(x)\psi(y)$ is a regular operation on two functions $\varphi(x)$ and $\psi(y)$. It is therefore extended onto arbitrary distributions $f(x) = [\varphi_n(x)]$, $g(y) = [\psi_n(y)]$ by assuming

$$f(x)\,g(y) = [\varphi_n(x)\,\psi_n(y)].$$

The distributions $f(x)$ and $g(y)$ being defined in O' and O'' respectively, their product is defined in O. The distributions $f(x)$ and $g(y)$ are q-dimensional and r-dimensional, their product is $(q+r)$-dimensional.

§ 12. Convolution by a smooth function vanishing outside an interval

First we shall show that there exist smooth functions vanishing outside a given interval I, but non-vanishing everywhere.

The function

$$\Omega(\xi) = \begin{cases} 0 & \text{for} \quad \xi \leqslant 0, \\ e^{-1/\xi} & \text{for} \quad \xi > 0 \end{cases}$$

is smooth, positive for $\xi > 0$, and vanishing for $\xi \leqslant 0$. The product

$$\Omega(\xi - a)\,\Omega(\beta - \xi)$$

is also smooth, positive for $a < \xi < \beta$ and vanishing elsewhere.

For any interval I:

$$a < x < b, \quad a = (a_1, \ldots, a_q), \quad b = (\beta_1, \ldots, \beta_q),$$

we define the function $\Omega_I(x)$ by the following formula:

$$\Omega_I(x) = \prod_{j=1}^{q} \Omega(\xi_j - a_j)\Omega(\beta_j - \xi_j).$$

This function has the required properties: is smooth, positive in I, and vanishing elsewhere.

If $f(x)$ is continuous or locally integrable in an open set O and $\omega(x)$ is continuous everywhere and vanishing outside an interval $a < x < b$, then by the *convolution* of $f(x)$ by $\omega(x)$ we understand the function

$$(1) \qquad f(x) * \omega(x) = \int_a^b f(x-t)\,\omega(t)\,dt,$$

defined in the open set O' of all points x such that the interval

$$x + a < t < x + b$$

is inside O. Convolution (1) can be written in the form

$$\int_{-\infty}^{\infty} f(x-t)\omega(t)\,dt \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(t)\omega(x-t)\,dt,$$

by adopting the convention that the product is equal to zero if one of the factors is equal to zero, no matter whether the second factor is defined or not.

The convolution of a continuous or locally integrable function $f(x)$ by a smooth function $\omega(x)$ (vanishing outside $a < x < b$) is a smooth function and

$$\big(f(x)*\omega(x)\big)^{(m)} = f(x)*\omega^{(m)}(x) \tag{2}$$

for every order m.

If $f(x)$ is smooth, then also

$$\big(f(x)*\omega(x)\big)^{(m)} = f^{(m)}(x)*\omega(x) \tag{3}$$

for every order m.

If $f_n(x) \rightrightarrows f(x)$ in an interval $a_0+a < x < b_0+b$, then

$$f_n(x)*\omega(x) \rightrightarrows f(x)*\omega(x) \tag{4}$$

in $a_0 < x < b_0$. Hence it follows that

12.1. *If $\varphi_n(x)$ is fundamental in O and $\omega(x)$ is smooth then $\varphi_n(x)*\omega(x)$ converges almost, uniformly in O'.*

In fact, let I': $a_0 < x < b_0$ be an interval inside O'; then the interval $I: a_0+a < x < b_0+b$ is inside O. Since $\varphi_n(x)$ is fundamental in O, we have

$$\varphi_n(x) = \Phi_n^{(k)}(x) \qquad \text{and} \qquad \Phi_n(x) \rightrightarrows \qquad \text{in } I.$$

Hence, by (3), (2) and (4),

$$\varphi_n(x)*\omega(x) = \big(\Phi_n(x)*\omega(x)\big)^{(k)} = \Phi_n(x)*\omega^{(k)}(x) \rightrightarrows \qquad \text{in } I'.$$

It follows from 12.1 and 3.1 that convolution by a smooth function vanishing outside an interval is a regular operation.

According to the general method, we define the convolution of an arbitrary distribution $f(x) = [\varphi_n(x)]$ by a smooth function $\omega(x)$, vanishing outside an interval I, by means of the formula

$$f(x)*\omega(x) = [\varphi_n(x)*\omega(x)].$$

Occasionally we shall also write $\omega(x)*f(x)$ instead of $f(x)*\omega(x)$.

§ 13. Calculations with distributions

In calculation various identities are useful, e.g.

$$
\begin{aligned}
\big(\varphi(x)-\psi(x)\big)+\psi(x) &= \varphi(x),\\
\lambda\big(\varphi(x)+\psi(x)\big) &= \lambda\varphi(x)+\lambda\psi(x),\\
\big(\omega(x)\varphi(x)\big)^{(e_j)} &= \omega^{(e_j)}(x)\varphi(x)+\omega(x)\varphi^{(e_j)}(x),\\
\varphi\big(\sigma(x)\big)^{(e_j)} &= \varphi'\big(\sigma(x)\big)\sigma^{(e_j)}(x),\\
\big(\omega(x)*\varphi(x)\big)^{(m)} &= \omega^{(m)}(x)*\varphi(x) = \omega(x)*\varphi^{(m)}(x).
\end{aligned}
\tag{1}
$$

All these formulae and many others can be extended onto distributions. We need not justify the correctness of this extension for each formula separately. We shall give a simple rule which permits to indicate a large class of formulas, valid both for smooth functions and for distributions. The rule is based on the concept of *iterations of operations*. For instance, the expression $\lambda\big(f(x)+g(x)\big)$ is an iteration of addition and multiplication (by the number λ).

Generally, by iteration of operations we understand the expression

$$A\Big(B\big(\varphi(x),\,\psi(x),\,\ldots\big),\;C\big(\chi(x),\,\vartheta(x),\,\ldots\big),\;\ldots\Big),$$

where A, B, C, ... are given operations. In each of the five examples quoted at the beginning of this Section there are equalities between iterations of operations, provided the identity operation $\mathscr{I}\big(\varphi(x)\big) = \varphi(x)$ is admitted. The identity operation is trivially a regular operation. In examples (1) there appear iterations of regular operations only. It follows directly from the definition of regular operations that iterations of regular operations of regular operations are again regular operations.

The meaning of formulas (1) is that the left and the right sides of each equality represent the same operation. All those operations are regular, and thus their extensions onto distributions are unique. This implies that the same formulas hold if we replace smooth functions $\varphi(x)$, $\psi(x)$ by distributions.

Also iterations of second order, i.e. iterations of iterations of regular operations, are regular operations, and so are iterations of an arbitrary finite order, i. e. any finite iterations of regular operations. Thus the following general rule holds:

13.1. *If an equality whose sides are both finite iterations of regular operations holds for smooth functions, then it also holds for arbitrary distributions.*

This rule is not merely of theoretical but above all of practical importance, for it allows to perform calculations on distributions in the same way as on smooth functions, provided all operations occurring in those calculations are regular.

§ 14. Delta-sequences and delta-distribution

If $\Omega_I(x)$ is the function defined in Section 12, the function

$$\omega_I(x) = \gamma^{-1}\Omega_I(x),$$

where

$$\gamma = \int_{-\infty}^{\infty} \Omega_I(x)\,dx$$

is smooth, positive in I, and vanishing elsewhere. Moreover, it has the property

$$\int_{-\infty}^{\infty} \omega_I(x)\,dx = 1.$$

Let a_n be positive numbers such that $a_n \to 0$. There exist smooth functions $\delta_n(x)$, non-negative for $|x| < a_n$ and vanishing elsewhere, such that

$$\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1.$$

The existence of such sequences is ensured by the preceding example. Any sequence $\delta_n(x)$ with the above properties will be called *δ-sequence.*

Every δ-sequence is fundamental. In fact, the sequence

$$\Delta_n(x) = \int_{-\infty}^{x} \delta_n(t)\,dt^2$$

converges uniformly everywhere and $\Delta_n^{(2)}(x) = \delta_n(x)$.

All δ-sequences are equivalent, for the interlaced sequence formed of two δ-sequences is again a δ-sequence.

Thus the δ-sequences determine a distribution

$$\delta(x) = [\delta_n(x)];$$

this is called the q-dimensional Dirac *delta-distribution.* The dimension of $\delta(x)$ is indicated by the dimension of the variable x.

If $\omega(x)$ is a smooth function, then $\omega(x)\delta(x)$ is a fundamental sequence, equivalent to $\omega(0)\delta_n(x)$. In fact, if $\varepsilon > 0$, there exists an index n_0 such that for $n > n_0$

$$|\omega(x) - \omega(0)| < \varepsilon \quad \text{in} \quad -a_n\mathbf{1} < x < a_n\mathbf{1}.$$

Hence

$$\left|\int_{-\infty}^{x} \big(\omega(t) - \omega(0)\big)\delta_n(t)\,dt\right| \leqslant \varepsilon \int_{-\infty}^{\infty} \delta_n(x)\,dx = \varepsilon,$$

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which proves that the integral converges uniformly to 0. Hence $\omega(x)\,\delta_n(x)-\omega(0)\,\delta_n(x)\sim 0$, and consequently

$$\omega(x)\,\delta_n(x)\sim\omega(0)\,\delta_n(x).$$

Since the left and the right side are fundamental sequences for the products $\omega(x)\,\delta(x)$ and $\omega(0)\,\delta(x)$ respectively, we obtain the formula

$$\omega(x)\,\delta(x)=\omega(0)\,\delta(x).$$

14.1. *If $\delta_n(x)$ is a δ-sequence and f is a continuous function in O, then the sequence of smooth functions*

$$f(x)*\delta_n(x)$$

converges to $f(x)$ almost uniformly in O.

In fact, let I be any interval inside O. For every positive number ε there is an index n_0 such that for $n>n_0$

$$|f(x-t)-f(x)|<\varepsilon \quad \text{for } x \text{ in } I \text{ and } -a_n1<t<a_n1.$$

Hence

$$|f(x)*\delta_n(x)-f(x)|\leqslant\int_{-\infty}^{\infty}|f(x-t)-f(x)|\,\delta_n(t)\,dt\leqslant\varepsilon$$

$$\text{for}\quad n\geqslant n_0 \text{ and } x \text{ in } I.$$

This proves that $f(x)*\delta_n(x)$ converges to $f(x)$ almost uniformly in O.

The following generalization of 14.1 is useful:

14.2. *If $\delta_n(x)$ is a δ-sequence and $f_n(x)$ is a sequence of continuous functions, convergent to $f(x)$ almost uniformly in O, then the sequence of smooth functions*

$$f_n(x)*\delta_n(x)$$

converges to $f(x)$ almost uniformly in O.

To prove this, remark that

$$f_n(x)*\delta_n(x)=f(x)*\delta_n(x)+\big(f_n(x)-f(x)\big)*\delta_n(x),$$

where the first member on the right side converges almost uniformly to $f(x)$, by 14.1. It suffices to prove that the sequence

$$\varphi_n(x)=\big(f_n(x)-f(x)\big)*\delta_n(x)$$

converges almost uniformly to 0. In fact, given any interval I inside O and any positive number $\varepsilon>0$, we have, for sufficiently large n,

$$|\varphi_n(x)|\leqslant\varepsilon*\delta_n(x)=\varepsilon\quad \text{in } I.$$

This Section will be completed by a simple remark on the product of delta distributions. The product

$$\delta_n(\xi_1) \dots \delta_n(\xi_q)$$

of one-dimensional δ-sequences is evidently a q-dimensional δ-sequence. Hence, by the definition of the product of distributions with separated variables, we obtain

$$\delta(x) = \delta(\xi_1) \dots \delta(\xi_q) \quad \text{for} \quad x = (\xi_1, \dots, \xi_q).$$

§ 15. Distributions in subsets

Any distribution in the open set O can be interpreted, if necessary, as a distribution in any open subset O', since the functions of any fundamental sequence representing $f(x)$ can be interpreted as functions in the subset. Thus every distribution defined in O is also defined in any open subset O'.

If we write

$$f(x) = g(x) \quad \text{in } O',$$

then we always understand that the open set O' is contained in each of the open sets where the distributions $f(x)$ and $g(x)$ are defined and that $f(x)$ and $g(x)$, when interpreted as distributions in O', are equal.

If we write the mere equality

$$f(x) = g(x),$$

we shall understand, if no particular explanation is added, that the distributions on both sides are equal in the common part of the open set where they are defined, and that this common part is not empty.

15.1. *If $f(x) = g(x)$ in every interval inside O, then $f(x) = g(x)$ in O.*

In fact, let $f(x) = [\varphi_n(x)]$ and $g(x) = [\psi_n(x)]$. The equality $f(x) = g(x)$ in every interval inside O implies that the sequences $\varphi_n(x)$, $\psi_n(x)$ satisfy conditions (E_1) and (E_2) in every interval inside an interval inside O, and consequently in any interval inside O. Thus the sequences are equivalent in O.

§ 16. Distributions as a generalization of the notion of continuous functions

Every continuous function may be considered as a distribution. In this way theory of distributions will embrace Classical Analysis.

In order to obtain the identification of continuous functions with distributions, we need two preparatory lemmas.

16.1. *If, in an interval $a < x < b$, $\varphi_n(x) \rightrightarrows 0$ and $\varphi_n^{(k)}(x) \rightrightarrows$, then $\varphi_n^{(k)}(x) \rightrightarrows 0$.*

This is true for $k = 0$. Argue by induction. Suppose that the assertion holds for an order k, and that $\varphi_n(x) \rightrightarrows 0$, $\varphi_n^{(k+e_j)} \rightrightarrows f(x)$ in $a < x < b$. Then

$$\varphi_n^{(k)}(x+\eta e_j) - \varphi_n^{(k)}(x) = \int_0^\eta \varphi_n^{(k+e_j)}(x+\zeta e_j)\,d\zeta \rightrightarrows \int_0^\eta f(x+\zeta e_j)\,d\zeta$$

in $a+|\eta|e_j < x < b-|\eta|e_j$. By the induction hypothesis, the last integral vanishes. The number η being arbitrary, we obtain $f(x) = 0$.

16.2. *Almost uniformly convergent sequences of smooth functions are equivalent iff they converge to the same continuous function.*

In fact, if sequences $\varphi_n(x)$ and $\psi_n(x)$ converge almost uniformly to $f(x)$, then they satisfy conditions (E_1) and (E_2) with $k = 0$. Thus $\varphi_n(x) \sim \psi_n(x)$. Conversely, if $\varphi_n(x) \sim \psi_n(x)$, then for every interval I inside O there exist smooth functions $\Phi_n(x)$ and $\Psi_n(x)$ and an order k such that conditions (E_1) and (E_2) are satisfied. Hence $\Phi_n(x) - \Psi_n(x) \rightrightarrows 0$ on I. By 16.1, $\varphi_n(x) - \psi_n(x) \rightrightarrows 0$ on I. Thus the limits of $\varphi_n(x)$ and $\psi_n(x)$ are equal.

Now we are in a position to establish the correspondence between continuous functions and some distributions.

By 14.1, for every continuous function $f(x)$ there exists a sequence of smooth functions $\varphi_n(x)$ which converges almost uniformly to $f(x)$. By 3.1, this sequence is fundamental. Thus to every continuous function $f(x)$ there corresponds a distribution $[\varphi_n(x)]$. By 16.2 the correspondence is one-to-one.

In the sequel we shall always identify the continuous function $f(x)$ with the distribution $[\varphi_n(x)]$.

In particular we can write, after 14.1,

$$f(x) = [f(x) * \delta_n(x)] \tag{1}$$

for every continuous function $f(x)$ and any δ-sequence $\delta_n(x)$.

Of course, also smooth functions $\varphi(x)$ are distributions, and for them we have a simpler identity

$$\varphi(x) = [\varphi(x)].$$

In particular, the zero distribution, i.e. the distribution identified with the function vanishing everywhere, will be denoted by 0.

By the identification assumed here distributions are a generalization of the notion of continuous function. This justifies using for them the notation $f(x)$, $g(x)$, ..., as for functions.

16.3. *The convolution $f(x) * \omega(x)$ of a distribution $f(x)$ by a smooth function $\omega(x)$ is a smooth function.*

In fact, let $f(x) = [\varphi_n(x)]$. By 12.1, the sequence $\varphi_n(x)*\omega(x)$ converges almost uniformly to a continuous function $g(x)$. Moreover, for every order m, the sequence $(\varphi_n(x)*\omega(x))^{(m)}$ also converges almost uniformly on account of 12.1 and formula (2) in Section 12. By a classical theorem, $g(x)$ has continuous first partial derivatives viz. the limit of $(\varphi_n(x)*\omega(x))_{(e_j)}$ is the jth derivative of $g(x)$. By the same argument, $g(x)$ has all second derivatives, all third derivatires, etc. Thus $g(x)$ is a smooth function. On the other hand, $f(x)*\omega(x) = [\varphi_n(x)*\omega(x)] = g(x)$ by the definition of convolution and identification of continuous functions with distributions.

Now, we can write the identity

$$\varphi(x) = \varphi(x)*\delta(x) \tag{2}$$

for every smooth function $\varphi(x)$. In fact, replacing in (1) $f(x)$ by $\varphi(x)$, we get

$$\varphi(x) = [\varphi(x)*\delta_n(x)] = \varphi(x)*[\delta_n(x)] = \varphi(x)*\delta(x).$$

We are also in a position to prove the following generalization of (1):

16.4. *If $\delta_n(x)$ is a δ-sequence and $f(x)$ any distribution, then*

$$f(x) = [f(x)*\delta_n(x)]. \tag{3}$$

In fact, for every interval I inside the set O, where $f(x)$ is defined, there exist an order k and a continuous function $F(x)$ such that $F^{(k)}(x) = f(x)$ in I. By (1),

$$F(x) = [F(x)*\delta_n(x)] \quad \text{in } I.$$

Hence, differentiating k times, we obtain (3) in I. By 15.1 formula (3) holds in the whole set O.

Since $0 = \varphi(x)*0$ for every smooth function $\varphi(x)$, it follows from (2) that $\delta(x)$ is not equal to the zero distribution when considered in the whole space. Observe, on the other hand, that

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0$$

(i.e. in the open set of all $x \neq 0$) since every δ-sequence $\delta_n(x)$ converges almost uniformly to 0 for $x \neq 0$.

§ 17. Operations on continuous functions

In Sections 5-12 we have defined several operations on distributions. Now, continuous functions are distributions, and thus operations defined earlier for distributions are also defined for continuous functions. However, the operations are also defined directly on continuous functions. The question arises whether the two definitions are compatible.

Before we prove the compatibility of direct operations and distributional operations, we shall use in this Section different symbols for them.

If A denotes a direct operation, the corresponding distributional operation will be denoted by $\tilde{A}$. This double notation was not necessary before the identification of continuous functions with distributions, since direct operations were performed on continuous functions and distributional operations on distributions, and no misunderstandig could arise.

17.1. *If* $A(\varphi, \psi, \ldots)$ *is a regular operation, then*

$$A(\varphi, \psi, \ldots) = \tilde{A}(\varphi, \psi, \ldots) \tag{1}$$

for smooth functions $\varphi, \psi, \ldots$

In fact, thanks to identification we can write $\varphi = [\varphi]$, $\psi = [\psi]$, ..., and $A(\varphi, \psi, \ldots) = [A(\varphi, \psi, \ldots)]$. On the other hand, by the definition of distributional operations, we have $\tilde{A}(\varphi, \psi, \ldots) = [A(\varphi, \psi, \ldots)]$. This proves (1).

Given a regular operation A, we shall say that continuous functions $f, g, \ldots$ satisfy the *continuity condition* for A if $A(f, g, \ldots)$ is defined directly for those functions and, moreover, there exist smooth functions $\varphi_n, \psi_n, \ldots$ such that $\varphi_n \rightrightarrows f$, $\psi_n \rightrightarrows g, \ldots$ and $A(\varphi_n, \psi_n, \ldots) \rightrightarrows A(f, g, \ldots)$.

17.2. *If continuous functions* $f, g, \ldots$ *satisfy the continuity condition for a regular operation* A, *then*

$$A(f, g, \ldots) = \tilde{A}(f, g, \ldots). \tag{2}$$

In fact, on account of identification, we then have $A(f, g, \ldots) = [A(\varphi_n, \psi_n, \ldots)]$. On the other hand, by the definition of distributional operations, $\tilde{A}(f, g, \ldots) = [A(\varphi_n, \psi_n, \ldots)]$. This proves (2).

The continuity condition is satisfied by all continuous functions in the case of operations introduced so far, except for derivation. Consequently, those operations coincide with ordinary operations on continuous functions. Moreover, all calculations on continuous functions, except for derivation, may be performed in the usual way.

It is easy to see that every function $f(x)$ which is continuous with its ordinary derivative $f^{(e_j)}(x)$ satisfies the continuity condition for derivation. Thus for such functions the ordinary derivative $f^{(e_j)}(x)$ coincides with the distributional one. Both notations, $f^{(e_j)}(x)$ and $\frac{\partial}{\partial \xi_j} f(x)$, may be used as equivalent. By induction we have more generally:

17.3. *If* $f(x)$ *is a continuous function and its ordinary derivative*

$$\frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_m}} f(x) \tag{3}$$

is continuous, and if all the derivatives occurring in the step-by-step differentiation in the arrangement indicated are continuous as well, then the derivative (3) *coincides with the distributional derivative of the same order.*

It follows from 8.1 that every continuous function has distributional derivatives of all orders. If such a derivative is continuous, and if all the derivatives of minor orders are continuous as well, it coincides with the ordinary derivative. However, it may happen that a distributional derivative of a continuous function $f(x)$ is a continuous function but the ordinary derivative of the same order does not exist at all, no matter what is the ordering of symbols $\dfrac{\partial}{\partial \xi_j}$.

For instance, if a continuous function $g(\xi)$ of one real variable is non-differentiable (in the ordinary sense), then the function

$$f(x) = g(\xi_1)+g(\xi_2)$$

does not have ordinary derivatives $\dfrac{\partial^2}{\partial\xi_1\partial\xi_2}f(x)$, $\dfrac{\partial^2}{\partial\xi_2\partial\xi_1}f(x)$. The corresponding distributional derivatives are equal, because the distributional derivatives do not depend on the ordering of symbols $\dfrac{\partial}{\partial\xi_j}$. To find the distributional derivative in question, let us remark that $\dfrac{\partial}{\partial\xi_1}\dfrac{\partial}{\partial\xi_2}g(\xi_1)=0$ in the ordinary and consequently in the distributional sense. Similarly we have $\dfrac{\partial}{\partial\xi_2}\dfrac{\partial}{\partial\xi_1}g(\xi_2)=0$ in the distributional sense. Since the ordering is irrelevant, we have also $\dfrac{\partial}{\partial\xi_1}\dfrac{\partial}{\partial\xi_2}g(\xi_2)=0$ in the distributional sense. Hence we obtain, as the distributional derivative, $\dfrac{\partial^2}{\partial\xi_1\partial\xi_2}f(x)=0$. It is interesting to note that neither $\dfrac{\partial}{\partial\xi_1}f(x)$ nor $\dfrac{\partial}{\partial\xi_2}f(x)$ are functions. This example shows that there are distributions which are not functions, but some of their derivatives are continuous functions.

17.4. *Every distribution in O is, in every interval I inside O, a derivative of some order of a continuous function.*

In fact, let $f(x) = [\varphi_n(x)]$. By (F_1), (F_2) there exist an order k, smooth functions $\Phi_n(x)$, and a continuous function $F(x)$ such that in I

$$\Phi_n^{(k)}(x) = \varphi_n(x) \quad \text{and} \quad \Phi_n(x) \rightrightarrows F(x).$$

Hence $F(x) = [\Phi_n(x)]$ in I and

$$f(x) = [\Phi_n^{(k)}(x)] = [\Phi_n(x)]^{(k)} = F^{(k)}(x) \quad \text{in } I.$$

§ 18. Locally integrable functions

As we have seen in Section 17, distributions are a generalization of continuous functions. Now we shall show that they embrace also a larger class of functions, viz. all locally integrable functions. Sections 18 and 19, which are concerned with those functions, can be omitted by readers not acquainted with the theory of Lebesgue Integral.

We recall that a function $f(x)$, defined in O, is said to be locally integrable in O iff the integral $\int_a^b f(t)\,dt$ exists for every interval $a < x < b$ inside O.

Observe first that if $f(x)$ is a continuous function in an interval I, then in I

(1) $$\Big(\int_{x_0}^{x} f(t)\,dt\Big)' = f(x) \qquad (x_0 \text{ in } I),$$

where the sign $'$ denotes derivation of order 1. If we do not assume the continuity of $f(x)$ but its integrability only, then the integral $\int_{x_0}^{x} f(t)\,dt$ is also a continuous function. In this case equality (1) holds almost everywhere, the derivative on the left side being defined as the usual limit (for $\alpha \to 0$, $\alpha > 0$) of the expression

(2) $$f_\alpha(x) = \frac{1}{\Delta x}\int_{x}^{x+\Delta x} f(t)\,dt,$$

where $\Delta x = (\alpha, \ldots, \alpha) = \alpha 1$ and $\dfrac{1}{\Delta x}$ denotes $\dfrac{1}{\alpha^q}$. The left side of (1) can also be interpreted as a distribution which is the distributional derivative of order 1 of the continuous function $\int_{x_0}^{x} f(t)\,dt$. It is easy to verify that this distribution does not depend on the choice of x_0 in I.

This suggests the following identification: A distribution is said to be equal to a function $f(x)$ locally integrable in O iff, for every interval I inside O, that distribution is the distributional derivative $\Big(\int_{x_0}^{x} f(t)\,dt\Big)'$ $(x_0$ in $I)$.

It follows from 15.1 that this distribution, if it exists, is determined uniquely by the locally integrable function $f(x)$. We shall prove that it always exists, for the distribution

(3) $$[f(x) * \delta_n(x)],$$

has the required property, $\delta_n(x)$ being any δ-sequence.

In fact, let I be any interval inside O and let

$$F(x) = \int_{x_0}^{x} f(t)\,dt \qquad (x_0 \text{ in } I).$$

By 14.1, the sequence $F(x)*\delta_n(x)$ converges to $F(x)$ almost uniformly in I. Hence, by the identification of continuous functions with distributions,

$$[F(x)*\delta_n(x)] = F(x) \quad \text{in } I,$$

and consequently $[F'(x)*\delta_n(x)] = F'(x)$, i.e.

$$[f(x)*\delta_n(x)] = F'(x) \quad \text{in } I.$$

Thus we have proved that every locally integrable function $f(x)$ can be identified with the distribution $[f(x)*\delta_n(x)]$.

If $f(x)$ is a continuous function, then $f(x)*\delta_n(x)$ converges almost uniformly to $f(x)$ by 14.1; thus the identification of integrable functions coincides in this case with the identification in Section 16.

The identification of locally integrable functions with distributions makes the following definition necessary: locally integrable functions $f(x)$ and $g(x)$ are equal iff they are equal as distributions, i.e. iff $\int_a^b f(t)\,dt = \int_a^b g(t)\,dt$ for every interval $a < t < b$ inside O, i.e. iff $f(x) = g(x)$ almost everywhere.

§ 19. Operations on locally integrable functions

As in the case of continuous functions, the question arises whether the distributional operations on locally integrable functions coincide with operations defined directly.

We shall say that a sequence of smooth functions $\varphi_n(x)$ is L-*convergent* to a locally integrable function $f(x)$ if it converges to $f(x)$ almost everywhere in O, and, moreover, if in every interval I inside O

$$(1) \qquad \int_a^x \varphi_n(t)\,dt \rightrightarrows \int_a^x f(t)\,dt \qquad (a \text{ in } I).$$

If $\varphi_n(x)$ is L-convergent to $f(x)$, then $\varphi_n(x)$ is fundamental and $[\varphi_n(x)] = f(x)$. In fact, (1) implies that $[\int_a^x \varphi_n(t)\,dt] = \int_a^x f(t)\,dt$. Hence, by differentiation of order 1, we get $[\varphi_n(x)] = f(x)$ in every interval I inside O, and consequently in the whole set O.

Given a regular operation A, we say that locally integrable functions $f, g, \ldots$ satisfy the *integrability condition* for A if the operation

$A(f, g, \ldots)$ is defined for those functions and, moreover, if there exist sequences of smooth functions $\varphi_n, \psi_n, \ldots$, L-convergent to $f, g, \ldots$ respectively, and such that $A(\varphi_n, \psi_n, \ldots)$ is L-convergent to $A(f, g, \ldots)$.

As in Section 15, let $\tilde{A}$ denote the distributional extension of the operation A.

19.1. *If locally integrable functions $f, g, \ldots$ satisfy the integrability condition for a regular operation A, then*

$$A(f, g, \ldots) = \tilde{A}(f, g, \ldots). \tag{2}$$

In fact, we then have for every interval I inside O

$$\int_a^x A(\varphi_n, \psi_n, \ldots)\,dt \rightrightarrows \int_a^x A(f, g, \ldots)\,dt \qquad (a \text{ in } I).$$

This implies that in I

$$\left[\int_a^x A(\varphi_n, \psi_n, \ldots)\,dt\right] = \int_a^x A(f, g, \ldots)\,dt$$

and by the identification principle

$$[A(\varphi_n, \psi_n, \ldots)] = A(f, g, \ldots) \quad \text{in } I.$$

On the other hand, by the definition of distributional operations,

$$[A(\varphi_n, \psi_n, \ldots)] = \tilde{A}(f, g, \ldots).$$

Hence equality (2) holds in I. Since I is arbitrary, (2) holds in O.

The integrability condition is always satisfied by all locally integrable functions in the case of operations introduced here, except for derivation. The proofs have nothing in common with the theory of distributions, and we need not enter into details. Consequently, all calculations on locally integrable functions, except for derivation, may be performed in the usual way.

It can happen that both the ordinary derivative of a locally integrable function and its distributional derivative exist but are different. For instance, the ordinary derivative of the Heaviside function of one real variable

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad x > 0 \end{cases}$$

is the zero distribution but the distributional derivative of $H(x)$ is equal to the one-dimensional Dirac delta distribution $\delta(x)$ for if $\delta_n(x)$ is any δ-sequence, then $\int_{-\infty}^x \delta_n(t)\,dt$ is L-convergent to $H(x)$ and consequently

$$\delta(x) = [\delta_n(x)] = \left[\int_{-\infty}^x \delta_n(t)\,dt\right]' = H'(x).$$

In the Theory of Distributions the ordinary derivative plays a minor role. Therefore, if no remark is added, the derivation of functions will always be understood in the distributional sense.

The only locally integrable functions $f(x)$ of one real variable which satisfy the integrability condition for the derivation of order *1* are absolutely continuous functions, i.e. functions with locally integrable derivative $f'(x)$, such that in every interval I inside O

$$f(x)-f(x_0) = \int_{x_0}^{x} f'(t)\,dt \qquad (x_0 \text{ in } I). \tag{3}$$

Thus the following statement holds:

19.2. *If $f(x)$ is an absolutely continuous function, then its distributional derivative $f'(x)$ coincides with its ordinary derivative.*

Analogous conditions may be stated for derivatives of higher orders, but we shall not enter into details here.

§ 20. Sequences of distributions

We say that a sequence of distributions $f_n(x)$ *converges* in O to a distribution $f(x)$ and we write

$$f_n(x) \to f(x) \text{ in } O \qquad \text{or} \qquad \lim_{n\to\infty} f_n(x) = f(x) \text{ in } O,$$

iff the distribution $f(x)$ is defined in O and, for every interval I inside O, there exist an order k and continuous functions $F(x)$ and $F_n(x)$ such that in I

$$\begin{aligned} F_n^{(k)}(x) &= f_n(x) \quad \text{for} \quad n > n_0, \\ F^{(k)}(x) &= f(x) \quad \text{and} \quad F_n(x) \rightrightarrows F(x). \end{aligned} \tag{1}$$

According to this definition the limit distribution $f(x)$ is defined in the whole set O, but this is not necessary for the distributions $f_n(x)$ (see Section 2).

It is useful to observe that the order k which occurs in (1) can be, if necessary, replaced by any order $l \geqslant k$. In fact, if conditions (1) hold, then also

$$\tilde{F}_n^{(l)}(x) = f_n(x) \quad \text{for} \quad n > n_0, \quad \tilde{F}^{(l)}(x) = f(x) \quad \text{and} \quad \tilde{F}_n(x) \rightrightarrows \tilde{F}(x),$$

where

$$\tilde{F}_n(x) = \int_{x_0}^{x} F_n(t)\,dt^{l-k}, \qquad \tilde{F}(x) = \int_{x_0}^{x} F(t)\,dt^{l-k} \qquad (x_0 \text{ in } I).$$

The limit, if it exists, is unique. In order to prove it, we need the following auxiliary theorem:

20.1. *If continuous functions $f_n(x)$ converge to $f(x)$ almost uniformly in O and if $f_n^{(m)}(x) = 0$, then $f^{(m)}(x) = 0$.*

By 14.1, for every interval I inside O, there exist smooth functions $\varphi_{rn}(x)$ such that

$$\varphi_{rn}(x) \rightrightarrows f_n(x) \quad \text{and} \quad \varphi_{rn}^{(m)}(x) = 0 \quad \text{in } I$$

Let r_n be such that $|\varphi_{r_n n}(x) - f_n(x)| < \frac{1}{n}$ in I. Then $\varphi_{r_n n}(x) \rightrightarrows f(x)$ and therefore $f(x) = [\varphi_{r_n n}(x)]$ in I. Differentiating m times, we obtain $f^{(m)}(x) = [\varphi_{r_n n}^{(m)}(x)] = 0$ in I. Since I is arbitrary, we have $f^{(m)}(x) = 0$ in the whole set O.

Now we are in a position to prove the uniqueness of the limit. Let I be any interval inside O. If $f_n(x)$ are distributions such that $f_n(x) \to f(x)$ and $f_n(x) \to g(x)$, then there exist continuous functions $F_n(x)$, $G_n(x)$ and orders k, l such that $F_n(x) \rightrightarrows F(x)$, $G_n(x) \rightrightarrows G(x)$ in I and

$$F_n^{(k)}(x) = f_n(x), \qquad F^{(k)}(x) = f(x),$$

$$\text{and} \qquad G_n^{(l)}(x) = f_n(x), \qquad G^{(l)}(x) = g(x).$$

We may assume that $k = l$ (for otherwise we could replace both orders by a greater order). Since $\big(F_n(x) - G_n(x)\big)^{(k)} = 0$ and $F_n(x) - G_n(x) \rightrightarrows F(x) - G(x)$, we have $\big(F(x) - G(x)\big)^{(k)} = 0$, on account of 20.1, which implies $f(x) = g(x)$ in I. Since I is arbitrary, the limit is unique.

Directly from the definition of the limit it follows that:

20.2. *If a sequence of continuous functions converges almost uniformly, then it converges also distributionally to the same limit.*

20.3. *If $f_n(x) \to f(x)$, then $f_{r_n}(x) \to f(x)$ for every sequence r_n of positive integers such that $r_n \to \infty$.*

20.4. *If $f_n(x) \to f(x)$ and $g_n(x) \to f(x)$, then the interlaced sequence*

$$f_1(x), g_1(x), f_2(x), g_2(x), \ldots$$

converges also to $f(x)$.

20.5. *If $f_n(x) \to f(x)$, then $\lambda f_n(x) \to \lambda f(x)$ for every number λ. If $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$, then $f_n(x) + g_n(x) \to f(x) + g(x)$.*

20.6. *If $f_n(x) \to f(x)$, then $f_n^{(m)}(x) \to f^{(m)}(x)$ for every order m.*

This simple theorem is of striking advantage for calculations on distributions, in contrast to the classical Differential Calculus, where additional restrictions are necessary.

20.7. *If $f_n(x) \to f(x)$ in every interval inside O, then $f_n(x) \to f(x)$ in the whole set O.*

For any interval I inside O there exists an interval I' inside O such that I is inside I'. Since $f_n(x) \to f(x)$ in I', there are an order k and continuous functions $F_n(x)$, $F(x)$ such that conditions (1) are satisfied. But this proves that $f_n(x) \to f(x)$ in O.

We say that a sequence of distributions $f_n(x)$ is *convergent* in O iff for every interval I inside O there exist an order k and continuous functions $F_n(x)$ such that

$$F_n^{(k)} = f_n(x) \quad \text{and} \quad F_n(x) \rightrightarrows \quad \text{in } I.$$

20.8. *If a sequence of distributions is convergent in O, then it converges to a distribution in O.*

Suppose that $f_n(x)$ is convergent in O. Let $\delta_n(x)$ be any δ-sequence. We shall prove that the sequence

$$\varphi_n(x) = f_n(x) * \delta_n(x)$$

is fundamental in O and that $f_n(x)$ converges to $[\varphi_n(x)]$.

In fact, let I be an arbitrary interval inside O and let I' be an interval inside O such that I is inside I'. There exist an order k and continuous functions $F_n(x)$, $F(x)$ such that

$$F_n^{(k)}(x) = f_n(x) \quad \text{and} \quad F_n(x) \rightrightarrows F(x) \quad \text{in } I'.$$

By 14.2, we have

(2) $$F_n(x) * \delta_n(x) \rightrightarrows F(x) \quad \text{in } I.$$

Since

$$\big(F_n(x) * \delta_n(x)\big)^{(k)} = \varphi_n(x),$$

the sequence $\varphi_n(x)$ is fundamental in O. It represents therefore a distribution $f(x)$ in O. By (2), we can write

$$F_n(x) * \delta_n(x) \to F(x) \quad \text{and} \quad [F_n(x) * \delta_n(x)] = F(x) \quad \text{in } I.$$

Hence, differentiating k times, we obtain

$$\varphi_n(x) \to F^{(k)}(x) \quad \text{and} \quad [\varphi_n(x)] = F^{(k)}(x) \quad \text{in } I.$$

Consequently

$$\varphi_n(x) \to f(x) \quad \text{in } I.$$

Since $F_n(x) - F_n(x) * \delta_n(x) \rightrightarrows 0$ in I, differentiating k times we obtain

$$f_n(x) - \varphi_n(x) \to 0 \quad \text{in } I.$$

Thus $f_n(x) \to f(x)$ in I. Since I is arbitrary, it follows from 20.7 that $f_n(x) \to f(x)$ in O.

§ 21. Convergence and regular operations

For distributions, the limit is commutative with all regular operations introduced so far. In other words, the following formulae are true:

$$\lim_{n\to\infty}\lambda f_n(x) = \lambda\lim_{n\to\infty} f_n(x),$$

$$\lim_{n\to\infty}\big(f_n(x)+g_n(x)\big) = \lim_{n\to\infty} f_n(x)+\lim_{n\to\infty} g_n(x),$$

$$\lim_{n\to\infty}\big(f_n(x)-g_n(x)\big) = \lim_{n\to\infty} f_n(x)-\lim_{n\to\infty} g_n(x),$$

$$\lim_{n\to\infty} f_n^{(m)}(x) = \big(\lim_{n\to\infty} f_n(x)\big)^{(m)},$$

$$\lim_{n\to\infty}\omega(x)f_n(x) = \omega(x)\cdot\lim_{n\to\infty} f_n(x),$$

$$\lim_{n\to\infty} f_n(x)g_n(y) = \lim_{n\to\infty} f_n(x)\cdot\lim_{n\to\infty} g_n(y),$$

$$\lim_{n\to\infty}\big(f_n(x)*\omega(x)\big) = \lim_{n\to\infty} f_n(x)*\omega(x).$$

In the case of substitution, the symbol $\lim\limits_{n\to\infty} f_n\big(\sigma(x)\big)$ has two interpretations: as the limit of the sequence $f_n\big(\sigma(x)\big)$ and as the substitution $y=\sigma(x)$ in the distribution $\lim\limits_{n\to\infty} f_n(y)$. The commutativity of the limit with the substitution means that both interpretations coincide. Similarly for translation.

The verification of commutativity is trivial for multiplication by a number, addition, subtraction, translation, derivation, multiplication of distributions with separated variables, and convolution by a smooth function vanishing outside an interval. The commutativity of the limit with product by a smooth function and with substitution results from the following two stronger theorems.

21.1. *If $\omega_n^{(m)}(x)$ converges almost uniformly to $\omega^{(m)}(x)$ for every order m, and if $f_n(x)\to f(x)$, then $\omega_n(x)f_n(x)\to\omega(x)f(x)$.*

For every interval I inside O, there are continuous functions $F_n(x)$, $F(x)$ and an order k such that $F_n(x)\rightrightarrows F(x)$, $F_n^{(k)}(x)=f_n(x)$, and $F^{(k)}(x)=f(x)$. Thus $\omega_n(x)F_n(x)\rightrightarrows\omega(x)F(x)$ in I. Since every uniformly convergent sequence is distributionally convergent, we can also write

$$\omega_n(x)\,F_n(x)\to\omega(x)\,F(x). \tag{4}$$

Similarly we have

$$\omega_n^{(e_j)}(x)\,F_n(x)\to\omega^{(e_j)}(x)\,F(x). \tag{5}$$

Differentiating (4), we obtain

$$\omega_n^{(e_j)}(x)F_n(x)+\omega_n(x)\,F^{(e_j)}(x)\to\omega^{(e_j)}(x)\,F(x)+\omega(x)\,F^{(e_j)}(x).$$

Hence, in view of (5),

$$\omega_n(x)F_n^{(e_j)}(x)\to\omega(x)F^{(e_j)}(x).$$

By induction we obtain

$$\omega_n(x)F_n^{(k)}(x)\to\omega(x)F^{(k)}(x),$$

i.e.

$$\omega_n(x)f_n(x)\to\omega(x)f(x) \tag{6}$$

in I. Since the interval I is arbitrary, (6) holds in the whole set O.

Another proof of 21.2 follows from the formula

$$\omega(x)\,\varphi^{(k)}(x)=\sum_{0\leqslant m\leqslant k}(-1)^m\binom{k}{m}\big(\omega^{(m)}(x)\,\varphi(x)\big)^{(k-m)},$$

where $\binom{k}{m}=\binom{\varkappa_1}{\mu_1}\cdots\binom{\varkappa_q}{\mu_q}$ and $(-1)^m=(-1)^{\mu_1+\ldots+\mu_q}$ $(k=(\varkappa_1,\ldots,\varkappa_q)$, $m=(\mu_1,\ldots,\mu_q))$. The verification of this formula for smooth functions $\omega(x)$ and $\varphi(x)$ is a question of routine calculations. If $\omega(x)$ is fixed, both sides of the formula are iterations of regular operations; thus the formula holds if $\varphi(x)$ is replaced by any distribution or continuous function. In particular, we have

$$\omega_n(x)\,F_n^{(k)}(x)=\sum_{0\leqslant m\leqslant k}(-1)^m\binom{k}{m}\big(\omega^{(m)}(x)\,F_n(x)\big)^{(k-m)},$$

and hence (6) follows in every interval I inside O, and consequently in the whole set O.

21.2. *If $\sigma_n^{(m)}(x)$ converges almost uniformly to $\sigma^{(m)}(x)$ for every order m ($\sigma_n(x)$ and $\sigma(x)$ having property* (1) *from Section* 10), *and $f_n(y)\to f(y)$, then $f_n(\sigma_n(x))\to f(\sigma(x))$.*

The proof of 21.2 will be based on formula (2) from Section 10. That formula involves regular operations only, and thus it holds also if $\Phi_n(y)$ is replaced by any distribution $f(y)$. Thus

$$f'(\sigma(x))=\frac{\dfrac{\partial}{\partial\xi_1}f(\sigma(x))\cdot\dfrac{\partial}{\partial\xi_1}\sigma(x)+\ldots+\dfrac{\partial}{\partial\xi_q}f(\sigma(x))\cdot\dfrac{\partial}{\partial\xi_q}\sigma(x)}{\left(\dfrac{\partial\sigma(x)}{\partial\xi_1}\right)^2+\ldots+\left(\dfrac{\partial\sigma(x)}{\partial\xi_q}\right)^2}. \tag{7}$$

Suppose that $\sigma(x)$ is defined in an open set O and that the values of $\sigma(x)$ are in an open set O' of real numbers y. The distribution $f(y)$

is supposed to be defined in O'. Let I be any interval inside O. The function $\sigma(x)$ maps I onto an interval I' inside O'. The sequence $\sigma_n^{(m)}(x)$ converges uniformly in I to $\sigma^{(m)}(x)$. There is an interval I'' inside O' such that I' is inside I'', and that the values of $\sigma_n(x)$ lie in I'' for sufficiently large indices n. For that interval I'' there exist functions $F_n(y), F(y)$ and a non-negative integer k such that $F_n(y) \rightrightarrows F(y)$ in I'', $F_n^{(k)}(y) = f_n(y)$, $F^{(k)}(y) = f(y)$. Evidently $F_n(\sigma_n(x)) \rightrightarrows F(\sigma(x))$ in I. Since uniformly convergent sequences are distributionally convergent, we can also write

$$F_n(\sigma_n(x)) \to F(\sigma(x)) \quad \text{in } I. \tag{8}$$

Applying formula (7) to distributions $F_n'(\sigma_n(x))$ and $F'(\sigma(x))$, we obtain in view of (8) and 21.1

$$F_n'(\sigma_n(x)) \to F'((\sigma(x)) \quad \text{in } I.$$

By induction we get

$$F_n^{(k)}(\sigma_n(x)) \to F^{(k)}(\sigma(x)) \quad \text{in } I,$$

i.e. $f_n(\sigma_n(x)) \to f(\sigma(x))$ in I. Since I is arbitrary, theorem 21.2 follows.

The question whether the limit is commutative with any regular operation will not be discussed here.

§ 22. Distributionally convergent sequences of smooth functions

First we shall prove that:

22.1. *A sequence of constant functions converges distributionally iff it converges in the ordinary sense.*

In fact, if constant functions converge in the ordinary sense, then they converge uniformly, and, by 20.2, also distributionally.

Conversely, suppose that a sequence c_n of constant functions converges distributionally. Then this sequence is bounded. For if otherwise there would exist a subsequence c_{r_n} such that $1/c_{r_n}$ converges in the usual sense to 0 and we would have $1 = \frac{1}{c_{r_n}} \cdot c_{r_n} \to 0$. Suppose that c_n does not converge in the ordinary sense. Then there exist two subsequences which converge to different limits. Thus those subsequences converge distributionally to different limits, which contradicts 20.3.

22.2. *A sequence of smooth functions $\varphi_n(x)$ is fundamental in O iff for every interval I inside O there exist continuous functions $F_n(x)$ and an order k such that in I*

$$F_n^{(k)}(x) = \varphi_n(x) \quad \text{and} \quad F_n(x) \rightrightarrows. \tag{1}$$

In fact, if $\varphi_n(x)$ is fundamental, there exist, for every interval I inside O, smooth functions $\Phi_n(x)$ and an order k such that in I

(2) $$\Phi_n(x) \rightrightarrows \quad \text{and} \quad \Phi_n^{(k)}(x) = \varphi_n(x).$$

Since smooth functions are continuous, the condition is satisfied.

Suppose, conversely, that (1) holds for every interval I inside O. Let I be fixed arbitrarily inside O and let I' be an interval inside O such that I is inside I' There exist functions $F_n(x)$ and an order k such that (1) holds in I'. Let

$$\Phi_{nr}(x) = \Big(F_n(x) - \int_{x_0}^{x} \varphi_n(t)\,dt^k\Big) * \delta_r(x) + \int_{x_0}^{x} \varphi_n(t)\,dt^k,$$

where x_0 is in I and $\delta_n(x)$ is a δ-sequence as in Section 14. Then $\Phi_{nr}^{(k)}(x) = \varphi_n(x)$ in I for sufficiently large r, say $r > p_n$. Moreover, by 14.1 we have $\Phi_{nr}(x) \rightrightarrows F_n(x)$ in I for $r \to \infty$. Let $F(x)$ denote the limit of $F_n(x)$. Since $F_n(x) \rightrightarrows F(x)$, there is a sequence of positive integers $r_n > p_n$ such that

$$\Phi_n(x) = \Phi_{nr_n}(x) \rightrightarrows F(x) \quad \text{in } I.$$

Evidently $\Phi_n^{(k)}(x) = \varphi_n(x)$ in I; thus the functions $\Phi_n(x)$ have the required properties.

22.3. *A sequence of smooth functions converges distributionally to a distribution $f(x)$ iff it is fundamental for $f(x)$.*

In fact, if $\varphi_n(x)$ is a fundamental sequence for $f(x)$, then for every interval I inside O there exist smooth functions $\Phi_n(x)$, a continuous function $F(x)$, and an order k such that

(3) $$\Phi_n(x) \rightrightarrows F(x), \quad \Phi_n^{(k)}(x) = \varphi_n(x) \quad \text{and} \quad F^{(k)}(x) = f(x) \quad \text{in } I.$$

The first two conditions follow from the definition of fundamental sequences. The third one is obtained by differentiating k times the equality $F(x) = [\Phi_n(x)]$, which follows from the first condition. Since smooth functions are continuous functions, (3) means that $\varphi_n(x) \to f(x)$ in O.

Conversely, if $\varphi_n(x) \to f(x)$ in O, then for every interval I inside O there exist functions $F_n(x)$, $F(x)$ and an order k such that in I

$$F_n(x) \rightrightarrows F(x), \quad F_n^{(k)}(x) = \varphi_n(x) \quad \text{and} \quad F^{(k)}(x) = f(x).$$

Thus, by 22.2 the sequence $\varphi_n(x)$ is fundamental. As we have just proved, every fundamental sequence converges to the distribution which it represents. This implies that $f(x) = [\varphi_n(x)]$.

§ 23. Locally convergent sequences of distributions

It may happen that we know the following property of a sequence of distributions $f_n(x)$: For every point x_0 in O there exists an interval inside O, containing x_0, in which $f_n(x)$ is convergent. The aim of this Section is to show that then $f_n(x)$ is convergent in O. We shall also state some important corollaries.

If continuous functions $\varphi_1(x)$ and $\varphi_2(x)$ are defined in sets O_1 and O_2, then their product $\varphi_1(x)\,\varphi_2(x)$ is defined in the common part of O_1 and O_2. We adopt, further on, the convention that this product is also defined and has the value 0 at all points where at least one of the factors $\varphi_1(x)$ and $\varphi_2(x)$ is defined and has the value 0.

In the following two lemmas, $\omega(x)$ denotes a smooth function defined everywhere and vanishing outside an interval I inside a given open set O. and I' is an interval inside O such that I is inside I'.

LEMMA 1. *If $\varphi_n(x)$ is a fundamental sequence in O, then $\omega(x)\,\varphi_n(x)$ is fundamental everywhere.*

In fact, there are smooth functions $\Phi_n(x)$ and an order k such that $\Phi_n^{(k)}(x) = \varphi_n(x)$ and $\Phi_n(x) \rightrightarrows$ in I'. Evidently $\omega(x)\,\Phi_n(x) \rightrightarrows$ everywhere. Thus the sequence $\omega(x)\,\Phi_n(x)$ is fundamental. Similarly $\omega^{(e_j)}(x)\Phi_n(x)$ is fundamental. Consequently the sequence

$$\omega(x)\,\Phi_n^{(e_j)}(x) = \big(\omega(x)\,\Phi_n(x)\big)^{(e_j)} - \omega^{(e_j)}(x)\,\Phi_n(x)$$

is fundamental everywhere. By induction, the sequence $\omega(x)\Phi_n^{(k)}(x)$, i.e. $\omega(x)\varphi_n(x)$ is fundamental everywhere.

By Lemma 1, if $f(x) = [\varphi_n(x)]$ in O, then the distribution $\omega(x) f(x) = [\omega(x)\,\varphi_n(x)]$ is defined everywhere.

LEMMA 2. *If a sequence of distributions $f_n(x)$ is convergent in O, then the sequence $\omega(x) f_n(x)$ converges everywhere.*

In fact, there are continuous functions $F_n(x)$ and an order k such that $F_n^{(k)}(x) = f_n(x)$ and $F_n(x) \rightrightarrows$ in I'. Evidently $\omega(x)\,F_n(x) \rightrightarrows$ everywhere. Thus $\omega(x)\,F_n(x)$ is distributionally convergent everywhere. Similarly $\omega_n^{(e_j)}(x)\,F_n(x)$ is distributionally convergent everywhere. Consequently the sequence

$$\omega(x)\,F_n^{(e_j)}(x) = \big(\omega(x)\,F_n(x)\big)^{(e_j)} - \omega^{(e_j)}(x)\,F_n(x)$$

is distributionally convergent everywhere. By induction, the sequence $\omega(x)\,F_n^{(k)}(x)$, i.e. $\omega(x) f_n(x)$, is convergent everywhere.

23.1. *If every point x_0 in O is in an interval I_0 such that a sequence of distributions $f_n(x)$ is convergent in I_0, then $f_n(x)$ is convergent in O.* In other words: *locally convergent sequences of distributions are convergent.*

In fact, let I be any interval inside O. There exists an interval J inside O such that I is inside J. We can divide J into a finite number of subintervals j (in this proof j will denote intervals, not numbers) such that every subinterval j is inside an interval I_j in which $f_n(x)$ is convergent.

Let $g_j(x)$ denote the characteristic function of j, i.e. a function such that

$$g_j(x) = \begin{cases} 1 & \text{on } j, \\ 0 & \text{outside } j. \end{cases}$$

Similarly let $g_J(x)$ denote the characteristic function of J. If $\delta_n(x)$ is a δ-sequence (see Section 14), we can choose an index p such that

$$\varphi_j(x) = g_j(x)*\delta_p(x) = 0 \quad \text{outside } I_j,$$

$$\varphi_J(x) = g_J(x)*\delta_p(x) = 1 \quad \text{on } I.$$

Evidently

$$g_J(x) = \sum_j g_j(x) \quad \text{and} \quad \varphi_J(x) = \sum_j \varphi_j(x).$$

Since $f_n(x)$ is convergent in I_j, the product $\varphi_j(x) f_n(x)$ is convergent everywhere, by Lemma 2. Since the number of intervals j is finite, also the sequence

$$\psi_n(x) = \sum_j \varphi_j(x) f_n(x) = \varphi_J(x) f_n(x)$$

is convergent everywhere. But $\psi_n(x) = f_n(x)$ in I, and thus $f_n(x)$ is convergent in I. Since I can be chosen arbitrarily inside O, $f_n(x)$ is convergent in O, by 20.7.

By 22.3, a sequence of smooth functions converges distributionally iff it is fundamental. Thus we immediately obtain the following corollary:

23.2. *If every point x_0 in O is in an interval I_0 such that $\varphi_n(x)$ is fundamental in I_n, then $\varphi_n(x)$ is fundamental in O.* In other words: *locally fundamental sequences are fundamental.*

We are also in a position now to prove the following important theorem:

23.3. *Let O be the union of open sets Θ. If in each of the sets Θ there is defined a distribution $f_\Theta(x)$, so that distributions on overlapping sets are equal one to another, then there exists a distribution $f(x)$ defined in the whole set O such that $f(x) = f_\Theta(x)$ in every set Θ.*

In fact, let $\delta_n(x)$ be a δ-sequence. For every fixed n, smooth functions $f_\Theta(x)*\delta_n(x)$ coincide one to another at the points where they are both defined. Thus those functions can be unified to a single function $\varphi_n(x)$ (depending on n), defined in the union of open sets in which $f_\Theta(x)*\delta_n(x)$ are defined. The sequence $\varphi_n(x)$ is fundamental in every interval which is inside at least one of the sets Θ. Since the union of all such intervals is O, $\varphi_n(x)$ is fundamental in O, by 23.2. The distribution $f(x) = [\varphi_n(x)]$ has the required property, for in every set Θ

$$f(x) = [f_\Theta(x)*\delta_n(x)] = f_\Theta(x).$$

§ 24. Distributions depending on a continuous parameter

We say that a continuous function $f_a(x)$, depending on a continuous parameter a, *converges* for $a \to a_0$ to $f(x)$ *uniformly* on a set I, and we write

$$f_a(x) \rightrightarrows f(x) \quad (a \to a_0) \qquad \text{on } I,$$

iff the function $f(x)$ is defined on I and, for any given number $\varepsilon > 0$, there is a number $\eta > 0$ such that for every a satisfying $|a-a_0| < \eta$ the function $f_a(x)$ is defined on the whole set I and satisfies there the inequality $|f_a(x)-f(x)| < \varepsilon$.

We say that the function $f_a(x)$ *converges* for $a \to a_0$ to $f(x)$ *almost uniformly* in an open set O iff $f_a(x) \rightrightarrows f(x)$ $(a \to a_0)$ on every interval inside O.

We say that a distribution $f_a(x)$, depending on a continuous parameter a, *converges* for $a \to a_0$ to a distribution $f(x)$ in an open set O iff $f(x)$ is defined in O, and if for every interval I inside O there exist an order k and continuous functions $F_a(x)$, $F(x)$ such that, for a sufficiently near to a_0,

$$F_a^{(k)}(x) = f_a(x), \qquad F^{(k)}(x) = f(x) \qquad \text{and} \qquad F_a(x) \rightrightarrows F(x) \quad (a \to a_0) \quad \text{in } I.$$

We then write

$$f_a(x) \to f(x) \quad (a \to a_0) \qquad \text{in } O,$$

or

$$f(x) = \lim_{a \to a_0} f_a(x) \qquad \text{in } O.$$

The limit $f(x)$, if it exists, is unique. The proof is similar to that for sequences.

In the above definition it is irrelevant whether a is a real or a complex parameter. It may also be a variable point of a multidimensional space. Then, of course, the symbol $|a-a_0|$ is to be read as the distance between the points a and a_0. Similarly we define the limit when $a_0 = \pm\infty$.

Of course,

24.1. *If a continuous function $f(x)$ with parameter converges almost uniformly, then it converges distributionally to the same limit.*

As for sequences, we can prove that the limit can be interchanged with all regular operations introduced here. Also the analogues of the theorems from Sections 21 and 23 remain true.

Now we can define the derivative of distributions in the same way as in the case of functions. In fact,

24.2. *For every distribution* $f(x)$,

$$f^{(e_j)}(x) = \lim_{a\to 0} \frac{f(x+ae_j)-f(x)}{a}.$$

Let I be any interval inside O and let I' be an interval inside O such that I is inside I'. Then there exist an order k and a continuous function $F(x)$ with a continuous derivative $F^{(e_j)}(x)$ such that $F^{(k)}(x) = f(x)$ in I'. Since in I

$$\frac{F(x+ae_j)-F(x)}{a} \rightrightarrows F^{(e_j)}(x) \quad \text{for} \quad a \to 0,$$

we have in I

$$\frac{f(x+ae_j)-f(x)}{a} = \left(\frac{F(x+ae_j)-F(x)}{a}\right)^{(k)} \to \left(F^{(e_j)}(x)\right)^{(k)} = f^{(e_j)}(x).$$

Since the interval I is arbitrary, the convergence holds in the whole set O.

§ 25. Multidimensional substitution

Let $\sigma_1(x), \ldots, \sigma_p(x)$ be smooth functions defined in an open subset O of the q-dimensional space such that the transformation

$$\sigma(x) = \left(\sigma_1(x), \ldots, \sigma_p(x)\right)$$

maps O into an open subset O' of the p-dimensional space, $p \leqslant q$, and at every point x of O at least one of the jacobians

$$J_{j_1,\ldots,j_p}(x) = \frac{\partial(\sigma_1, \ldots, \sigma_p)}{\partial(\xi_{j_1}, \ldots, \xi_{j_p})} = \begin{vmatrix} \dfrac{\partial\sigma_1}{\partial\xi_{j_1}}, \ldots, \dfrac{\partial\sigma_1}{\partial\xi_{j_p}} \\ \cdots\cdots\cdots \\ \dfrac{\partial\sigma_p}{\partial\xi_{j_1}}, \ldots, \dfrac{\partial\sigma_p}{\partial\xi_{j_p}} \end{vmatrix} \qquad (j_1 < \ldots < j_p)$$

does not vanish, i.e.

$$J(x) = \sum_{j_1<\ldots<j_p} J_{j_1,\ldots,j_p}(x)^2 > 0 \quad \text{in } O.$$

We are going to show that the substitution $\varphi(\sigma(x))$, where $\varphi(y)$ is a smooth function defined in O', is a regular operation on $\varphi(y)$ ($\sigma(x)$ being kept fixed). The proof is similar to that of an analoguous statement in Section 10.

Observe first that if, for some smooth functions $\Phi_n(y)$, the sequence $\Phi_n(\sigma(x))$ is fundamental in an open set, so is the sequence $\Phi_n^{(e_j)}(\sigma(x))$. In fact, from

$$\Phi_n(\sigma(x))^{(e_i)} = \sum_{j=1}^{p} \Phi_n^{(e_j)}(\sigma(x)) \cdot \frac{\partial \sigma_j(x)}{\partial \xi_i}$$

we find by algebraic calculations

$$J_{j_1,\dots,j_p}(x)\cdot \Phi_n^{(e_j)}(\sigma(x)) = J_{n,j;j_1,\dots,j_p}(x),$$

where

$$J_{n,j;j_1,\dots,j_p}(x) = \frac{\partial(\sigma_1, \dots, \sigma_{j-1}, \Phi_n(\sigma), \sigma_{j+1}, \dots, \sigma_p)}{\partial(\xi_{j_1}, \dots, \xi_{j_p})}.$$

Hence

$$\Phi_n^{(e_j)}(\sigma(x)) = \frac{1}{J(x)} \sum_{j_1<\dots<j_p} J_{j_1,\dots,j_p}(x)\cdot J_{n,j;j_1,\dots,j_p}(x),$$

which proves the fundamentality of $\Phi_n^{(e_j)}(x)$.

By induction, if a sequence $\Phi_n(\sigma(x))$ is fundamental in an open set, so is the sequence $\Phi_n^{(k)}(\sigma(x))$ for every order k.

Every point x_0 in O is contained in an interval I_0 inside O such that the transformation $\sigma(x)$ maps I_0 into an interval I_0' inside O'. Now let $\Phi_n(y)$ be smooth functions such that, for an order k,

$$\Phi_n^{(k)}(y) = \varphi_n(y) \quad \text{and} \quad \Phi_n(y) \rightrightarrows \quad \text{in } I_0'.$$

Then $\Phi_n(\sigma(x)) \rightrightarrows$ in I_0. Consequently the sequence $\Phi_n^{(k)}(\sigma(x))$, i.e. $\varphi_n(\sigma(x))$, is fundamental in I_0. By 23.2, $\varphi_n(\sigma(x))$ is fundamental in O.

Observe that the hypothesis on $\sigma(x)$ and the above proof can be simplified in the case $q = p$. It suffices then to deal with one jacobian only,

$$\frac{\partial(\sigma_1, \dots, \sigma_q)}{\partial(\xi_1, \dots, \xi_q)},$$

which should be different from 0 in O.

We have proved that substitution is a regular operation. It can therefore be extended onto distributions $f(y) = [\varphi_n(y)]$ defined in O', by assuming

$$f(\sigma(x)) = [\varphi_n(\sigma(x))].$$

When $p = 1$, the above definition coincides with that in Section 10. When $p = q$ and $\sigma(x) = x+h$, it coincides with the definition of translation in Section 8. In the case where $f(y)$ is a continuous or locally integrable function, the distributional substitution $f(\sigma(x))$ coincides with the ordinary substitution of functions, provided $p \leqslant q$. If $p > q$, then, in contrast to the case of functions, the substitution $f(\sigma(x))$ is not always feasible.

Theorem 21.2 remains true also for multidimensional substitutions.

§ 26. Distributions constant in some variables

A distribution $f(x)$ in O is said to be *constant in variables* $\xi_{p+1}, \ldots, \xi_q$ or *independent of* $\xi_{p+1}, \ldots, \xi_q$ $(0 \leqslant p < q)$ iff it can be represented in the form $[\varphi_n(x)]$ where the smooth functions $\varphi_n(x)$ are constant in $\xi_{p+1}, \ldots, \xi_q$.

It follows immediately from the definition that

26.1. *If* $f(x)$ *is constant in* $\xi_{p+1}, \ldots, \xi_q$, *then* $f^{(e_j)}(x) = 0$ *for* $j = p+1, \ldots, q$.

The converse statement is not true, even for functions, in the case of an arbitrary open set Θ.

In fact, let O be a two-dimensional set, defined by the inequality $\xi_1 < |\xi_2|$ (fig. 1) and let

$$f(x) = f(\xi_1, \xi_2) = \begin{cases} 0 & \text{for} \quad \xi_1 < 0, \\ \xi_1^2 & \text{for} \quad 0 \leqslant \xi_1 < \xi_2, \\ -\xi_1^2 & \text{for} \quad 0 \leqslant \xi_1 < -\xi_2 \end{cases}$$

(fig. 2). The function $f(x)$ is continuous in O, and

$$f^{(e_2)}(x) = 0 \quad \text{in } O.$$

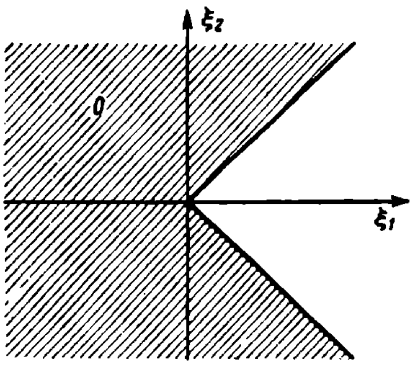

Fig. 1

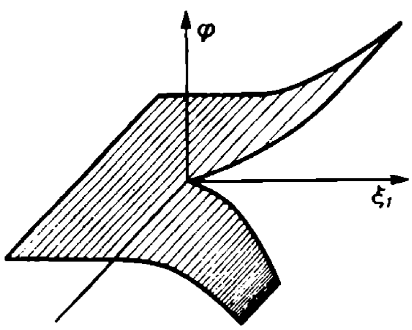

Fig. 2

It is also easy to see that $f(x)$ is constant in ξ_2 in every interval I in O, but it is not constant in ξ_2 in the whole set O.

The converse statement is true, for functions and for distributions, in the case of an open interval:

26.2. *If $f^{(e_j)}(x) = 0$ for $j = p+1, \ldots, q$ in an interval I, then $f(x)$ is constant in $\xi_{p+1}, \ldots, \xi_q$ in I.*

In fact, for any δ-sequence of functions $\delta_n(x)$ vanishing outside $|x| < a_n$ $(a_n \to 0)$, we have $\big(f(x)*\delta_n(x)\big)^{(e_j)} = f^{(e_j)}(x)*\delta_n(x) = 0$ in the interval I_n such that the distance of points in I_n from points outside I is greater than a_n. Thus the smooth functions $\varphi_n(x) = f(x)*\delta_n(x)$ are constant in $\xi_{p+1}, \ldots, \xi_q$ in I_n. Since $f(x) = [\varphi_n(x)]$, the distribution $f(x)$ is constant in $\xi_{p+1}, \ldots, \xi_q$ in I.

For any order $k = (\varkappa_1, \ldots, \varkappa_q)$, k_p will denote the order

$$k_p = (\varkappa_1, \ldots, \varkappa_p, 0, \ldots, 0) = k - \varkappa_{p+1}e_{p+1} - \ldots - \varkappa_q e_q.$$

The following lemma plays fundamental part in the investigation of distributions constant in $\xi_{p+1}, \ldots, \xi_q$:

26.3. *If $\varphi_n(x)$ are smooth functions constant in $\xi_{p+1}, \ldots, \xi_q$ and, for an interval I_0, there exist an order k and smooth functions $\Phi_n(x)$ such that*

$$\Phi_n^{(k)}(x) = \varphi_n(x), \qquad \Phi_n(x) \rightrightarrows \quad \text{in } I_0,$$

then for every interval I inside I_0 there exist smooth functions $\Psi_n(x)$ constant in $\xi_{p+1}, \ldots, \xi_q$ and such that

$$\Psi_n^{(k_p)}(x) = \varphi_n(x), \qquad \Psi_n(x) \rightrightarrows \quad \text{in } I.$$

If $\varkappa_q > 0$, let

$$\bar{\Phi}_n(x) = \frac{1}{\eta}\big(\Phi_n(x+\eta e_q) - \Phi_n(x)\big).$$

We have

$$\bar{\Phi}_n^{(k-e_q)}(x) = \varphi_n(x) \quad \text{and} \quad \bar{\Phi}_n(x) \rightrightarrows.$$

By induction we obtain smooth functions $\tilde{\Phi}_n(x)$ such that

$$\tilde{\Phi}_n^{(k-\varkappa_q e_q)}(x) = \varphi_n(x) \quad \text{and} \quad \tilde{\Phi}_n(x) \rightrightarrows.$$

The functions

$$\tilde{\Psi}_n(x) = \tilde{\Phi}_n(\xi_1, \ldots, \xi_{q-1}, \gamma) \quad (\gamma \text{ constant})$$

are constant in ξ_q,

$$\text{(1)} \qquad \tilde{\Psi}_n^{(k-\varkappa_q e_q)}(x) = \varphi_n(x) \quad \text{and} \quad \tilde{\Psi}_n(x) \rightrightarrows.$$

If $\varkappa_q = 0$ we can write directly $\tilde{\Psi}_n(x) = \Phi_n(\xi_1, \ldots, \xi_{q-1}, \gamma)$ and then (1) also holds.

Similarly, in the case of $p < q-1$, we obtain smooth functions $\bar{\Psi}_n(x)$ constant in ξ_{q-1}, ξ_q, and such that

$$\bar{\Psi}_n^{(k-\varkappa_{q-1}e_{q-1}-\varkappa_q e_q)}(x) = \varphi_n(x) \quad \text{and} \quad \bar{\Psi}_n(x) \rightrightarrows.$$

By induction, there exist smooth functions $\Psi_n(x)$ constant in $\xi_{p+1}, \ldots, \xi_q$ and such that

$$\Psi_n^{(k-\varkappa_{p+1}e_{p+1}-\ldots-\varkappa_q e_q)}(x) = \varphi_n(x) \quad \text{and} \quad \Psi_n(x) \rightrightarrows.$$

If the number η is sufficiently small in all the inductive steps, the last conditions hold in I.

26.4. *If a distribution $f(x)$ constant in $\xi_{p+1}, \ldots, \xi_q$ is, in an interval I_1, the derivative of an order k of a continuous function, then in every interval I inside I_1 the distribution $f(x)$ is the derivative of the order k_p of a continuous function constant in $\xi_{p+1}, \ldots, \xi_q$.*

Let $\delta_n(x)$ be any δ-sequence. If $f(x) = F^{(k)}(x)$ in I_1, then the smooth functions

$$\varphi_n(x) = f(x) * \delta_n(x), \qquad \Phi_n(x) = F(x) * \delta_n(x)$$

satisfy the hypotheses of Lemma 26.3, I_0 being an interval inside I_1 such that I is inside I_0. Thus there exist smooth functions $\Psi_n(x)$ constant in $\xi_{p+1}, \ldots, \xi_q$ such that $\Psi_n^{(k_p)}(x) = \varphi_n(x)$ nad $\Psi_n(x) \rightrightarrows G(x)$ in I. The continuous function $G(x)$ is also constant in $\xi_{p+1}, \ldots, \xi_q$. Since $G(x) = [\psi_n(x)]$, we have

$$G^{(k_p)}(x) = [\varphi_n(x)] = [\Phi_n(x)]^{(k)} = F^{(k)}(x) = f(x) \qquad \text{in } I.$$

26.5. *A distribution $f(x)$ is, in an interval I_0, constant in $\xi_{p+1}, \ldots, \xi_q$ iff, in every interval I inside I_0, $f(x)$ is a derivative of a continuous function constant in $\xi_{p+1}, \ldots, \xi_q$.*

In fact, if $f(x) = F^{(k)}(x)$ in I and $F^{(e_j)}(x) = 0$ for $j = p+1, \ldots, q$, then $f^{(e_j)}(x) = \left(F^{(e_j)}(x)\right)^{(k)} = 0$ in I. Since I is arbitrary, we have $f^{(e_j)}(x) = 0$ in the whole interval I_0, $j = p+1, \ldots, q$. By 26.2, $f(x)$ is constant in $\xi_{p+1}, \ldots, \xi_q$ in I_0.

The remaining part of theorem 26.5 follows from 17.4 and 26.4.

Of course, all the considerations of this Section remain true if we replace $\xi_{p+1}, \ldots, \xi_q$ by an arbitrary set of variables $\xi_{j_1}, \ldots, \xi_{j_r}$ $(1 \leqslant r \leqslant q)$.

§ 27. Dimension of distributions

Distributions defined in an open subset of the q-dimensional space are called *q-dimensional distributions* or distributions of q variables. To emphasize the number of variables, if necessary, we write $f(\xi_1, \ldots, \xi_q)$ instead of $f(x)$. Now we are going to examine relations between p-dimensional distributions and q-dimensional distributions constant in $\xi_{p+1}, \ldots, \xi_q$ $(p < q)$.

Every function $\varphi(\xi_1, \ldots, \xi_p)$ of p variables determines uniquely a corresponding function of q variables $\varphi(\xi_1, \ldots, \xi_q)$ whose value at a point $(\xi_1, \ldots, \xi_q)$ is, for all real $\xi_{p+1}, \ldots, \xi_q$, equal to the value of $\varphi(\xi_1, \ldots, \xi_p)$

at the point $(\xi_1, \ldots, \xi_p)$. Thus, if a p-dimensional function $\varphi(\xi_1, \ldots, \xi_p)$ is defined in an open subset O' of the p-dimensional space, then the corresponding q-dimensional function $\varphi(\xi_1, \ldots, \xi_q)$ is defined in the open set O of all points $(\xi_1, \ldots, \xi_q)$ of the q-dimensional space such that $(\xi_1, \ldots, \xi_p)$ is in O', and $\varphi(\xi_1, \ldots, \xi_q)$ is constant in $\xi_{p+1}, \ldots, \xi_q$.

It is easy to see that, if $\varphi_n(\xi_1, \ldots, \xi_p)$ is a sequence of p-dimensional smooth functions fundamental in O', then the sequence $\varphi_n(\xi_1, \ldots, \xi_q)$ of the corresponding q-dimensional smooth functions is fundamental in O. The converse follows easily from 26.3. Thus:

27.1. *A sequence of p-dimensional smooth functions* $\varphi_n(\xi_1, \ldots, \xi_p)$ *is fundamental in* O' *iff the sequence of corresponding q-dimensional functions* $\varphi_n(\xi_1, \ldots, \xi_q)$ *is fundamental in* O.

Hence, by the definition of equivalent sequences,

27.2. *Two sequences of p-dimensional functions* $\varphi_n(\xi_1, \ldots, \xi_p)$ *and* $\psi_n(\xi_1, \ldots, \xi_p)$ *are equivalent in* O' *iff the corresponding sequences of q-dimensional functions* $\varphi_n(\xi_1, \ldots, \xi_q)$ *and* $\psi_n(\xi_1, \ldots, \xi_q)$ *are equivalent in* O.

By 27.1 and 27.2, every p-dimensional distribution $f(\xi_1, \ldots, \xi_p) = [\varphi_n(\xi_1, \ldots, \xi_p)]$ in O' determines a corresponding q-dimensional distribution $f(\xi_1, \ldots, \xi_q) = [\varphi_n(\xi_1, \ldots, \xi_q)]$ in O, constant in $\xi_{p+1}, \ldots, \xi_q$, and this correspondence is one-to-one. Moreover, every distribution in O, constant in $\xi_{p+1}, \ldots, \xi_q$, corresponds to a distribution in O'.

The question arises whether operations performed on p-dimensional distributions yield the same result as operations performed on corresponding q-dimensional distributions. In order to answer this question, denote by

$$B\big(\varphi(\xi_1, \ldots, \xi_p)\big) = \varphi(\xi_1, \ldots, \xi_q)$$

the q-dimensional smooth function corresponding to a p-dimensional smooth function $\varphi(\xi_1, \ldots, \xi_p)$. By definition, B is an operation performed on p-dimensional smooth functions, its result being a q-dimensional smooth function. This operation is regular. Thus it is extended onto distributions $f(\xi_1, \ldots, \xi_p) = [\varphi_n(\xi_1, \ldots, \xi_p)]$ by assuming

$$B\big(f(\xi_1, \ldots, \xi_p)\big) = \big[B\big(\varphi_n(\xi_1, \ldots, \xi_p)\big)\big].$$

By definition, $B\big(f(\xi_1, \ldots, \xi_p)\big)$ is the q-dimensional distribution $f(\xi_1, \ldots, \xi_q)$ corresponding to the p-dimensional distribution $f(\xi_1, \ldots, \xi_p)$.

Suppose that another regular operation $A(\varphi, \psi, \ldots)$ is given and that

$$B\big(A(\varphi, \psi, \ldots)\big) = A\big(B(\varphi), B(\psi), \ldots\big).$$

This equality is a precise formulation of the fact that the operation A performed on p-dimensional smooth functions yields an analogous result to that obtained on the corresponding q-dimensional functions.

Both sides of the above equality being iterations of regular operations, the same formula holds for distributions:

$$B(A(f, g, \ldots)) = A(B(f), B(g), \ldots).$$

Our result can be formulated more intuitively as follows:

Every regular operation performed on p-dimensional distributions yields an analogous result when performed on the corresponding q-dimensional distributions provided the same situation holds for smooth functions.

The following theorem shows that the limit of a sequence of p-dimensional distributions exists iff it exists for the corresponding q-dimensional distributions and that, moreover, the limits correspond each to another.

27.3. *A sequence of p-dimensional distributions $f_n(\xi_1, \ldots, \xi_p)$ converges in O' to $f(\xi_1, \ldots, \xi_p)$ iff the sequence of the corresponding q-dimensional distributions $f_n(\xi_1, \ldots, \xi_q)$ converges in O to the distribution $f(\xi_1, \ldots, \xi_q)$ corresponding to $f(\xi_1, \ldots, \xi_p)$.*

It is evident that the convergence of the p-dimensional sequence implies the convergence of the q-dimensional sequence to the corresponding limit.

Conversely, suppose that in O

$$f_n(\xi_1, \ldots, \xi_q) \to f(\xi_1, \ldots, \xi_q). \tag{2}$$

Let I' be any interval inside O', let I be the interval of all points $(\xi_1, \ldots, \xi_q)$ such that $(\xi_1, \ldots, \xi_q)$ is in I' and $|\xi_j| < 1$ for $j = p+1, \ldots, q$, and let I_0 be any interval inside O such that I is inside I_0. It follows from (2) that all the distributions $f_n(\xi_1, \ldots, \xi_q)$ are derivatives of a fixed order $k = (\varkappa_1, \ldots, \varkappa_q)$ of continuous functions in I_0. By 26.3 there exist functions $F_n(\xi_1, \ldots, \xi_q)$ constant in $\xi_{p+1}, \ldots, \xi_q$ such that in I

$$F_n^{(kp)}(\xi_1, \ldots, \xi_q) = f_n(\xi_1, \ldots, \xi_q),$$

i.e. in I', for $k' = (\varkappa_1, \ldots, \varkappa_p)$,

$$F_n^{(k')}(\xi_1, \ldots, \xi_p) = f_n(\xi_1, \ldots, \xi_p).$$

Let $\Phi_n(\xi_1, \ldots, \xi_q)$ be smooth functions such that

$$\Phi_n(\xi_1, \ldots, \xi_p) - F_n(\xi_1, \ldots, \xi_p) \rightrightarrows 0, \tag{3}$$

and let

$$\varphi_n(\xi_1, \ldots, \xi_p) = \Phi_n^{(k')}(\xi_1, \ldots, \xi_p).$$

Differentiating (3) k' times, we obtain

$$\varphi_n(\xi_1, \ldots, \xi_p) - f_n(\xi_1, \ldots, \xi_p) \to 0. \tag{4}$$

Hence it follows, by the part of 27.3 just proved, that in I

$$\varphi_n(\xi_1, \ldots, \xi_q) - f_n(\xi_1, \ldots, \xi_q) \to 0.$$

Consequently, by (2),

$$\varphi_n(\xi_1, \ldots, \xi_q) \to f(\xi_1, \ldots, \xi_q),$$

and by 22.3

$$[\varphi_n(\xi_1, \ldots, \xi_q)] = f(\xi_1, \ldots, \xi_q).$$

Hence, for the corresponding p-dimensional functions and distribution, we have in I'

$$[\varphi_n(\xi_1, \ldots, \xi_p)] = f(\xi_1, \ldots, \xi_p),$$

and by 22.3

$$\varphi_n(\xi_1, \ldots, \xi_p) \to f(\xi_1, \ldots, \xi_p).$$

Thus, by (4),

$$f_n(\xi_1, \ldots, \xi_p) \to f(\xi_1, \ldots, \xi_p).$$

Since the interval I' is arbitrary, the convergence holds in O'.

If no ambiguity occurs, q-dimensional distributions constant in $\xi_{p+1}, \ldots, \xi_q$ can be denoted by the symbol $f(\xi_1, \ldots, \xi_p)$ like p-dimensional distributions. A similar convention is widely used for functions.

Of course, all the considerations of this Section remain true if we replace $\xi_{p+1}, \ldots, \xi_q$ by an arbitrary set of variables $\xi_{j_1}, \ldots, \xi_{j_r}$ $(1 \leqslant r \leqslant q)$

§ 28. Distributions with vanishing m-th derivatives

In order to find a general form of a distribution satisfying the condition $f^{(m)}(x) = 0$, we shall prove three auxiliary theorems: 28.1, 28.2 and 28.3.

28.1. *If $f(x)$ is a distribution such that $f^{(\mu e_j)}(x) = 0$ in the interval $a - \varepsilon e_j < x < b + \varepsilon e_j$, then, for $|\eta| < \varepsilon$,*

$$f(x+\eta e_j) = f(x) + \frac{\eta}{1!} f^{(e_j)}(x) + \ldots + \frac{\eta^{\mu-1}}{(\mu-1)!} f^{(\mu e_j - e_j)}(x)$$

in $a < x < b$.

In fact, if $\delta_n(x)$ is a δ-sequence and $\varphi_n(x) = f(x) * \delta_n(x)$, then $f(x) = [\varphi_n(x)]$ and $\varphi_n^{(\mu e_j)}(x) = 0$. Thus the above formula follows from the analoguous Taylor expansion of $\varphi_n(x+\eta e_j)$, by adding the brackets [ ].

28.2. *If $f^{(m+e_j)}(x) = 0$ in O, $m = (\mu_1, \ldots, \mu_q)$, $\mu_j = 0$, then in every interval I inside O the distribution $f(x)$ can be represented in the form*

$$f(x) = g(x) + h(x),$$

where $g^{(e_j)}(x) = 0$ and $h^{(m)}(x) = 0$ in I. Moreover, if $f(x)$ is a smooth continuous or integrable function in I, we may assume that $g(x)$ and $h(x)$ are also such functions.

In fact, the interval I is inside an interval I_0 inside O. If $f^{(m+e_j)}(x) = 0$ in O, then, by 26.2, $f^{(m)}(x)$ is constant in ξ_j in I_0. By 26.5, there exist an order $k = (\varkappa_1, \ldots, \varkappa_q)$, $\varkappa_j = 0$, and a continuous function $F(x)$, constant in ξ_j, such that $F^{(k)}(x) = f^{(m)}(x)$ in I. We may assume that $k \geqslant m$. The distributions $g(x) = F^{(k-m)}(x)$ and $h(x) = f(x) - g(x)$ have the required properties.

If $f(x)$ is a smooth, continuous or integrable function, we can obtain $g(x)$ directly from $f(x)$, by replacing in $f(x)$ the variable ξ_j by a constant γ. Then $g(x)$ is smooth, continuous or (with properly chosen γ) integrable, respectively, and so is $h(x) = f(x) - g(x)$. Moreover, $g^{(e_j)}(x) = 0$. It remains to verify that $h^{(m)}(x) = 0$, i.e. that $f^{(m)}(x) = g^{(m)}(x)$. This is clear when $f(x)$ is smooth. If it is continuous, there exists a sequence of smooth functions $\varphi_n(x)$, almost uniformly convergent to $f(x)$, such that $\varphi_n^{(m+e_j)}(x) = 0$. Replacing in $\varphi_n(x)$ the variable ξ_j by γ we get a sequence of smooth functions $\psi_n(x)$ almost uniformly convergent to $g(x)$. Since

$$f(x) = [\varphi_n(x)], \qquad g(x) = [\psi_n(x)]$$

$$\text{and} \qquad \varphi_n^{(m)}(x) = \psi_n^{(m)}(x),$$

it follows that

$$f^{(m)}(x) = [\varphi_n^{(m)}(x)] = [\psi_n^{(m)}(x)] = g^{(m)}(x).$$

If $f(x)$ is integrable, the proof is the same but the almost uniform convergence must be replaced by L-convergence.

28.3. *If $f^{(m+e_j)}(x) = 0$ in O, $m = (\mu_1, \ldots, \mu_q)$, $\mu_j > 0$, then in every interval I inside O the distribution $f(x)$ can be represented in the form*

$$f(x) = \xi_j g(x) + h(x),$$

where $g^{(m)}(x) = 0$ and $h^{(m)}(x) = 0$ in I. Moreover, when $f(x)$ is a smooth, continuous or integrable function, we may assume that also $f(x)$ and $g(x)$ are such functions.

In fact, let

$$g(x) = \frac{1}{\mu_j \eta}\big(f(x+\eta e_j) - f(x)\big), \qquad h(x) = f(x) - \xi_j g(x). \tag{1}$$

For η sufficiently small, $g(x)$ and $h(x)$ are defined in I. Moreover, if $f(x)$ is a smooth, continuous or integrable function, so are $g(x)$ and $h(x)$. Since $f^{(m+e_j)}(x) = 0$, we have $f^{(m)}(x+\eta e_j) = f^{(m)}(x)$ in I, by 28.1. This implies that $g^{(m)}(x) = 0$. Differentiating the second formula of (1), we find that

$$h^{(m)}(x) = f^{(m)}(x) - \mu_j g^{(m-e_j)}(x) = f^{(m)}(x) + \frac{1}{\eta} f^{(m-e_j)}(x) - \frac{1}{\eta} f^{(m-e_j)}(x+\eta e_j).$$

Since $f^{(m+e_j)}(x) = 0$, the right side vanishes, on account of 28.1.

28.4. *The equality $f^{(m)}(x) = 0$ holds in O iff in every interval I inside O the distribution $f(x)$ can be represented in the form*

$$(2) \qquad f(x) = \sum_{i=0}^{\mu_1 - 1} \xi_1^i f_{1i}(x) + \ldots + \sum_{i=0}^{\mu_q - 1} \xi_q^i f_{qi}(x),$$

where the distributions $f_{ji}(x)$ are constant in ξ_i. Moreover, if $f(x)$ is a smooth, continuous or integrable function, we may assume that all the coefficients $f_{ji}(x)$ are such functions. (In formula (2), if $\mu_j = 0$ for some j, then the corresponding sum must be replaced by 0).

In fact, it is a question of an easy verification that, if (2) holds in I, then $f^{(m)}(x) = 0$ in I. Since I is arbitrary, we have $f^{(m)}(x) = 0$ in O. Conversely, if $f^{(m)}(x) = 0$ in O, we prove (2) by induction. First we remark that (2) is trivially satisfied in the case of $m = 0$. Suppose that it is satisfied for some $m \geqslant 0$. It suffices to show that the corresponding formula holds also for $m + e_j$. In fact, if $\mu_j = 0$, the statement follows from 28.2; if $\mu_j > 0$, the statement follows from 28.3.

Let us remark that representation (2) is not unique. For instance, if $m = (1, 1)$ and $f(x) = \xi_1 + \xi_2$, we can also write $f(x) = (\xi_1 + 1) + (\xi_2 - 1)$.

CONTENTS

ERRATA

Page and line	Instead of	Read
14^{1}	$(x)\,\varphi(y)$	$(x)\,\psi(y)$
21^{6}	$(\varphi_n(x) * \omega(x))_{(e_j)}$	$(\varphi_n(x) * \omega(x))^{(e_j)}$
21^{7}	derivatires	derivatives
22^{15-16}	smooth functions $\varphi_n, \psi_n, \ldots$ such that $\varphi_n \rightrightarrows f$, $\psi_n \rightrightarrows g, \ldots$ and $A(\varphi_n, \psi_n, \ldots) \rightrightarrows A(f, g, \ldots)$.	sequences of smooth functions $\varphi_n, \psi_n, \ldots$ almost uniformly convergent to $f, g, \ldots$ respectively, and such that also $A(\varphi_n, \psi_n, \ldots)$ converges almost uniformly to $A(f, g, \ldots)$.
33^{7}	I'	I'.
34^{14}	set O.	set O,
43_{11}	26.3	26.4