POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CX

A. JANKOWSKI

Algebras of the cohomology operations in some cohomology theories

WARSZAWA 1974
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

Contents

0. Introduction

After the papers of Conner and Floyd [7], Nowikow [13] and others it became clear that the unitary cobordism theory $U^*(\ )$ plays a very important role in algebraic topology.

It turned out that

(i) the theory $U^*(\ )$ has interesting relations with some other cohomology theories,

(ii) the algebra of the cohomology operations $\mathscr{A}_U$ has a fairly simple structure.

The purpose of this paper is to determine the structure of the algebras of the cohomology operations in some cohomology theories.

The main tool in the study of these algebras is their connection with the algebra $\mathscr{A}_U$.

In Section 3 we deal with the theory $BP^*(\ ,Z_p)$, i.e. with the cohomology theory represented by the Brown–Peterson spectrum and with Z_p-coefficients. We establish several properties of the spectrum BP_p, which represents this theory. Here the main result is Proposition 3.9, which characterizes the spectrum BP_p in terms of k-invariants.

Moreover, we consider the structure of the algebra $\mathscr{A}_{BP(\ Z_p)}$. This algebra is certainly quite similar to that of the theory $BP^*_{(p)}(\ )$ and full information concerning its structure can be derived from [14] and Proposition 2.11. There are, however, some differences:

(i) there is an additive base of $\mathscr{A}_{BP(\ ,Z_p)}$ over Λ'_p which in some sense comes from the Steenrod algebra $\mathscr{A}_p$,

(ii) the coefficient ring Λ'_p is a left normal subalgebra of $\mathscr{A}_{BP(\ ,Z_p)}$ and

$$\mathscr{A}_{BP(\ ,Z_p)}//\Lambda'_p \simeq Z_p]\beta[\otimes \mathfrak{A}_p,$$

where $\mathfrak{A}_p$ is the algebra of reduced powers.

In Section 4 we consider the algebra of cohomology operations in cohomology theory $SU^*(\ )$. Here our results are far from complete. We have proved that there is a homomorphism

$$\tilde{\varkappa}: \mathscr{A}_{SU} \to \mathscr{A}_U$$

such that the kernel of $\tilde{\varkappa}$ is the torsion part $\operatorname{Tor}\mathscr{A}_{SU}$ and that this torsion part consists of elements of order 2. Moreover, if the set of operations in $\mathscr{A}_{SU}$ has the property that its image generates im $\tilde{\varkappa}$ as a Γ' module, then this set generates $\mathscr{A}_{SU}$ as a Γ-module. We have been able to get more information concerning the algebra $\mathscr{A}_{SU}$ after localization at the odd primes.

The results of Section 2, which is devoted to the comparison of the algebras of operations in the cohomology theories $h^*(\ )$ and $h^*(\ ,Z_p)$, were estabilished also by Uchida [16]. Our approach is very similar and we have included this section because some details are needed in Section 3.

Although all the proofs in this paper are fairly simple, they are based on more difficult results which we either recall very briefly or do not recall at all. To fill this gap we are careful to give a reference every time it may be needed.

I am greatly indebted to V. M. Buhštaber, A. S. Miščenko and P. S. Novikov for the helpful conversations I had with them during the preparation of this paper.

1. Preliminaries

In the sequel we shall consider some cohomology theories on the category $\mathscr{F}$ of finite CW complexes. All the cohomology theories we shall deal with are represented by ring-like spectra which belong to the homotopy category of spectra. A convenient category of spectra $\mathscr{S}$ was defined in [3]. Here we recall some basic definitions and properties of the category $\mathscr{S}$; the numbers in parentheses refer to the corresponding statements of the mimeographed notes cited above.

One gets the category $\mathscr{S}$ from $\mathscr{F}$ by applying three constructions:

- *stabilization* $\mathscr{F}\mapsto\mathscr{F}_s$ (C.2),
- *completion* $\mathscr{F}_s\mapsto\mathscr{F}_{sw}$ (B.3),
- *homotopy* $\mathscr{F}_{sw}\mapsto\mathscr{F}_{swh}=\mathscr{S}$ (C.9).

The second point needs some comment. The objects of the completed category are directed diagrams of finite spectra (objects of $\mathscr{F}_s$) in which all arrows are just embeddings of sub-spectra. The set of morphisms is defined as a limit in an appropriate sense.

$\mathscr{F}$ is a topological category (with a compact open topology in the set of morphisms) and the operations of stabilization and completion lead from a topological category to a topological one. Therefore $\mathscr{F}_{sw}$ is a topological category and the notion of homotopy is clear.

Note that the category $\mathscr{S}$ is different from that defined earlier by E. Spanier and J. H. C. Whitehead.

The set of morphisms in $\mathcal{S}$ is equipped with the natural structure of an abelian group and is denoted by $\{X, Y\}$. We have some operations in $\mathcal{S}$

— *suspension* S	(C.13),
— *cone* C	(D.11),
— *smash product* $\wedge$	(D.1),
— *identification to a point* $/$	(G.1).

The suspension functor S is an automorphism of the category $\mathcal{S}$. We convert $\mathcal{S}$ into a graded category $\mathcal{S}^*$ as follows: let

$$\{X, Y\}_n = \{X, Y\}^{-n} = \{S^n X, Y\}, \tag{E.1}$$

the elements of $\{X, Y\}_n$ being called *maps of degree n*. Then

$$\{X, Y\}^* = \bigoplus_n \{X, Y\}^n \tag{E.2}$$

is the set of morphisms in the graded category.

If $A \subset X$ is an inclusion of spectra, we have a sequence of maps

$$A \to X \to X/A \to A, \tag{J.2}$$

where the last map is of degree -1, which we call a *standard exact triangle*. For an arbitrary spectrum Y the sequences

$$\ldots \to \{Y, A\}^* \to \{Y, X\}^* \to \{Y, X/A\}^* \to \{Y, A\}^* \to \ldots \tag{J.14}$$

and

$$\to \{A, Y\}^* \to \{X/A, Y\}^* \to \{X, Y\}^* \to \{A, Y\}^* \to \ldots \tag{J.15}$$

are exact.

The triangle

$$A \to B \to C \to A \tag{J.4}$$

is called *exact* if it is isomorphic to some standard exact triangle.

The basic fact concerning exact triangles is that, given a map

$$A \xrightarrow{f} B,$$

we can include it into exact triangles:

$$\begin{gathered} A \xrightarrow{f} B \to C \to A, \\ D \to A \xrightarrow{f} B \to D. \end{gathered} \tag{J.11}$$

Sometimes we shall call a spectrum an object which in the terminology of [3] is called a *pre-spectrum* (H.17). This, in view of (H.19) and (H.20), is quite admissible.

Let $h^*(\ )$ be a cohomology theory. The algebra of the stable cohomology operations in the theory $h^*(\ )$ on the category $\mathscr{F}$ will be denoted by $\mathscr{A}_h$.

Let H be the spectrum which represents $h^*(\ )$ and let $\{H_n\}$ be the filtration of H by finite spectra. It is not difficult to see that

$$\mathscr{A}_h = \lim_{\substack{\leftarrow \\ n}} \{H_n, H\}^*,$$

which in general is not equal to

$$\{H, H\}^*$$

It turns out, however, that in all cases considered in this paper the equality

$$\mathscr{A}_h = \{H, H\}^*$$

holds.

Let Λ be the coefficient ring of the theory $h^*(\ )$. There is a homomorphism

$$\Lambda \to \mathscr{A}_h,$$

since each element λ of Λ can be considered as a stable operation: multiplication on λ. Recall that the algebra $\mathscr{A}_h$ has the following structures:

(i) multiplication — a composition of operations,

(ii) the left Λ-module structure — composition with multiplication on the elements of Λ on the left,

(iii) the right Λ-module structure — composition with multiplication on the elements of Λ on the right,

(iv) if $\mathscr{A}_h$ is a flat left Λ-module, then there is a diagonal map

$$\mathscr{A}_h \to \mathscr{A}_h \otimes_\Lambda \mathscr{A}_h$$

which is a left Λ-module homomorphism; here $\otimes_\Lambda$ is a tensor product of the left Λ-modules.

2. Generalized cohomology theories with a coefficient group Z_p

Let $\{h^n, \sigma_n\}$ be the generalized cohomology theory on the category $\mathscr{F}$. We shall briefly recall the definition and properties of the cohomology groups $h^n(\ , Z_p)$, the groups $h^n(\ )$ with the coefficient Z_p. For details we refer the reader to [2].

Let M_p be the co-Moore space, $M_p = S^1 \cup_p D^2$. The generalized cohomology theory $\{h^n(\ , Z_p), \sigma_n\}$ is defined as follows.

For a space X we put

$$h^n(X, Z_p) = h^{n+2}(X \wedge M_p).$$

The suspension homomorphism σ_n is a composition

$$h^n(X, Z_p) = h^{n+2}(X \wedge M_p) \xrightarrow{\sigma_{n+2}} h^{n+3}(S(X \wedge M_p)) \to h^{n+3}(SX \wedge M_p)$$
$$= h^{n+1}(SX, Z_p)$$

and, for a map $f\colon X \to Y$, we define $f^*\colon h^n(Y, Z_p) \to h^n(X, Z_p)$ to be the composition

$$h^n(Y, Z_p) = h^{n+2}(Y \wedge M_p) \xrightarrow{(f\wedge \mathrm{id})^*} h^{n+2}(X \wedge M_p) = h^n(X, Z_p).$$

Let

$$S^1 \xrightarrow{\varphi} M_p \xrightarrow{\psi} S^2$$

be the cofibre sequence, where φ is the canonical inclusion and ψ is the collapsing map.

The cohomology operation

$$\beta\colon h^n(X, Z_p) \to h^{n+1}(X, Z_p)$$

is defined as the composition

$$h^n(X, Z_p) = h^{n+2}(X \wedge M_p) \xrightarrow{(\mathrm{id}\wedge\varphi)^*} h^{n+2}(X \wedge S^1) \xrightarrow{\sigma} h^{n+3}(X \wedge S^2)$$
$$\xrightarrow{(\mathrm{id}\wedge\psi)^*} h^{n+3}(X \wedge M_p) = h^{n+1}(X, Z_p).$$

Assume that h^* is a multiplicative cohomology theory with commutative and associative multiplication and with unit $1 \in \Lambda^0$. The question of the existence of compatible multiplication in $h^*(\ , Z_p)$ was discussed in detail in [2]. We assume in the sequel that, for a fixed prime p, the cohomology theory h^* satisfies the conditions of Theorems 10.6 and 10.7 of [2].

ASSUMPTION 1. 1. *There is admissible commutative and associative multiplication in* $h^*(\ , Z_p)$.

2. $h^*(pt, Z_p)$ *is a* Z_p*-module.*

The coefficient ring $h(pt, Z_p)$ will be denoted by Λ_p.

Remark. We shall deal mainly with the unitary cobordism theory. In this case there does not exist an admissible commutative and associative multiplication in $U^*(\ , Z_2)$. This can be seen from what follows: let $\bar{\eta} \in \{S^2M_2, S^2\}$ be the generator considered in [2], Proposition 6.3. The necessary condition for the existence of a "good" multiplication in $U^*(\ , Z_2)$ is that $\bar{\eta}^*\colon U^*(S^2) \to U^*(S^2M_2)$ be a zero map. If that was true it should also be true for the complex K-theory by [7], Theorem 10.1. But it was shown in [2] that for the complex K-theory $\bar{\eta}^*$ is not zero.

Therefore we shall exclude the case of $p = 2$ from our considerations.

ASSUMPTION 2. *p is an odd prime.*

Under this assumption we can simplify our discussion observing that in this case the unique admissible commutative and associative multiplication in $h^*(\ ,Z_p)$ is the composition

$$\begin{aligned} h^i(X,Z_p)\otimes h^j(Y,Z_p) &= h^{i+2}(X\wedge M_p)\otimes h^{j+2}(Y\wedge M_p) \\ &\xrightarrow{\mu} h^{i+j+4}(X\wedge M_p\wedge Y\wedge M_p)\xrightarrow{T^*} h^{i+j+4}(X\wedge Y\wedge M_p\wedge M_p) \\ &\xrightarrow{(\mathrm{id}\wedge\alpha)^*} h^{i+j+4}(X\wedge Y\wedge S^2M_p)\xrightarrow{\sigma^{-1}\sigma^{-2}} h^{i+j+2}(X\wedge Y\wedge M_p) = h^{i+j}(X\wedge Y, Z_p), \end{aligned}$$

where μ is the multiplication in h^* and

$$\alpha\colon S^2M_p\to M_p\wedge M_p$$

is a properly chosen stable map such that the diagram

$$\begin{tikzcd} & M_p\wedge M_p \arrow[dl, "\mathrm{id}\wedge\psi"'] & \\ M_p\wedge S^2 & & M_p\wedge S^2 \arrow[ll, "\mathrm{id}"'] \arrow[ul, "\alpha"'] \end{tikzcd}$$

is stably homotopy commutative. The details can be found in [2].

We want to describe the relation between the algebras of cohomology operations on theories h^* and $h^*(\ ,Z_p)$. This result was obtained independently by Uchida [16] but, since we need some extra information, we shall sketch the proofs here.

We need the following result:

LEMMA 2.1. *Let*

$$u\colon X\wedge Y\to\Sigma$$

be a duality map. Then u induces a map

$$\varkappa_u\colon\{A\wedge Y, B\}\to\{A, B\wedge X\}$$

which is natural in all variables and is an isomorphism if A, B are arbitrary and X, Y are finite spectra.

This is K.13 of [3]; the direct proof is given in [16].

Let K_p be the Moore complex $K'(Z_p, 2) = S^2\cup_p D^3$,

$$u\colon K_p\wedge M_p\to S^4$$

the duality map and

$$S^2 \xrightarrow{\psi'} K_p \xrightarrow{\varphi'} S^3$$

the standard cofibre sequence which is 4-dual to that of M_p.

As the immediate corollary of the above lemma we get the following proposition:

PROPOSITION 2.2. *If the cohomology theory h^* is represented by the spectrum $\boldsymbol{H}$, then the cohomology theory $h^*(\ ,Z_p)$ is represented by the spectrum*

$$S^{-2}\boldsymbol{H}\wedge K_p.$$

Let

$$a'\colon K_p\wedge K_p \to K_p\wedge S^2$$

be the stable map 8-dual to α. It follows from the known properties of duality that the diagram

$$\begin{tikzcd}
 & K_p\wedge K_p \arrow[dr, "a'"] & \\
K_p\wedge S^2 \arrow[ur, "\mathrm{id}\wedge\psi'"] \arrow[rr, "\mathrm{id}"] & & K_p\wedge S^2
\end{tikzcd}$$

is stably homotopy commutative.

Let $\mu\colon \boldsymbol{H}\wedge\boldsymbol{H}\to\boldsymbol{H}$ be the multiplication in the spectrum $\boldsymbol{H}$ which induces the multiplication in the cohomology theory h^*. From the naturality of the map $\varkappa_u$ easily follows:

LEMMA 2.3. *The multiplication in $h^*(\ ,Z_p)$ is induced by the composition*

$$S^{-2}\boldsymbol{H}\wedge K_p\wedge S^{-2}\boldsymbol{H}\wedge K_p \to S^{-2}\boldsymbol{H}\wedge S^{-2}\boldsymbol{H}\wedge K_p\wedge K_p$$
$$\xrightarrow{\mu\wedge a'} S^{-4}\boldsymbol{H}\wedge K_p\wedge S^2 \to S^{-2}\boldsymbol{H}\wedge K_p.$$

Recall that an algebra of all stable cohomology operations in the cohomology theory h^* on the category $\mathscr{S}_0$ is denoted by $\mathscr{A}_h$.

For $\theta\in\mathscr{A}_h$ define

$$\Phi(\theta)\colon h^*(\ ,Z_p)\to h^*(\ ,Z_p)$$

by the formula

$$\Phi(\theta)_X = \theta_{X\wedge M_p}.$$

It is clear that $\Phi(\theta)$ is a stable cohomology operation in $h^*(\ ,Z_p)$; therefore we got a map

$$\Phi\colon \mathscr{A}_h\to\mathscr{A}_{h(\ ,Z_p)}.$$

We have the following proposition:

PROPOSITION 2.4. *The map Φ is a degree preserving homomorphism. Operations belonging to* $\operatorname{im}\Phi$ *commute with the operation* $\beta \in \mathscr{A}^1_{h(\ ,Z_p)}$.

Proof. It is clear that Φ is homomorphism. In order to prove that $\Phi(\theta)$ commutes with β consider the diagram

$$\begin{tikzcd}
h^n(X, Z_p) \arrow[r, phantom, "="] \arrow[d, "\Phi(\theta)"] & h^{n+2}(X \wedge M_n) \arrow[r, "(\mathrm{id} \wedge \varphi)^*"] \arrow[d, "\theta"] & h^{n+2}(X \wedge S^1) \arrow[r, "\sigma"] \arrow[d, "\theta"] \arrow[dr, phantom, "I"] & {} \\
h^{n+k}(X, Z_p) \arrow[r, phantom, "="] & h^{n+k+2}(X \wedge M_p) \arrow[r, "(\mathrm{id} \wedge \varphi)^*"] & h^{n+k+2}(X \wedge S^1) \arrow[r, "\sigma"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & h^{n+3}(X \wedge S^2) \arrow[r, "(\mathrm{id} \wedge \psi)^*"] \arrow[d, "\theta"] & h^{n+3}(X \wedge M_p) \arrow[r, phantom, "="] \arrow[d, "\theta"] & h^{n+1}(X, Z_p) \arrow[d, "\Phi(\theta)"] \\
{} \arrow[r] & h^{n+k+3}(X \wedge S^2) \arrow[r, "(\mathrm{id} \wedge \psi)^*"] & h^{n+k+3}(X \wedge M_p) \arrow[r, phantom, "="] & h^{n+k+1}(X, Z_p).
\end{tikzcd}$$

All the squares commute except I, which commutes up to the sign $(-1)^k$. It follows that

$$\Phi(\theta)\beta = (-1)^k \beta \Phi(\theta),$$

where k is a degree of $\Phi(\theta)$.

The following lemma gives the structure of $h^*(K_p, Z_p)$ as the A_p-module.

LEMMA 2.5. *$h^*(K_p, Z_p)$ is a free A_p-module on generators $a \in h^2(K_p, Z_p)$ and $b \in h^3(K_p, Z_p)$.*

Proof. The cofibration

$$S^2 \xrightarrow{\psi'} K_p \xrightarrow{\varphi'} S^3$$

leads to an exact triangle

$$\begin{tikzcd}
h(S^3, Z_p) \arrow[dr, "\varphi'^*"'] & & h(S^2, Z_p) \arrow[ll, "\delta"'] \\
& h(K_p, Z_p) \arrow[ur, "\psi'^*"'] &
\end{tikzcd}$$

where δ is the suspension homomorphism followed by multiplication on p. It follows from Assumption 1 that $\delta = 0$ and we get a short exact sequence

$$0 \to h^*(S^3, Z_p) \to h^*(K_p, Z_p) \to h^*(S^2, Z_p) \to 0$$

of A_p-modules. Now $h^*(S^i, Z_p)$ is a free A_p-module on one generator $a_i \in h^i(S^i, Z_p)$ and the lemma follows.

Assumption 3. *Λ is of finite type.*

Lemma 2.6. *For any spectrum $\boldsymbol{X}$ we have an isomorphism*

$$h^*(K_p \wedge \boldsymbol{X}, Z_p) = h^*(K_p, Z_p) \otimes_{\Lambda_p} h^*(\boldsymbol{X}, Z_p).$$

Proof. Let $\boldsymbol{X}_n$ be the filtration of $\boldsymbol{X}$ by finite spectra. It follows from Assumption 3 that $h^*(\boldsymbol{X}_n, Z_p)$ are finite in each dimension. Therefore from [3] H.4 we have

$$h^*(\boldsymbol{X}, Z_p) = \varprojlim_n h^*(\boldsymbol{X}_n, Z_p)$$

and

$$h^*(K_p \wedge \boldsymbol{X}, Z_p) = \varprojlim_n h^*(K_p \wedge \boldsymbol{X}_n, Z_p)$$

because in both cases the Rlim term is zero.

Applying Lemma 2.5 we get

$$h^*(K_p \wedge \boldsymbol{X}_n, Z_p) = h^*(K_p, Z_p) \otimes_{\Lambda_p} h^*(\boldsymbol{X}_n, Z_p),$$

and the lemma follows by passing to the inverse limit, which in this case commutes with $\otimes_{\Lambda_p}$.

It will be convenient to consider the complex K_p as the spectrum. With this understood, we have the equivalence of the spectra

$$S^{-2}(K_p \wedge \boldsymbol{H}) \to (S^{-2}K_p) \wedge \boldsymbol{H}.$$

Let us denote by Ω a composed map

$$\begin{aligned} h^*(S^{-2}K_p, Z_p) \otimes_{\Lambda_p} h^*(\boldsymbol{H}, Z_p) &\to h^*(S^{-2}K_p \wedge \boldsymbol{H}, Z_p) \\ &\to h^*\big(S^{-2}(K_p \wedge \boldsymbol{H}), Z_p\big) \to h^*\big(S^{-2}(\boldsymbol{H} \wedge K_p), Z_p\big) \\ &\to \lim h^*\big(S^{-2}(\boldsymbol{H}_n \wedge K_p), Z_p\big) = \mathscr{A}_{h(\ ,Z_p)}. \end{aligned}$$

It is not difficult to see, by using the Rlim argument and the associativity and commutativity of the multiplication in $h^*(\ , Z_p)$, that the following proposition holds.

Proposition 2.7. *The map*

$$\Omega\colon h^*(S^{-2}K_p, Z_p) \otimes_{\Lambda_p} h^*(\boldsymbol{H}, Z_p) \to \mathscr{A}_{h(\ ,Z_p)}$$

is a Λ_p-module homomorphism and an isomorphism.

We have an element $\varrho \in h^0(\boldsymbol{H}, Z_p)$, the reduction mod$p$ operation, represented by the map

$$\mathrm{id} \wedge \psi'\colon \boldsymbol{H} \wedge S^2 \to \boldsymbol{H} \wedge K_p.$$

Lemma 2.8. *The following relations hold:*

$$\Omega(\sigma^{-2}a, \varrho) = 1, \quad (\sigma^{-2}b, \varrho) = \beta.$$

Proof. The elements $\sigma^{-2}a, \sigma^{-2}b$ are represented by the maps

$$\Sigma \wedge K_p \xrightarrow{j\wedge \mathrm{id}} \boldsymbol{H} \wedge K_p,$$

$$\Sigma \wedge K_p \xrightarrow{\mathrm{id}\wedge\varphi'} \Sigma \wedge S^3 \xrightarrow{\mathrm{id}\wedge S\psi'} \Sigma \wedge SK_p \xrightarrow{j\wedge\mathrm{id}} \boldsymbol{H} \wedge SK_p \to S\boldsymbol{H} \wedge K_p,$$

where

$$j\colon \Sigma \to \boldsymbol{H}$$

is the unit map.

Therefore the element $\Omega(\sigma^{-2}a, \varrho)$ is represented by the composition

$$\boldsymbol{H} \wedge S^2K_p \to \boldsymbol{H} \wedge S^2 \wedge \Sigma \wedge K_p \xrightarrow{\mathrm{id}\wedge\psi'\wedge j\wedge\mathrm{id}} \boldsymbol{H} \wedge K_p \wedge \boldsymbol{H} \wedge K_p$$
$$\to \boldsymbol{H} \wedge \boldsymbol{H} \wedge K_p \wedge K_p \xrightarrow{\mu\wedge\alpha'} \boldsymbol{H} \wedge S^2K_p.$$

Since $\mu(\mathrm{id} \wedge j) = \mathrm{id}$ and $\alpha'(\psi' \wedge \mathrm{id}) = \mathrm{id}$, we get

$$\Omega(\sigma^{-2}a, \varrho) = \mathrm{id}.$$

Similarly, the element $\Omega(\sigma^{-2}b, \varrho)$ is represented by the composition

$$\boldsymbol{H} \wedge S^2K_p \to \boldsymbol{H} \wedge S^2 \wedge \Sigma \wedge K_p \xrightarrow{\mathrm{id}\wedge\psi'\wedge j\wedge S\psi'\varphi'} \boldsymbol{H} \wedge K_p \wedge \boldsymbol{H} \wedge SK_p$$
$$\to \boldsymbol{H} \wedge \boldsymbol{H} \wedge K_p \wedge SK_p \xrightarrow{\mu\wedge S\alpha'} \boldsymbol{H} \wedge S^3K_p \to S\boldsymbol{H} \wedge S^2K_p$$

and $S\alpha'(S\psi'\varphi' \wedge \psi') = S\psi'\varphi' \wedge \mathrm{id}$. Hence this composition is just $S^2(\mathrm{id} \wedge S\psi'\varphi')$ and this map represents β.

From the cofibre sequence of spectra

$$S^2\boldsymbol{H} \to \boldsymbol{H} \wedge K_p \to S^3\boldsymbol{H}$$

we get an exact sequence

$$0 \to h^*(\boldsymbol{H}) \otimes Z_p \xrightarrow{m} h^*(\boldsymbol{H}, Z_p) \xrightarrow{n} \mathrm{Tor}(h^*(\boldsymbol{H}), Z_p) \to 0.$$

LEMMA 2.9. *The relation*

$$m(\mathrm{id} \otimes 1_p) = \varrho$$

holds.

ASSUMPTION 4. 1. *The map*

$$h^*(\boldsymbol{H}) \to \mathscr{A}_h$$

is an isomorphism.

2. $\mathrm{Tor}(h^*(\boldsymbol{H}), Z_p) = 0$.

Let us define the maps

$$\Phi_i\colon \mathscr{A}_h \to \mathscr{A}_{h(\ ,Z_p)},$$

$$\Psi_i\colon \mathscr{A}_h \to h^*(S^{-2}K_p, Z_p) \otimes_{A_p} h^*(\boldsymbol{H}, Z_p)$$

for $i = 0, 1$ by the formulas

$$\Phi_0 = \Phi, \quad \Phi_1(\theta) = \beta\Phi(\theta),$$

$$\Psi_0(\theta) = \sigma^{-2}a \otimes m(\theta \otimes 1_p), \quad \Psi_1(\theta) = \sigma^{-2}b \otimes m(\theta \otimes 1_p).$$

LEMMA 2.10. *The diagrams*

$$\begin{tikzcd}
h^*(S^{-2}K_p, Z_p) \otimes_{\Lambda_p} h^*(H, Z_p) \arrow[r, "\Omega"] & \mathscr{A}_{h(\ ,Z_p)} \\
 & \mathscr{A}_h \arrow[u, "\Phi_i"'] \arrow[ul, "\Psi_i"]
\end{tikzcd}$$

are commutative for $i = 0, 1$.

Proof. The proof is quite similar to that of Lemma 2.8 but somewhat longer.

Let $Z_p]\beta[$ be the exterior algebra on one generator β in dimension 1 and let

$$\lambda\colon Z_p]\beta[\to h^*(S^{-2}K_p, Z_p)$$

be a map such that

$$\lambda(1) = \sigma^{-2}a, \quad \lambda(\beta) = \sigma^{-2}b.$$

Let

$$Z_p]\beta[\otimes \mathscr{A}_h$$

be the tensor product of algebras and define a map

$$\Omega_1\colon Z_p]\beta[\otimes \mathscr{A}_h \to \mathscr{A}_{h(\ ,Z_p)}$$

as the composition

$$Z_p]\beta[\otimes \mathscr{A}_n \xrightarrow{\lambda\otimes\pi} h^*(S^{-2}K_p, Z_p) \otimes_{\Lambda_p} (\mathscr{A}_h \otimes Z_p)$$
$$\xrightarrow{\mathrm{id}\otimes m} h(S^{-2}K_p, Z_p) \otimes_{\Lambda_p} h^*(H, Z_p) \xrightarrow{\Omega} \mathscr{A}_{h(,\ Z_p)}.$$

With the material discussed in this section at hand, it is easy to show the following proposition:

PROPOSITION 2.11. *The map*

$$\Omega_1\colon Z_p]\beta[\otimes \mathscr{A}_h \to \mathscr{A}_{h(\ ,Z_p)}$$

is an isomorphism of algebras.

3. Cohomology theory $BP^*(\ ,Z_p)$

Let $U^*(\ )$ be the unitary cobordism theory represented by the spectrum MU. Recall that

$$\Lambda = U^*(pt) = Z[Y_1, \ldots, Y_i, \ldots], \quad \dim Y_i = -2i.$$

The algebra of cohomology operations in $U^*(\ )$ was determined by Novikov [13]. Additively

$$\mathscr{A}_U = \Lambda \hat{\otimes} \mathscr{S},$$

where $\mathscr{S}$ is a subalgebra of $\mathscr{A}_U$ and is a free Z-module generated by the operations S_ω.

More information about the algebra $\mathscr{A}_U$ the reader may find in [13] and [11].

We shall denote by $U^*_{(p)}(\ )$ the unitary cobordism theory localized at a prime p. Recall that

$$U^*_{(p)}(X) = U^*(X) \otimes Z_{(p)},$$

where $Z_{(p)}$ denotes the integers localized at the prime ideal (p). This theory is represented by some spectrum $MU_{(p)}$, the spectrum MU localized at the prime p. For details we refer the reader to [1].

The structure of the algebra $\mathscr{A}_{U_{(p)}}$ was determined in [1]. We have

$$\mathscr{A}_{U_{(p)}} = \mathscr{A}_U \otimes Z_{(p)}$$

and the map

$$\mathscr{A}_U \to \mathscr{A}_{U_{(p)}},$$

which is defined in an obvious way, is

$$\theta \mapsto \theta \otimes 1$$

under the identification above.

We may introduce the coefficient group Z_p into $U^*_{(p)}(\ )$ but we get nothing new. In fact the following lemma holds.

LEMMA 3.1. *There is a natural isomorphism*

$$\tau_p\colon\ U^*(\ , Z_p) \to U_{(p)}(\ , Z_p)$$

of cohomology theories.

Proof. We have a natural transformation

$$\tau\colon\ U^*(\ ) \to U^*_{(p)}(\ ) = U^*(\ ) \otimes Z_{(p)}$$

defined by

$$\tau(a) = a \otimes 1.$$

Transformation τ induces a transformation τ_p

$$\begin{tikzcd}
U^*(X, Z_p) \arrow[r, "\tau_p"] \arrow[d, equal] & U^*_{(p)}(X, Z_p) \arrow[d, equal] \\
U^*(X \wedge M_p) \arrow[r, "\tau"] \arrow[dr, "\mathrm{id} \otimes 1"'] & U^*_{(p)}(X \wedge M_p) \arrow[d, equal] \\
& U^*(X \wedge M_p) \otimes Z_{(p)}
\end{tikzcd}$$

Since $U^*(X \wedge M_p)$ is a Z_p-module, the homomorphism $\mathrm{id} \otimes 1$ is an isomorphism. Therefore τ_p is an isomorphism.

It follows from the above lemma and the results of Section 2 that there is a commutative diagram of the algebra homomorphisms

$$\begin{tikzcd}
\mathscr{A}_U \arrow[rr] \arrow[dr, "\Phi"'] & & \mathscr{A}_{U_{(p)}} \arrow[dl, "\Phi'"] \\
& \mathscr{A}_{U(\ ,Z_p)} &
\end{tikzcd}$$

LEMMA 3.2. *If the operation θ is multiplicative, then the operations $\Phi(\theta), \Phi'(\theta)$ are multiplicative, too.*

Proof. This is an easy consequence of the fact that in the case under consideration the multiplication in $U^*(\ , Z_p)$ is induced by the multiplication in $U^*(\ )$ and a map

$$\alpha\colon S^2M_p \to M_p \wedge M_p.$$

It was shown by Quillen, [14], that there exists a multiplicative idempotent

$$\xi \in \mathscr{A}^0_{U_{(p)}}$$

which splits the theory $U^*_{(p)}(\ )$ into the sum of cohomology theories isomorphic to $BP^*_{(p)}$, where $BP^*_{(p)}(\ )$ is a cohomology theory represented by the Brown–Peterson spectrum BP localized at the prime p.

For the definition and properties of the spectrum BP we refer the reader to [4].

There is an analogous splitting of the theory $U^*(\ , Z_p)$.

PROPOSITION 3.3. *The cohomology operation*

$$\Phi'(\xi) \in \mathscr{A}^0_{U(\ ,Z_p)}$$

is a multiplicative idempotent which splits $U^*(\ , Z_p)$ into the sum of cohomology theories isomorphic to $BP^*(\ , Z_p)$.

Proof. We know from Lemma 3.2 that $\Phi'(\xi')$ is a multiplicative idempotent and it is a standard fact that $\operatorname{im}\Phi'(\xi)$ is a cohomology theory.

Let $\Lambda' = BP_{(p)}(pt) = Z_{(p)}[Y_1, \ldots, Y_i, \ldots]$, $\dim Y_i = -2(p^i-1)$. Λ' is a subring of $\Lambda \otimes Z_{(p)}$.

Quillen's result is stated as follows. There is a natural isomorphism

$$\Lambda_{(p)} \otimes_{\Lambda'} BP^*_{(p)}(X) \xrightarrow{\cong} U^*_{(p)}(X)$$

and $BP^*_{(p)}(X) \hookrightarrow U^*_{(p)}(X)$ is the image of ξ_X.

Applying this to $X \wedge M_p$, we get

$$U^*(X, Z_p) = U^*_{(p)}(X \wedge M_p) = \Lambda_{(p)} \otimes_{\Lambda'} BP^*_{(p)}(X \wedge M_p),$$

where

$$BP^*_{(p)}(X\wedge M_{(p)}) \hookrightarrow U^*_{(p)}(X\wedge M_p)$$

is the image of $\xi X\wedge M_p$. It is easy to see that under the identification $U^*_{(p)}(X\wedge M_p) = U^*(X, Z_p)$ we have

$$\operatorname{im}\Phi'(\xi)_X = \operatorname{im}\xi X\wedge M_p = BP^*_{(p)}(X\wedge M_p).$$

In order to complete the proof we have to show that

$$BP^*_{(p)}(X\wedge M_p) \simeq BP^*(X, Z_p)$$

with a shift of indices. To do this observe that the map

$$BP\wedge K_p \to BP_{(p)}\wedge K_p$$

is a homotopy equivalence of spectra because it induces an isomorphism of homotopy groups. This, together with Proposition 2.2, completes the proof.

We shall denote the spectrum $S^{-2}BP\wedge K_p$ which represents $BP^*(\ ,Z_p)$ by BP_p and similarly for $S^{-2}MU\wedge K_p$.

Recall Quillen's result on the structure of the algebra $\mathscr{A}_{BP_{(p)}}$ as stated in [17]. There is a coalgebra $\mathscr{R}$ freely generated over Z by the operations r_E and

$$\mathscr{A}_{BP_{(p)}} = \Lambda'\hat{\otimes}\mathscr{R}.$$

This result allows us to apply Proposition 2.11 in the case of the $BP^*_{(p)}(\ )$ theory, and we get the following proposition.

PROPOSITION 3.4. *There is an isomorphism of algebras*

$$\mathscr{A}_{BP(\ ,Z_p)} \simeq Z_p]\beta[\otimes\mathscr{A}_{BP_{(p)}}.$$

Since the multiplicative structure of the algebra $\mathscr{A}_{BP_{(p)}}$ is known, the above proposition describes $\mathscr{A}_{BP(\ ,Z_p)}$ completely. However, we shall give simpler description of this algebra later.

Let $\mathscr{A}_p$ be the Steenrod algebra, that is the algebra of operations in ordinary cohomology theory with Z_p coefficients. The structure of this algebra is known, [12], and we assume that the reader is familiar with that paper.

Recall, [4], that $H^*(BP, Z_p)$ is an $\mathscr{A}_p$ module with one generator $1\in H^0(BP, Z_p)$ and is isomorphic to

$$\mathscr{A}_p/(Q_0).$$

Let $\mathscr{L}\subset\mathscr{A}_p$ be the two-sided ideal generated by Q_i, $i\geqslant 1$. The following lemma gives the structure of the cohomology of BP_p.

LEMMA 3.5. (i) *$H^*(BP_p, Z_p)$, as an $\mathscr{A}_p$ module, is isomorphic to*

$$\mathscr{A}_p/\mathscr{L}$$

with generator $1 \in H^0(BP_p, Z_p)$.

(ii) *$H^*(BP_p, Z_q) = 0$ for q prime to p.*

Proof.

$$H^i(K_p, Z_p) = \begin{cases} Z_p & \text{generated by } u \text{ for } i = 2, \\ Z_p & \text{generated by } v \text{ for } i = 3, \\ 0 & \text{for } i \neq 2, 3 \end{cases}$$

with $Q_0 u = v$ and all other operations acting trivially. We have an isomorphism

$$H^*(BP_p, Z_p) \simeq H^*(BP, Z_p) \otimes H^*(K_p, Z_p)$$

with a shift of dimensions. Let $1 \in H^0(BP_p, Z_p)$ be an element which goes to $1 \otimes u$ under this isomorphism, and define a map

$$\mathscr{A}_p \to H^*(BP_p, Z_p)$$

by $\theta \to \theta(1)$.

It is clear that this map is an epimorphism and that the ideal $\mathscr{L}$ lies in the kernel. To see that $\mathscr{L}$ is equal to the kernel it is sufficient to observe that the elements $\mathscr{P}^R(1)$, $\mathscr{P}^R Q_0(1)$ are independent in $H^*(BP_p, Z_p)$. This completes the proof, for the second part is clear.

Brown and Peterson, [4], provided us with some homotopy decomposition of the spectrum BP. Making use of their result, we are able to construct an analogous filtration of the spectrum BP_p.

Let $\mathscr{R}$ be the set of finite sequences $R = (r_i)$, $r_i \geqslant 0$, and let

$$\dim R = \sum_i 2r_i(p^i - 1), \qquad l(R) = \sum_i r_i.$$

Let W_s be a Z_p-vector space with base $\{R; l(R) = s\}$.

PROPOSITION 3.6. *There exists a homotopy decomposition*

$$\{Y_s, k_R^s \in H^*(Y_s, Z_p); l(R) = s+1\}$$

of the spectrum BP_p such that:

(i) $Y_0 = K(W_0)$,

(ii) *the fibre of $Y_s \to Y_{s-1}$ is $K(W_s)$,*

(iii) *the k-invariants satisfy the relations*

$$\sum_j Q_j k^s_{R-\Delta_j} = 0 \quad \text{for } l(R) = s+2,$$

(iv) *if* $i\colon K(W_s)\to Y_s$ *is the fibre of* $\pi_s\colon Y_s\to Y_{s-1}$, *then*

$$i^* k^s_R = \sum_j Q_j \beta_{R-\Delta_j},$$

where $\beta_T \in H\big(K(W_s), Z_p\big)$ *is the part of the fundamental class corresponding to the generator* T *of* W_s.

Proof. Let $\{X_s, \tilde{k}^s_R\}$ be the homotopy decomposition of the spectrum BP given in [4]. We shall multiply this decomposition on K_p. But in order to show (iii) and (iv) we have to write down some details.

Let V_s be a free abelian group generated by the set $\{R;\ l(R) = s\}$. There is a homotopy equivalence of spectra

$$\varphi_s\colon S^{-2}K(V_s)\wedge K_p \to K(W_s)$$

such that

$$\varphi_s^* \beta_R = a_R \otimes u,$$

where a_R, is a part of the reduction mod p of the fundamental class corresponding to the generator R.

We are going to define inductively the sequence of spaces Y_s and k-invariants k^s_R such that conditions (i)–(iv) are satisfied, and, moreover, to define a sequence of homotopy equivalences of spectra

$$\psi_s\colon S^{-2}X_s \wedge K_p \to Y_s$$

which commutes with the projections and is such that

$$\psi_s^* k^s_R = \bar{k}^s_R \otimes u;$$

here $\bar{k}^s_R$ is a reduction mod p of the integral class $\tilde{k}^s_R$.

For $s = 0$ we put

$$\begin{aligned} Y_0 &= K(W_0),\\ \psi_0 &= \varphi_0\colon S^{-2}K(V_0)\wedge K_p \to K(W_0),\\ k^0_{\Delta_j} &= Q_j \beta_0. \end{aligned}$$

We have

$$\begin{aligned} \psi_0^* k^0_{\Delta_j} &= \psi_0^* Q_j \beta_0 = Q_j(a_0\otimes u) = Q_j a_0 \otimes u\\ &= (Q_0 c\mathscr{P}^{\Delta_j} a_0 - c\mathscr{P}^{\Delta_j} Q_0 a_0)\otimes u = Q_0 c\mathscr{P}^{\Delta_j} a_0 \otimes u = \bar{k}^0_{\Delta_j}\otimes u. \end{aligned}$$

Condition (iii) in this case is

$$Q_i Q_j + Q_j Q_i = 0, \qquad Q_j^2 = 0$$

and this relations hold in $\mathscr{A}_p$.

Assume now that $Y_{s-1}, k_R^{s-1}, \psi_{s-1}$, etc. are defined and satisfy all the conditions. Let

$$f_{s-1}\colon Y_{s-1} \to SK(W_s)$$

be a map such that

$$f_{s-1}^* \sigma \beta_R = k_R^{s-1}$$

and

$$g_{s-1}\colon X_{s-1} \to SK(V_s)$$

a map such that

$$g_{s-1}^* \sigma \tilde{\alpha}_R = \tilde{k}_R^{s-1}.$$

The square

$$\begin{tikzcd}
Y_{s-1} \arrow[r, "f_{s-1}"] & SK(W_s) & \\
S^{-2}X_{s-1} \wedge K_p \arrow[u, "\psi_{s-1}"] \arrow[rr, "g_{s-1} \wedge \mathrm{id}"] & & S^{-1}K(V_s) \wedge K_p \arrow[ul, "\varphi_s"']
\end{tikzcd}$$

is homotopy commutative.

The spectrum X_s is by definition the kernel of the map g_{s-1}; therefore $S^{-2}X_s \wedge K_p$ is the kernel of $g_{s-1} \wedge \mathrm{id}$. It follows that the diagram above can be completed by the dotted arrows to the diagram

$$\begin{tikzcd}
Y_s \arrow[r, dashed, "\pi"] & Y_{s-1} \arrow[r, "f_{s-1}"] & SK(W_s) \\
S^{-2}X_s \wedge K_p \arrow[u, dashed, "\psi_s"] \arrow[r, "\pi \wedge \mathrm{id}"] & S^{-2}X_{s-1} \wedge K_p \arrow[u, "\psi_{s-1}"] \arrow[r, "g_{s-1} \wedge \mathrm{id}"] & S^{-1}K(V_s) \wedge K_p, \arrow[u, "\varphi_s"]
\end{tikzcd}$$

where Y_s is the kernel of f_{s-1}. The map ψ_s is a homotopy equivalence by the "five lemma".

Now choose elements k_R^s such that

$$\psi_s^* k_R^s = \bar{k}_R^s \otimes u.$$

We have

$$\begin{aligned}
\psi_s^* \Big(\sum_j Q_j k_{R-\Delta_j}^s \Big) &= \sum_j Q_j \psi_s^* k_{R-\Delta_j}^s = \sum_j Q_j (\bar{k}_{R-\Delta_j}^s \otimes u) = \Big(\sum_j Q_j \bar{k}_{R-\Delta_j}^s \Big) \otimes u \\
&= \Big(\sum_j (Q_0 c\mathscr{P}^{\Delta_j} - c\mathscr{P}^{\Delta_j} Q_0) \bar{k}_{R-\Delta_j}^s \Big) \otimes u = \Big(\sum_j Q_0 c\mathscr{P}^{\Delta_j} \bar{k}_{R-\Delta_j}^s \Big) \otimes u \\
&= \Big(Q_0 \sum c\mathscr{P}^{\Delta_j} \bar{k}_{R-\Delta_j}^s \Big) \otimes u = (Q_0 \bar{k}_R^s) \otimes u = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}\varphi_s^* i^* k_R^s &= i^* \psi_s^* k_R^s = i^* (\bar{k}_R^s \otimes u) = (i^* \bar{k}_R^s) \otimes u \\ &= \sum_j (Q_0 c\mathscr{P}^{\Delta_j} \tilde{a}_{R-\Delta_j}) \otimes u = \sum_j Q_j (\tilde{a}_{R-\Delta_j} \otimes u) = \sum_j Q_j \varphi_s^* \beta_{R-\Delta_j} \\ &= \varphi_s^* \Big(\sum_j Q_j \beta_{R-\Delta_j} \Big).\end{aligned}$$

Here we have used some relations between classes $\bar{k}_R^s$, see [4], p. 152. Because ψ_s^*, φ_s^* are isomorphisms, this completes the proof.

We want to know to what extent conditions (i)–(iv) of Proposition 3.6 determine the spectrum BP_p. In this connection we have the following result.

PROPOSITION 3.7. *Each spectrum Y which has a homotopy decomposition satisfying conditions* (i)–(iv) *of Proposition* 3.6 *has a cohomology satisfying* (i) *and* (ii) *of Lemma* 3.5.

Proof. As in the proof of Proposition 3.6, we shall follow the arguments of [4].

Let $\mathscr{B}$ be a subalgebra of $\mathscr{A}_p$ generated by Q_i, $i \geqslant 1$. We have a $\mathscr{B}$-free resolution of Z_p:

$$\to \mathscr{B} \otimes W_s \xrightarrow{d_s} \mathscr{B} \otimes W_{s-1} \to \ldots \mathscr{B} \otimes W_0 \to Z_p \to 0,$$

where

$$d_s(1 \otimes R) = \sum_j Q_j \otimes (R - \Delta_j).$$

Tensoring this resolution over $\mathscr{B}$ by $\mathscr{A}_p$, which is a free $\mathscr{B}$-module, we get a resolution

$$\to N_s \xrightarrow{d_s} N_{s-1} \to \ldots \to N_0 \to \mathscr{A}_p / \mathscr{L} \to 0,$$

where

$$N_s = \mathscr{A}_p \otimes W_s = H^*(K(W_s), Z_p)$$

and d_s is an $\mathscr{A}_p$ homomorphism defined by the formula

$$d_s(1 \otimes R) = \sum_j Q_j \otimes (R - \Delta_j).$$

Define a map

$$\tau_{s+1} \colon N_{s+1} \to H^*(Y_s, Z_p)$$

by the formula

$$\tau_{s+1}(1 \otimes R) = k_R^s.$$

We are going to show inductively that

(a) coker $\tau_{s+1} \simeq \mathscr{A}_p/\mathscr{L}$,
(b) the sequence

$$N_{s+2} \xrightarrow{d_{s+2}} N_{s+1} \xrightarrow{\tau_{s+1}} H(Y_s, Z_p)$$

is exact.

For $s = 0$ we have

$$Y_0 = K(W_0)$$

and

$$\tau_{s+1} = d_1;$$

therefore (a) and (b) are satisfied.

Suppose that (a) and (b) are true for $s-1$ and consider the diagram

$$\begin{tikzcd}
N_{s+2} \arrow[r, "d_{s+2}"] & N_{s+1} \arrow[dr, "d_{s+1}"] \arrow[dd, "\tau_{s+1}"] & & \\
& & N_s \arrow[r, "\tau_s"] & H^*(Y_{s-1}, Z_p) \\
H^*(Y_{s-1}, Z_p) \arrow[r, "\pi^*"] & H^*(Y_s, Z_p) \arrow[ur, "i^*"'] & &
\end{tikzcd}$$

with the exact rows. The triangle is commutative because

$$i^*\tau_{s+1}(1 \otimes R) = i^* k_R^s = \sum_j Q_j \beta_{R-\Delta_j} = \sum_j Q_j \otimes (R - \Delta_j);$$

the last equation holds under the identification

$$\mathscr{A}_p \otimes W_s \simeq H^*\big(K(W_s), Z_p\big).$$

Moreover, $\tau_{s+1}\, d_{s+2} = 0$ because

$$\tau_{s+1} d_{s+2}(1 \otimes R) = \tau_{s+1} \sum_j Q_j \otimes (R - \Delta_j) = \sum_j Q_j \tau_{s+1}(1 \otimes R - \Delta_j)$$
$$= \sum_j Q_j k^s_{R-\Delta_j} = 0.$$

From these observations and the inductive assumption it easily follows that the sequence

$$N_{s+2} \xrightarrow{d_{s+2}} N_{s+1} \xrightarrow{\tau_{s+1}} H^*(Y_s, Z_p)$$

is exact and that $\ker i^*$ is mapped isomorphically onto coker τ_{s+1}. Therefore

$$\mathscr{A}_p/\mathscr{L} \simeq \operatorname{im} \pi^* = \ker i^* \simeq \operatorname{coker} \tau_{s+1}.$$

Thus we have proved condition (i) and this finishes the proof, for (ii) is clear.

Let $Y_R \in A'_p = Z_p[Y_1, \ldots, Y_i, \ldots]$ denote the element

$$Y_1^{r_1} \ldots Y_i^{r_i}.$$

Note that $\dim Y_R = -\dim R$. Let $\Lambda^{p,q}$ be the Z_p vector space with base $\{Y_R;\ \dim R = -(p+q),\ l(R) = p\}$. We have

$$\Lambda_p' = \bigoplus_{p,q} \Lambda^{p,q}$$

and it is clear that Λ_p' has become a bigraded ring.

Proposition 3.6 has the following corollary.

COROLLARY 3.8. *There is a spectral sequence*

$$\{E_r^{s,t}, d_r\} \Rightarrow BP^*(X, Z_p)$$

such that

$$E_1^{s,t} = \bigoplus_{i+j=t} \Lambda^{s,j} \otimes H^i(X, Z_p)$$

and the differential d_1 on the homogeneous elements is given by the formula:

$$d_1(Y_R \otimes x) = \sum_j Y_{R+\Delta_j} \otimes Q_j x.$$

Proof. Filtration of Proposition 3.6 defines an exact couple such that

$$E_1^{s,t} = \{S^{-(s+t)}X, K(W_s)\},$$
$$D_1^{s,t} = \{S^{-(s+t)}X, Y_s\}.$$

Each element of $E_1^{s,t}$ can be uniquely written in the form

$$x = \sum_{R, l(R)=s} x_R,$$

where

$$x_R \in H^{\dim R + s + t}(X, Z_p).$$

Let a map

$$\mu\colon E_1^{s,t} \to \bigoplus_{i+j=t} \Lambda^{s,j} \otimes H^i(X, Z_p)$$

be defined by

$$\mu(x) = \sum_R Y_R \otimes x_R.$$

It is clear that μ is an isomorphism. Let

$$\bar{d}_1\colon \bigoplus_{i+j=t} \Lambda^{s,j} \otimes H^i(X, Z_p) \to \bigoplus_{i+j=t} \Lambda^{s,j} \otimes H^i(X, Z_p)$$

be defined on the homogeneous elements by the formula

$$\bar{d}_1(Y_R \otimes x) = \sum_j Y_{R+\Delta_j} \otimes Q_j x.$$

We shall show that

$$\bar{d}_1\mu = \mu d_1.$$

A homogeneous element $x \in E_1^{s,t}$ is of the form

$$x = f^*\beta, \quad f^*\beta_S = 0 \quad \text{for } S \neq R,$$

where

$$f\colon S^{-(s+t)}X \to K(W_s)$$

and $\beta = \sum_{l(R)=s} \beta_R$ is a fundamental class.

Let $(d_1x)_T$, $l(T) = s+1$, be the T component of d_1x in $E_1^{s+1,t}$. We have

$$(d_1x)_T = f^*i^*k^*\beta_T = f^*\sum Q_j\beta_{T-\Delta_j} = \sum Q_jx_{T-\Delta_j}.$$

But $x_{T-\Delta_j} = 0$ for $T-\Delta_j \neq R$ and we get

$$(d_1x)_T = \begin{cases} 0 & \text{for } T-\Delta_j \neq R, \\ Q_jx & \text{for } T-\Delta_j = R. \end{cases}$$

Therefore

$$\mu d_1 x = \sum_j Y_{R+\Delta_j} \otimes Q_jx = d_1(Y_R \otimes x) = \bar{d}_1\mu(x),$$

and it follows that the first differential is as stated.

The following proposition gives the characterization of the spectrum BP_p in terms of the k-invariants.

PROPOSITION 3.9. *Let Y be the spectrum which has a homotopy decomposition satisfying* (i)–(iv) *of Proposition* 3.6 *and, moreover,*

(v) Y *is a ring-like spectrum.*

Then the spectra Y and BP_p are homotopy equivalent.

Proof. Let

$$f_0\colon BP \to K(Z_p) = Y_0$$

be a map such that

$$f_0^*\beta_0 = 1 \in H^0(BP, Z_p).$$

Since $H^{\mathrm{odd}}(BP, Z_p) = 0$, the map f_0 can be lifted to Y, for all k_R^s are odd-dimensional.

We get a map

$$f\colon BP \to Y$$

such that

$$f^*1 = 1 \in H^0(BP, Z_p).$$

The cofibre sequence

$$BP \xrightarrow{p} BP \to BP_p \to BP$$

leads to the exact sequence

$$\{BP, Y\}^* \to \{BP_p, Y\}^* \to \{BP, Y\}^* \xrightarrow{p} \{BP, Y\}^*.$$

It follows from (v) that $\{X, Y\}^*$ consists of elements of order p for each finite spectrum X. Therefore, the application of Theorem 1 of [10] shows that

$$\{BP, Y\}^*$$

consists of elements of order p.

We see that the map

$$f\colon BP \to Y$$

can be lifted so as to give a map

$$\bar{f}\colon BP_p \to Y$$

such that the diagram

$$\begin{tikzcd}
BP_p \arrow[dr, "\bar{f}"] & \\
& Y \\
BP \arrow[uu] \arrow[ur, "f"'] &
\end{tikzcd}$$

is commutative.

Hence

$$\bar{f}^*1 = 1 \in H^0(BP_p, Z_p).$$

Now, Lemma 3.5 and Proposition 3.7 show that

$$\bar{f}^*\colon H^*(Y, Z_p) \to H^*(BP_p, Z_p)$$

is an isomorphism.

The application of the Bockstein spectral sequence shows that there are no elements of order p^2 in $H^*(Y, Z)$ and $H^*(BP_p, Z)$. Moreover, (ii) of Lemma 3.5 shows that there are no elements of finite order prime to p or of infinite order in these groups.

Therefore

$$\bar{f}^*\colon H^*(Y, Z) \to H^*(BP_p, Z)$$

is an isomorphism and the general coefficient theorem shows that

$$\bar{f}_*\colon H_*(BP_p, Z) \to H_*(Y, Z)$$

is an isomorphism. Here the reader has to note that

$$H^{ev}(BP_p, Z) = H^{ev}(Y, Z) = 0.$$

Since the spectra BP_p and Y are connected, the map f is a homotopy equivalence.

This concludes the proof.

We complete our study of the properties of the spectrum BP_p with the following proposition.

PROPOSITION 3.10. *The Hurewicz homomorphism*

$$\pi_*(BP_p) \to H_*(BP_p)$$

is a zero map, except in the dimension zero.

Proof. We have a commutative diagram

$$\begin{tikzcd}
H(BP_{(p)}, Z) \arrow[r, "p"] & H(BP_{(p)}, Z) \arrow[r] & H(BP_p, Z) & \\
& \pi_*(BP_{(p)}) \arrow[u, "\chi_1"] \arrow[r] & \pi_*(BP_p) \arrow[u, "\chi_2"] \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

It is known that the polynomial generators of

$$\Lambda' = Z_{(p)}[\ldots, Y_i, \ldots]$$

can be so chosen that all their Chern numbers are divisible on p. Therefore

$$\operatorname{im}\chi_1 \subset \operatorname{im} p$$

and we get $\chi_2 = 0$.

Now we are going to relate the algebras $\mathscr{A}_{BP(\ ,Z_p)}$ and $\mathscr{A}_{U(\ ,Z_p)}$.

LEMMA 3.11. *There exists a homomorphism of the algebras*

$$i\colon \mathscr{A}_{BP(\ ,Z_p)} \to \mathscr{A}_{U(\ ,Z_p)}$$

which is a monomorphism and commutes with diagonal maps.

Proof. There exist multiplicative natural transformations

$$BP(\ ,Z_p) \xrightarrow{\varepsilon_1} U(\ ,Z_p) \xrightarrow{\varepsilon_2} BP(\ ,Z_p)$$

such that

$$\varepsilon_1\varepsilon_1 = \mathrm{id}.$$

For $\theta \in \mathscr{A}_{BP(\ ,Z_p)}$ we define

$$i(\theta) = \varepsilon_1\theta\varepsilon_2.$$

To see that i is a monomorphism observe that it has an inverse. Now

$$i(\theta_1\theta_2) = \varepsilon_1\theta_1\theta_2\varepsilon_2 = \varepsilon_1\theta_1\varepsilon_2\varepsilon_1\theta_2\varepsilon_2 = i(\theta_1)i(\theta_2)$$

and

$$\begin{aligned} i(\theta)(xy) &= (\varepsilon_1\theta\varepsilon_2)(xy) = \varepsilon_1\theta\varepsilon_2(xy) \\ &= \varepsilon_1\theta(\varepsilon_2x)(\varepsilon_2y) = \sum(\varepsilon_1\theta_j'\varepsilon_2x)(\varepsilon_1\theta_j''\varepsilon_2y) \end{aligned}$$

for each x, y; here

$$\psi(\theta) = \sum \theta_j' \otimes \theta_j''$$

is a diagonal map in $\mathscr{A}_{BP(\ ,Z_p)}$. Because the operations in $\mathscr{A}_{U(\ ,Z_p)}$ can be distinguished by their values on cohomology classes, we get

$$\psi i(\theta) = i \otimes i\psi(\theta).$$

There is a remarkable relation between the algebras $\mathscr{A}_p$ and $\mathscr{A}_{BP(\ ,Z_p)}$ due to Landweber [11].

Recall that for each multiplicative cohomology theory $h^*(\ )$ such that $h^i(pt) = 0$ for $i > 0$ there exists a multiplicative transformation

$$\chi\colon h^*(\ ) \to H^*\big(\ , h^0(pt)\big),$$

called the *Hopf homomorphism.*

Let

$$\varrho_p\colon U^*(\ , Z_p) \to H^*(\ , Z_p)$$

be the Hopf homomorphism for $U^*(\ , Z_p)$; it is a reduction mod p of the Hopf homomorphism

$$\varrho\colon U^*(\ ) \to H^*(\ , Z).$$

We can, in view of Proposition 2.11, restate Lanweber's result as follows.

LEMMA 3.12. *There exists a map*

$$\psi\colon \mathscr{A}_p \to \mathscr{A}_{U(\ ,Z_p)}$$

such that

(i) *ψ is an algebra homomorphism and commutes with diagonal maps,*

(ii) $\ker\psi = \mathscr{L}$,

(iii) $\varrho_p \psi(\theta) = \theta\varrho_p$,

(iv) $\psi(Q_0) = \beta$.

Proof. Landweber showed that there exists a homomorphism of the Hopf algebras

$$\Gamma\colon \mathscr{A}_p \to Z_p \otimes \mathscr{S}$$

defined as follows.

The map

$$1 \otimes \varrho\colon Z_p \otimes \mathscr{A}_U \to Z_p \otimes H^*(MU)$$

maps the subalgebra $Z_p \otimes \mathscr{S}$ isomorphically onto

$$Z_p \otimes H^*(MU) = H^*(MU, Z_p)$$

and we put

$$\Gamma(\theta) = (1\otimes\varrho)^{-1}\theta(1).$$

Now it follows from Proposition 2.11 that $\mathscr{A}_{U(\ ,Z_p)}$ contains a subalgebra $Z_p]\beta[\otimes\mathscr{S}$, and it is easy to see that ϱ_p maps this subalgebra isomorphically onto $H^*(MU_p, Z_p)$; the diagram

$$\begin{tikzcd}
Z_p\otimes\mathscr{S} \arrow[d, "1\otimes\varrho"] & Z_p]\beta[\otimes\mathscr{S} \arrow[l] \arrow[d, "\varrho_p"] \\
H^*(MU, Z_p) & H^*(MU_p, Z_p) \arrow[l]
\end{tikzcd}$$

is commutative and, moreover,

$$\varrho_p(\beta) = v = Q_0(1).$$

Therefore all the statements of the lemma follow from the corresponding statements of Theorem 8.1 of [11].

Let

$$\bar{\varrho}\colon BP^*(\ ,Z_p)\to H^*(\ ,Z_p)$$

be the Hopf homomorphism in the $BP^*(\ ,Z_p)$ theory. It is not hard to see that

$$\bar{\varrho}\varepsilon_2 = \varrho, \qquad \varrho\varepsilon_1 = \bar{\varrho}.$$

PROPOSITION 3.13. *There exists a map*

$$\bar{\psi}\colon \mathscr{A}_p\to\mathscr{A}_{BP(\ ,Z_p)}$$

such that

(i) $\bar{\psi}$ *commutes with diagonal maps*,

(ii) $\ker\bar{\psi} = \mathscr{L}$,

(iii) $\bar{\varrho}\bar{\psi}(\theta) = \theta\bar{\varrho}$,

(iv) $\bar{\psi}(Q_0) = \beta$,

(v) $\mathscr{A}_{BP(\ ,Z_p)} \simeq \Lambda'_p\otimes\operatorname{im}\bar{\psi}$.

Proof. Let

$$j\colon \mathscr{A}_{U(\ ,Z_p)}\to\mathscr{A}_{BP(\ ,Z_p)}$$

be the left inverse of the map i defined in Lemma 3.11. We have

$$j(\theta) = \varepsilon_2\theta\varepsilon_1.$$

The map j is not the algebra homomorphism but commutes with diagonal maps and

$$j(\beta) = \beta.$$

We define a map $\bar{\psi}$ as the composition

$$\bar{\psi} = j\psi.$$

It follows that

$$\bar{\psi}(Q_0) = \beta$$

and let

$$T^R \in \mathscr{A}_{BP(\ ,Z_p)}$$

be $\bar{\psi}(\mathscr{P}^R)$.

Consider the commutative diagram

$$\begin{tikzcd}
& \mathscr{A}_p \arrow[dl, "\bar{\psi}"'] \arrow[dr, "\psi"] & \\
\mathscr{A}_{BP(\ ,Z_p)} \arrow[d, "\bar{\varrho}"'] & & \mathscr{A}_{U(\ ,Z_p)} \arrow[ll, "j"] \arrow[d, "\varrho_p"] \\
H^*(BP_p, Z_p) & & H^*(MU_p, Z_p) \arrow[ll, "\varepsilon_1^*"']
\end{tikzcd}$$

Since

$$\varrho_p\psi(\mathscr{P}^R) = \mathscr{P}^R(1), \qquad \varrho_p\psi(Q_0) = Q_0(1)$$

and

$$\varepsilon_1^*(1) = 1 \in H^*(BP_p, Z_p),$$

we see that

$$\bar{\varrho}\bar{\psi}(\mathscr{P}^R), \quad \bar{\varrho}\bar{\psi}(Q_0), \quad \bar{\varrho}\bar{\psi}(Q_0\mathscr{P}^R)$$

is a Z_p-base of $H^*(BP_p, Z_p)$.

Therefore the standard spectral sequence argument shows that the set of the operations

$$T^R, \beta T^R$$

is a Λ'_p-base of $\mathscr{A}_{BP(\ ,Z_p)}$. This shows (ii) and (v). The other statements are clear.

Let us denote by $\mathscr{T}$ the coalgebra

$$\bar{\psi}\big(\mathscr{A}_p/(Q_0)\big).$$

COROLLARY 3.14. *Additively*

$$\mathscr{A}_{BP(\ ,Z_p)} \simeq \Lambda'_p]\beta[\otimes\mathscr{T}$$

This result is analogous to Proposition 3.4, the difference being that the coalgebra $\mathscr{T}$ comes from $\mathscr{A}_p$. I do not know how to compute the multiplicative structure of the algebra $\mathscr{A}_{BP(\ ,Z_p)}$ in terms of the operations T^R. The methods used by Adams and Quillen do not apply. We have the following result, which shows that the situation is somewhat different from that for $U^*(\ )$ and $BP^*_{(p)}(\ )$:

PROPOSITION 3.15. *Λ'_p is a left normal subalgebra of $\mathscr{A}_{BP(\ ,Z_p)}$ and the map*

$$\bar{\psi}:\mathscr{A}_p\to\mathscr{A}_{BP(\ ,Z_p)}$$

induces the isomorphism

$$\bar{\psi}: Z_p]\beta[\otimes\mathfrak{A}_p\to\mathscr{A}_{BP(\ ,Z_p)}//\Lambda'_p.$$

Proof. We must show that, the right ideal

$$\Lambda'_p\mathscr{A}$$

is a left ideal as well. It clearly consists of elements of the from

$$\sum_R\lambda_R T^R+\beta\sum_S\lambda_S T^S$$

with $\dim\lambda_R$, $\dim\lambda_S<0$, and it is sufficient to show that

$$T^S\lambda$$

belongs to $\Lambda'_p\mathscr{A}$ for arbitrary S and $\dim\lambda<0$. But

$$T^S\lambda=\sum_{S=S_1+S_2}T^{S_1}(\lambda)T^{S_2}$$

and $T^S\lambda=0$ provided $|S|\geqslant\dim\lambda>0$. Therefore

$$T^S\lambda=\sum_{S=S_1+S_2,\,|S_1|<\dim\lambda}T^{S_1}(\lambda)T^{S_2}\in\Lambda'_p\mathscr{A}.$$

The skeletal filtration of the spectrum BP_p induces the filtration

$$\mathscr{A}_{BP(\ ,Z_p)}=F^0\supset F^{-1}\supset\ldots,$$

and it follows from the triviality of the Atiyah–Hirzebruch spectral sequence that the map

$$\bar{\varrho}:\mathscr{A}_{BP(\ ,Z_p)}\to H^*(BP_p,Z_p)$$

is an epimorphism with kernel F^{-1}. Moreover, the standard argument shows that

$$F^{-1}=\Lambda'_p\mathscr{A}.$$

Hence in order to show that the composite map

$$\mathscr{A}_p\xrightarrow{\bar{\psi}}\mathscr{A}_{BP(\ ,Z_p)}\to\mathscr{A}_{BP(\ ,Z_p)}//\Lambda'_p$$

is a homomorphism we have to show that

$$\bar{\varrho}(\bar{\psi}(\theta_1\theta_2))=\bar{\varrho}(\bar{\psi}(\theta_1)\bar{\psi}(\theta_2)).$$

To this end note that:

(i) for $\alpha \in \mathscr{A}_{U(\ ,Z_p)}$

$$\bar{\psi}(\alpha) = \varepsilon_2 \alpha \varepsilon_1,$$

(ii) for $\alpha \in \mathscr{A}_{BP(\ ,Z_p)} \bar{\varrho}(\alpha) = \alpha^*$, where we consider α as a map

$$\alpha: BP_p \to BP_p,$$

(iii) if $\alpha = \psi(\theta) \in \mathscr{A}_{U(\ ,Z_p)}$, then $\alpha^* 1 = \theta 1 \in H^*(MU_p, Z_p)$.

Now let $\psi(\theta_1) = \alpha_1$, $\psi(\theta_2) = \alpha_2$. Then

$$\begin{aligned}\bar{\varrho}\big(\bar{\psi}(\theta_1)\bar{\psi}(\theta_2)\big) &= (\varepsilon_2 \alpha_1 \varepsilon_1 \varepsilon_2 \alpha_2 \varepsilon_1)^* 1 = \varepsilon_1^* \alpha_2^* \varepsilon_2^* \varepsilon_1^* \cdot \alpha_1^* 1 = \varepsilon_1^* \alpha_2^* \varepsilon_2^* \varepsilon_1^* \theta_1 1 \\ &= \theta_1(\varepsilon_1^* \alpha_2^* \varepsilon_2^* \varepsilon_1^* 1) = \theta_1(\varepsilon_1^* \alpha_2^* 1) = \theta 1(\varepsilon_1^* \theta_2 1) = \theta_1 \theta_2 \varepsilon_1^* 1 = \theta_1 \theta_2 1\end{aligned}$$

and

$$\begin{aligned}\bar{\varrho}\big(\bar{\psi}(\theta_1 \theta_2)\big) &= (\varepsilon_2^+ \alpha_1 \alpha_2 \varepsilon_1^*) 1 = \varepsilon_1^* \alpha_2^* \alpha_1^* \varepsilon_2^* 1 = \varepsilon_1^* \alpha_2^* \alpha_1^* 1 = \varepsilon_1^* \alpha_2^* \theta_1 1 \\ &= \varepsilon_1^* \theta_1 \alpha_2^* 1 = \varepsilon_1^* \theta_1 \theta_2 1 = \theta_1 \theta_2 1.\end{aligned}$$

Moreover, the composition above is epimorphic and its kernel is $\mathscr{L}$. It follows that $\bar{\psi}$ induces the desired isomorphism. The proof of Proposition 3.15 is concluded.

4. Cohomology theories $SU^*(\ )$ and $W^*(\ )$

The aim of this section is to compare the algebras $\mathscr{A}_{SU}$ and $\mathscr{A}_U$. We shall need some cohomology theory $W^*(\ )$, which lies between $SU^*(\ )$ and $U^*(\ )$.

Recall, [13], that there are operations

$$\Delta, \psi, \partial \in \mathscr{A}_U$$

which satisfy the following relations:

(i) $\Delta \psi = 1$,

(ii) $\partial^2 = 0$,

(iii) $\Delta \partial = 0$, $\partial \psi = 0$.

Let $\pi = [\psi, \Delta] = \psi \Delta - \Delta \psi = \psi \Delta - 1$. This operation is an idempotent element of $\mathscr{A}_U$. We denote by $W^*(\ )$ a generalized cohomology theory $\operatorname{im} \pi$.

Let W be the spectrum defined as follows:

$$W_{2n} = MSU(n-1) \wedge CP(2), \qquad W_{2n+1} = SW_{2n}$$

and $\varepsilon_{2n+1}: S^2 W_{2n} \to W_{2(n+1)}$ is a composition

$$S^2 W_{2n} = S^2 MSU(n-1) \wedge CP(2) \xrightarrow{\varepsilon_{2n+1} \wedge \mathrm{id}} MSU(n) \wedge CP(2) = W_{2(n+1)}.$$

The space W_{2n} can be identified with the space

$$M(\xi_{n-1}\oplus\eta),$$

where ξ_{n-1} is the classifying bundle over $BSU(n-1)$ and η is a Hopf bundle over $S^2 = CP(1)$.

Therefore there is a map

$$\mu_{2n}\colon W_{2n}\to MU(n)$$

induced by the map which classifies the bundle $\xi_{n-1}\otimes\eta$ over $SU(n-1)\times \times CP(1)$.

Clearly the diagrams

$$\begin{tikzcd} S^2W_{2n} \arrow[r, "S^2\mu_{2n}"] \arrow[d, "\varepsilon_{2n+1}"] & S^2MU(n) \arrow[d, "\tilde{\varepsilon}_{2n+1}"] \\ W_{2(n+1)} \arrow[r, "\mu_{2(n+1)}"] & MU(n+1) \end{tikzcd}$$

are commutative and we get a map of spectra

$$\mu\colon \boldsymbol{W}\to \boldsymbol{MU}.$$

PROPOSITION 4.1. *The cohomology theory* $W^*(\ )$ *is represented by the spectrum* $\boldsymbol{W}$ *defined above.*

Proof. Let us denote, for a moment, the cohomology theory represented by the spectrum $\boldsymbol{W}$ by $H^*(\ ,\boldsymbol{W})$.

The map $\mu\colon W\to MU$ induces the transformation

$$\mu\colon H^*(\ ,\boldsymbol{W})\to U^*(\ ).$$

We have to show first that μ maps $H^*(\ ,W^*)$ into $W^*(\ )$. It is sufficient to show that

$$\Delta\mu = 0$$

as the cohomology operation from $H^*(\ ,\boldsymbol{W})$ to $U^*(\ )$. The spectrum $\boldsymbol{W}$ has no torsion in homology and MU has no torsion in homotopy. Therefore it is sufficient to show that $\Delta\mu$ acts trivially on the spheres, [10]. But this was done in [6], Theorem 17.1.

It follows that μ induces a natural transformation

$$\bar{\mu}\colon H^*(\ ,\boldsymbol{W})\to W^*(\ )$$

and it was shown in [6], Section 15 and Theorem 17.1 that

$$\bar{\mu}\colon H(pt, W)\to W^*(pt)$$

is an isomorphism. Hence the proof of the proposition is completed.

PROPOSITION 4.2. *There exists a multiplicative monomorphism*

$$i\colon \mathscr{A}_W\to\mathscr{A}_U.$$

The operation ∂ is in the image of i; therefore there is a unique operation $\partial \in \mathscr{A}_W$ with the property $\partial^2 = 0$ which maps to $\partial \in \mathscr{A}_U$.

We shell need later some information about the homology groups

$$H_*\big(W^*(X), \partial\big).$$

The groups $H_*\big(W^*(pt), \partial\big)$ were completely determined in [6]. In spite of $W^*(\ )$ not being multiplicative, $H_*\big(W^*(pt), \partial\big)$ turned out to be a Z_2-algebra.

LEMMA 4.3. *If the connected spectrum X has no torsion in cohomology, then*

$$H_*\big(W^*(X), \partial\big)$$

is an $H_*\big(W^*(pt), \partial\big)$-module and the multiplication is generated by that in $U^*(\ )$.

Proof. Recall that there is a transformation

$$ch_U\colon U^*(X) \to H^*(X, Q) \otimes \Lambda,$$

which commutes with the action of $\mathscr{A}_U$ if we make $\mathscr{A}_U$ acting on $H^*(X, Q) \otimes \Lambda$ by $\theta(x \otimes \lambda) = x \otimes \theta\lambda$.

For a complex without torsion in cohomology ch_U is a monomorphism.

We know from [6] that if $a, b \in \Lambda$ and $\Delta a = \Delta b = 0$, then

$$\partial(ab) = a\partial b + \partial ab - [CP(1)]\partial a \partial b$$

and if, moreover, $\partial a = 0$ then

$$\Delta(ab) = 0.$$

We shall show that if for $x \in U(X)$, $a \in \Lambda$, $\Delta x = \Delta a = 0$, then

(i) if $\partial a = 0$, then $\partial(ax) = a\partial x$ and $\Delta(ax) = 0$,

(ii) if $\partial x = 0$, then $\partial(ax) = \partial a x$ and $\Delta(ax) = 0$.

Let $t_i \in H^*(X, Q)$ be a Q base of $H^*(X, Q)$ and let

$$ch_U x = \sum t_i \otimes a_i, \qquad a_i \in \Lambda \otimes Q.$$

If $\Delta x = 0$, then

$$ch_U \Delta x = \sum t_i \otimes \Delta a_i = 0$$

and we get $\Delta a_i = 0$; similarly if $\partial x = 0$ we have $\partial a_i = 0$.

Now in case (i)

$$\begin{aligned} ch_U \Delta(ax) &= \Delta\Big(\sum t_i \otimes a_i\Big) = \sum t_i \otimes \Delta(aa_i) = 0, \\ ch_U\big(\partial(ax) - a\partial x\big) &= \partial \sum t_i \otimes aa_i - a\Big(\sum t_i \otimes \partial a_i\Big) \\ &= \sum t_i \otimes \big(\partial(aa_i) - a\partial a_i\big) = 0. \end{aligned}$$

Case (ii) is similiar.

It easily follows from (i), (ii) that multiplying in $U^*(\ )$ elements which represents homology classes we get a well defined $H_*(W^*(pt), \partial)$-module structure in $H_*(W^*(X), \partial)$.

In the sequel let $\bar{\xi}_k$ denote the classifying $U(k)$-bundle and ξ_k the classifying $SU(k)$-bundle. Let

$$\bar{\varkappa}_n\colon BSU(n)\to BU(n)$$

be a map which induces the classifying $SU(n)$-bundle and put

$$\varkappa_n = M(\bar{\varkappa}_n)\colon MSU(n)\to MU(n).$$

The sequence $\{\varkappa_n\}$ is a map of spectra

$$\varkappa\colon MSU\to MU.$$

We shall denote by the same letter a transformation of the cohomology theories

$$\varkappa\colon SU^*(\ )\to U^*(\ )$$

induced by $\varkappa$.

Let the coefficient ring $SU^*(pt)$ be denoted by Γ. The structure of Γ was completely determined in [6]. We are going to apply the results of the paper cited above to give some information about the algebra $\mathscr{A}_{SU}$.

LEMMA 4.4. *There exists an operation*

$$\partial_1\colon U^*(\ )\to SU^*(\ )$$

of degree 2 *such that*

$$\varkappa\partial_1 = \partial.$$

Proof. Let $\bar{\eta}$ be a line bundle over $BU(n)$ such that

$$c_1(\bar{\eta}) = -c_1.$$

The bundle $\bar{\xi}_n\oplus\bar{\eta}$ has a unique $SU(n+1)$-bundle structure; therefore there is a bundle map

$$\begin{tikzcd}
\bar{\xi}_n\oplus\bar{\eta} \arrow[r, "d"] \arrow[d] & \xi_{n+1} \arrow[d] \\
BU(n) \arrow[r, "d'"] & BSU(n+1).
\end{tikzcd}$$

Let $i\colon \bar{\xi}_n\to\bar{\xi}_n\oplus\bar{\eta}$ be the standard inclusion and define

$$\partial_1 = M(di)\colon MU(n)\to MSU(n+1).$$

These maps clearly commute with $\bar{\varepsilon}_n$, $\hat{\varepsilon}_n$ and therefore define a map of spectra

$$\partial_1\colon MU\to MSU$$

of degree 2.

We want to show that $\varkappa\partial_1 = \partial$. It is sufficient to prove that this equality holds on Λ. Hence we have to show that the operations $\varkappa\partial_1$ and ∂ act in the same way on the Chern numbers.

The formula which relates the Chern numbers of M^{2n} and ∂M^{2n}, [6], when rewritten in the terms of the dual Chern numbers, becomes

$$\bar{c}_{i_1}\ldots\bar{c}_{i_r}(\partial M^{2n}) = -\bar{c}_1\bar{D}_{i_1}\ldots\bar{D}_{i_r}(M^{2n}),$$

where

$$\bar{D}_i = \bar{c}_i - \bar{c}_1\bar{c}_{i-1}.$$

Using the multiplicativity of the Thom classes in cohomology, it is not difficult to see that

$$\varphi_H^{-1}M(di)\varphi_H(c_{i_1}\ldots c_{i_r}) = -c_1D_{i_1}\ldots D_{i_r};$$

here c_i are the universal Chern classes,

$$D_i = c_i - c_1c_{i-1}$$

and φ_H are the Thom homomorphisms in the bundles $\bar{\xi}_n$ and $\bar{\xi}_{n+1}$ respectively.

Now the standard argument shows that the dual Chern numbers of $\varkappa\partial_1M^{2n}$ and ∂M^{2n} are equal.

LEMMA 4.5. *There are no elements of infinite filtration in* $U^*(MSU)$ *and* $SU^*(MSU)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
SU^*(MSU) \arrow[d, "ch_{SU}"] & U^*(MSU) \arrow[l, "\theta"'] \arrow[d, "ch_U"] & U^*(MU) \arrow[l, "\varkappa^*"'] \arrow[d, "ch_U"] \\
H(MSU, Q)\otimes\Gamma & H(MSU, Q)\otimes\Lambda \arrow[l, "\mathrm{id}\otimes\partial"'] & H(MU, Q)\otimes\Lambda \arrow[l, "\varkappa^*"']
\end{tikzcd}$$

It follows from [6] that the homomorphism $\mathrm{id}\otimes\partial$ in the bottom row is an epimorphism, and $\varkappa^*$ in the bottom row is also an epimorphism.

We know that $U^*(MU)$ does not contain elements of infinite filtration. Therefore the lemma follows by applying Theorem 1 of [10] twice.

Consider the maps

$$MSU(n) \xleftarrow{\bar{s}_{2n-1}} S^2MSU(n-1) \xrightarrow{\mathrm{id}\wedge i} MSU(n-1)\wedge CP(2),$$

where

$$i\colon S^2\to CP(2)$$

is a standard inclusion.

They induce the maps of spectra

$$MSU \xleftarrow{\bar{s}} MSU \xrightarrow{\bar{e}} W$$

and the map $\bar{\varepsilon}$ is an equivalence. Therefore we can define a map of degree 0

$$\varrho = \bar{\varrho}\bar{\varepsilon}^{-1}\colon MSU \to W,$$

and we shall denote by ϱ the transformation

$$\varrho\colon SU^*(\ ) \to W^*(\ )$$

induced by this map.

LEMMA 4.6. *The diagram*

$$\begin{tikzcd}
MSU(k-1) \wedge CP(2) \arrow[r, "\mu_{2k-1}"] & MU(k) \\
MSU(k-1) \wedge S^2 \arrow[u, "\mathrm{id} \wedge i"] \arrow[r, "\bar{\varepsilon}_{2k-1}"] & MSU(k) \arrow[u, "\varkappa_k"']
\end{tikzcd}$$

is commutative.

Proof. The lemma follows immediately from the definitions of the maps μ and $\varkappa$ because the Hopf bundle over $CP(1)$, when considered over a point, is a trivial line bundle.

COROLLARY 4.7. *The diagram*

$$\begin{tikzcd}
& SU^*(\ ) \arrow[dl, "\varrho"'] \arrow[dr, "\varkappa"] & \\
W^*(\ ) \arrow[rr, "\mu"'] & & U^*(\ )
\end{tikzcd}$$

is commutative.

Let us define a map

$$\tilde{\varrho}\colon \mathscr{A}_{SU} \to \mathscr{A}_W$$

as follows.

Given $\theta \in \mathscr{A}_{SU}$, let $\tilde{\varrho}(\theta)$ be a map unique up to homotopy and such that the diagram

$$\begin{tikzcd}
MSU \wedge CP(2) \arrow[r, "\theta \wedge \mathrm{id}"] \arrow[d, "\mathrm{id}"] & MSU \wedge CP(2) \arrow[d, "\mathrm{id}"] \\
W \arrow[r, "\tilde{\varrho}\theta"] & W
\end{tikzcd}$$

is commutative.

PROPOSITION 4.8. *The map*

$$\tilde{\varrho}\colon \mathscr{A}_{SU} \to \mathscr{A}_W$$

is a homomorphism of algebras and the diagrams

$$\begin{tikzcd}
SU^*(\ ) \arrow[r, "\theta"] \arrow[d, "\varrho"] & SU^*(\ ) \arrow[d, "\varrho"] \\
W^*(\ ) \arrow[r, "\tilde{\varrho}\theta"] & W^*(\ )
\end{tikzcd}$$

are commutative.

Proof. The first statement is clear. In order to prove the second one, consider the diagram

$$\begin{tikzcd}
S^2MSU \arrow[rrrrrr, "\theta\wedge\mathrm{id}"] \arrow[ddrr, "\bar{\varepsilon}"] \arrow[dddddd, "\mathrm{id}\wedge i"'] & & & & & & S^2MSU \arrow[ddll, "\varepsilon"'] \arrow[dddddd, "\mathrm{id}\wedge i"] \\
& & & \mathrm{II} & & & \\
& & MSU \arrow[rr, "\theta"] \arrow[dd, "\varrho"'] & & MSU \arrow[dd, "\varrho"] & & \\
& \mathrm{I} & & & & \mathrm{I} & \\
& & W \arrow[rr, "\tilde{\varrho}\theta"] & & W & & \\
& & & \mathrm{III} & & & \\
MSU\wedge CP(2) \arrow[uurr, "\mathrm{id}"] \arrow[rrrrrr, "\theta\wedge\mathrm{id}"] & & & & & & MSU\wedge CP(2) \arrow[uull, "\mathrm{id}"']
\end{tikzcd}$$

In this diagram the squares I, II, and III are commutative.

Hence

$$\tilde{\varrho}\theta\varrho\bar{\varepsilon} \sim \varrho\theta\bar{\varepsilon}$$

and, since $\bar{\varepsilon}$ is an equivalence of spectra, we get

$$\tilde{\varrho}\theta\varrho = \varrho\theta.$$

Let a map

$$\tilde{\varkappa}\colon \mathscr{A}_{SU} \to \mathscr{A}_U$$

be defined as the composition

$$\mathscr{A}_{SU} \xrightarrow{\tilde{\varrho}} \mathscr{A}_W \xrightarrow{i} \mathscr{A}_U.$$

The map $\tilde{\varkappa}$ is clearly a homomorphism of a algebras.

LEMMA 4.9. *The diagrams*

$$\begin{tikzcd}
SU^*(\ ) \arrow[r, "\theta"] \arrow[d, "\varkappa"] & SU^*(\ ) \arrow[d, "\varkappa"] \\
U^*(\ ) \arrow[r, "\tilde{\varkappa}\theta"] & U^*(\ )
\end{tikzcd}$$

are commutative.

Proof. This is an immediate consequence of Propositions 4.2, 4.9 and Corollary 4.8.

LEMMA 4.10. *The diagram*

$$\begin{tikzcd}
W^*(MSU) \arrow[rr, "\mu"] & & U^*(MSU) \\
& \mathscr{A}_{SU} \arrow[ul, "\varrho"'] \arrow[ur, "\varkappa"] \arrow[dl, "\tilde{\varrho}"] \arrow[dr, "\tilde{\varkappa}"'] & \\
\mathscr{A}_W \arrow[uu, "\varrho^*"] \arrow[rr, "\iota"] & & \mathscr{A}_U \arrow[uu, "\varkappa^*"']
\end{tikzcd}$$

is commutative.

LEMMA 4.11. *There is an exact sequence*

$$0 \to SU^{\text{odd}}(MSU) \xrightarrow{\theta} SU^{\text{ev}}(MSU) \xrightarrow{\varrho} W^{\text{ev}}(MSU) \xrightarrow{\beta} SU^{\text{ev}}(MSU) \xrightarrow{\theta} SU^{\text{odd}}(MSU) \to 0,$$

where

(i) *the homomorphism θ of degree -1 is a multiplication on $\gamma \in \Gamma^{-1}$*,

(ii) $\deg \beta = 2$.

Proof. The cofibre sequence

$$S^2 \to CP(2) \to S^4$$

leads to the cofibre sequence of spectra

$$MSU \xrightarrow{\varrho} W \xrightarrow{\beta} MSU \xrightarrow{\theta} MSU.$$

It follows from Lemma 4.5 and Theorem 1 of [10] that $U^{\text{odd}}(MSU) = 0$. Therefore $W^{\text{odd}}(MSU) = 0$ and the long exact sequence induced by the cofibre sequence above splits. (i) was proved in [6], Theorem 16.6, and (ii) is clear.

Let us denote by Γ' the image of Γ in Λ.

PROPOSITION 4.12. *The homomorphism*

$$\tilde{\varkappa}\colon \mathscr{A}_{SU} \to \mathscr{A}_U$$

has the following properties:

(i) $\ker \tilde{\varkappa} = \operatorname{Tor} \mathscr{A}_{SU}$,

(ii) $\operatorname{Tor} \mathscr{A}_{SU}$ *does not contain elements of an odd order or of order* 4,

(iii) *if $\{T_a\}$ is a set of operations in $\mathscr{A}_{SU}$ such that operations $\tilde{\varkappa} T_a$ generate* $\operatorname{im} \tilde{\varkappa}$ *as a Γ'-module, then $\{T_a\}$ generate $\mathscr{A}_{SU}$ as a Γ-module.*

Proof. The exact sequence of Lemma 4.12 gives, by using Lemma 4.5, an exact sequence

$$0 \to \mathscr{A}_{SU}^{\text{odd}} \xrightarrow{\theta} \mathscr{A}_{SU}^{\text{ev}} \xrightarrow{\varrho} W^{\text{ev}}(MSU) \xrightarrow{\beta} \mathscr{A}_{SU}^{\text{ev}} \xrightarrow{\theta} \mathscr{A}_{SU}^{\text{odd}} \to 0,$$

Now it follows from Lemma 4.5 that $U^*(MSU)$ does not contain elements of finite order. Therefore $W^{\mathrm{ev}}(MSU)$ does not contain elements of finite order. Hence

$$\mathrm{Tor}\ \mathscr{A}_{SU}^{\mathrm{ev}} \subset \mathrm{im}\,(\mathscr{A}_{SU}^{\mathrm{odd}} \xrightarrow{\theta} \mathscr{A}_{SU}^{\mathrm{ev}}).$$

Since θ is the multiplication on the element $\gamma\in\Gamma^{-1}$, which is of order 2,

$$\mathrm{Tor}\ \mathscr{A}_{SU}^{\mathrm{ev}} = \mathrm{im}\,\theta$$

consists of elements of order 2.

On the other hand, the map

$$\mathscr{A}_{SU}^{\mathrm{ev}} \xrightarrow{\theta} \mathscr{A}_{SU}^{\mathrm{odd}}$$

is an epimorphism, and thus

$$\mathrm{Tor}\ \mathscr{A}_{SU}^{\mathrm{odd}} = \mathscr{A}_{SU}^{\mathrm{odd}}$$

consists of elements of order 2.

It follows from Lemma 4.10 that

$$\ker\tilde{\varkappa} \subset \ker\varrho$$

for μ is a monomorphism. Since $\mathscr{A}_U$ does not contain elements of finite order,

$$\ker\tilde{\varkappa} \supset \mathrm{Tor}\ \mathscr{A}_{SU}.$$

Therefore

$$\ker\tilde{\varkappa} = \mathrm{Tor}\ \mathscr{A}_{SU}.$$

Now, let T_a be a set of elements such that each element of $\mathrm{im}\,\tilde{\varkappa}$ can be written in the form

$$\sum \gamma_a\tilde{\varkappa}(T_a),$$

where $\gamma_a\in\Gamma'$. Applying Lemma 4.10 again, we see that each element of $\mathscr{A}_{SU}^{\mathrm{ev}}$ is of the form

$$\sum \tilde{\gamma}_a T_a + \theta(x),$$

where $x\in\mathscr{A}_{SU}^{\mathrm{odd}}$. The element x can be written in the form

$$x = \theta\Big(\sum \tilde{\mu}_a T_a + \theta(y)\Big)$$

and we get

$$\sum \tilde{\gamma}_a T_a + \theta(x) = \sum \tilde{\gamma}_a T_a + \sum \gamma^2 \tilde{\mu}_a T_a$$

for $\theta^3 = 0$.

Similar arguments show that the elements in $\mathscr{A}_{SU}^{\mathrm{odd}}$ can be written in the form

$$\sum \gamma\tilde{\gamma}_a T_a.$$

This concludes the proof.

It is known that the study of $SU^*(\ )$ becomes much simpler after localization at the odd primes. We shall follow this line to get some more information about the algebra $\mathscr{A}_{SU}$.

Let l be the set consisting of odd primes and let $U_l^*(\ )$ be the unitary cobordism theory localized at l and similarly for $SU^*(\ )$.

LEMMA 4.13. *The map*

$$\varkappa_l\colon \Gamma_l \to \Lambda_l$$

is a monomorphism.

Proof. We know that $\ker\varkappa$ consists of elements of order 2; therefore $\ker\varkappa \otimes Q_l = 0$ and the lemma follows.

Recall that there exists an operation

$$\chi \in \mathscr{A}_U$$

of degree -2 such that

$$[\chi, \partial] = 2.$$

LEMMA 4.14. *The operation*

$$\tfrac{1}{2}\partial\chi \in \mathscr{A}_{Ul}$$

is an idempotent element and

$$\mathrm{im}(\tfrac{1}{2}\partial\chi) \simeq SU_l^*(\ ).$$

Proof. We have

$$(\tfrac{1}{2}\partial\chi)(\tfrac{1}{2}\partial\chi) = \tfrac{1}{4}\partial(2+\partial\chi)\chi = \tfrac{1}{2}\partial\chi;$$

therefore $\frac{1}{2}\partial\chi$ is an idempotent.

We know from Lemma 4.4 that

$$\mathrm{im}\,\tfrac{1}{2}\partial\chi \in \mathrm{im}\,\varkappa,$$

if $x \in \mathrm{im}\,\varkappa$

$$\tfrac{1}{2}\partial\chi x = \tfrac{1}{2}(2+\chi\partial)x = x$$

because $\partial\varkappa = 0$.

Hence

$$\mathrm{im}\,\varkappa = \mathrm{im}\,\tfrac{1}{2}\partial\chi$$

and the lemma follows because $\varkappa_l$ is a monomorphism.

Let the transformation

$$\tau\colon U_l^*(\ )\to SU_l^*(\ )$$

be defined as the composition

$$\tau = \varkappa_l^{-1}\tfrac{1}{2}\partial\chi.$$

Clearly τ is the transformation of the cohomology theories and

$$\tau\varkappa_l = \mathrm{id}.$$

Now define a map

$$\tilde{\tau}\colon \mathscr{A}_{U_l}\to\mathscr{A}_{SU_l}$$

by the formula

$$\tilde{\tau}(\theta) = \tau\theta\varkappa_l.$$

PROPOSITION 4.15. *The map* $\tilde{\tau}\colon \mathscr{A}_{U_l}\to\mathscr{A}_{SU_l}$ *is a right inverse of the map* $\tilde{\varkappa}_l\colon \mathscr{A}_{SU_l}\to\mathscr{A}_{U_l}$.

5. Final remarks

We are going to list in this section some questions related to the material discussed in Sections 3 and 4 which we left unsolved.

5.1. Let $\tilde{\mathscr{A}}_h$ be the subset of $\mathscr{A}_h$ consisting of the operations which are Λ-homomorphisms. $\tilde{\mathscr{A}}_h$ is clearly a subalgebra of $\mathscr{A}_h$.

In the case of the $U^*(\ )$ theory we have

$$\tilde{\mathscr{A}}_U = \Lambda.$$

This can be seen as follows. If $\theta\in\tilde{\mathscr{A}}_U$, then for $\lambda\in\Lambda$

$$\theta\lambda = \theta(\lambda 1) = \lambda\theta(1).$$

Since the operations in $\mathscr{A}_U$ are determined by their action on Λ, we get

$$\theta = \theta(1)\in\Lambda.$$

Question. What is $\tilde{\mathscr{A}}_{U(\ ,Z_p)}$?

It can be shown fairly easily that the Riemann–Roch transformation

$$\mu_c\colon U^*(\ )\to K^*(\ )$$

can be reduced mod$p$ to give a transformation

$$\bar{\mu}\colon U^*(\ ,Z_p)\to K^*(\ ,Z_p)$$

and that the map

$$\bar{\mu}\colon U^*(\ ,Z_p)\otimes_{A_p}Z_p[t,t^{-1}]\to K^*(\ ,Z_p)$$

is an isomorphism (here K^* is Z-graded).

The map $\bar{\mu}$ induces a map

$$\tilde{\mu}\colon \tilde{\mathscr{A}}_{U(\ ,Z_p)}\to \mathscr{A}_{K(\ ,Z_p)}.$$

Question. Is the map $\tilde{\mu}$ an epimorphism?

Note that $U^*_{(p)}$ determines $K^*_{(p)}$ together with the action of the operations ψ^k, $(k,p)=1$, but in a certain not "natural" way. The map

$$\tilde{\mu}\colon \tilde{\mathscr{A}}_{U_{(p)}}\to \mathscr{A}_{K_{(p)}}$$

is not epimorphic.

5.2. In Section 3 we gave a method of determining the action of the algebra $\mathscr{A}_{BP(\ ,Z_p)}$ on the coefficient ring A'_p. That method, however, is not satisfactory. Therefore we have the following question.

Question. Does there exist a more effective way of determining the action of the operations T^R on some properly chosen set of polynomial generators of the ring A'_p?

5.3. It follows from Proposition 3.4 that there are operations r_R reduced mod$p$ in the algebra $\mathscr{A}_{BP(\ ,Z_p)}$. They can be written in the form

$$r_R=\sum a_{R,S}T^S,$$

where $a_{R,S}$ belong to A'_p.

Question. Is it true that

$$a_{R,S}=\begin{cases}1 & \text{for } R=S,\\ 0 & \text{otherwise?}\end{cases}$$

5.4. Sullivan in [15] gave the geometric interpretation of the homology theory $U_*(\ ,Z_p)$ dual in the sense of Whitehead to the theory $U^*(\ ,Z_p)$ considered in Section 3. $U_*(X,Z_p)$ is the set of the bordism classes of the singular p-manifolds in X.

Question. Does there exist a class of p-manifolds such that $BP_*(\ ,Z_p)$ is the set of bordism classes of the p-manifolds belonging to this class?

The fact that one can choose as the polynomial generators of the ring Λ in dimensions $2(p^i-1)$ manifolds which admit Z_p-action with a trivial normal bundle to the fixed point set suggests that the class we interested in consists of p-manifolds with some sort of Z_p-action.

5.5. We showed at the end of Section 4 that there exists an idempotent element of $\mathscr{A}_{U_l}$ such that its image is isomorphic to $SU_l^*(\ )$.

Question. Is that idempotent a multiplicative operation? If not, does there exist a multiplicative idempotent such that its image is isomorphic to $SU_l^*(\ )$?

References

[1] J. F. Adams, *Lectures on generalized cohomology*, Category Theory, Homology Theory and Their Applications III, Lecture Notes in Math. 99 (1969).

[2] S. Araki and H. Toda, *Multiplicative structures in* Mod$q$ *cohomology theories*, I and II, Osaka J. Math. 2 (1965), p. 71-116 and 3 (1966), p. 81-120.

[3] J. M. Boardman, *Stable homotopy theory*, mimeographed notes, University of Warwick, 1965.

[4] E. H. Brown and F. P. Peterson, *A spectrum whose Z_p cohomology is the algebra of reduced p-th powers*, Topology 5 (1966), p. 149–154.

[5] J. M. Cohen, *The Hurewicz homomorphism on MU*, Invent. Math. 10 (1970), p. 177–186.

[6] P. E. Conner and E. E. Floyd, *Torsion in SU-bordism*, Mem. Amer. Math. Soc. 60 (1966).

[7] — *Relations of cobordism to K-theory*, Lecture Notes in Math. 28 (1966).

[8] A. Jankowski, *Les applications d'opérations homotopiques au problème du prolongement*, C. R. Acad. Sci. Paris 264 (1967), p. 315-318.

[9] — *Remarks on generalized homology theories*, Comm. Math. 15 (1971), p. 185–199.

[10] — *Elements of the infinite filtration in generalized cohomology theories on the category of spectra*, Mat. Zam. 11 (1972), p. 699–704.

[11] P. S. Landweber, *Cobordism operations and Hopf algebra*, Trans. Amer. Math. Soc. 129 (1967), p. 94–110.

[12] J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. 67 (1958), p. 150–171.

[13] S. P. Novikov, *The method of algebraic topology from the vievpoint of cobordism theories*, Izv. Akad. Nauk SSSR, ser. mat. 31 (1967), p. 855–981.

[14] D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. 75 (1969), p. 1293–1298.

[15] D. Sullivan, *Geometric topology*, I, mimeographed notes, 1970.

[16] F. Uchida, *Stable operations in* Mod$p$ *cohomology theories*, Osaka J. Math. 6 (1969), p. 93–106.

[17] R. Zahler, *The Adams–Novikov spectral sequence for the spheres*, The University of Chicago, Thesis (1970).