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Logarithmic and antilogarithmic mappings

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Euler in his paper *De la controverse entre Mrs. Leibniz and Bernoulli sur les logarithmes des nombres négatifs and imaginaires* (Mémoires de l'Académie des Sciences de Berlin 5 (1749), 139–171, in: *Opera*, (1) 17, 195–232; cf. C. G. Fraser [1]) considered the rule $d(\log x) = dx/x$. He rejected an earlier suggestion of Leibniz that this rule is only valid for positive real values of x with the following observation:

"(...) Car, comme ce calcul roule sur les quantités variables, c. à d. sur des quantités considérées en général, s'il n'était pas vrai généralement qu'il fût $d \cdot lx = dx/x$, quelque quantité qu'on donne à x, soit positive ou négative, ou même imaginaire, on ne pourrait jamais se servir de cette règle, la vérité du calcul différentiel étant fondée sur la généralité des règles qu'il renferme." ([1])

([1]) For, as this calculus concerns variable quantities, that is, quantities considered in general, if it were not generally true that $d \cdot lx = dx/x$, whatever value we give to x, either positive, negative, or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains.

Introduction

Let X be a D-algebra, i.e. a commutative algebra with a right invertible operator $D \in L(X)$ such that the domain of D is a subalgebra of X. Let $\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}$ be a multifunction defined as follows:

$$\Omega u = \{x \in \operatorname{dom} D : Du = uDx\} \quad \text{for } u \in \operatorname{dom} D.$$

This multifunction is well-defined. Any invertible selector L of Ω is said to be a *logarithmic mapping* and $E = L^{-1}$ is said to be an *antilogarithmic mapping*.

In the author's paper [8] it was shown that in a wide class of D-algebras over $\mathbb{R}$ all considerations can be reduced to three cases: either right invertible operators under consideration satisfy the classical Leibniz condition for products, or are reducible by a substitution to multiplicative operators, or logarithmic and antilogarithmic mappings do not exist. An analogue of power functions was defined by means of the introduced logarithmic and antilogarithmic mappings.

We have studied properties of logarithmic and antilogarithmic mappings in arbitrary D-algebras over $\mathbb{R}$ (cf. the author's preprint [9]). These mappings are applied to solving linear equations with coefficients belonging to X in an explicit form. In particular, this method permits us to obtain, in the Leibniz case, solutions of linear equations with scalar coefficients under weaker assumptions than those admitted in the author's previous works (cf. for instance, Przeworska-Rolewicz [1], [2]).

We have seen that our considerations lead to non-linear problems, which can be solved in linear spaces without any topology. This is so because logarithmic and antilogarithmic mappings are non-linear. However, with the help of these non-linear mappings we obtain solutions to linear equations.

We shall extend these notions and the results already obtained to linear spaces over an arbitrary scalar field $\mathcal{F}$ of characteristic zero. We shall examine the relations between the complex and real cases.

It will be shown that in algebras with left invertible and invertible operators one can define in a similar way logarithmic and antilogarithmic mappings and derive analogous results. As a conclusion, we shall obtain properties of logarithms and antilogarithms induced by operators having either finite nullity or finite deficiency.

It should be pointed out that, due to the presented approach, we derive results unknown even in the classical case of the operator $D = d/dt$.

0. Preliminaries

Let X be a linear space over a field $\mathcal{F}$ of scalars of characteristic zero. In the sequel we shall write 1 for the unit in $\mathcal{F}$ and $1/n$ for $(1/n)\cdot 1$ ($n\in\mathbb{N}$).

Write

$$L(X) = \text{ the set of all linear operators with domains and ranges in } X;$$
$$L_0(X) = \{A\in L(X) : \operatorname{dom} A = X\}.$$

If X is a commutative algebra over $\mathcal{F}$ with an $D\in L(X)$ such that $x,y\in \operatorname{dom} D$ implies $xy\in\operatorname{dom} D$, then we shall write $D\in \mathsf{A}(X)$. If $D\in\mathsf{A}(X)$ then set

$$f_D(x,y) = D(xy) - c_D(xDy + yDx) \quad \text{for } x,y\in\operatorname{dom} D,$$

where c_D is a scalar depending on D only. Clearly, f_D is a bilinear (i.e. linear in each variable) symmetric form. This form is called a *non-Leibniz component*. Non-Leibniz components have been introduced for right invertible operators $D\in \mathsf{A}(X)$ (cf. Przeworska-Rolewicz [1], Section 6.1). Recall that in the case of a right invertible $D\in\mathsf{A}(X)$ the algebra X is said to be a *D-algebra*.

By definition of the non-Leibniz component, if $D\in\mathsf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D(xDy + yDx) + f_D(x,y) \quad \text{for } x,y\in\operatorname{dom} D.$$

Non-Leibniz components for powers of D are determined by recursive (equivalent) formulae. Namely, for all $k\in\mathbb{N}$ and $x,y\in\operatorname{dom} D^k$ such that $f_D(x,y)\in \operatorname{dom} D^k$ we have $xy\in\operatorname{dom} D^k$ and

$$D^k(xy) = c_D^k(xD^ky + yD^kx) + f_D^{(k)}(x,y), \tag{0.1}$$

where $f_D^{(1)} = f_D$ and for $k=2,3,\dots,$

$$\begin{aligned} f_D^{(k)}(x,y) = c_D^k[(Dx)D^{k-1}y + (D^{k-1}x)Dy] \\ + c_D^{k-1}[f_D(x,D^{k-1}y) + f_D(D^{k-1}x,y)] + Df_D^{(k-1)}(x,y)\end{aligned} \tag{0.2}$$

or

$$\begin{aligned} f_D^{(k)}(x,y) = c_D^k[(Dx)D^{k-1}y + (D^{k-1}x)Dy] \\ + c_D[f_D^{(k-1)}(x,Dy) + f_D^{(k-1)}(Dx,y)] + D^{k-1}f_D(x,y).\end{aligned} \tag{0.2$'$}$$

Thus we have

$$c_{D^k} = c_D^k, \quad f_{D^k} = f_D^{(k)} \quad \text{for } k\in\mathbb{N}.$$

The proof (by induction) is the same as that given in the author's book [1], Section 6.1, for D-algebras, since the right invertibility of D was not used. Suppose that $D\in\mathsf{A}(X)$ and $p\neq 0$ is a fixed scalar. Then $pD\in\mathsf{A}(X)$ and

$$c_{pD} = c_D, \quad f_{pD}^{(k)} = p^k f_D^{(k)} \quad \text{for } k\in\mathbb{N}. \tag{0.3}$$

Suppose that $D_1, D_2 \in \mathsf{A}(X)$, the superposition $D = D_1 D_2$ exists (hence belongs to $R(X)$) and $D_1 D_2 \in \mathsf{A}(X)$. Then

$$\begin{aligned} c_{D_1 D_2} &= c_{D_1} c_{D_2}, \\ f_{D_1 D_2}(x,y) &= f_{D_1}(x,y) + D_1 f_{D_2}(x,y) \\ &\quad + c_{D_1} c_{D_2}[(D_1 x)(D_2 y) + (D_2 x)(D_1 y)] \end{aligned} \tag{0.4}$$

for $x, y \in \operatorname{dom} D = \operatorname{dom} D_1 \cap D_2$. If $D_1 D_2 = D_2 D_1$, we can write the non-Leibniz component for the superposition in a symmetrized form:

$$f_{D_1 D_2}(x,y) = \tfrac{1}{2}[(D_1 + I) f_{D_2}(x,y) + (D_2 + I) f_{D_1}(x,y)], \tag{0.5}$$

since $yx = xy$.

Let $D \in \mathsf{A}(X)$. The operator of *multiplication* by a fixed element $a \in X$ will be denoted by the same letter. Thus we shall mean by Aa the *superposition* of two operators:

$$(Aa)x = A(ax) \quad \text{for } A \in L(X),\ a, x \in X.$$

Let $D \in \mathsf{A}(X)$. We shall consider the following sets:

- the set of all *multiplicative* mappings (not necessarily linear) with domains and ranges in X:

$$M(X) = \{A : \operatorname{dom} A \subset X \text{ and } A(xy) = A(x)A(y) \text{ for } x, y \in \operatorname{dom} A\};$$

- the set of all *multiplicative linear* operators belonging to $L(X)$:

$$ML(X) = M(X) \cap L(X);$$

- the set $I(X)$ of all *invertible* elements belonging to X;
- the set of all elements from $E \subset X$ having n-th *roots*:

$$I_n(E) = \{x \in E : \exists_{y \in I(E)}\ y^n = x\} \quad (n \in \mathbb{N});$$

If $x \in I_n(E)$ and $y^n = x$ then we write $y = x^{1/n}$. Recall that

- $R(X)$ is the set of all right invertible operators belonging to $L(X)$;
- $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$ is the set of all right inverses to $D \in R(X)$;
- $\mathcal{F}_D = \{F \in L_0(X) : F^2 = F,\ FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D} FR = 0\}$ is the set of all *initial* operators for $D \in R(X)$;
- $\Lambda(X)$ is the set of all left invertible operators belonging to $L_0(X)$ (we assume that $\operatorname{dom} T = X$ for $T \in \Lambda(X)$);
- $\mathcal{L}_T = \{S \in L(X) : ST = I\}$ is the set of all left inverses to $T \in \Lambda(X)$;
- $\mathcal{G}_T = \{G \in L(X) : G^2 = G, \ker G = TX \text{ and } \exists_{S \in \mathcal{L}_T} SG = 0\}$ is the set of all *co-initial* operators for $T \in \Lambda(X)$;
- $\mathcal{I}(X) = R(X) \cap \Lambda(X)$.

Clearly, if $\ker D \neq \{0\}$ then the operator D is right invertible, but not invertible. Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$. Elements of the kernel of $D \in R(X)$ are said to be *constants*.

Clearly, if $D \in \mathcal{I}(X)$ then $\mathcal{F}_D = \mathcal{G}_D = \{0\}$ and $\mathcal{R}_D = \mathcal{L}_D = \{D^{-1}\}$.

Recall also that any solution of the equation

$$Dx = y, \quad y \in X, \ D \in R(X),$$

is of the form $x = Ry + z$, where $R \in \mathcal{R}_D$ and $z \in \ker D$ is arbitrary. This form is *independent* of the choice of R. Indeed, if $R' \in \mathcal{R}_D$ and $R' \neq R$ then for all $x \in X$ we have $R'x - Rx \in \ker D$. Thus, a change of R implies only a change of the constant z (cf. Przeworska-Rolewicz [1], Section 2.1).

The equation

$$Tx = y, \quad y \in TX, \ T \in \Lambda(X),$$

has a unique solution $x = Sy$ which is independent of the choice of $S \in \mathcal{L}_T$. Indeed, by definition, $x = STx = Sy$ and $\ker T = \{0\}$ (cf. Przeworska-Rolewicz [1], Section 2.5).

The sets defined above are, in a sense, dual. Namely, if $D \in R(X)$ and $F \in \mathcal{F}_D$ corresponds to an $R \in \mathcal{R}_D$, i.e. $FR = 0$, then $R \in \Lambda(X)$ and $F \in \mathcal{G}_D$ corresponds to $D \in \mathcal{L}_R$.

It is well-known that F is an initial operator for a $D \in R(X)$ if and only if there is an $R \in \mathcal{R}_D$ such that $F = I - RD$ on $\operatorname{dom} D$ (cf. Przeworska-Rolewicz [1], Section 2.2).

1. Basic equation. Logarithms and antilogarithms

Definition 1.1. Suppose that $D \in \mathsf{A}(X)$. Let $\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}$ be the multifunction defined by

$$\Omega u = \{x \in \operatorname{dom} D : Du = uDx\} \quad \text{for } u \in \operatorname{dom} D. \tag{1.1}$$

The equation

$$Du = uDx, \quad (u, x) \in \operatorname{graph} \Omega, \tag{1.2}$$

is said to be the *basic equation*. Clearly,

$$\Omega^{-1}x = \{u \in \operatorname{dom} D : Du = uDx\} \quad \text{for } x \in \operatorname{dom} D. \tag{1.3}$$

The multifunction Ω is well-defined. We start with right invertible operators. Left invertible and invertible operators will be considered later.

Proposition 1.1. *Let $D \in R(X)$. Then*

$$\operatorname{dom} \Omega \supset \{x = R(u^{-1}Du) + z : u \in I(X) \cap \operatorname{dom} D; z \in \ker D; R \in \mathcal{R}_D\} \neq \emptyset.$$

Proof. Let $u \in I(X) \cap \operatorname{dom} D$, $z \in \ker D$ and $R \in \mathcal{R}_D$. If $Du = uDx$ then $Dx = u^{-1}Du$. Hence $x = R(u^{-1}Du) + z$ and $x \in \operatorname{dom} \Omega$. ∎

Proposition 1.2. *If $D \in R(X) \cap ML(X)$ then*

$$\operatorname{dom} \Omega \supset \{x = uRu^{-1} + z : u \in I(X) \cap \operatorname{dom} D; z \in \ker D; R \in \mathcal{R}_D\} \neq \emptyset.$$

If X has a unit $e \in \operatorname{dom} D$ then $e \in \operatorname{dom} \Omega$.

P r o o f. Let $u \in I(X) \cap \operatorname{dom} D$, $z \in \ker D$ and $R \in \mathcal{R}_D$. Since $D \in ML(X)$, we have $Dx = u^{-1}Du = (DRu^{-1})(Du) = D(uRu^{-1})$. Hence $x = uRu^{-1} + z$ and $x \in \operatorname{dom} \Omega$. If X has a unit $e \in \operatorname{dom} D$ then

$$\Omega e = \{x \in \operatorname{dom} D : De = eDx\} = \{x = e + z : z \in \ker D\} = e + \ker D \neq \emptyset. \ \blacksquare$$

Proposition 1.3. *If* $D \in \mathsf{A}(X)$, $u \in I(X)$ *and* $x, y \in \Omega u$, *then* $x - y \in \ker D$.

P r o o f. By definition we have $Dx = u^{-1}Du$ and $Dy = u^{-1}Du$. Therefore $D(x - y) = 0$. ■

Proposition 1.4. *For any* $E \subset X$ *we write* $\Omega(E) = \bigcup_{u \in E} \Omega u$. *If* $D \in R(X)$ *then*

(i) $\Omega(\ker D) \subset \ker D$;
(ii) $zu \in I(X) \cap \operatorname{dom} \Omega$ *for* $z \in I(X) \cap \ker D$ *and* $u \in I(X) \cap \operatorname{dom} \Omega$.

P r o o f. (i) Let $z \in \ker D$. Then $zDx = Dz = 0$, which implies that $z \in \operatorname{dom} D$ and $\Omega z \subset \ker D$.

(ii) Suppose that z and u satisfy the assumptions of (ii). By definitions, there is an $x \in \operatorname{dom} D$ such that $DLu = Dx = u^{-1}Du$ and $Dz = 0$. Hence

$$D(zu) = c_D(zDu + uDz) + f_D(u, z) = c_D zDu + f_D(u, z),$$

which implies

$$\begin{aligned}(zu)^{-1}D(zu) &= z^{-1}u^{-1}c_D zDu + z^{-1}u^{-1}f_D(u, z)\\ &= c_D u^{-1}Du + z^{-1}u^{-1}f_D(u, z) = c_D Dx + z^{-1}u^{-1}f_D(u, z).\end{aligned}$$

We therefore conclude that there is a $y \in \operatorname{dom} D$ such that $Dy = (zu)^{-1}D(zu)$. Namely,

$$y = c_D x + R[z^{-1}u^{-1}f_D(u, z)] \quad \text{for an } R \in \mathcal{R}_D.$$

Thus $zu \in I(X) \cap \operatorname{dom} \Omega$. ■

Suppose that $(u, x) \in \operatorname{graph}\Omega$, L is a selector of Ω and E is a selector of Ω^{-1}. By definition, $Lu \in \operatorname{dom} \Omega^{-1}$, $Ex \in \operatorname{dom} \Omega$ and

$$Du = uDLu, \qquad DEx = (Ex)Dx.$$

Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and let $u \in I(X)$. Then these equations can be written in equivalent forms:

$$Lu = R(u^{-1}Du) + FLu, \qquad [I - (Dx)R]Ex = FEx.$$

Definition 1.2. Any invertible selector L of Ω is said to be a *logarithmic mapping* and its inverse $E = L^{-1}$ is said to be an *antilogarithmic mapping*. If $(u, x) \in \operatorname{graph} \Omega$ and L is an invertible selector of Ω then the element Lu is said to be a *logarithm* of u and Ex is said to be an *antilogarithm* of x. We denote by $G[\Omega]$ the set of all pairs (L, E), where L is an invertible selector of Ω and $E = L^{-1}$.

Clearly, by definition, we have

$$ELu = u, \quad LEx = x \quad \text{for all } (L,E) \in G[\Omega],\ (u,x) \in \operatorname{graph} \Omega, \tag{1.4}$$

$$DEx = (Ex)Dx, \qquad Du = uDLu. \tag{1.5}$$

PROPOSITION 1.5. *Let $D \in R(X)$. A logarithm of zero is not defined. If $(L,E) \in G[\Omega]$ then $L(\ker D \setminus \{0\}) \subset \ker D$ and $E(\ker D) \subset \ker D$. In particular, $E(0) \in \ker D$.*

Proof. If $x = L(0)$ then $0 = D(0) = 0Dx$. Hence $L(0)$ is not defined. Suppose that $z \in \ker D$ and $z \neq 0$. Then, by Proposition 1.4, $Lz \in \ker D$. Let $x = Ez$. Then $DEz = (Ez)Dz = 0$, which implies $Ez \in \ker D$. In particular, $E(0) \in \ker D$. ■

DEFINITION 1.3. A logarithmic mapping L is said to be of *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in \operatorname{dom} \Omega$.

PROPOSITION 1.6. *Let $D \in R(X)$. Let $(L,E) \in G[\Omega]$. If L is of exponential type then*

$$E(x+y) = (Ex)(Ey) \quad \textit{for } x, y \in \operatorname{dom} \Omega^{-1}.$$

If $L \in M(X)$ then $E \in M(X)$.

Proof. Let $(L,E) \in G[\Omega]$. Let $x, y \in \operatorname{dom} \Omega$ and $u = Lx$, $v = Ly$. Then $u, v \in \operatorname{dom} \Omega$ and $x = Eu$, $y = Ev$. If L is of exponential type then, by (1.4),

$$(Ex)(Ey) = uv = EL(uv) = E(Lu + Lv) = E(x+y).$$

If $L \in M(X)$ then

$$(Ex)(Ex) = uv = EL(uv) = E[(Lu)(Lv)] = E(xy),$$

i.e. $E \in ML(X)$. ■

For $A \in L(X)$ write

$$V_A^r(X) = \{a \in X : I - Aa \text{ is invertible}\}, \tag{1.6}$$

$$V_A^l(X) = \{a \in X : I - aA \text{ is invertible}\}. \tag{1.7}$$

Clearly, $\lambda e \in V_A^r(X) \cap V_A^l(X)$ for all $\lambda \in v_{\mathcal{F}} A$, where

$$v_{\mathcal{F}} A = \{0 \neq \lambda \in \mathcal{F} : I - \lambda A \text{ is invertible}\}. \tag{1.8}$$

PROPOSITION 1.7. *If $D \in R(X)$, $x \in \operatorname{dom} \Omega^{-1}$, $x_1 = Dx \in V_R^r(X)$ for an $R \in \mathcal{R}_D$ and F is an initial operator for D corresponding to R then*

$$E = (I - Rx_1)^{-1} FE \quad \textit{and} \quad (L,E) \in G[\Omega],\ E = L^{-1}.$$

Moreover, $FEx \neq 0$ whenever $x \in \operatorname{dom} \Omega^{-1}$.

Proof. The operator $(I - Rx_1)^{-1} F$ is well-defined by our assumptions. Since $0 = DFE = D(I - Rx_1)E = DE - DRx_1E = DE - x_1E$, for $x \in \operatorname{dom} \Omega^{-1}$ we have $DEx = (Ex)Dx = x_1Ex$. Hence the basic equation is satisfied. Moreover, for every $x \in \operatorname{dom} \Omega^{-1}$ there is a unique u such that $Lu = x$, namely, $u = Ex$.

Let $x \in \operatorname{dom} \Omega^{-1}$. If $Ex = 0$ then $FEx = 0$. If $FEx = 0$ then we have $Ex = (I - Rx_1)^{-1} FEx = 0$. But, by Proposition 1.5, $Ex \neq 0$ whenever $x \in \operatorname{dom} \Omega^{-1}$. Hence $FEx \neq 0$. ∎

PROPOSITION 1.8. *If $D \in R(X)$, $x \in \operatorname{dom} \Omega^{-1}$, $x_1 = Dx \in V_R^l(X)$ for an $R \in \mathcal{R}_D$ and F is an initial operator for D corresponding to R then*

$$E = R(I - x_1 R)^{-1}(x_1 FE) + FE \quad \text{and} \quad (L, E) \in G[\Omega],\ E = L^{-1}.$$

Moreover, $FEx \neq 0$ *whenever* $x \in \operatorname{dom} \Omega^{-1}$.

Proof. By our assumptions, $DE = (I - x_1 R)^{-1}(x_1 FE) + DFE = (I - x_1 R)^{-1}(x_1 FE)$. Hence $x_1 FE = (I - x_1 R) DE = DE - x_1 RDE = DE - x_1(I - F)E = DE - x_1 E + x_1 FE$. This implies $DE = x_1 E$. Further arguments are the same as in the proof of Proposition 1.7. ∎

PROPOSITION 1.9. *Let X be a D-algebra with unit $e \in \ker D$. Then $z = Ee \in \ker D$ and $Lz = e$.*

Proof. By our assumptions, $D \in R(X)$. By definition, $DEe = (Ee)De = 0$. Hence $z = Ee \in \ker D$ and $Lz = LEe = e$. ∎

DEFINITION 1.4. By $\mathbf{Lg}(D)$ we denote the class of those D-algebras with unit $e \in \operatorname{dom} \Omega$ for which there exist invertible selectors of Ω, i.e. there exist $(L, E) \in G[\Omega]$.

THEOREM 1.1. *Let $X \in \mathbf{Lg}(D)$. Let $R \in \mathcal{R}_D$ and $g = Re$. Then $g \in \operatorname{dom} \Omega$ if and only if $g \in I(X)$.*

In other words: *a logarithm of $g = Re$ exists if and only if g is invertible.*

Proof. By our assumptions, $De = 0$, $Dg = e$, $D^2 g = 0$ and $e \in \operatorname{dom} \Omega$. Let $(L, E) \in G[\Omega]$.

Sufficiency. If $u \in I(X) \cap \operatorname{dom} \Omega$ then the basic equation $Du = uDx$ may be written as $Dx = u^{-1} Du$, which implies that $x \in \operatorname{dom} D$ and $x = Lu$. In particular, if $g \in I(X)$ then $DLg = g^{-1} Dg = g^{-1}$, which implies

$$Lg = Rg^{-1} + z, \qquad \text{where } z \in \ker D.$$

Necessity. Suppose that $g \in \operatorname{dom} \Omega$. Then the element Lg is well-defined and $e = Dg = gDLg$. Hence $g \in I(X)$ and $g^{-1} = DLg$. ∎

COROLLARY 1.1. *Suppose that all assumptions of Theorem* 1.1 *are satisfied. Let $R_1 \neq R$ be a right inverse of D. Then Theorem* 1.1 *holds for $g_1 = R_1 e$.*

Proof. Clearly, $g_1 e - g = R_1 e - Re = z \in \ker D$ (cf. Przeworska-Rolewicz [1], Section 2.1). This implies that $Dg_1 = D(Re + z) = e$, as before.

COROLLARY 1.2. *Suppose that all assumptions of Theorem* 1.1 *are satisfied. If $g \in \operatorname{dom} \Omega$ then there is no $\delta \in \operatorname{dom} \Omega \setminus \{0\}$ such that $\delta g = 0$, i.e. g is not a zero divisor.*

Proof. By Theorem 1.1, $g \in I(X)$. If there is a $\delta \neq 0$ such that $\delta g = 0$ then $\delta = g^{-1} g \delta = 0$, a contradiction. ∎

EXAMPLE 1.1. Suppose that X is a D-algebra with unit e. Let $(L, E) \in G[\Omega]$ and $R \in \mathcal{R}_D$. Let $g = Re \in \operatorname{dom} \Omega$ and let $v = Lg = LRe$. Then $e = DRe = Dg = gDv$, by our definitions. By Theorem 1.1, $g \in I(X)$ and $DLg = g^{-1}$, i.e. $DL(Re) = (Re)^{-1}$. Note that the element $g = Re$ plays a role of an *argument*. In the classical case of $\mathcal{F} = \mathbb{R}$, $X = C(\mathbb{R}_+)$ with pointwise multiplication, $D = d/dt$ and $R = \int_0^t$ we have $g = R1 = t$ and

$$\frac{d}{dt} \ln t = \frac{1}{t} \quad (\text{for } t \neq 0).$$

PROPOSITION 1.10. *Let $X \in \mathbf{Lg}(D)$, $R \in \mathcal{R}_D$, $\lambda \in v_{\mathcal{F}} R$ and $e \in \operatorname{dom} \Omega^{-1}$. Set $g = Re$. Then $\lambda g \in \operatorname{dom} \Omega^{-1}$ and there are $(L, E) \in G[\Omega]$ such that*

$$E(\lambda g) = (I - \lambda R)^{-1} z = e_\lambda z \in \ker(D - \lambda I) \quad \textit{for all } z \in \ker D.$$

Proof. Elements of the form $u = e_\lambda z = (I - \lambda R)^{-1} z$ are well-defined for all $z \in \ker D$ and $(D - \lambda I)u = D(I - \lambda R)u = Dz = 0$. Moreover, $Du = \lambda u = u\lambda e = u\lambda DRe = uD(\lambda g)$, which implies that $\lambda g \in \operatorname{dom} \Omega^{-1}$ and there are $(L, E) \in G[\Omega]$ such that $e_\lambda z = u = E(\lambda g)$. Clearly, $L(e_\lambda z) = \lambda g$. ∎

Recall that elements of the form $e_\lambda z$ are called *exponentials* for D (cf. Przeworska-Rolewicz [1]). Indeed, these elements belong to $\ker(D - \lambda I)$, hence they are eigenvectors of D.

PROPOSITION 1.11. *Let $X \in \mathbf{Lg}(D)$. If $(L, E), (L', E') \in G[\Omega]$ and $(u, x) \in \operatorname{graph} \Omega$ then*

$$L'u = Lu + z, \quad E'(x + z) = Ex \quad \textit{for a } z \in \ker D.$$

Proof. Let $y = L'u$. Then $y \in \Omega u$. By definition and Proposition 1.3, we have $Lu - L'u = x - y = z \in \ker D$. This implies that $Ex = u = E'L'u = E'y = E'(x + z)$. ∎

PROPOSITION 1.12. *Let $X \in \mathbf{Lg}(D)$, $(L, E) \in G[\Omega]$ and $(u, x) \in \operatorname{graph} \Omega$. If $L'u = Lu + z$ and $E'(x + z) = Ex$ for a $z \in \ker D$ then $(L', E') \in G[\Omega]$.*

Proof. By our assumptions, $Dz = 0$, $x = Lu$ and $y = Ex$. Hence

$$Du = uDLu = uD(Lu + z) = uDL'u,$$
$$DE'(x + z) = DEx = (Ex)Dx = E'(x + z)D(x + z),$$

i.e. $y = L'u$ and $E'(x + z)$ satisfy the basic equation. Thus $y \in \operatorname{dom} \Omega$. By our assumption, L is invertible and $L'u = Lu + z$. This means that for every fixed $z \in \ker D$ and for every $y \in \operatorname{dom} \Omega$ the equation $Lu = y - z$ has a unique solution, which implies that L' is invertible. Since $y = L'u = Lu + z = x + z$, we have $E'y = E'(x + z) = u = Ex$ and $y = L'E'(x + z) = L'E'y$, $u = E'(x + z) = E'y = E'L'u$. We therefore conclude that $(L', E') \in G[\Omega]$. ∎

PROPOSITION 1.13. *Let $X \in \mathbf{Lg}(D)$ and $(L, E) \in G[\Omega]$. Suppose that F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Let $L' = R^m D^m L$ and*

$E' = ER^m D^m$ *for a fixed* $m \in \mathbb{N}$. *Let* $x = Lu \in \operatorname{dom} D^m$. *Then* $FD^j L' = 0$ *for* $j = 0, 1, \ldots, m-1$ *and* $(L', E') \in G[\Omega]$, *where*

$$L' = \Big(I - \sum_{j=0}^{m-1} R^j F D^j\Big) L, \qquad E'\Big(I - \sum_{j=0}^{m-1} R^j F D^j\Big) = E.$$

Proof. Let $j = 0, 1, \ldots, m-1$. By definition, $FD^j L' = FD^j R^m D^m L = FR^{m-j} L = 0$. By the Taylor formula,

$$R^m D^m = I - \sum_{j=0}^{m-1} R^j F D^j \quad \text{ on } \operatorname{dom} D^m.$$

Hence for $u \in \operatorname{dom} \Omega$ and $x = Lu \in \operatorname{dom} D^m$ we have

$$E'L'u = E'R^m D^m Lu = ELu = u,$$
$$L'E'R^m D^m x = L'Ex = R^m D^m LEx = R^m D^m x. \ \blacksquare$$

Propositions 1.11–1.13 imply

COROLLARY 1.3. *Let* $X \in \mathbf{Lg}(D)$. *Then logarithms and antilogarithms are uniquely determined up to a constant.*

DEFINITION 1.5. Let $X \in \mathbf{Lg}(D)$ and $(L, E) \in G[\Omega]$. Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Fix $m \in \mathbb{N}$. If $FD^j L = 0$ for $j = 0, 1, \ldots, m-1$, then (L, E) is said to be *m-normalized* by R and we write $(L, E) \in G_{R,m}[\Omega]$.

THEOREM 1.2. *If* $X \in \mathbf{Lg}(D)$ *then there are* $R \in \mathcal{R}_D$ *and* $(L, E) \in G_{R,1}[\Omega]$ *such that for* $u, v \in I(X) \cap \operatorname{dom} \Omega$ *and* $x = Lu, y = Lv$,

(1.9) $$L(uv) = c_D(Lu + Lv) + R[u^{-1}v^{-1} f_D(u, v)],$$

(1.10) $$(Ex)(Ey) = E\{c_D(x + y) + R[(Ex)^{-1}(Ey)^{-1} f_D(Ex, Ey)]\}.$$

Proof. Let $u, v \in I(X) \cap \operatorname{dom} \Omega$. Then $Du = uDLu$, $Dv = vDLv$ and

$$\begin{aligned} DL(uv) &= (uv)^{-1} D(uv) = u^{-1}v^{-1}[c_D(uDv + vDu) + f_D(u, v)] \\ &= c_D(v^{-1}Dv + u^{-1}Du) + u^{-1}v^{-1} f_D(u, v) \\ &= c_D(DLu + DLv) + u^{-1}v^{-1} f_D(u, v) \\ &= D\{c_D(Lu + Lv) + R[u^{-1}v^{-1} f_D(u, v)]\}, \end{aligned}$$

for some $R \in \mathcal{R}_D$. Hence

$$L(uv) = c_D(Lu + Lv) + R[u^{-1}v^{-1} f_D(u, v)] + z, \qquad \text{where } z \in \ker D.$$

If $z = 0$ then L is the selector we are looking for. Moreover, if $x = Lu$ and $y = Lv$, then

$$\begin{aligned} (Ex)(Ey) &= uv = EL(uv) = E\{c_D(Lu + Lv) + R[u^{-1}v^{-1} f_D(u, v)]\} \\ &= E\{c_D(x + y) + R[(Ex)^{-1}(Ey)^{-1} f_D(Ex, Ey)]\}. \end{aligned}$$

Let $z \neq 0$. Let F be an initial operator for D corresponding to R, i.e. $FR = 0$. Clearly, $FL(uv) = c_D F(Lu + Lv) + Fz = F[c_D(Lu + Lv)] + z$. Hence $z = F[L(uv) - c_D(Lu)(Lv)]$. Let $L' = (I - F)L$ and $E = E'(I - F)$. Proposition 1.13 for $m = 1$ implies that $(L', E') \in G[\Omega]$ and $FL' = 0$. Hence

$$\begin{aligned} F[L'(uv) - c_D(L'u + L'v)] &= F[(I - F)L(uv) - c_D(I - F)(Lu + Lv)] \\ &= (I - F)F[L(uv) - c_D(Lu + Lv)] = 0. \end{aligned}$$

We therefore conclude that (L', E') satisfies all the required conditions. ■

COROLLARY 1.4. *Suppose that* $c_D \neq 0$, $X \in \mathbf{Lg}(D)$ *and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *Let* $n \geq 2$. *Then* $u^n \in I(X) \cap \operatorname{dom} \Omega$ *if and only if* $u \in I(X) \cap \operatorname{dom} \Omega$. *In that case*,

$$Lu^n = \Big(c_D \sum_{k=0}^{n-2} c_D^k + 2c_D^n\Big) Lu + R\Big[u^{-n} \sum_{k=0}^{n-2} c_D^k u^k f_D(u, u^{n-1-k})\Big].$$

P r o o f. Let $n = 2$. *Sufficiency* is already proved by Theorem 1.2 (without the assumption that $c_D \neq 0$).

Necessity. Suppose that $u^2 \in \operatorname{dom} \Omega$. This means that there is an $x \in \operatorname{dom} D$ such that $Du^2 = u^2 Dx$. Let $y = \frac{1}{2c_D}[x - Rf_D(u, u)]$, where $R \in \mathcal{R}_D$. Clearly, $y \in \operatorname{dom} D$ and

$$\begin{aligned} 2c_D Dy + f_D(u, u) = Dx = u^{-2} Du^2 &= Dx = u^{-2}[2c_D u Du + f_D(u, u)] \\ &= 2c_D u^{-1} Du + f_D(u, u), \end{aligned}$$

which implies $Dy = u^{-1} Du$. Hence $u \in \operatorname{dom} \Omega$.

Using the same arguments, by an easy induction we get the required formula for all $n \geq 2$. ■

Corollary 1.4 permits us to calculate nth roots of elements $u \in I_n(\operatorname{dom} \Omega)$, i.e. elements $u^{1/n}$, by means of their logarithms in the same manner as in the classical case.

THEOREM 1.3. *Let* $X \in \mathbf{Lg}(D)$ *and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *Then* L *is of exponential type if and only if* X *is a Leibniz* D*-algebra*, *i.e. if* $c_D = 1$ *and* $f_D = 0$.

P r o o f. *Necessity*. Suppose that L is of exponential type, i.e. $L(uv) = Lu + Lv$ for all $u, v \in \operatorname{dom} \Omega$. By definition, we have $uDLu = Du$, $vDLv = Dv$ and

$$D(uv) = uvDL(uv) = uvD(Lu + Lv) = v(uDLu) + u(vDLv) = vDu + uDv.$$

Sufficiency. Suppose that $c_D = 1$ and $f_D = 0$. Hence $D(xy) = xDy + yDx$ for $x, y \in \operatorname{dom} D$. Then, by our assumption and (1.9), $L(uv) = Lu + Lv$. Hence L is of exponential type. ■

PROPOSITION 1.14. *Let* $X \in \mathbf{Lg}(D)$, $D \in ML(X)$ *and* $(L, E) \in G[\Omega]$. *Then* L *and* E *are multiplicative.*

P r o o f. Let $u, v \in \operatorname{dom} \Omega$. Then $x = Lu$, $y = Lv \in \operatorname{dom} \Omega^{-1}$ and $u = Ex$, $v = Ey$. By our assumptions,

$$D[(Ex)(Ey)] = (DEx)(DEy) = [(Ex)Dx][(Ey)Dy] \\ = (Ex)(Ey)(Dx)(Dy) = (Ex)(Ey)D(xy).$$

Hence $E(xy) = (Ex)(Ey)$. This implies that

$$L(uv) = L[(Ex)(Ey)] = LE(xy) = xy = (Lu)(Lv). \ \blacksquare$$

PROPOSITION 1.15. *Let* $X \in \mathbf{Lg}(D)$ *and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *Suppose that either* X *is a Leibniz* D*-algebra or* $D \in ML(X)$. *Then* $(L', E') \in G[\Omega]$, *where*

$$L'u = L(zu), \quad E'x = zEx \quad \textit{for } (u, x) \in \operatorname{graph} \Omega, \ z \in \ker D.$$

P r o o f. Let $z' = Lz$, i.e. $z = Ez'$. By Proposition 1.5, $z' \in \ker D$. Suppose that X is a Leibniz D-algebra. By Theorem 1.3, L is of exponential type. Hence $L(zu) = Lu + Lz$ for $u \in \operatorname{dom} \Omega$ and $E(x + z') = (Ez')Ex = zEx$ for $x \in \operatorname{dom} \Omega^{-1}$. Proposition 1.12 now implies that $E(x + z') = zEx$, $L'u = L(zu)$ and $(L', E') \in G[\Omega]$.

Suppose now that $D \in ML(X)$. By Proposition 1.6, $L, E \in M(X)$, and

$$E(z'x) = (Ez')Ex = zEx, \qquad L(zu) = (Lz)(Lu).$$

Hence

$$DE(z'x) = D[(Ez')(Ex)] = D(zEx) = (Dz)DEx = 0$$

and

$$DL(zu) = D[(Lz)(Lu)] = (DLz)DLu = 0. \ \blacksquare$$

PROPOSITION 1.16. *If* $X \in \mathbf{Lg}(D)$, $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$ *and*

$$g_D(u) = u^{-1} f_D(u, e) \quad \textit{for } u \in I(X) \cap \operatorname{dom} \Omega \tag{1.11}$$

then

$$(1 - c_D)Lu = c_D Le + R g_D(u) \quad \textit{for } u \in I(X) \cap \operatorname{dom} \Omega. \tag{1.12}$$

P r o o f. Put $v = e$ in (1.9). Then $Lu = L(ue) = c_D(Lu + Le) + R[u^{-1} f_D(u, e)] = c_D(Lu + Le) + R g_D(u)$, which implies (1.12). $\blacksquare$

THEOREM 1.4. *Let* $X \in \mathbf{Lg}(D)$ *and let* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *Then the mapping* g_D *defined by* (1.11) *is not constant if and only if* $c_D \neq 1$.

P r o o f. Suppose that $X \in \mathbf{Lg}(D), c_D \neq 1$ and g_D is constant, i.e. there is an $a \in X$ such that $g_D(u) = a$ for all $I(X) \cap \operatorname{dom} \Omega$. Then, by Proposition 1.16,

$$Lu = \frac{1}{1 - c_D}(c_D Le + Ra) \quad \text{for all } u \in I(X) \cap \operatorname{dom} \Omega,$$

i.e. L is constant. This contradicts our assumption that L is invertible.

Conversely, put $c_D = 1$ in (1.12). Then we get $R g_D(u) = -Le$, which implies $g_D(u) = DR g_D(u) = -DLe = -De$ for all $u \in I(X) \cap \operatorname{dom} \Omega$. $\blacksquare$

PROPOSITION 1.17. *Let* $X \in \mathbf{Lg}(D)$. *Let* F *be an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$. *Let* $(L, E) \in G_{R,1}[\Omega]$. *Then*

(i) $(1 - 2c_D)Le = Rf_D(e, e)$;
(ii) $c_D = 1/2$ *implies* $f_D(e, e) = 0$;
(iii) $c_D \neq 1/2$ *implies* $Le = \frac{1}{1-2c_D} Rf_D(e, e)$ *and* $FLe = 0$;
(iv) $c_D \neq 1/2$ *and* $f_D(e, e) = 0$ *implies* $Le = 0$;
(v) $c_D \neq 1/2$ *and* $e \in \ker D$ *implies* $Le = 0$;
(vi) $Le = e + z$, *where* $z \in \ker D$;
(vii) *if* X *is a Leibniz* D*-algebra then* $Le = 0$;
(viii) $(1 - 2c_D)De = f_D(e, e)$;
(ix) $c_D \neq 1/2$ *implies* $De = \frac{1}{1-2c_D} f_D(e, e)$;
(x) $c_D \neq 1/2$ *implies* $Le = e - Fe$.

Proof. Put $u = e$ in (1.12). Then $(1 - c_D)Le = c_D Le + Rf_D(e, e)$, which implies (i). Acting on both sides of (i) by the operator F we get

$$(1 - 2c_D)FLe = FRf_D(e, e) = 0.$$

If $c_D = 1/2$ then $f_D(e, e) = DRf_D(e, e) = D(1 - 2c_D)Le = 0$. Let $c_D \neq 1/2$. Since (L, E) is 1-normalized, we have $FLe = 0$ and

$$Le = \frac{1}{1 - 2c_D} Rf_D(e, e) + FLe = \frac{1}{1 - 2c_D} Rf_D(e, e).$$

If, in addition, $f_D(e, e) = 0$ then $Le = 0$. If $c_D \neq 1/2$ and $e \in \ker D$ then $Le \in \ker D$, which implies $Le = FLe = 0$. By definition, $De = eDLe$, which implies $D(Le - e) = 0$. Hence $Le = e + z$, where $z \in \ker D$. If X is a Leibniz D-algebra then $c_D = 1$ and $e \in \ker D$. Hence, by (v), $Le = 0$. Since $De = DLe$, (i) implies $f_D(e, e) = DRf_D(e, e) = (1 - 2c_D)DLe = (1 - 2c_D)De$. Thus for $c_D \neq 1/2$ we get (ix). Moreover, $Le = \frac{1}{1-2c_D} Rf_D(e, e) = RDe = e - Fe$. ∎

THEOREM 1.5. *Suppose that* $X \in \mathbf{Lg}(D)$ *and* $c_D = 0$. *Let* $\alpha \in \mathcal{F} \setminus \{0\}$ *and let* $u \in \operatorname{dom} \Omega$. *If* $\alpha \neq 1$ *then* $\alpha u \notin \operatorname{dom} \Omega$.

Proof. Let $(L, E) \in G_{R,1}[\Omega]$ for an $R \in \mathcal{R}_D$. Suppose that $\alpha \in \mathcal{F} \setminus \{0\}$ and $u \in \operatorname{dom} \Omega$. By our assumption, $\alpha u \in \operatorname{dom} \Omega$. Since $c_D = 0$, (1.12) implies that $Lu = Rg_D(u)$. Hence

$$\begin{aligned} L(\alpha u) &= Rg_D(\alpha u) = R[(\alpha u)^{-1} f_D(\alpha u, e)] \\ &= R[\alpha^{-1} \alpha u^{-1} f_D(u, e)] = R[u^{-1} f_D(u, e)] = Rg_D(u) = Lu. \end{aligned}$$

Hence $u = ELu = EL(\alpha u) = \alpha u$, which implies $\alpha = 1$. ∎

COROLLARY 1.5. *Suppose that all assumptions of Theorem* 1.5 *are satisfied. Then* $-e \notin \operatorname{dom} \Omega$.

THEOREM 1.6. *Suppose that* $X \in \mathbf{Lg}(D)$ *and* $c_D \neq 0$. *Let* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *Let* $\alpha \in \mathcal{F} \setminus \{0\}$ *and* $u \in I(X) \cap \operatorname{dom} \Omega$. *Then* $\alpha u \in I(X) \cap \operatorname{dom} \Omega$

and

$$L(\alpha u) = Lu + L(\alpha e) - Le = Lu + z, \qquad \textit{where } z \in \ker D.$$

Proof. Since $e \in \operatorname{dom} \Omega$, we have $\alpha e \in \operatorname{dom} \Omega$. Indeed, by definition,

$$DL(\alpha e) = (\alpha e)^{-1} D(\alpha e) = \alpha^{-1}\alpha e De = De,$$

which implies $L(\alpha e) = e + z'$, where $z' \in \ker D$. By Proposition 1.17(vi), $Le = e + z''$, where $z'' \in \ker D$. Now, (1.9) and (1.12) imply that

$$\begin{aligned} L(\alpha u) = L(\alpha e u) &= c_D[Lu + L(\alpha e)] + R[(\alpha e)^{-1} u^{-1} f_D(\alpha e, u)] \\ &= c_D[Lu + L(\alpha e)] + R[u^{-1} f_D(u, e)] = c_D Lu + c_D L(\alpha e) + R g_D(u) \\ &= Lu - Le + L(\alpha e) = Lu - e - z'' + e + z' = Lu + z, \end{aligned}$$

where $z = z' - z'' \in \ker D$. ∎

Corollary 1.6. *Suppose that* $X \in \mathbf{Lg}(D)$ *and* $c_D \neq 0$. *Let* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *If* $u \in I(X) \cap \operatorname{dom} \Omega$ *then* $-u \in I(X) \cap \operatorname{dom} \Omega$ *and*

$$L(-u) = c_D(Lu + Le) + R g_D(u).$$

In particular, $-e \in I(X) \cap \operatorname{dom} \Omega$ *and* $L(-e) = Le$.

Proof. By Theorem 1.6, $-e \in I(X) \cap \operatorname{dom} \Omega$. Corollary 1.4 and Proposition 1.17(i) imply that

$$\begin{aligned} Le = L[(-e)(-e)] &= 2c_D L(-e) + R[(-e)^{-2} f_D(-e, -e)] \\ &= 2c_D L(-e) + R f_D(e, e) = 2c_D L(-e) + (1 - 2c_D) Le, \end{aligned}$$

which implies $2c_D L(-e) = 2c_D Le$. Since $c_D \neq 0$, we get $L(-e) = Le$. Suppose that $u \in I(X) \cap \operatorname{dom} \Omega$. Again by (1.9),

$$\begin{aligned} L(-u) = L(-eu) &= c_D[Lu + L(-e)] + R[u^{-1}(-e)^{-1} f_D(u, -e)] \\ &= c_D(Lu + Le) + R g_D(u). \;∎ \end{aligned}$$

Corollary 1.7. *Suppose that all assumptions of Corollary* 1.6 *are satisfied and* X *is a Leibniz* D*-algebra. Then* $L(-u) = Lu$ *for* $u \in I(X) \cap \operatorname{dom} \Omega$.

Proof. Since X is a Leibniz D-algebra, we have $c_D = 1$ and $f_D = 0$. By Proposition 1.17(vii), we have $Le = 0$. Corollary 1.6 now implies the assertion.

Corollary 1.8. *Suppose that* $\mathcal{F} = \mathbb{C}$, $X \in \mathbf{Lg}(D)$ *and* $c_D \neq 0$. *Let* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *If* $u \in I(X) \cap \operatorname{dom} \Omega$ *then* $iu \in I(X) \cap \operatorname{dom} \Omega$ *and*

$$L(iu) = \frac{1}{2}(Lu + Le) + \frac{1}{2c_D} R[g_D(u) - u^{-2} f_D(u, u)].$$

Proof. Corollaries 1.4 and 1.6 imply that

$$\begin{aligned} c_D(Lu + Le) + R g_D(u) = L(-u) &= L[(iu)(iu)] \\ &= 2c_D L(iu) + R[(iu)^{-2} f_D(iu, iu)] \\ &= 2c_D L(iu) + R[u^{-2} f_D(u, u)]. \end{aligned}$$

Since $c_D \neq 0$, we get the required formula for $L(iu)$. ∎

Corollary 1.9. *Suppose that all assumptions of Corollary* 1.8 *are satisfied and* X *is a Leibniz* D*-algebra. Then* $L(iu) = \frac{1}{2}Lu = \frac{1}{2}L(-u)$ *for all* $u \in I(X) \cap \operatorname{dom} \Omega$. *In particular*, $L(ie) = 0$.

Proof. By our assumptions, we have $c_D = 1$, $f_D = 0$ and $Le = 0$. Now, Corollaries 1.7 and 1.8 imply our statement (cf. the proof of Corollary 1.7). ∎

Proposition 1.18. *Let* X *be a* D*-algebra with* $c_D \neq 1$. *Suppose that the mapping* g_D *defined by* (1.11) *is not constant. Define the mapping* A *by*

$$Au = \frac{1}{1 - c_D}[c_D e + R g_D(u)] \quad \text{for } u \in I(X) \cap \operatorname{dom} \Omega, \text{ where } R \in \mathcal{R}_D.$$

If A *is invertible then there is a* $(L, E) \in G_{R,1}[\Omega]$ *such that* $L = A$ *and* $E = A^{-1}$, *i.e.* $X \in \mathbf{Lg}(D)$.

Proof. By definition, we have

$$DAu = \frac{1}{1 - c_D}[c_D De + g_D(u)] \quad \text{for } u \in I(X) \cap \operatorname{dom} \Omega.$$

On the other hand, $Du = D(ue) = c_D(uDe + Du) + f_D(u, e)$, which implies $(1 - c_D)Du = c_D uDe + f_D(u, e)$. Since $c_D \neq 1$ we get

$$u^{-1}Du = \frac{1}{1 - c_D}[c_D e + u^{-1} f_D(u, e)] = \frac{1}{1 - c_D}[c_D De + g_D(u)] = DAu.$$

Hence $Au \in \operatorname{dom} \Omega^{-1}$ and there is an $(L, E) \in G[\Omega]$ such that $x = Au \in \operatorname{dom} \Omega^{-1}$ and $Du = uDx$. This implies that there is an $(L, E) \in G[\Omega]$ such that $x = Lu$ and $u = Ex$, i.e. $L = A$ and $E = A^{-1}$. By definition, (L, E) is 1-normalized. ∎

Example 1.2. Let X be a D-algebra with unit e and with the product rule

$$D(xy) = (Fx)Dy + (Fy)Dx \quad \text{for } x, y \in \operatorname{dom} D, \tag{1.13}$$

where F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. In the classical case of the operator $D = d/dt$ in the space $X = C[a, b]$ over $\mathbb{R}$ with pointwise multiplication and with $R = \int_a^t$ (i.e. $(Fx)(t) = x(a)$ for $x \in X$) this is the so-called *point derivation*. Clearly, here $c_D = 1$ and $f_D(x, y) = (Fx - x)Dy + (Fy - y)Dx$. We shall consider two cases:

(i) $e \in \ker D$. Then $Fe = e$ and $f_D(u, e) = 0$ for $u \in \operatorname{dom} D$. Hence the mapping g_D is constant.

(ii) By (1.13), for $x = y = e$ we get $De = De^2 = 2(Fe)De$, which implies $(e - 2Fe)De = 0$. Suppose that $2Fe = e$. Then $RDe = e - Fe = 2Fe - Fe = Fe = e - RDe$. Hence $RDe = \frac{1}{2}e$, which implies $De = \frac{1}{2}De$. Thus $De = 0$ and $e \in \ker D$.

It is well-known that $\operatorname{dom} D = RX \oplus \ker D$ (cf. Przeworska-Rolewicz [1]). Suppose that $x, y \in RX$. Then $Fx = Fy = 0$ and $D(xy) = 0$. Hence $RX \notin \mathbf{Lg}(D)$. If $x, y \in \ker D$ then $Fx = x$ and $Fy = y$. Hence $\ker D$ is a Leibniz D-algebra.

2. Logarithms and antilogarithms of higher order

We shall now consider logarithmic and antilogarithmic mappings induced by powers D^n ($n \in \mathbb{N}$) of the right invertible operator D under consideration. These mappings are said to be logarithmic and antilogarithmic mappings *of order* n. The corresponding logarithms and antilogarithms are said to be logarithms and antilogarithms *of order* n.

THEOREM 2.1. *For* $n \in \mathbb{N}$ *define a multifunction*

$$\Omega_n : \operatorname{dom} D^n \to 2^{\operatorname{dom} D^n}$$

by

$$\Omega_n u = \{x \in \operatorname{dom} D^n : D^n u = u D^n x\} \quad \text{for } x \in \operatorname{dom} D^n.$$

For $(L, E) \in G[\Omega]$ *write* $\Omega_1 = \Omega$, $L_1 = L$ *and* $E_1 = E$. *Then* $\operatorname{dom} \Omega_{n+1} \subset \operatorname{dom} \Omega_n$ ($n \in \mathbb{N}$). *If* $X \in \mathbf{Lg}(D)$ *then* $X \in \mathbf{Lg}(D^n)$ *for all* $n \in \mathbb{N}$. *Moreover, if* $(L_1, E_1) \in G_{R,1}[\Omega_1]$ *for an* $R \in \mathcal{R}_D$ *then for every* $n \in \mathbb{N}$ *there exists an* $(L_n, E_n) \in G_{R,n}[\Omega_n]$ *such that for all* $(u, x), (v, y) \in \operatorname{graph} \Omega_n$ *and* $n \geq 2$,

$$L_n u = c_D L_{n-1} u + R^n[c_D(DL_1 u)D^{n-1} L_{n-1} u + u^{-1} f_D(u, D^{n-1} L_{n-1} u)], \tag{2.1}$$

and for all $n \in \mathbb{N}$,

$$L_n(uv) = c_D^n(L_n u + L_n v) + R^n[u^{-1} v^{-1} f_D^{(n)}(u, v)], \tag{2.2}$$

$$(E_n x)(E_n y) = E_n\{c_D^n(x + y) + R^n[(Ex)^{-1}(Ey)^{-1} f_D^{(n)}(Ex, Ey)]\}, \tag{2.3}$$

and

$$(1 - 2c_D^n) L_n e = R^n f_D^{(n)}(e, e). \tag{2.4}$$

Proof. For all $n \in \mathbb{N}$ we have $D^n \in R(X)$ and $R^n \in \mathcal{R}_{D^n}$ whenever $D \in R(X)$ and $R \in \mathcal{R}_D$. We shall prove by induction that there is an $(L_n, E_n) \in G_{R,n}[\Omega_n]$ such that (2.1) holds. For $n = 1$ this is proved in Theorem 1.2. Suppose this formula to be true for a fixed $n \in \mathbb{N}$. By our assumption, we have $FD^j L_n = 0$ for $j = 0, 1, \ldots, n-1$. Let $u \in I(X) \cap \operatorname{dom} \Omega_{n+1}$ and let $x = L_n u$. Then $D^n u = u D^n x$. By definition, $DL_1 u = u^{-1} Du$. Let $y = R^{n+1}(u^{-1} D^{n+1} u)$. Then

$$\begin{aligned} D^{n+1} y &= u^{-1} D^{n+1} u = u^{-1} D(D^n u) = u^{-1} D(u D^n x) \\ &= u^{-1}\{c_D[u D^{n+1} x + (Du) D^n x] + f_D(u, D^n x)\} \\ &= c_D[D^{n+1} x + (u^{-1} Du) D^n x] + u^{-1} f_D(u, D^n x) \\ &= D^{n+1}\{c_D L_n u + c_D R^{n+1}[(DL_1 u) D^n L_n u + u^{-1} f_D(u, D^n L_n u)]\}, \end{aligned}$$

which implies that $y \in \operatorname{dom} \Omega$ and

$$y = c_D L_n u + R^{n+1}[c_D(DL_1) D^n L_n u + u^{-1} f_D(u, D^n L_n u)] + w, \quad \text{where } w \in \ker D^n.$$

If $w = 0$ then y is of the form (2.1). Suppose that $w \neq 0$. By our assumption, $FD^j y = c_D FD^j L_n u = 0$ for $j = 0, 1, \ldots, n-1$. Put $y' = R^{n+1}D^{n+1}y$ (cf. Proposition 1.13). Then $FD^j y' = FR^{n+1-j}D^{n+1}y = 0$ for $j = 0, 1, \ldots, n$, i.e. y' is of the form (2.1).

We still have to prove that $y = L_{n+1}u$, where L_{n+1} is an invertible selector of Ω_{n+1}. The basic equation $D^{n+1}u = uD^{n+1}y$ together with the conditions $FD^j y = 0$ for $j = 0, 1, \ldots, n$ can be written as follows:

$$y = R^{n+1}(u^{-1}D^{n+1}u) + \sum_{j=0}^{n} R^j FD^j y = R^{n+1}(u^{-1}D^{n+1}u).$$

Hence for every $u \in I(X) \cap \operatorname{dom} \Omega_{n+1}$ there is a unique $y \in \operatorname{dom} \Omega_{n+1}^{-1}$. This implies that there exists an invertible selector E_{n+1} of Ω_{n+1}^{-1} and $L_{n+1} = E_{n+1}^{-1}$.

As in the proof of Theorem 1.2, by an application of (0.1), we prove (2.2). Putting $x = Lu$ and $y = Lv$ in (2.2) we get (2.3). Putting $u = v = e$ in (2.2) we obtain (2.4). ■

COROLLARY 2.1. *Logarithms and antilogarithms of order* n (*if they exist*) *are uniquely determined up to a polynomial of degree* $n-1$.

Note. A Leibniz D-algebra X is not a Leibniz D^n-algebra for $n \geq 2$. Indeed, by the Leibniz formula, if $n \geq 2$ then, in general,

$$f_{D^n}(x, y) = f_D^{(n)}(x, y) = \sum_{k=1}^{n-1} \binom{n}{k} (D^k x)(D^{n-k} y) \neq 0 \quad (x, y \in \operatorname{dom} D^n).$$

COROLLARY 2.2. *Let* $X \in \mathbf{Lg}(D)$ *be a Leibniz* D*-algebra and* $(L_1, E_1) \in G_{R,1}[\Omega_1]$ *for an* $R \in \mathcal{R}_D$. *Then* $L_n e = 0$ *for all* $n \in \mathbb{N}$.

Proof. Since X is a Leibniz D-algebra, $e \in \ker D$. By the Leibniz formula,

$$f_D^{(n)}(e, e) = \sum_{k=1}^{n-1} \binom{n}{k} (D^k e)(D^{n-k} e) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Since $c_D = 1$, (2.4) now implies $L_n e = 0$ $(n \in \mathbb{N})$. ■

COROLLARY 2.3. *Suppose that* $X \in \mathbf{Lg}(D)$, $(L_1, E_1) \in G_{R,1}[\Omega_1]$ *for an* $R \in \mathcal{R}_D$ *and* F *is an initial operator for* D *corresponding to* R. *Then*

(i) $(1 - 2c_D^n)FL_n e = 0$ *for all* $n \in \mathbb{N}$;
(ii) $(1 - 2c_D^n)D^n e = f_D^{(n)}(e, e)$ *for all* $n \in \mathbb{N}$;
(iii) $c_D^n = 1/2$ *for an* $n \in \mathbb{N}$ *implies* $f_D^{(n)}(e, e) = 0$;
(iv) $c_D^n \neq 1/2$ *for all* $n \in \mathbb{N}$ *implies* $FL_n e = 0$ *for all* $n \in \mathbb{N}$.

Proof. Since $FR = 0$, acting on both sides of (2.4) by F we get (i). By definitions, $D^n e = eD^n L_n e$, and (2.4) implies that

$$(1 - 2c_D^n)D^n e = (1 - 2c_D^n)D^n L_n e = D^n R^n f_D^{(n)}(e, e) = f_D^{(n)}(e, e)$$

for $n \in \mathbb{N}$. If $c_D^n = 1/2$ for an $n \in \mathbb{N}$ then, by (ii), $f_D^{(n)}(e,e) = 0$. If $c_D^n \neq 1/2$ for all $n \in \mathbb{N}$ then (i) and (ii) imply (iv). ∎

THEOREM 2.2. *Let $X \in \mathbf{Lg}(D)$ and $D \in ML(X)$. Then $X \in \mathbf{Lg}(D^n)$ for all $n \in \mathbb{N}$ and there is an $(L_n, E_n) \in G[\Omega_n] \cap ML(X)$ such that for some $R \in \mathcal{R}_D$, all $(u,x) \in \operatorname{graph} \Omega_n$ and all $n \in \mathbb{N}$,*

$$L_n u = \prod_{j=0}^{n-1} R^j L_1 u, \tag{2.5}$$

$$\prod_{j=0}^{n-1} E_n(R^j x) = E_n\Big(\prod_{j=0}^{n-1} R^j x\Big) = E_1 x. \tag{2.6}$$

Proof. Since $D \in ML(X) \cap R(X)$, also $D^n \in ML(X) \cap R(X)$ and $R^n \in \mathcal{R}_{D^n}$ whenever $R \in \mathcal{R}_D$ and $n \in \mathbb{N}$. Proposition 1.16 implies that $L_1 = L$ and $E_1 = E \in ML(X)$. Theorem 2.1 implies that $X \in \mathbf{Lg}(D)$. If $(L_n, E_n) \in G[\Omega]$ then $L_n, E_n \in ML(X)$, since $D^n \in ML(X)$. We shall prove by induction that there exists an $(L_n, E_n) \in G[\Omega_n]$ such that (2.5) and (2.6) hold. For $n = 1$, (2.5) is trivial. Suppose (2.5) to be true for a fixed $n \in \mathbb{N}$. Let $u \in \operatorname{dom} \Omega_{n+1}$ and $y_n = L_n u$. Then, by definition, $Du = uDy_1$, $D^n u = uD^n y_n$ and

$$\begin{aligned} D^{n+1}u &= D(D^n u) = D(uD^n y) = (Du)D^{n+1} y_n = (uDy_1)D^{n+1} y_n \\ &= u(D^{n+1}R^n y_1)D^{n+1} y_n = uD^{n+1}[(R^n y_1) y_n] \\ &= uD^{n+1}\Big[(R^n y_1)\prod_{j=0}^{n-1} R^j L_1 u\Big] = uD^{n+1}\prod_{j=0}^{n} R^j L_1 u = uD^{n+1} y_{n+1}. \end{aligned}$$

Hence $y_{n+1} = L_{n+1} u$ is of the form (2.5). Clearly, for all $n \in \mathbb{N}$,

$$E_1 x = u = E_n L_n u = E_n\Big(\prod_{j=0}^{n-1} R^j L_1 u\Big) = E_n\Big(\prod_{j=0}^{n-1} R^j x\Big) = \prod_{j=0}^{n-1} E_n R^j x. \ ∎$$

LEMMA 2.1. *Let X be a D-algebra with $c_D \neq 0$. Then for all $u \in I(X) \cap \operatorname{dom} D$,*

$$Du^{-1} = c_D^{-1} u^{-1}[De - c_D u^{-1} Du - f_D(u, u^{-1})].$$

Proof. Let $u \in I(X) \cap \operatorname{dom} D$. Then

$$De = D(uu^{-1}) = c_D(uDu^{-1} + u^{-1}Du) + f_D(u, u^{-1}),$$

which implies the required formula. ∎

THEOREM 2.3. *Let $\mathcal{F} = \mathbb{C}$, $X \in \mathbf{Lg}(D)$, $c_D \neq 0$ and $(L_n, E_n) \in G[\Omega_n]$ $(n = 1, 2)$. If $u \in I(X) \cap \operatorname{dom} \Omega_2$ then $iu \in I(X) \cap \operatorname{dom} \Omega_2$ and*

$$L_1(iu) = c_D L_2 u + R^2[(DL_1 u)(De - c_D L_1 u) + (e - DL_1 u) f_D(u, u^{-1})], \tag{2.7}$$

where $R \in \mathcal{R}_D$.

P r o o f. Let $u \in I(X) \cap \operatorname{dom} \Omega_2$. Then, by definitions and Lemma 2.1,

$$\begin{aligned} D^2 L_1(iu) &= D[DL_1(iu)] = D[(iu)^{-1} D(iu)] = D[-iu^{-1} iDu] \\ &= D(u^{-1} Du) = c_D[u^{-1} D^2 u + (Du^{-1}) Du] + f_D(u, u^{-1}) \\ &= c_D D_2^L u + (DL_1 u)[De - c_D DL_1 u - f_D(u, u^{-1})] + f_D(u, u^{-1}) \\ &= c_D D^2 L_2 u + (DL_1 u)(De - c_D DL_1 u) + (e - DL_1 u) f_D(u, u^{-1}). \end{aligned}$$

By Corollary 2.1, we get (2.7) for an $R \in \mathcal{R}_D$. ∎

Corollary 2.4. *Suppose that all assumptions of Theorem* 2.3 *are satisfied and X is a Leibniz D-algebra. Then*

$$L(iu) = L_2 u - R^2 (DL_1 u)^2, \quad \text{where } R \in \mathcal{R}_D.$$

Example 2.1. Let $X \in \mathbf{Lg}(D)$ be a Leibniz D-algebra. By (2.1), if $(L_n, E_n) \in G_{R,n}[\Omega_n]$ for an $R \in \mathcal{R}_D$ and $(u, x) \in \operatorname{graph} \Omega_n$ ($n \in \mathbb{N}$) then

$$L_n u = L_{n-1} u + R^n [(DL_1 u) D^{n-1} L_{n-1} u].$$

Example 2.2. Let $\mathcal{F} = \mathbb{R}$, $X = C(\mathbb{R})$ and $D = d/dt$. Then X with pointwise multiplication is a Leibniz D-algebra. By Theorem 2.1, $X \in \mathbf{Lg}(D^n)$. Recall that, by Corollary 2.1, logarithms and antilogarithms of order n are uniquely determined up to a polynomial of degree $n-1$. Let $u(t) = e^{\lambda t}$. Fix $n \in \mathbb{N}$. Let $(L_n, E_n) \in G[\Omega_n]$. Since $D^n u = \lambda^n u$, by definition we have $D^n L_n u = u^{-1} D^n u = \lambda^n$. This implies that

$$L_n u = \frac{1}{n!} (\ln u + a)^n, \quad \text{where } a \in \mathbb{R} \text{ is arbitrary},$$

i.e.

$$L_n e^{\lambda t} = \frac{1}{n!} (\lambda t + a)^n.$$

Let $(Fx)(t) = x(0)$ for $x \in X$. Clearly, since $a = 0$, we get $F L_n e^{\lambda t} = \frac{1}{n!} a^n = 0$. Moreover, again by definition, we have

$$E_n \left[\frac{1}{n!} (\lambda t + a)^n \right] = e^{\lambda t}.$$

Example 2.3. Denote by $C(E; \mathcal{F})$ the space of all functions continuous on a set E and with values in $\mathcal{F}$ (considered as a linear space over $\mathcal{F}$). In this example E will be a closed interval in $\mathbb{R}$.

Fix $\varepsilon > 0$. Let $X = C([\varepsilon, \pi - \varepsilon]; \mathbb{R})$ and $D = d/dt$. Then X with pointwise multiplication is a Leibniz D-algebra belonging to $\mathbf{Lg}(D)$. By similar arguments to those used in Example 2.2, for $u(t) = \sin t$ we find that

$$L_1u = \ln u, \qquad E_1 \ln u = u,$$
$$L_j u = -\frac{1}{j!}(\arcsin u)^j = \frac{t^j}{j!} \qquad (j = 2k,\ k \in \mathbb{N}),$$
$$L_j u = \int_{\varepsilon}^{t} \frac{(t-s)^j}{j!} \ln\sin s\, ds \qquad (j = 2k+1,\ k \in \mathbb{N}).$$

We also have $L_2 \cot t = 2\ln u$.

Similarly, in the space $Y = C([-\pi/2+\varepsilon, \pi/2-\varepsilon]; \mathbb{R})$ for $v(t) = \cos t$ we find that

$$L_1v = \ln v, \qquad E_1 \ln v = v,$$
$$L_j v = -\frac{1}{j!}(\arccos v)^j = \frac{t^j}{j!} \qquad (j = 2k,\ k \in \mathbb{N}),$$
$$L_j v = \int_{-\pi+\varepsilon}^{t} \frac{(t-s)^j}{j!} \ln\cos s\, ds \qquad (j = 2k+1,\ k \in \mathbb{N}).$$

Also $L_2 \tan t = 2\ln v$.

Let now

$$X' = C([\varepsilon, \pi-\varepsilon]; \mathbb{C}), \qquad Y' = C([-\pi/2+\varepsilon, \pi/2-\varepsilon]; \mathbb{C}).$$

By definitions, $X \subset X'$ and $Y \subset Y'$. Clearly, by the *De Moivre formula*, we get

$$i \arcsin u = L_1(iu + \sqrt{1-u^2}) \quad \text{in } X',$$
$$i \arccos v = L_1(v + i\sqrt{1-v^2}) \quad \text{in } Y'.$$

THEOREM 2.4. *Let $n \in \mathbb{N}$. Suppose that $X \in \mathbf{Lg}(D^n)$, $(L_n, E_n) \in G[\Omega_n]$, $a \in X$ and $R^n a \in \operatorname{dom} \Omega_n^{-1}$ for an $R \in \mathcal{R}_D$. Then*

(i) *$R^n a + z \in \operatorname{dom} \Omega_n^{-1}$ for all $z \in \ker D^n$;*

(ii) *$x \in \ker(D^n - a)$ if and only if $x = E_n(R^n a + z)$, where $z \in \ker D^n$ is arbitrary, i.e. $z = \sum_{k=0}^{n-1} R^k z_k$, where $z_0, \ldots, z_{n-1} \in \ker D$ are arbitrary;*

(iii) *If $y \in \operatorname{dom} \Omega_n$ and there is a $v \in \operatorname{dom} \Omega_n$ such that*

$$c_D^n L_n v + f_D^{(n)}(E_n(R^n a), v) = L_n y - c_D^n R^n a, \tag{2.8}$$

then $x' = E_n R^n(a+v)$ is a particular solution of the equation

$$D^n x = ax + y, \qquad y \in \operatorname{dom} \Omega_n. \tag{2.9}$$

Proof. Let $z \in \ker D^n$. Let $b = R^n a + z$. Then $D^n b = D^n R^n a + D^n z = a$ and $D^n E_n b = (E_n b) D^n b = a(E_n b)$. Hence $x = E_n b \in \ker(D^n - a)$. Conversely, suppose that $x \in \ker(D^n - a)$. Then $D^n x = ax = xD^n(R^n a + z)$, where $z \in \ker D^n$ is arbitrary. This implies that $x = E_n(R^n a + z)$. Recall that $z \in \ker D^n$ if and only if $z = \sum_{k=0}^{n-1} R^k z_k$, where $z_0, \ldots, z_{n-1} \in \ker D$ are arbitrary.

Having thus proved (i) and (ii), consider the non-homogeneous equation (2.9). Let v satisfy (2.8). Let $x' = E_n R^n(a+v)$. Then

$$\begin{aligned}(D^n x' - ax') &= E_n L_n (D^n x' - ax') = E_n L_n [D^n E_n R^n(a+v) - aE_n R^n(a+v)]\\ &= E_n L_n [E_n R^n(a+v) D^n R^n(a+v) - aE_n R^n(a+v)]\\ &= E_n L_n [E_n R^n(a+v)v]\\ &= c_D^n [L_n E_n R^n(a+v) + L_n v] + f_D^{(n)}(E_n(R^n a), v)\\ &= E_n [c_D^n (R^n a + L_n v) + f_D^{(n)}(E_n(R^n a), v)] = E_n L_n y = y. \ \blacksquare\end{aligned}$$

COROLLARY 2.5. *Suppose that all assumptions of Theorem* 2.4 *are satisfied and there is a* $v \in \operatorname{dom} \Omega_n$ *satisfying* (2.8). *Then all solutions of* (2.9) *are of the form*

(2.10) $$x = E_n R^n(a+v) + E_n(R^n a + z), \quad \text{where } z \in \ker D^n \text{ is arbitrary.}$$

In the next sections we shall examine the cases when particular solutions of (2.9) can be found in an explicit form, without looking for solutions of (2.8).

3. Reduction theorems

We shall now consider a class of D-algebras which are important in some applications and have non-Leibniz components of a particular form.

To begin with, recall the following

THEOREM 3.0 (cf. Targonski [1]). *Let* X *be a commutative algebra and let* $A \in L(X)$ *satisfy the so-called* Bourlet condition:

$$A(xy) = f(x, y, Ax, Ay) \quad \text{for } x, y \in \operatorname{dom} A$$

(cf. C. Bourlet, C. R. Acad. Sci. Paris 124 (1827)). *If* f *is an entire function of its variables, then* f *is necessarily of the form*

$$f(x, y, Ax, Ay) = \alpha(Ax)(Ay) + \beta(xAy + yAx) + \gamma xy,$$

where α, β, γ *are scalars.*

If X *has a unit* e *and there are no zero divisors in* X, *then* A *is either a scalar multiple of the identity or belongs to one of the following three types*:

(i) $A_0(xy) = \alpha(A_0 x)(A_0 y)$, *where* $A_0 = A - \mu I$ (μ *is a scalar depending on* α, β, γ);

(ii) $A(xy) = \frac{1}{2}(xAy + yAx)$;

(iii) $A_1(xy) = xA_1 y + yA_1 x$, *where* $A_1 = A + \gamma I$.

Note that in the case (i), $A_1 = \alpha A_0 = \alpha(A - \mu)$. In the case (ii), A is multiplication by a fixed element. Namely, $Ax = xAe$ for all $x \in \operatorname{dom} A$. In the case (iii) we have $e \in \ker A_1$.

In the sequel we shall not always need the assumption that the algebras under consideration are without zero divisors and that the coefficients α, β, γ are scalars. Nevertheless, the Targonski theorem motivates in a sense the following:

DEFINITION 3.1. Denote by $\mathcal{A}_D(a,c,d)$ the set of those D-algebras X with unit $e \in \operatorname{dom} D$ and with $\ker D \neq \{0\}$ without zero divisors for which the following condition is satisfied:

$$D(xy) = c(xDy + yDx) + d(Dx)(Dy) + axy \quad \text{for } x, y \in \operatorname{dom} D,$$

where $a, c, d \in X$ depend on D only and do not vanish simultaneously (cf. Targonski [1] for scalar a, c, d).

Clearly, for $X \in \mathcal{A}_D(a,c,d)$ the *non-Leibniz component* is of the form

$$f_D(x,y) = (c-e)(xDy + yDx) + d(Dx)(Dy) + axy$$

and $c_D = 1$.

Let $r \in \mathcal{F}$. In the sequel we shall write r instead of re whenever it does not lead to any misunderstanding.

PROPOSITION 3.1. *Let $X \in \mathcal{A}_D(a,c,d)$. Suppose that $X \in \mathbf{Lg}(D)$, $u, v \in I(X) \cap \operatorname{dom} \Omega$ and $(L,E) \in G[\Omega]$. If $Hu = dDLu + c$ then*

$$H(uv) = (Hu)Hv + h, \quad \text{where } h = ad - c^2 + c. \tag{3.1}$$

Proof. We have

$$\begin{aligned} DL(uv) &= (uv)^{-1}D(uv) \\ &= u^{-1}v^{-1}[c(uDv + vDu) + d(Du)Dv + auv] \\ &= c(v^{-1}Dv + u^{-1}Du) + d(u^{-1}Du)(v^{-1}Dv) + au^{-1}uv^{-1}v \\ &= c(DLu + DLv) + d(DLu)DLv + a \end{aligned}$$

and

$$\begin{aligned} H(uv) &= dDL(uv) + c = d[c(DLu + DLv) + d(DLu)(DLv) + a] + c \\ &= d^2(DLu)DLv + cd(DLu + DLv) + ade + c \\ &= (dDLu + c)(dDLv + c) + (ad - c^2 + c) = (Hu)(Hv) + h. \ \blacksquare \end{aligned}$$

Consider some particular cases.

EXAMPLE 3.1. Let $X \in \mathcal{A}_D(a,c,d)$ and $u, v \in I(X) \cap \operatorname{dom} D$.

(i) If $a = d = 0$ and $c = e$ then X is a Leibniz D-algebra, $e \in \ker D$ and $H = I$.

(ii) If $a = 0$, $c = e$, $d = d_0 e$ and $d_0 \in \mathcal{F} \setminus \{0\}$ then X is a *quasi-Leibniz D-algebra*, i.e.

$$D(xy) = xDy + yDx + d_0(Dx)Dy \quad \text{for } x, y \in \operatorname{dom} D,$$
$$H = d_0 DL + e, \quad H(uv) = d_0(Hu)(Hv).$$

(iii) If $a = d = 0$ and $c = \frac{1}{2}e$ then X is a *simple Duhamel D-algebra*: $D(xy) = xDy = yDx$, i.e.

$$D(xy) = \tfrac{1}{2}(xDy + yDx) \quad \text{for } x, y \in \operatorname{dom} D \text{ and } H = I.$$

(iv) If $a = c = 0$ and $d = e$ then $D \in ML(X)$.

Several examples of Leibniz, quasi-Leibniz and simple Duhamel D-algebras are given by the author in [1], Section 6.1. An example of a D-algebra with a multiplicative D is the following:

Example 3.2. Let $X = (s)_{\mathcal{F}}$ be the algebra of all sequences $\{x_n\}$ such that $x_n \in \mathcal{F}$ for $n \in \mathbb{N}$ with coordinatewise multiplication. Let D be the operator of *forward shift*: $D\{x_n\} = \{x_{n+1}\}$. Clearly, $D \in ML(X)$. Let R be the operator of *backward shift*: $R\{x_n\} = \{x_{n-1}\}$, where we set $x_{n-k} = 0$ for $k \geq n$. Then $DR = I$, i.e. $D \in R(X)$. By definition, X is a D-algebra with unit $e = \{e_n\}$, where $e_n = 1$ for $n \in \mathbb{N}$. Hence $X \in \mathcal{A}_D(0, 0, e)$.

Consider now examples of D-algebras with non-scalar coefficients a, c, d:

Example 3.3. Let X and D be as in Example 3.2. Let $p = \{p_n\} \in X$ and $p_0 = 1$. Set $D_p x = Dx - px = \{x_{n+1} - p_n x_n\}$ for $x = \{x_n\} \in X$. It is easy to verify that $D_p \in R(X)$ and that $R_p \in \mathcal{R}_{D_p}$, where

$$R_p = \{y_n\}, \quad y_0 = 1, \quad y_2 = x_1 \quad \text{and} \quad y_n = \sum_{k=1}^{n-1} p_{n-1} \ldots p_{k+1} x_k.$$

Moreover, $\ker D_p = \{z = r\{p_0 \ldots p_{n-1}\} : r \in \mathcal{F}\}$, $e = \{e_n\}$, $e_n = 1$ for $n \in \mathbb{N}$ and

$$D_p(xy) = p(xD_p y + yD_p x) + (D_p x)(D_p y) + (p^2 - p)xy \quad \text{for } x, y \in X.$$

Here we have $c = p$, $d = e$ and $a = p^2 - p$, hence $h = ad - c^2 + c = 0$.

Example 3.4. Let X be as in Example 3.2. Set $D\{x_n\} = \{(n+1)x_{n+1}\}$ and $R\{x_n\} = \left\{\frac{1}{n} x_{n-1}\right\}$ for $\{x_n\} \in X$, where we assume (as in Example 3.2) $x_{n-k} = 0$ for $k \geq n$. It is easy to verify that $D \in R(X)$ and $R \in \mathcal{R}_D$. An initial operator for D corresponding to R is defined by $F\{x_n\} = x_1\{\delta_{1n}\}$. Moreover, we have

$$D(xy) = \left\{\frac{1}{n+1}\right\}(Dx)Dy, \quad \text{where } x = \{x_n\},\ y = \{y_n\} \in X.$$

Thus here we have $a = c = 0$ and $d = \left\{\frac{1}{n+1}\right\}$. Hence $h = 0$.

Corollary 3.1. *Let $X \in \mathcal{A}_D(a, c, d)$ and $(L, E) \in G[\Omega]$. If $Le = 0$ then $ad = 0$.*

Proof. If $Le = 0$ then $He = dDLe + c = c$, and (3.1) implies that $c = He = He^2 = (He)(He) + h = c^2 + (ad - c^2 + c) = ad + c$. Hence $ad = 0$. ∎

Proposition 3.2. *Let $X \in \mathcal{A}_D(a, c, d)$. If $D^0 = dD + cI$ then $X \in \mathcal{A}_{D^0}(h, 0, e)$, where $h = ad - c^2 + c$.*

Proof. By definition, $dD = D^0 - cI$ and hence for $x, y \in \operatorname{dom} D$,

$$\begin{aligned} D^0(xy) &= dD(xy) + cxy = d[c(xDy + yDx) + d(Dx)Dy + axy] + cxy \\ &= c(xdDy + ydDx) + (dDx)dDy + (ad + c)xy \\ &= c[x(D^0y - cy) + y(D^0x - cx)] + (D^0x - cx)(D^0y - cy) + (ad + c)xy \\ &= c(xD^0y + yD^0x - xD^0y - yD^0x) + (D^0x)D^0y \\ &\quad + (ad + c - 2c^2 + c^2)xy \\ &= (D^0x)(D^0y) + hxy \end{aligned}$$

(cf. also Lausch and Przeworska-Rolewicz [2]). ■

Corollary 3.2. *Let $X \in \mathcal{A}_D(a, c, d)$ and $d \in I(X)$. If $D^0 = dD + cI$ and $a = cd^{-1}(c - e)$ then $D^0 \in ML(X)$, i.e. $X \in \mathcal{A}_{D^0}(0, 0, e)$.*

Indeed, if $a = cd^{-1}(c - e)$ then $h = ad - c^2 + c = c(c - e) - c^2 + c = 0$.

Proposition 3.3. *Let $X \in \mathcal{A}_D(a, c, d)$ and $a = cd^{-1}(c - e)$, $d \in I(X)$. If the operator $dI + Rc$ is invertible for an $R \in \mathcal{R}_D$ then $D^0 = dD + cI \in R(X) \cap ML(X)$ and $R^0 = d(dI + Rc)^{-1}Rd^{-1} \in \mathcal{R}_{D^0}$, i.e. $X \in \mathcal{A}_{D^0}(0, 0, e)$. In particular, if*

(3.2) $\quad c = \alpha e, \quad d = \beta e, \quad$ *where* $\alpha, \beta \in \mathcal{F} \setminus \{0\}$, $-\alpha/\beta \in v_{\mathcal{F}}R$ *for an* $R \in \mathcal{R}_D$,

then $D^0 \in R(X) \cap ML(X)$.

Proof. By our assumptions, the operator $(dI + Rc)^{-1}$ is well-defined and

$$\begin{aligned} D^0R^0 &= (dD + cI)d(dI + Rc)^{-1}Rd^{-1} = dD(dI + Rc)d^{-1}d(dI + Rc)^{-1}Rd^{-1} \\ &= dDRd^{-1} = dId^{-1} = I. \end{aligned}$$

By Corollary 3.2, $D^0 \in ML(X)$, hence $X \in \mathcal{A}_{D^0}(0, 0, e)$. Clearly, if (3.2) is satisfied then the operator $dI + cR = \frac{1}{\beta}\big(I + \frac{\alpha}{\beta}R\big)$ is invertible. ■

Note. If $c = 0$, then the operator $dI + Rc$ is automatically right invertible. Similar results can be obtained by substitution $D_0 = Dd + cI$.

Proposition 3.4. *Let $X \in \mathcal{A}_D(0, e, \delta e)$ with $\delta \in \mathcal{F} \setminus \{0\}$. If $1/\delta \in v_{\mathcal{F}}R$ for an $R \in \mathcal{R}_D$ then $D^0 = \delta D + I \in ML(X) \cap R(X)$ and $R^0 = \frac{1}{\delta}R \in \mathcal{R}_{D^0}$, hence $X \in \mathcal{A}_{D^0}(0, 0, \delta e)$.*

Proof. Here we have $a = 0$, $c = e$ and $d = \delta e$. Hence $h = ad - c^2 + c = 0$. Further arguments are the same as in the proof of Proposition 3.3. ■

Proposition 3.5. *Suppose that $X \in \mathcal{A}_D(a, c, d)$, $d \in I(X)$, there is an $R \in \mathcal{R}_D$ such that the operator $dI + Rc$ is invertible and $a = d^{-1}c(c - e)$. Let $D^0 = dD + cI$,*

$$\Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0u = uD^0y\} \quad \textit{for } u \in \operatorname{dom} D^0.$$

If $(L^0, E^0) \in G[\Omega^0]$ then $X \in \mathbf{Lg}(D)$ and $(L, E) \in G_{R,1}[\Omega]$, where

$$Lu = R(D^0L^0u - d^{-1}c) \quad \textit{for } (u, x) \in \operatorname{graph} \Omega^0.$$

P r o o f. By our assumptions and Proposition 3.3, $D^0 \in ML(X) \cap R(X)$. By Proposition 1.14, $L^0, E^0 \in ML(X)$. Let $(u,x) \in \operatorname{graph} \Omega$ and $(u,y) \in \operatorname{graph} \Omega^0$. Then $x = Lu$, $u = Ex$, $y = L^0 u$, $u = E^0 y$ and

$$duDx = dDu = (dD^0 - cI)u = dD^0 u - cu = duDy - cu.$$

Hence for $u \in I(X)$,

$$DLu = Dx = D^0 y - d^{-1}c = D^0 L^0 u - d^{-1}c = DR(D^0 L^0 u - d^{-1}c),$$

which implies $Lu = R(D^0 L^0 u - d^{-1}c) + z$, where $z \in \ker D$. Arguing as before, without loss of generality we may put $z = 0$ (cf. Proposition 1.13 and Corollary 1.3). Then $Ex = u = ELu = ER(D^0 L^0 u - d^{-1}c) = E^0 L^0 u = E^0 y$. It is easy to verify that for every x there is a unique u such that $Lu = x$, namely, $u = E^0(Dx + Dd^{-1}c)$. Hence L is an invertible selector of Ω. ∎

Theorem 3.1. *Suppose that $X \in \mathcal{A}_D(0,c,0)$ and $e - c$ is not a zero divisor. If either $c = \frac{1}{2}e$ or $e \in \ker D$ or $e - 2c$ is not a zero divisor then $X \notin \mathbf{Lg}(D)$.*

P r o o f. By our assumption, $D(xy) = c(xDy + yDx)$ for $x, y \in \operatorname{dom} D$. Then $De = De^2 = 2ceDe = 2cDe$, i.e.

$$(e - 2c)De = 0.$$

Suppose that $e \in \ker D$ and $(L,E) \in G[\Omega]$. Then for every $x \in [\operatorname{dom} \Omega \setminus \ker D] \cap I(X)$ we have $Dx = D(xe) = c(xDe + eDx) = cxDe + cDx$, i.e. $(e - c)Dx = 0$ and $DLx = (e - c)x^{-1}xDx = 0$, which implies $DLx = 0$. Hence $Lx = z \in \ker D$ and $x = Ez \in \ker D$, which contradicts our assumption that $x \notin \ker D$.

If $e - 2c$ is not a zero divisor then, by (3.2), we get $De = 0$, i.e. $e \in \ker D$. Again we conclude that invertible selectors of Ω do not exist.

Suppose then that $c = 1/2$. Let $(L,E) \in G[\Omega]$. Let $u \in \operatorname{dom} \Omega$ and $x = Lu$. Then $u = Ex$. Since here $c_D = 1/2$ and $f_D = 0$, we get

$$Lu = L(ue) = L[(Ex)(Ee)] = LE\big[\tfrac{1}{2}(x + Le)\big] = \tfrac{1}{2}(x + Le) = \tfrac{1}{2}Lu + \tfrac{1}{2}Le.$$

Hence $Lu = Le$ for all u, i.e. L is a constant mapping. This contradicts our assumption that L is invertible. ∎

Example 3.5. Let $X = C([0,T];\mathbb{C})$. Consider the so-called *Pommiez operator*:

$$(Dx)(t) = \begin{cases} \dfrac{x(t) - x(0)}{t} & \text{if } t \geq 0, \\ x'(0) & \text{if } t = 0. \end{cases}$$

Note that $\operatorname{dom} D = \{x \in C[0,T] : \exists x'(0)\}$. The operator D is right invertible and has a right inverse defined by $(Rx)(t) = tx(t)$ for $x \in X$. An initial operator F corresponding to R is $(Fx)(t) = x(0)$. Clearly, X is a D-algebra with respect to pointwise multiplication. This algebra has a unit $e \in \ker D$, namely the function $e(t) \equiv 1$. We shall show that in the case of the Pommiez operator invertible selectors of Ω do not exist.

Indeed, let $x, y \in \operatorname{dom} D$. It is easy to verify that

$$D(xy) = \tfrac{1}{2}[(x + Fx)Dy + (y + Fy)Dx] \quad \text{for } x, y \in \operatorname{dom} D.$$

Suppose that $(L, E) \in G[\Omega]$. Let $x, y \in RX$. Since $FR = 0$, we have $Fx = Fy = 0$ and

$$D(xy) = \tfrac{1}{2}(xDy + yDx) \quad \text{for } x, y \in RX.$$

By the same arguments as in the proof of Theorem 3.1 for $c = \frac{1}{2}e$, we conclude that invertible selectors of Ω do not exist on $RX \subset \operatorname{dom} D$.

PROPOSITION 3.6. *Suppose that* $X \in \mathcal{A}_D(a, c, 0)$ *and* $2c - e \in I(X)$. *Let* $b = a(2c - e)^{-1}$. *Suppose that the operator* $I + Rb$ *is invertible for an* $R \in \mathcal{R}_D$. *Then*

(i) $D^0 = D + b \in R(X)$, $R^0 = (I + Rb)^{-1}R \in \mathcal{R}_{D^0}$ *and* $X \in \mathcal{A}_D(0, c, 0)$;
(ii) $X \in \mathbf{Lg}(D)$ *only if* $c = e$.

P r o o f. If $I+Rb$ is invertible for an $R \in \mathcal{R}_D$ then $D^0 = D+b = D(I+Rb)^{-1} \in R(X)$ and $R^0 = (I + Rb)^{-1}R \in \mathcal{R}_{D^0}$. Moreover, for $x, y \in \operatorname{dom} D^0 = \operatorname{dom} D$ we have

$$\begin{aligned} D^0(xy) &= D(xy) + bxy = c(xDy + yDx) + axy + bxy \\ &= c[x(D^0 - b)y + y(D^0 - b)x] + (a + b)xy \\ &= c(xD^0y + yD^0x) + (a + b - 2bc)xy = c(xD^0y + yD^0x) \end{aligned}$$

for $a = b(2c - e) = 2bc - b$. Hence $X \in \mathcal{A}_{D^0}(0, c, 0)$. By Theorem 3.1, invertible selectors of Ω exist only if $c = e$. ∎

Propositions 3.5, 3.6 and Theorem 3.1 imply that any algebra $X \in \mathcal{A}_D(a, c, d)$ belongs to one of three types: either X is a Leibniz algebra, or D may be reduced by a substitution to a multiplicative operator, or logarithmic and antilogarithmic mappings do not exist (cf. Przeworska-Rolewicz [8] in the case $\mathcal{F} = \mathbb{R}$).

We shall now construct logarithmic mappings induced by polynomials in a right invertible operator D. Recall that, by Theorem 2.1, if $X \in \mathbf{Lg}(D)$ then $X \in \mathbf{Lg}(D^n)$ for all $n \in \mathbb{N}$.

THEOREM 3.2. *Let* $X \in \mathbf{Lg}(D)$, $(L_n, E_n) \in G[\Omega_n]$ *and let*

$$P(D) = \sum_{k=0}^{N} p_k D^k, \qquad p_0, \ldots, p_N \in X,\ p_N = e.$$

Set

$$D^0 = P(D), \qquad \Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D^N \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0 u = uD^0 y\} \quad \text{for } u \in \operatorname{dom} D^0.$$

Let Ω_k *be as in Theorem* 2.1 *and let* $(L_k, E_k) \in G[\Omega_k]$ *for* $k = 1, \ldots, N$. *Suppose* *that the operator*

$$P(I,R)=\sum_{k=0}^{N} p_k R^{N-k}$$

is invertible for an $R\in\mathcal{R}_D$ *and*

$$L^0u = R^N[P(I,R)]^{-1}\Big(p_0u^{-1}+\sum_{k=1}^{N} p_k D L_k u\Big) \quad \text{for } u\in \operatorname{dom}\Omega_N\cap I(X). \tag{3.3}$$

If the mapping L^0 *is invertible then* $X\in\mathbf{Lg}(D^0)$ *and* $(L^0,E^0)\in G_{R,N}[\Omega^0]$, *where* $E^0=(L^0)^{-1}$.

P r o o f. By our assumptions, $D^0=P(D)\in R(X)$ and $R^0=R^N[P(I,R)]^{-1}\in\mathcal{R}_{D^0}$ (cf. Przeworska-Rolewicz [1]). Let $(u,x)\in \operatorname{graph}\Omega^0$. Then

$$\begin{aligned} D^0x = u^{-1}D^0u = u^{-1}P(D)u = u^{-1}\sum_{k=0}^{N} p_k D^k u \\ = \sum_{k=0}^{N} p_k u^{-1} D^k u = p_0u^{-1}+\sum_{k=1}^{N} p_k D^k L_k u, \end{aligned}$$

which implies that x is of the form (3.3), i.e. $x=L^0u$. If L^0 is invertible and $E^0=(L^0)^{-1}$ then $(L^0,E^0)\in G[\Omega^0]$. Let F be an initial operator for D corresponding to R. Clearly, since $FR=0$, we have

$$FD^jL^0u=0 \quad (j=0,1,\ldots,N-1). \ \blacksquare$$

N o t e. If $p_N\neq e$, but $p_N\in I(X)$, then by the substitution $p'_k=p_N^{-1}p_k$ we obtain a polynomial $P'(D)=\sum_{k=0}^{N} p'_k D^k$ satisfying the conditions of Theorem 3.2.

THEOREM 3.3. *Let* $X\in\mathbf{Lg}(D)$, $(L_n,E_n)\in G[\Omega_n]$ *and let*

$$Q(D)=\sum_{k=0}^{N} Q_k D^k, \quad Q_k\in L(X)\ (k=0,1,\ldots,N-1),\ Q_N=I. \tag{3.4}$$

Set

$$\begin{gathered} D^0=Q(D), \quad \Omega^0:\operatorname{dom}D^0=\operatorname{dom}D^N\to 2^{\operatorname{dom}D^0}, \\ \Omega^0u=\{y\in\operatorname{dom}D^0: D^0u=uD^0y\} \quad \text{for } u\in\operatorname{dom}D^0. \end{gathered}$$

Let Ω_k *be as in Theorem* 2.1 *and let* $(L_k,E_k)\in G[\Omega_k]$ *for* $k=1,\ldots,N$. *Suppose that the operator*

$$Q(I,R)=\sum_{k=0}^{N} Q_k R^{N-k} \tag{3.5}$$

is invertible for an $R\in\mathcal{R}_D$ *and*

$$L^0u=R^N[Q(I,R)]^{-1}u^{-1}[Q(I,R)]^{-1}[u^{-1}Q(D)u] \quad \text{for } u\in\operatorname{dom}\Omega^0\cap I(X). \tag{3.6}$$

If the mapping L^0 *is invertible then* $X \in \mathbf{Lg}(D^0)$ *and* $(L^0, E^0) \in G_{R,N}[\Omega^0]$, *where* $E^0 = (L^0)^{-1}$.

Proof. By our assumptions, we conclude that $D^0 = Q(D) \in R(X)$ and $R^0 = R^N[Q(I,R)]^{-1}$ (cf. Przeworska-Rolewicz [1], Section 3). Let L^0 be defined by (3.6). Since, by our definitions, $Q(D)R^N = Q(I,R)$, we find, for all $u \in I(X) \cap \operatorname{dom} \Omega^0$ and for $x = L^0 u$ that

$$\begin{aligned} D^0 x = D^0 L^0 u = Q(D)L^0 u = Q(D)R^N[Q(I,R)]^{-1}[u^{-1}Q(D)u] \\ = Q(I,R)[Q(I,R)]^{-1}[u^{-1}Q(D)u] = u^{-1}Q(D)u = u^{-1}D^0 u. \end{aligned}$$

This proves that L^0 is a selector of Ω^0. Further arguments are the same as in the proof of Theorem 3.2. ∎

THEOREM 3.4. *Let* $X \in \mathbf{Lg}(D)$, $(L_n, E_n) \in G[\Omega_n]$ *and let*

$$(3.7) \qquad Q\langle D\rangle = \sum_{k=0}^{N} D^k Q_k, \qquad Q_k \in L(X),\ Q_k X \in \operatorname{dom} D^N\ (k = 0, 1, \ldots, N-1),\ Q_N = I.$$

Set

$$D^0 = Q\langle D\rangle, \qquad \Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D^N \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0 u = uD^0 y\} \quad \text{for } u \in \operatorname{dom} D^0.$$

Let Ω_k *be as in Theorem* 2.1 *and let* $(L_k, E_k) \in G[\Omega_k]$ *for* $k = 1, \ldots, N$. *Suppose that the operator*

$$(3.8) \qquad Q\langle I, R\rangle = \sum_{k=0}^{N} R^{N-k} Q_k$$

is invertible for an $R \in \mathcal{R}_D$, $(L_N, E_N) \in G_{R,N}[\Omega_N]$ *and*

$$(3.9) \qquad L^0 u = [Q\langle I,R\rangle]^{-1} R^N [u^{-1}(Q\langle I,R\rangle u) D^N L_N(Q\langle I,R\rangle u)] \quad \text{for } u \in \operatorname{dom} \Omega^0 \cap I(X).$$

If the mapping L^0 *is invertible then* $X \in \mathbf{Lg}(D^0)$ *and* $(L^0, E^0) \in G_{R,N}[\Omega^0]$, *where* $E^0 = (L^0)^{-1}$.

Proof. By our assumptions, we have $D^0 = Q\langle D\rangle \in R(X)$ and $R^0 = [Q\langle I,R\rangle]^{-1} R^N$ (cf. Przeworska-Rolewicz [1], Section 3). The *similarity property*: $Q\langle D\rangle = D^N Q\langle I,R\rangle$ (cf. Przeworska-Rolewicz [1], Corollary 3.2.1) implies that for $x = L^0 u$, where L^0 is defined by (3.9), we get

$$\begin{aligned} D^0 x = u^{-1} D^0 u = u^{-1} Q\langle D\rangle u = u^{-1} D^N Q\langle I,R\rangle u \\ = u^{-1}(Q\langle I,R\rangle u) D^N L_N(Q\langle I,R\rangle u). \end{aligned}$$

Further arguments are the same as in the proofs of Theorems 3.2 and 3.3. ∎

Note. Combining Theorems 3.3 and 3.4 with Proposition 2.1 we obtain similar results (under appropriate assumptions) for operators of the form

$$D^M Q(D), \quad Q(D)D^M, \quad D^M Q\langle D\rangle, \quad Q\langle D\rangle D^M \quad (M \in \mathbb{N}).$$

All these operators are right invertible (cf. Przeworska-Rolewicz [1], Section 3). Nguyen Van Mau [2] proved that the *resolving operators* $Q(I,R)$ and $Q\langle I,R\rangle$ are simultaneously invertible (left invertible, right invertible) or not. In the same paper he considered operators of a more general form:

$$Q[D] = \sum_{m=0}^{M}\sum_{n=0}^{N} D^m A_{mn} D^n \quad \text{with } A_{mn} \in L(X),$$

which are again right invertible (under appropriate assumptions). Thus, applying his results, one can construct logarithmic mappings for the operator $Q[D]$ (provided that they exist) in a similar manner to Theorems 3.3 and 3.4.

PROPOSITION 3.7. *Suppose that* $X \in \mathbf{Lg}(D') \cap \mathbf{Lg}(D'')$, *both superpositions* $D'D''$, $D''D'$ *exist and* $D'D'' = D''D' = D$ *on* $\operatorname{dom} D = \operatorname{dom} D' \cap \operatorname{dom} D''$. *Let* $(L', E') \in G[\Omega']$ *and* $(L'', E'') \in G[\Omega'']$, *where*

$$\Omega' : \operatorname{dom} D' \to 2^{\operatorname{dom} D'}, \quad \Omega' u = \{x \in \operatorname{dom} D' : D'u = uD'x\}, \quad u \in \operatorname{dom} D',$$

$$\Omega'' : \operatorname{dom} D'' \to 2^{\operatorname{dom} D''}, \quad \Omega'' u = \{x \in \operatorname{dom} D'' : D''u = uD''x\}, \quad u \in \operatorname{dom} D''.$$

Write

$$\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}, \quad \Omega u = \{x \in \operatorname{dom} D : Du = uD'x\}, \quad u \in \operatorname{dom} D.$$

Let $u \in I(X) \cap \operatorname{dom} \Omega$. *Let* $R' \in \mathcal{R}_{D'}$ *and* $R'' \in \mathcal{R}_{D''}$. *Then*

(i) *there is a selector* L *of* Ω *of the form*

$$Lu = c_{D'} L'' u + R[c_{D'}(D'L'u)D''L''u + u^{-1} f_{D'}(u, D''L''u)],$$

where $R = R'R'' \in \mathcal{R}_D$;

(ii) *if* $c_{D'} \neq 0$ *then* L *is invertible*;

(iii) *if* $c_{D'} = 0$, *but the mapping* f *defined by* $f(u) = f_{D'}(u, D''L''u)$ *for* $u \in \operatorname{dom} \Omega$ *is invertible*, *then* L *is invertible*;

(iv) *if* L *is invertible and* $E = L^{-1}$ *then* $(L, E) \in G_{R,1}[\Omega]$;

(v) *if* $e \in \ker D''$ *and* $L''e = 0$ *then* $Le = 0$.

Proof. (i) Let $(u, x) \in \operatorname{graph} \Omega$ and $u \in I(X)$. Then, by definition,

$$\begin{aligned} DLu &= Dx = u^{-1}Du = u^{-1}D'D''u \\ &= u^{-1}D'(uu^{-1}D''u) = u^{-1}D'(uD''L''u) \\ &= u^{-1}\{c_{D'}[(D'u)D''L''u + uD'D''L''u] + f_{D'}(u, D''L''u)\} \\ &= c_{D'}[(u^{-1}D'u)D''L''u + DL''u] + u^{-1} f_{D'}(u, D''L''u) \\ &= c_{D'}[(D'L'u)D''L''u + DL''u] + u^{-1} f_{D'}(u, D''L''u), \end{aligned}$$

which implies that L is of the required form (cf. (0.4)).

(ii) Let $c_{D'} \neq 0$. For every $x \in \operatorname{dom} \Omega^{-1}$ and for every fixed $x'' \in \operatorname{dom} \Omega''$ such that $D''x'' \in I(X)$ there is a unique $x' \in \operatorname{dom} \Omega'^{-1}$ satisfying $x = c_{D'}x'' + R[c_{D'}(D'x')D''x'']$ such that $F'x' = 0$, where $F' \in \mathcal{F}_{D'}$ corresponds to R'. Namely, $x' = R'[(D''x'')^{-1}D(c_{D'}^{-1}x - x'')]$. Having thus uniquely determined x', we conclude that $u = E'x'$ is a unique solution of the equation $Lu = x$.

(iii) and (iv) are obvious.

(v) Suppose that $e \in \ker D$. Then $L'e, L''e \in \ker D$, which implies $D'L'e = D''L''e = 0$. Hence $Le = c_{D'}L''e = 0$. ∎

Since, by our assumptions, $D'D'' = D''D' = D$ on $\operatorname{dom} D$, interchanging the roles of D', D'', we conclude that there is a selector $\widetilde{L}$ of Ω of the form

$$\widetilde{L}u = c_{D''}L'u + R''R'[c_{D''}(D'L'u)D''L''u + f_{D''}(u, D'L'u)]$$

such that $\widetilde{L}u = 0$, provided that $e \in \ker D$ and $L'e = 0$. If we apply (0.5) instead of (0.4) then we obtain a symmetrical form of selectors L (but more complicated).

COROLLARY 3.3. *Suppose that all assumptions of Proposition* 3.7 *are satisfied. Suppose*, *moreover*, *that* X *is a Leibniz* D'-*algebra and* D''-*algebra. Then* $X \in \mathbf{Lg}(D), (L, E) \in G_{R,1}[\Omega]$, *where* $E = L^{-1}$ *and*

$$L(uv) = Lu + Lv + Rd(u, v),$$

where

$$d(u, v) = (D'L'u)D''L''v + (D''L''u)D'L'v, \qquad u, v \in \operatorname{dom} \Omega,$$

and

$$(Ex)(Ey) = E[x + y + Rd(Ex, Ey)], \qquad x, y \in \operatorname{dom} \Omega^{-1}.$$

P r o o f. By our assumption, $c_{D'} = 1$. Proposition 3.7(ii), (iv) now implies that $X \in \mathbf{Lg}(D)$ and $(L, E) \in G[\Omega]$. For all $x, y \in \operatorname{dom} D$ we have

$$\begin{aligned} D(xy) &= D'D''(xy) = D'(xD''y + yD''x) \\ &= (D'x)D''y + xD'D''y + (D'y)D''x + yD'D''x \\ &= xDy + yDx + (D'x)D''y + (D''x)D'y. \end{aligned}$$

Hence for $u, v \in I(X) \cap \operatorname{dom} \Omega$ we get

$$\begin{aligned} DL(uv) &= (uv)^{-1}D(uv) = (uv)^{-1}[uDv + vDu + (D'u)D''v + (D''u)D'v] \\ &= v^{-1}Dv + u^{-1}Du + (u^{-1}D'u)(v^{-1}D''v) + (u^{-1}D''u)(v^{-1}D'v)] \\ &= DLu + DLv + (D'L'u)D''L''v + (D''L''u)D'L'v \\ &= DLu + DLv + d(u, v). \end{aligned}$$

Further arguments are the same as in the proof of Theorem 1.2. ∎

EXAMPLE 3.6. Let $E = [0, a] \times [0, b] \in \mathbb{R}^2$, $X = C(E; \mathbb{R})$, $D' = \partial/\partial t$ and $D'' = \partial/\partial s$. Then X is a Leibniz D'-algebra and D''-algebra with respect to pointwise multiplication, i.e. $c_{D'} = c_{D''} = 1$ and $f_{D'} = f_{D''} = 0$. This algebra has a unit e. Namely, $e(t, s) \equiv 1$. Moreover, $e \in \ker D'$ and $e \in \ker D''$. Note that both $\ker D'$ and $\ker D''$ are infinite-dimensional. The operator $D = \partial^2/\partial t \partial s$, as a superposition of right invertible operators, is again right invertible. It is well-known that $\partial^2 x/\partial t\partial s = \partial^2 x/\partial s\partial t$, provided that both these derivatives are continuous. Several examples of right inverses to the operator D can be found in the author's book [1]. The operator D satisfies all conditions of Corollary 3.3.

PROPOSITION 3.8. *Suppose that* $X \in \mathbf{Lg}(D_{(j)})$ $(j = 1, \ldots, n)$ *and the operator*

$$D = \sum_{j=1}^{n} D_{(j)} \quad \text{with } \operatorname{dom} D = \bigcap_{j=1}^{n} \operatorname{dom} D_{(j)}$$

is right invertible. Let $(L_{(j)}, E_{(j)}) \in G[\Omega_{(j)}]$, *where*

$$\Omega_{(j)} : \operatorname{dom} D_{(j)} \to 2^{\operatorname{dom} D_{(j)}},$$
$$\Omega_{(j)} u = \{x \in \operatorname{dom} D_{(j)} : D_{(j)} u = u D_{(j)} x\}, \quad u \in \operatorname{dom} D.$$

Write

$$\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}, \quad \Omega u = \{x \in \operatorname{dom} D : Du = uDx\}, \quad u \in \operatorname{dom} D.$$

Then

(i) *there is a selector* L *of* Ω *of the form*

$$Lu = \sum_{j=1}^{n} L_{(j)} u \quad \text{for } u \in I(X) \cap \operatorname{dom} \Omega;$$

(ii) *if* L *is invertible and* $E = L^{-1}$ *then* $(L, E) \in G[\Omega]$;

(iii) *if* $L_{(j)} e = 0$ *for* $j = 1, \ldots, n$ *then* $Le = 0$.

Proof. Let $u \in I(X) \cap \operatorname{dom} \Omega$. By definition, we have

$$DLu = u^{-1} Du = u^{-1} \sum_{j=1}^{n} D^2_{(j)} u = \sum_{j=1}^{n} u^{-1} D^2_{(j)} u$$
$$= \sum_{j=1}^{n} DL_{(j)} u = D \sum_{j=1}^{n} L_{(j)},$$

which implies (i). Statements (ii) and (iii) are obvious. ■

Note. Proposition 3.8 can be applied to construct logarithmic mappings for partial differential operators of the first order with variable coefficients: $D = \sum_{j=1}^{n} a_j(t)\partial/\partial t$. To do this, one can apply the construction of right inverses for D given by Virsik [1] (in a more general case over $\mathbb{R}$). This construction, however, is too complicated to be given here as an example.

PROPOSITION 3.9. *Suppose that* $X \in \mathbf{Lg}(D_{(j)})$ $(j = 1, \ldots, n)$ *and the operator*

$$D = \sum_{j=1}^{n} D^2_{(j)} \quad \text{with } \operatorname{dom} D = \bigcap_{j=1}^{n} \operatorname{dom} D_{(j)}$$

is right invertible. Let $(L_{(j)}, E_{(j)}) \in G[\Omega_{(j)}]$, *where*

$$\Omega_{(j)} : \operatorname{dom} D_{(j)} \to 2^{\operatorname{dom} D_{(j)}},$$
$$\Omega_{(j)} u = \{x \in \operatorname{dom} D_{(j)} : D_{(j)} u = u D_{(j)} x\}, \qquad u \in \operatorname{dom} D.$$

Write

$$\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}, \quad \Omega u = \{x \in \operatorname{dom} D : Du = uDx\}, \quad u \in \operatorname{dom} D.$$

Then

(i) *there is a selector* L *of* Ω *of the form*

$$Lu = \sum_{j=1}^{n} \{c_{D_{(j)}}[L_{(j)} u + R_{(j)}^2 (D_{(j)} L_{(j)} u)^2] + u^{-1} f_{D_{(j)}}(u, L_{(j)} u)\}$$

for $u \in I(X) \cap \operatorname{dom} \Omega$ *and an* $R \in \mathcal{R}_D$;

(ii) *if* L *is invertible and* $E = L^{-1}$ *then* $(L, E) \in G[\Omega]$;

(iii) *if* $L_{(j)} e = 0$ *for* $j = 1, \ldots, n$ *then* $Le = 0$.

P r o o f. Let $u \in I(X) \cap \operatorname{dom} \Omega$. By Theorem 2.1, we have

$$DLu = u^{-1} Du = u^{-1} \sum_{j=1}^{n} D_{(j)}^2 u = \sum_{j=1}^{n} u^{-1} D_{(j)}^2 u$$
$$= D \sum_{j=1}^{n} \{c_{D_{(j)}}[L_{(j)} u + R_{(j)}^2 (D_{(j)} L(j) u)^2] + u^{-1} f_{D_{(j)}}(u, L_{(j)} u)\},$$

which implies (i). Statements (ii) and (iii) are obvious. ■

Corollary 3.4. *Suppose that all assumptions of Proposition* 3.9 *are satisfied and* X *is a Leibniz* $D_{(j)}$*-algebra* ($j = 1, \ldots, n$). *Then there is a selector* L *of* Ω *such that*

$$L(uv) = Lu + Lv + R\delta(u, v),$$

where

$$\delta(u, v) = 2 \sum_{j=1}^{n} (D_{(j)} L_{(j)} u) D_{(j)} L_{(j)} v, \qquad u, v \in \operatorname{dom} \Omega.$$

If L *is invertible then* $X \in \mathbf{Lg}(D)$ *and* $(L, E) \in G[\Omega]$, *where* $E = L^{-1}$ *and*

$$(Ex)(Ey) = E[x + y + R\delta(Ex, Ey)], \qquad x, y \in \operatorname{dom} \Omega^{-1}.$$

P r o o f. It is easy to verify that

$$D(xy) = xDy + yDx + \sum_{j=1}^{n} (D_{(j)} x)(D_{(j)} y) \quad \text{for } x, y \in \operatorname{dom} D.$$

Let $u, v \in I(X) \cap \operatorname{dom} \Omega$. Then

$$\begin{aligned} DL(uv) &= (uv)^{-1}D(uv) = (uv)^{-1}\Big[uDv + vDu + 2\sum_{j=1}^{n}(D_{(j)}u)D_{(j)}v\Big] \\ &= u^{-1}Du + v^{-1}Dv + 2\sum_{j=1}^{n}(u^{-1}D_{(j)}u)D_{(j)}v \\ &= DLu + DLv + 2\sum_{j=1}^{n}(D_{(j)}L_{(j)}u)D_{(j)}L_{(j)}v = DLu + DLv + \delta(u,v). \end{aligned}$$

Further arguments are the same as in the proof of Theorem 1.2. ■

EXAMPLE 3.7. Let $E \subset \mathbb{R}^n$ be a domain with boundary S of Lyapunov type. Let $X = C(\overline{E}; \mathbb{R})$ and let $D_{(j)} = \partial/\partial t_j$ $(j = 1, \ldots, n)$, $t = (t_1, \ldots, t_n) \in E$. Clearly, X is a Leibniz $D_{(j)}$-algebra with unit $e(t) \equiv 1$ with respect to pointwise multiplication. The Laplace operator $\Delta = \sum_{j=1}^{n} D_{(j)}^2 = \sum_{j=1}^{n} \partial^2/\partial t_j^2$ is right invertible in X (cf. Przeworska-Rolewicz [1]). The operator Δ satisfies all conditions of Corollary 3.4.

To conclude Sections 1–3, note that the existence (or non-existence) of logarithmic and antilogarithmic mappings is a property of an algebraic structure *together* with a right invertible operator, and not a property of the operator only.

In the next sections we shall consider particular properties in Leibniz D-algebras and in algebras with a multiplicative operator D, and formulae for solutions to linear equations.

4. Multiplicative case

We shall see that in the case of a D-algebra with a multiplicative right invertible operators D some pathologies may appear. One of them has been shown in Theorem 1.5.

LEMMA 4.1. *Suppose that* X *is a commutative algebra* (*over* $\mathcal{F}$) *with unit* e, $A \in M(X)$, $A \neq 0$, $\operatorname{dom} A$ *has no zero divisors and* $e \in \operatorname{dom} A$. *Then*

(i) $Ae = e$;
(ii) $Ax \in I(X)$ *and* $Ax^{-1} = (Ax)^{-1}$ *for* $x \in I(X) \cap \operatorname{dom} A$;
(iii) $Ax^n = (Ax)^n$ *for all* $x \in \operatorname{dom} A$ *and* $n \in \mathbb{N}$;
(iv) $Ax \in I_2(\operatorname{dom} A)$ *and* $Ax^{1/2} = (Ax)^{1/2}$ *for* $x \in I_2(\operatorname{dom} A)$.

Proof. (i) Since $Ae = Ae^2 = (Ae)(Ae)$, we find that $(Ae)(Ae - e) = 0$. Since $\operatorname{dom} A$ has no zero divisors, we have either $Ae = 0$ or $Ae = e$. Suppose then that $Ae = 0$. Then $Ax = A(xe) = (Ax)(Ae) = 0$ for all $x \in \operatorname{dom} A$, which contradicts our assumption that $A \neq 0$. Thus $Ae = e$.

(ii) Let $x \in I(X) \cap \operatorname{dom} A$. Then $e = Ae = A(xx^{-1})$, which implies that $Ax \in I(X)$ and $(Ax)^{-1} = Ax^{-1}$.

(iii) is proved by an easy induction.

(iv) Let $x \in I_2(\operatorname{dom} A)$. Then there is a y such that $y^2 = x$, i.e. $y = x^{1/2}$. Hence $(Ay)^2 = Ay^2 = Ax$, which implies that $Ax \in I_2(\operatorname{dom} A)$ and $Ax^{1/2} = (Ax)^{1/2}$. ∎

PROPOSITION 4.1. *Suppose that X is a D-algebra with unit $e \in \operatorname{dom} D$, $D \in ML(X)$ and* $\operatorname{dom} D$ *has no zero divisors. Then*

(i) $De = e$;
(ii) $Dx \in I(X)$ *and* $Dx^{-1} = (Dx)^{-1}$ *for* $x \in I(X) \cap \operatorname{dom} D$;
(iii) $Dx^n = (Dx)^{-1}$ *for* $x\in \operatorname{dom} D$ *and* $n \in \mathbb{N}$;
(iv) $Dx \in I_2(\operatorname{dom} D)$ *and* $Dx^{1/2} = (Dx)^{1/2}$ *for* $x \in I_2(\operatorname{dom} D)$;
(v) $\ker D \cap I(X) = \emptyset$ *and* $(\operatorname{dom} D)(\ker D) \subset \ker D$.

P r o o f. Assertions (i)–(iv) are immediate consequences of Lemma 4.1.

(v) Suppose that $z \in \ker D \cap I(X)$. By (i), $De = e$. Since $Dz = 0$, we get $e = De = D(zz^{-1}) = (Dz)Dz^{-1} = 0$, a contradiction. Let now $x \in \operatorname{dom} D$ and $z \in \ker D$. Then $D(xz) = (Dx)(Dz) = 0$, which implies $xz \in \ker D$. ∎

In this section we make use of the following condition:

[**M**] $X \in \mathbf{Lg}(D)$, $D \in ML(X)$ *and* $\operatorname{dom} D$ *has no zero divisors.*

By Condition [**M**], $X \in \mathcal{A}_D(0,0,e) \cap \mathbf{Lg}(D)$. Since $\ker D \subset \operatorname{dom} D$, the property that $\ker D$ has no zero divisors assumed in the definition of the class $\mathcal{A}_D(a,c,d)$ is automatically satisfied.

PROPOSITION 4.2. *Suppose that Condition* [**M**] *holds*, $(L,E)\in G[\Omega]$ *and* $e \in \operatorname{dom} \Omega^{-1}$. *Then*

(i) $e \in \operatorname{dom} \Omega \cap \operatorname{dom} \Omega^{-1}$ *and* $Ee = Le = e$;
(ii) $Ex \in I(X)$ *and* $Ex^{-1} = (Ex)^{-1}$ *for all* $x \in I(X) \cap \operatorname{dom} \Omega^{-1}$;
(iii) $Ex^n = (Ex)^n$ *for all* $x \in \operatorname{dom} \Omega^{-1}$ *and* $n \in \mathbb{N}$;
(iv) $Ex \in I_2(\operatorname{dom} \Omega^{-1})$ *and* $Ex^{1/2} = (Ex)^{1/2}$ *for* $x \in I_2(\operatorname{dom} \Omega^{-1})$;
(v) *if* $\alpha, \beta \in \mathcal{F}$ *and* αe, $\beta e \in \operatorname{dom} \Omega^{-1}$ *then* $E(\alpha\beta e) = E(\alpha e)E(\beta e)$;
(vi) $0 \notin \operatorname{dom} \Omega^{-1}$.

P r o o f. By Proposition 1.14, L and E are multiplicative. Statements (i)–(iii) are proved in Lemma 4.1(i)–(iii). Moreover, if $Ee = e$ then $Le = e$.

(iv) Let $x \in I_2(\operatorname{dom} \Omega^{-1})$. By Proposition 4.1, $x^{1/2} \in \operatorname{dom} D$ and $Dx^{1/2} = (Dx)^{1/2}$. Hence $Ex = E(x^{1/2}x^{1/2}) = (Ex^{1/2})^2$, i.e. $Ex \in I_2(\operatorname{dom} \Omega^{-1})$ and $Ex^{1/2} = (Ex)^{1/2}$.

(v) Let $\alpha, \beta \in \mathcal{F}$. Then $E(\alpha\beta e) = E(\alpha e\beta e) = E(\alpha e)E(\beta e)$.

(vi) By Proposition 1.9, $z = E(0) \in \ker D$. By definition, $Dz = DE(0) = E(0)De = ze = z$, which implies $z = 0$, i.e. $E(0) = 0$. Hence $L0 = LE0 = 0$, i.e. $0 \in \operatorname{dom} \Omega$. This contradicts Proposition 1.5. ∎

Definition 4.1. Suppose that $\mathcal{F} = \mathbb{C}$ or $\mathbb{R}$, Condition **[M]** holds and $(L, E) \in G[\Omega]$. We say that E is *multiplicative with exponent* λ if there is a $\lambda \in \mathbb{R}_+ \setminus \{0\}$ such that

$$E(\alpha e) = \alpha^\lambda e \quad \text{for } \alpha \in \mathbb{R}_+ \setminus \{0\}.$$

Proposition 4.3. *Suppose that all assumptions of Definition* 4.1 *are satisfied*, E *is multiplicative with exponent* $\lambda > 0$, $x \in \operatorname{dom} \Omega^{-1}$ *and* $\alpha \in \mathbb{R}_+ \setminus \{0\}$. *If* $\alpha x \in \operatorname{dom} \Omega^{-1}$ *then* $\alpha = 1$.

Proof. Let $\alpha \in \mathbb{R}_+ \setminus \{0\}$. Let $x, \alpha x \in \operatorname{dom} \Omega^{-1}$. By Proposition 4.2(i), $e \in \operatorname{dom} \Omega^{-1}$. Suppose that $\alpha e \in \operatorname{dom} \Omega^{-1}$. By Proposition 4.2(v), $E(\alpha x) = E(\alpha e)Ex$. By Definition 4.1, $\alpha e = LE(\alpha e) = L(\alpha^\lambda e)$. By Theorem 1.5, $\alpha^\lambda = 1$, i.e. $\lambda = 0$. This contradicts our assumptions. Hence $\alpha = 1$. ∎

Note. Theorem 1.5 and Proposition 4.3 show that a few results of the author's papers [8] and [9] are trivial.

Theorem 4.1. *Suppose that Condition* **[M]** *holds. Let*

$$Y = \{x \in \operatorname{dom} \Omega^{-1} : u = Ex \in I(X)\}.$$

Then

$$Y \cap \ker D = \{0\}, \quad \text{i.e.} \quad D|_Y \text{ is invertible.} \tag{4.1}$$

Proof. By our assumption, $D \in ML(X)$. Hence $c_D = 0$ and $f_D(u, v) = (Du)(Dv)$ for $u, v \in \operatorname{dom} D$. Let $(L, E) \in G_{R,1}[\Omega]$, $u \in I(X) \cap \operatorname{dom} \Omega$ and $x = Lu$. Then $u = Ex \in I(X)$. By definition, $DEx = (Ex)Dx$. By (1.10),

$$\begin{aligned} Ex = E(ex) = (Ee)(Ex) &= E\{R[e^{-1}(Ex)^{-1}(De)(DEx)]\} \\ &= ER[e(Ex)^{-1}e(Ex)Dx] = E(RDx) = E(x - Fx), \end{aligned}$$

where F is an initial operator for D corresponding to R. Hence $x = LEx = LE(x - Fx) = x - Fx$. Thus $Fx = 0$ and $x \in RX$. Since $\operatorname{dom} D = RX \oplus \ker D$ (cf. Przeworska-Rolewicz [1]), the arbitrariness of $x \in Y$ implies (4.1). ∎

Proposition 4.4. *Suppose that Condition* **[M]** *holds and* $(L, E) \in G[\Omega]$. *Then* $E(-e)$ *and* $L(-e)$ *do not exist*, *i.e.* $-e \notin \operatorname{dom} \Omega \cap \operatorname{dom} \Omega^{-1}$.

Proof. By Theorem 1.4, $-e \notin \operatorname{dom} \Omega$. By our assumptions, if $r = E(-e)$ then

$$r^2 = E(-e)E(-e) = E[(-e)(-e)] = E(e^2) = Ee = e.$$

Hence either $r = e$ or $r = -e$.

Suppose that $r = e$. Then $E(-e) = r = eEe$ and $e = Le = Lr = LE(-e) = -e$, which leads to a contradiction.

Suppose that $r = -e$. Then $E(-e) = r = -e = -Ee$ and $-e = -LE(-e) = L(-Ee) = L(-e)$, which implies $-e \in \operatorname{dom} \Omega$, a contradiction. ∎

Corollary 4.1. *Suppose that* $\mathcal{F} = \mathbb{C}$, *Condition* **[M]** *holds and* $(L, E) \in G[\Omega]$. *Then* $E(ie)$ *and* $L(ie)$ *do not exist*, *i.e.* $ie \notin \operatorname{dom} \Omega \cap \operatorname{dom} \Omega^{-1}$.

Proof. Suppose $E(ie)$ exists. By Proposition 4.2(iii), $E(ie)E(ie) = E(i^2e) = E(-e)$. This contradicts Proposition 4.4. A similar proof for $L(ie)$. ■

LEMMA 4.2. *Suppose that $D \in ML(X)$ and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Then*

(i) *$R(xy) = (Rx)(Ry) - F[(Rx)(Ry)]$ for $x, y \in X$;*

(ii) *if $F \in ML(X)$ then $R \in ML(X)$;*

(iii) *if $F \in ML(X)$ then $F[(Rx)(Ry)] = 0$ for $x, y \in X$;*

(iv) *if $R \in ML(X)$ and F is almost averaging, i.e. $F(zx) = zFx$ for $z \in \ker D$ and $x \in X$, then $F \in ML(X)$.*

Proof. (i) Recall that $FR = 0$. Let $x, y \in X$. We have $DR(xy) = xy = (DRx)(DRy) = D[(Rx)(Ry)]$. Hence $D[R(xy) - (Rx)(Ry)] = 0$, i.e. $R(xy) = (Rx)(Ry) + z$, where $z = F[(Rx)(Ry)] \in \ker D$.

(ii) If $F \in ML(X)$ then $R(xy) = (Rx)(Ry) - F[(Rx)(Ry)] = (Rx)(Ry) - (FRx)(FRy) = (Rx)(Ry)$.

(iii) If $R \in ML(X)$ then $R(xy) = (Rx)(Ry)$. Hence we have $F[(Rx)(Ry)] = (FRx)(FRy) = 0$.

(iv) Let $x, y \in \operatorname{dom} D$. Then $x = Ru + z_1$ and $y = Rv + z_2$, where $u, v \in X$, $z_1 = Fx$, $z_2 = Fy \in \ker D$ and

$$\begin{aligned} F(xy) &= F[(Ry + z_1)(Rv + z_2)] = F[(Ru)(Rv)] + F(z_1Ru + z_2Rv) + F(z_1z_2) \\ &= FR(uv) + z_1FRv + z_2FRu + z_1Fz_2 = z_1z_2 = (Fx)(Fy). \ ■ \end{aligned}$$

PROPOSITION 4.5. *Let Condition* **[M]** *hold. Suppose that $(L, E) \in G[\Omega]$, $F \in ML(X)$ is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $x \in V_R^l(X) \cap \operatorname{dom} \Omega$. Then $Ex = (I - xR)^{-1}z$, where $z \in \ker D$ is arbitrary.*

Proof. Since $F \in ML(X)$, Lemma 4.2(ii) implies that $R \in ML(X)$. Let $u = Ex$. Then $Du = uDx = (DRu)(Dx) = D(xRu)$ and $D(I - xR)u = 0$, which implies $u = xRu + z$, where $z \in \ker D$ is arbitrary. Since $x \in V_R^l(X) \cap \operatorname{dom} D$, the operator $I - xR$ is invertible and $Ex = u$ is of the required form (cf. (1.7) and Propositions 1.7, 1.8). ■

PROPOSITION 4.6. *Suppose that Condition* **[M]** *holds, $a \in X$ and there is an $R \in \mathcal{R}_D$ such that $Ra, g = Re \in \operatorname{dom} \Omega^{-1}$. Let $(L, E) \in G[\Omega]$. Then*

$$x = E(Ra)E(g + z) \in \ker(D - a) \quad \textit{for all } z \in \ker D. \tag{4.2}$$

Proof. By our assumption, $D \in ML(X)$. Since $D(g + z) = DRe + Dz = e$, we get

$$\begin{aligned} Dx &= D[E(Ra)E(g + z)] = [DE(Ra)][DE(g + z)] \\ &= E(Ra)(DRa)E(g + z)e = aE(Ra)E(g + z) = ax. \ ■ \end{aligned}$$

PROPOSITION 4.7. *Suppose that all assumptions of Proposition 4.6 are satisfied and $a, Ra \in I(X)$. If $x \in \ker(D - a)$ then x is of the form (4.2).*

P r o o f. Proposition 4.2 and our assumptions imply that $E(Ra) \in I(X)$ and $[E(Ra)]^{-1} = E[(Ra)^{-1}]$. Write $u = x[E(Ra)]^{-1}$. Then $x = E(Ra)u$ and

$$\begin{aligned} aE(Ra)(Du-u) &= aE(Ra)Du - aE(Ra)u = [DE(Ra)]Du - aE(Ra)u \\ &= D[E(Ra)u] - aE(Ra)u = Dx - ax = 0. \end{aligned}$$

Since $a, Ra \in I(X)$, we conclude that $Du = u = ue = uD(g+z)$, where $z \in \ker D$. Hence $u = E(g+z)$ and x is of the form (4.2). ∎

An immediate consequence of Propositions 4.6 and 4.7 is

COROLLARY 4.2. *Suppose that Condition* **[M]** *holds*, $(L,E) \in G[\Omega]$, $a \in I(X)$ *and there is an* $R \in \mathcal{R}_D$ *such that* $g = Re \in \operatorname{dom} \Omega^{-1}$ *and* $Ra \in I(X) \cap \operatorname{dom} \Omega^{-1}$. *Then* $x \in \ker(D-a)$ *if and only if* $x = E(Ra)E(g+z)$, *where* $z \in \ker D$.

PROPOSITION 4.8. *Suppose that* $\mathcal{F} = \mathbb{C}$ *or* $\mathbb{R}$, *Condition* **[M]** *holds*, $(L,E) \in G[\Omega]$, E *is multiplicative with exponent* $\lambda > 0$, $R \in \mathcal{R}_D$, $g = Re \in I(X) \cap \operatorname{dom} \Omega^{-1}$, $\alpha \in \mathcal{F}$ *and* $a = \alpha e$. *If* $\alpha g = \alpha Re \in \operatorname{dom} \Omega^{-1}$ *then*

(i) $\alpha = 1$, *i.e.* $Ra = g$;
(ii) $x \in \ker(D-I)$ *if and only if*

$$x = E[g(g+z)], \quad \textit{where } z \in \ker D \textit{ is arbitrary.} \tag{4.3}$$

P r o o f. (i) is immediate by our assumptions and Proposition 4.3.
(ii) Put $a = e$ in Corollary 4.2. Since E is multiplicative, by (4.2), we get

$$x = E(Re)E(g+z) = E(g)E(g+z) = E[g(g+z)], \quad \text{where } z \in \ker D.$$ ∎

THEOREM 4.2. *Suppose that all assumptions of Corollary* 4.2 *are satisfied and* $1 \in v_{\mathcal{F}}R$ (*i.e. the operator* $I-R$ *is invertible*, *cf.* (1.8)). *Then all solutions of the equation*

$$Dx = ax + y, \quad y \in X, \tag{4.4}$$

are of the form

$$x = x_0 + x_1, \quad \textit{where } x_0 \in \ker D \textit{ and} \tag{4.5}$$

$$x_1 = E(Ra)(I-R)^{-1}Ry_a, \quad y_a = a^{-1}E(Ra)^{-1}y. \tag{4.6}$$

P r o o f. By our assumption, the operator $I-R$ is invertible. We define $u = (I-R)^{-1}Ry_a = R(I-R)^{-1}y_a$. Then

$$Du - u = DR(I-R)^{-1}y_a - R(I-R)^{-1}y_a = (I-R)(I-R)^{-1}y_a = y_a.$$

Let $x_1 = E(Ra)u$. We get

$$\begin{aligned} Dx_1 - ax_1 &= D[E(Ra)u] - aE(Ra)u = [DE(Ra)]Du - aE(Ra)u \\ &= aE(Ra)Du - aE(Ra)u = aE(Ra)(Du-u) = aE(Ra)y_a \\ &= aE(Ra)a^{-1}E(Ra)^{-1}y = y. \end{aligned}$$

Hence x_1 is a particular solution of (4.4). Corollary 4.2 now implies that all solutions of that equation are of the form (4.5)–(4.6). ∎

An immediate consequence of this theorem and Proposition 4.8 is

COROLLARY 4.3. *Let $\mathcal{F}=\mathbb{C}$ or $\mathbb{R}$. Suppose that Condition* **[M]** *holds*, $(L,E)\in G[\Omega], E$ *is multiplicative with exponent* $\lambda>0$, $R\in\mathcal{R}_D$ *and* $g=Re\in I(X)\cap \operatorname{dom}\Omega^{-1}$. *If* $1\in v_{\mathcal{F}}R$ *then all solutions of the equation*

$$(D-I)x=y,\qquad y\in X, \tag{4.7}$$

are of the form $x=x_0+x_1$, *where*

$$x_0=E[g(g+z)]\in\ker(D-\lambda I),\qquad z\in\ker D \text{ is arbitrary}, \tag{4.8}$$

$$x_1=E(g)(I-R)^{-1}R[E(g)^{-1}y]. \tag{4.9}$$

Theorems 2.2 and 4.2 imply

COROLLARY 4.4. *Suppose that Condition* **[M]** *holds*, $(L_n,E_n)\in G[\Omega_n]$ *for all* $n\in\mathbb{N}$, $a\in I(X)$ *and there is an* $R\in\mathcal{R}_D$ *such that* $g_n=R^ne\in\operatorname{dom}\Omega_n^{-1}$, $R^na\in I(X)\cap\operatorname{dom}\Omega_n^{-1}$ *and* $1\in v_{\mathcal{F}}R^n$. *Then all solutions of the equation*

$$D^nx=ax+y,\qquad y\in X, \tag{4.10}$$

are of the form $x=x_0+x_1$, *where*

$$x_0=E_n(R^na)E_n(R^ne+z)\in\ker(D^n-a),\qquad z\in\ker D^n \text{ is arbitrary},$$

i.e.

$$z=\sum_{k=0}^{n-1}R^kz_k,\qquad z_0,\dots,z_{n-1} \text{ are arbitrary},$$

and

$$x_1=E_n(R^na)(I-R^n)^{-1}R^ny_{a,n},\quad \text{where}\quad y_{a,n}=a^{-1}[E_n(R^na)]^{-1}y. \tag{4.11}$$

Note. If F is a multiplicative initial operator for D corresponding to R then, by Lemma 4.2, R is multiplicative, $R^na=(Ra)^n$ and $g_n=R^ne=(Re)^n=g^n$. By Theorem 2.2, also $E_n\in ML(X)$. Thus $E_n(R^na)=E_n[(Ra)^n]=[E_n(Ra)]^n$ and $E_n(R^ne+z)=E_n(g^n+z)$.

5. Leibniz case

In this section we make use of the following condition:

$$[\mathbf{L}]\qquad X\in\mathbf{Lg}(D) \text{ is a Leibniz } D\text{-algebra.}$$

If Condition **[L]** is satisfied then, by definition, $X\in\mathcal{A}_D(0,e,0)$ and $e\in\ker D\subset\operatorname{dom}\Omega$. By Theorem 1.3, L is of exponential type whenever $(L,E)\in G[\Omega]$. By Corollaries 1.7 and 1.8, $-u, iu\in I(X)\cap\operatorname{dom}\Omega$ whenever $u\in I(X)\cap\operatorname{dom}\Omega$, and $L(-u)=Lu$, $L(iu)=\frac12L(-u)=\frac12Lu$. Recall that in Corollary 1.9 we have $\mathcal{F}=\mathbb{C}$. By Corollary 2.2, if $(L_1,E_1)\in G_{R,1}[\Omega]$ for an $R\in\mathcal{R}_D$, then $L_ne=0$ for all $n\in\mathbb{N}$. If $\mathcal{F}=\mathbb{C}$ then, by Corollary 2.4, $L_1(iu)=L_2u-R^2(DL_1u)^2$.

Proposition 5.1. *Suppose that Condition* [**L**] *holds and* $(L,E)\in G[\Omega]$. *Then*

(i) $Ex, E(-x)\in I(X)$ *and* $(Ex)^{-1}=E(-x)$ *for* $x\in\operatorname{dom}\Omega^{-1}$;
(ii) $E(nx)=(Ex)^n$ *for* $x\in\operatorname{dom}\Omega^{-1}$ *and* $n\in\mathbb{N}$;
(iii) $Ex\in I_n(\operatorname{dom}\Omega^{-1})$ *and* $(Ex)^{1/n}=E\big(\frac{1}{n}x\big)$ *for* $x\in\operatorname{dom}\Omega^{-1}$ *and* $n\in\mathbb{N}$;
(iv) $(Ex)^q=E(qx)$ *for* $x\in\operatorname{dom}\Omega^{-1}$ *and* $q\in\mathbb{Q}$;
(v) $Lu^q=qLu$ *for* $u\in\operatorname{dom}\Omega$ *and* $q\in\mathbb{Q}$;
(vi) $L(uv^{-1})=Lu-Lv$ *for* $u,v\in I(X)\cap\operatorname{dom}\Omega$;
(vii) $E(0)=e$.

Proof. (i) Let $x\in\operatorname{dom}\Omega^{-1}$. By Theorem 1.3 and Proposition 1.6, we have $e=E(0)=E(x-x)=(Ex)E(-x)$. Hence both Ex and $E(-x)$ belong to $I(X)$ and $(Ex)^{-1}=E(-x)$. Assertion (ii) is proved by an easy induction.

(iii) Let $n\in\mathbb{N}$. Suppose that $x\in\operatorname{dom}\Omega^{-1}$. Then we have

$$Ex=E\left(n\frac{1}{n}x\right)=\left[E\left(\frac{1}{n}x\right)\right]^n.$$

Hence $Ex\in I_n(\operatorname{dom}\Omega^{-1})$ and $(Ex)^{1/n}=E\big(\frac{1}{n}x\big)$. Assertions (ii) and (iii) imply (iv).

(v) Let $u\in\operatorname{dom}\Omega$ and $q\in\mathbb{Q}$. Then $x=Lu$ and $u=Ex$. Hence, by (iv), $L(u^q)=L(Ex)^q=LE(qx)=qx=qLu$.

(vi) Let $u,v\in I(X)\cap\operatorname{dom}\Omega$, $x=Lu$ and $y=Lv$. Then $u=Ex$ and $v=Ey$. Since L is of exponential type, by (i),

$$\begin{aligned}L(uv^{-1})&=Lu+L(v^{-1})=Lu+L[(Ey)^{-1}]\\&=Lu+LE(-y)=Lu-y=Lu-Lv.\end{aligned}$$

(vii) Proposition 1.17(vii) implies that $Le=0$. Thus, by definition, $E(0)=ELe=e$. ∎

Theorem 5.1. *Suppose that Condition* [**L**] *holds and* X *is a complete linear metric space over* $\mathbb{R}$, $(L,E)\in G[\Omega]$, L *is continuous on* $\operatorname{dom}\Omega$ *and* E *is continuous on* $\operatorname{dom}\Omega^{-1}$. *Then*

$$(Ex)^r=E(rx),\quad Lu^r=rLu\quad\textit{for}\quad (u,x)\in\operatorname{graph}\Omega,\ r\in\mathbb{R}\setminus\{0\}.$$

Proof. Fix $r\in\mathbb{R}\setminus\{0\}$. Then there is a sequence $\{q_n\}\subset\mathbb{Q}$, $q_n>0$, such that $\lim_{n\to\infty}|q_n-r|=0$. Now, our assumptions and assertions (i), (iv) and (v) of Proposition 5.1 imply the conclusion. ∎

Definition 5.1. Suppose that [**L**] holds and $(L,E)\in G[\Omega]$. Then L is said to be *natural* if $L(p_ne)=e\ln p_n$, where p_n is the nth prime ($n\in\mathbb{N}$).

Proposition 5.2. *Suppose that* [**L**] *holds*, $(L,E)\in G[\Omega]$ *and* L *is natural. Then*

$$L(qe)=e\ln q,\quad E(e\ln q)=qe\quad\textit{for } 0<q\in\mathbb{Q}.$$

Proof. By Proposition 5.1, $L\big(\frac{p}{q}e\big)=L[(pe)(qe)^{-1}]=L(pe)-L(qe)$ and $L(p^qe)=L(pe)^q=qLu$ for positive $p,q\in\mathbb{Q}$. There are positive integers M and

N such that $q = M/N$ and

$$N = \prod_{j=1}^{n_N} p_{n_j}^{r_j}, \qquad M = \prod_{k=1}^{m_M} p_{m_k}^{s_k},$$

where $n_j, m_k, r_j, s_k \in \mathbb{N}$ and $n_j \neq m_k$ $(j = 1, \ldots, n_N; k = 1, \ldots, m_M)$. This implies that

$$\begin{aligned} L(qe) &= L\Big(\frac{M}{N}e\Big) = L(Me) - L(Ne) = \sum_{j=1}^{n_N} r_j L(p_{n_j}e) - \sum_{k=1}^{m_M} s_k L(p_{m_k}e) \\ &= e\Big[\sum_{j=1}^{n_N} r_j \ln p_{n_j} - \sum_{k=1}^{m_M} s_k \ln p_{m_k}\Big] = e \ln \frac{M}{N} = e \ln q. \end{aligned}$$

Moreover, $E(e \ln q) = EL(qe) = qe$. ∎

COROLLARY 5.1. *Suppose that Condition* [**L**] *holds*, X *is a complete linear metric space over* $\mathbb{R}$, $(L, E) \in G[\Omega]$, L *is natural and continuous on* $\operatorname{dom}\Omega$ *and* E *is continuous on* $\operatorname{dom}\Omega^{-1}$. *Then*

$$L(ru) = Lu + e \ln r, \qquad rEx = E(x \ln r) \qquad \text{for } (u, x) \in \operatorname{graph}\Omega,\ r \in \mathbb{R}_+ \setminus \{0\}. \tag{5.1}$$

Proof. Let $0 < q \in \mathbb{Q}$ and $(u, x) \in \operatorname{graph}\Omega$. Then

$$\begin{aligned} L(qu) &= L(qeu) = L(qe) + Lu = Lu + e \ln q, \\ qEx &= qeEx = E(e \ln q)Ex = E(xe \ln q) = E(x \ln q). \end{aligned}$$

Having thus proved (5.1) for $0 < q \in \mathbb{Q}$, we prove it for $r \in \mathbb{R}_+ \setminus \{0\}$ by similar arguments to those used in the proof of Theorem 5.1. ∎

THEOREM 5.2. *Suppose that Condition* [**L**] *holds*, $(L, E) \in G[\Omega]$, $R \in \mathcal{R}_D$ *and* $g = Re \in I(X)$. *Then*

(i) *all monomials*, *i.e. elements of the form* $R^n z$, *where* $z \in \ker D$, $n \in \mathbb{N}$, *belong to* $\operatorname{dom}\Omega$ *and*

$$L(R^n z) = nLg + Lz + L\frac{1}{n!}e \quad \textit{for } z \in \ker D,\ n \in \mathbb{N};$$

(ii) *all monomials are invertible*;

(iii) *if* L *is natural*, *i.e.* $L(ne) = e \ln n$ *for* $n \in \mathbb{N}$, *then*

$$L(R^n z) = nLg + Lz - e \ln n! \quad \textit{for } z \in \ker D,\ n \in \mathbb{N}.$$

Proof. Since X is a Leibniz D-algebra, we have $e \in \ker D$ and

$$R^n z = zR^n e = z\frac{(Re)^n}{n!} = z\frac{g^n}{n!} \quad \text{for } z \in \ker D,\ n \in \mathbb{N} \tag{5.2}$$

(cf. Przeworska-Rolewicz and von Trotha [1], also Przeworska-Rolewicz [1], Section 6.2). Since, by Theorem 1.3, L is of exponential type, we get

$$L(R^n z) = L\left(z\frac{g^n}{n!}\right) = Lg^n + Lz + L\left(\frac{1}{n!}\right) = nLg + Lz + L\left(\frac{1}{n!}e\right).$$

(ii) is an immediate consequence of (i) and Theorem 1.1.

(iii) If L is natural, then, by Proposition 5.2, $L\left(\frac{1}{n!}e\right) = e\ln\frac{1}{n!} = -e\ln n!$. ∎

Corollary 5.2. *Suppose that all assumptions of Theorem* 5.2 *are satisfied. Then no monomial is a zero divisor.*

This follows immediately from Theorem 5.2, since invertible elements cannot be zero divisors (cf. also Corollary 1.2).

Proposition 5.3. *Suppose that Condition* **[L]** *holds*, $(L,E) \in G[\Omega]$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* $g = Re \in I(X)$. *Then*

(i) $D^n g^{-1} = (-1)^n n! g^{-n+1}$ *for* $n \in \mathbb{N}$;

(ii) $D^n Lg = (-1)^{n-1}(n-1)! g^{-n}$ *for* $n \in \mathbb{N}$;

(iii) *if* $F \in ML(X)$ *then* Fg^{-1} *does not exist.*

Proof. By definition, $Dg = DRe = e \in \ker D$ and $Dg^n = ng^{n-1}Dg = ng^{n-1}$ for all $n \in \mathbb{N}$. Since $g \in I(X)$, we have $0 = De = D(gg^{-1}) = gDg^{-1} + g^{-1}Dg = gDg^{-1} + g^{-1}$, which implies that $Dg^{-1} = g^{-2}$. We shall prove by induction that

$$Dg^{-n} = -ng^{-(n+1)} \quad \text{for } n \in \mathbb{N}. \tag{5.3}$$

Indeed, for $n = 1$, (5.3) is already proved. Suppose it is true for a fixed $n \in \mathbb{N}$. Then

$$\begin{aligned} Dg^{-(n+1)} &= D(g^{-n}g^{-1}) = g^{-1}Dg^{-n} + g^{-n}Dg^{-1} \\ &= g^{-1}(-ng^{-(n+1)}) - g^{-n}g^{-2} = -(n+1)g^{-(n+2)}. \end{aligned}$$

We prove (i) also by induction. For $n = 1$, (i) is already proved. Suppose now (i) to be true for a fixed $n \in \mathbb{N}$. Then

$$D^{n+1}g^{-1} = (-1)^n n! D(g^{-n+1}) = (-1)^n n![-(n+1)]g^{-n} = (-1)^{n+1}(n+1)! g^{-n}.$$

Now we prove (ii) by induction. Let $n = 1$. We have shown in Example 1.1 that $DLg^{-1} = g^{-1}$. Suppose (ii) to be true for a fixed $n \in \mathbb{N}$. Then

$$D^n Lg = D^{n-1}DLg = D^{n-1}g^{-1} = (-1)^{n-1}(n-1)! g^{-n+1}.$$

(iii) Since $e \in \ker D$, $g \in I(X)$, $FR = 0$ and $F \in ML(X)$, we get $e = Fe = F(gg^{-1}) = (Fg)(Fg^{-1}) = (FRe)(Fg^{-1}) = 0$, which contradicts the assumption that e is the unit of X. ∎

Note that in the classical case $Fg^{-1} = \frac{1}{t}|_{t=0}$, hence it does not exist (cf. Example 1.1).

Theorem 5.3. *Suppose that Condition* **[L]** *holds*, $(L,E) \in G[\Omega]$, $a \in X$ *and* $Ra \in \operatorname{dom}\Omega^{-1}$ *for an* $R \in \mathcal{R}_D$. *Let* $x \in I(X) \cap \operatorname{dom}\Omega$. *Then* $x \in \ker(D - a)$ *if* *and only if*

$$x = zE(Ra), \quad \textit{where } z \in \ker D \textit{ is arbitrary.} \tag{5.4}$$

Proof. *Sufficiency*. Suppose that x is of the form (5.4). Since X is a Leibniz D-algebra and $Dz = 0$ we have

$$\begin{aligned} Dx &= D[zE(Ra)] = (Dz)E(Ra) + zDE(Ra) = zDE(Ra) = zE(Ra)DRa \\ &= zE(Ra)a = ax, \end{aligned}$$

i.e. $x \in \ker(D - a)$.

Necessity. Suppose that $x \in \ker(D - a)$. Hence $Dx = ax$. Since $x \in I(X)$, we find that $DLx = x^{-1}Dx = x^{-1}ax = x^{-1}xa = a$. By Proposition 1.5, $Lz \in \ker D$ whenever $z \in \ker D$. Since L is of exponential type, we get

$$Lx = Ra + z' = LE(Ra) + Lz = L[zE(Ra)],$$

where $z, z' = Lz \in \ker D$ are arbitrary. This implies that x is of the form (5.4). ∎

Corollary 5.3. *Suppose that Condition* [**L**] *holds*, $(L, E) \in G[\Omega]$ *and there are* $\lambda \in \mathcal{F}$ *and* $R \in \mathcal{R}_D$ *such that* $\lambda g = \lambda Re \in \operatorname{dom} \Omega^{-1}$. *Let* $x \in I(X) \cap \operatorname{dom} \Omega$. *Then* $x \in \ker(D - \lambda I)$ *if and only if*

$$x = zE(\lambda g), \quad \textit{where } z \in \ker D \textit{ is arbitrary.}$$

Proof. We apply Theorem 5.3 for $a = \lambda e$. Then $Ra = \lambda Re = \lambda g$. ∎

Proposition 5.4. *Suppose that Condition* [**L**] *holds and there is a* $u \in \operatorname{dom} \Omega$ *such that* $Du = u$. *Then* $g = Re \in \operatorname{dom} \Omega^{-1}$ *for every* $R \in \mathcal{R}_D$.

Proof. Fix $R \in \mathcal{R}_D$. By our assumptions, $Du = u = ue = uDg$, which implies that $g \in \operatorname{dom} \Omega^{-1}$ and $Lu = g$, i.e. $u = E(g)$ for a $(L, E) \in G[\Omega]$. ∎

Corollary 5.4. *Suppose that Condition* [**L**] *holds*, $(L, E) \in G[\Omega]$, $R \in \mathcal{R}_D$, $\lambda \in v_{\mathcal{F}}R$, $\lambda g = \lambda Re \in \operatorname{dom} \Omega^{-1}$ *and* F *is an initial operator for* D *corresponding to* R. *Then*

$$FE(\lambda g) = e, \quad FL[zE(\lambda g)] = Lz \quad \textit{for all } z \in \ker D. \tag{5.5}$$

If F *is almost averaging*, *i.e.* $F(zx) = zFx$ *for all* $x \in \operatorname{dom} D$ *and* $z \in \ker D$, *then*

$$F[zE(\lambda g)] = z \quad \textit{for all } z \in \ker D. \tag{5.6}$$

Proof. Since $e \in \ker D$, we have $Fe = e$. By Proposition 1.10, for $u = E(\lambda g)$ we have

$$FE(\lambda g) = Fu = u - RDu = u - \lambda Ru = (I - \lambda R)u = e.$$

Let $z \in \ker D$. Then $Fz = z$. Since L is of exponential type and $Lz \in \ker D$, we find that

$$FL[zE(\lambda g)] = F[Lz + LE(\lambda g)] = FLz + FLe = Lz + Le = L(ze) = Lz.$$

If F is almost averaging then $F[zE(\lambda g)] = zFE(\lambda g) = ze = z$. ∎

N o t e. If Condition [**L**] holds and $\dim\ker D = 1$, then all $F \in \mathcal{F}_D$ are almost averaging (cf. Przeworska-Rolewicz [3]).

THEOREM 5.4. *Suppose that all assumptions of Theorem* 5.3 *are satisfied. Then the general solution of the equation*

$$Dx + ax = y, \quad y \in \operatorname{dom}\Omega, \tag{5.7}$$

is of the form

$$x = E(Ra)R[E(-Ra)y] + zE(Ra), \quad \text{where } z \in \ker D \text{ is arbitrary}, \tag{5.8}$$

or in an equivalent form

$$x = E(Ra)[RE(Ly - Ra) + z], \quad \text{where } z \in \ker D \text{ is arbitrary}. \tag{5.9}$$

P r o o f. All solutions of the homogeneous equation $Dx = ax$ are determined by Theorem 5.3. We are looking for a particular solution of (5.7) which is of the form $x = uE(Ra)$, where $u \in \operatorname{dom}\Omega$ is to be determined. We have

$$Dx = (Du)E(Ra) + uDE(Ra) = (Du)E(Ra) + uE(Ra)DRa = E(Ra)(Du + au).$$

Hence

$$y + auE(Ra) = y + ax = Dx = E(Ra)(Du + au) = auE(Ra) + E(Ra)Du$$

and $E(Ra)Du = y$, i.e.

$$E(-Ra)y = E(-Ra)E(Ra)yDu = E(Ra - Ra)Du = E(0)Du = eDu = Du.$$

This implies that $u = R[E(-Ra)y] + z$, where $z \in \ker D$ is arbitrary. We therefore conclude that all solutions of (5.7) are of the form

$$x = uE(Ra) = E(Ra)\{R[E(-Ra)y] + z\} = E(Ra)R[E(-Ra)y] + zE(Ra),$$

i.e. of the form (5.8).

Since $E(-Ra)y = E(-Ra)ELy = E(Ly - Ra)$, we obtain from (5.8) the form (5.9). ■

COROLLARY 5.5. *Suppose that all assumptions of Theorem* 5.3 *are satisfied and* $Ra \in I(X)$. *If* $F \in ML(X)$ *is an initial operator for* D *corresponding to* R *then the initial value problem for* (5.7) *with the condition*

$$Fx = x_0, \quad \text{where } x_0 \in \ker D \text{ is given}, \tag{5.10}$$

has a unique solution of the form

$$\begin{aligned} x &= E(Ra)R[E(-Ra)y] + x_0[FE(-Ra)]E(Ra) \\ &= E(Ra)[RE(Ly - Ra) + x_0FE(-Ra)]. \end{aligned} \tag{5.11}$$

P r o o f. If $F \in ML(X) \cap \mathcal{F}_D$ corresponds to R, i.e. $FR = 0$, then

$$\begin{aligned} x_0 = Fx &= F\{E(Ra)R[E(-Ra)y] + zE(Ra)\} \\ &= [FE(Ra)]FR[E(-Ra)y] + F[zE(Ra)] = (Fz)FE(Ra) = zFE(Ra). \end{aligned}$$

By our assumption and Proposition 5.1(i), $E(Ra) \in I(X)$ and $[E(Ra)]^{-1} = E(-Ra)$. Since $F \in ML(X)$, we find that

$$e = Fe = F[E(Ra)E(-Ra)] = [FE(Ra)][FE(-Ra)],$$

which implies that $FE(Ra) \in I(X)$ and

$$z = x_0[FE(Ra)]^{-1} = x_0 FE(-Ra).$$

Now, (5.8) implies that the problem (5.7), (5.10) has a unique solution of the form (5.11). ∎

THEOREM 5.5. *Let Condition* **[L]** *hold. Suppose that* $(L, E) \in G[\Omega]$, $a_1, \ldots, a_n \in X$ *and there is an* $R \in \mathcal{R}_D$ *such that* $Ra_j \in \operatorname{dom} \Omega^{-1}$ $(j = 1, \ldots, n)$. *Then the general solution of the equation*

$$\Big[\prod_{j=1}^{n}(D - a_j)\Big]x = y, \quad y \in \operatorname{dom} \Omega, \tag{5.12}$$

is of the form

$$x = E(Ra_n)[RE(Ly_{n-1} - Ra_n) + z_n], \tag{5.13}$$

where

$$y_0 = y, \quad y_j = E(Ra_j)[RE(Ly_{j-1} - Ra_j)] + z_j \quad (j = 1, \ldots, n),$$
$$z_1, \ldots, z_n \in \ker D \text{ are arbitrary.}$$

Proof (induction). For $n=1$, (5.13) is proved by Theorem 5.4, formula (5.9). Suppose (5.13) to be true for fixed $n \in \mathbb{N}$. Consider the equation

$$\Big[\prod_{j=1}^{n+1}(D - a_j)\Big]x = y, \quad \text{where } Ra_{n+1} \in \operatorname{dom} \Omega^{-1},\ y \in \operatorname{dom} D. \tag{5.14}$$

Write

$$y_n = (D - a_{n+1})x. \tag{5.15}$$

Clearly, y_n satisfies (5.12). By our inductive assumption, y_n is of the form (5.13). On the other hand, again by (5.9), we get from (5.15),

$$x = E(Ra_{n+1})[RE(Ly_n - Ra_{n+1})] + z_{n+1}, \quad \text{where } z_{n+1} \in \ker D \text{ is arbitrary.}$$

This proves that (5.14) has all solutions of the required form. ∎

COROLLARY 5.6. *Suppose that Condition* **[L]** *holds. Let* $m \in \mathbb{N}$. *Suppose that* $(L_m, E_m) \in G[\Omega_m]$, $a_1, \ldots, a_n \in X$ *and there is an* $R \in \mathcal{R}_D$ *such that* $R^m a_j \in \operatorname{dom} \Omega_m^{-1}$ $(j = 1, \ldots, n)$. *Then the general solution of the equation*

$$\Big[\prod_{j=1}^{n}(D^m - a_j)\Big]x = y, \quad y \in \operatorname{dom} \Omega_m, \tag{5.16}$$

is of the form

$$x = E_m(R^m a_n)[R^m E_m(L_m y_{n-1} - R^m a_n) + z_n], \tag{5.17}$$

where

$$y_0 = y, \quad y_j = E_m(R^m a_j)[R^m E_m(L_m y_{j-1} - R^m a_j)] + z_j \quad (j = 1, \ldots, n),$$
$$z_1, \ldots, z_n \in \ker D^m \text{ are arbitrary.}$$

To prove this corollary it is enough to apply Theorems 2.1 and 5.5 with D^m replacing D.

Solutions of equations with polynomials in D having scalar coefficients can be obtained in a simpler form, as is shown by the following

PROPOSITION 5.5. *Suppose that* $[\mathbf{L}]$ *holds*, $(L, E) \in G[\Omega]$ *and there are* $\lambda \in \mathcal{F}$ *and* $R \in \mathcal{R}_D$ *such that* $\lambda g = \lambda Re \in \operatorname{dom} \Omega^{-1}$. *Let* $n \in \mathbb{N}$ *and* $x \in I(X) \cap \operatorname{dom} \Omega$. *Then* $x_0 \in \ker(D - \lambda I)^n$ *if and only if*

$$x_0 = zE(\lambda g), \tag{5.18}$$

where $z = \sum_{k=0}^{n-1} z_k R^k e \in \ker D^n$ *and* $z_0, \ldots, z_{n-1}$ *are arbitrary.*

Proof (induction). For $n = 1$, (5.18) is proved in Corollary 5.3. Suppose (5.18) to be true for a fixed $n \in \mathbb{N}$. Consider the equation

$$(D - \lambda I)^{n+1} x = 0. \tag{5.19}$$

By our inductive assumption, it can be written as $(D - \lambda I)x = x_0$, where x_0 is of the form (5.18). By (5.8),

$$x = E(\lambda g)R[E(-\lambda g)x_0] + z_n E(\lambda g), \quad \text{where } z_n \in \ker D \text{ is arbitrary.}$$

Hence

$$\begin{aligned} x &= E(\lambda g)R[E(-\lambda g)zE(\lambda g)] + z_n E(\lambda g) = E(\lambda g)Rz \\ &= \Big(R\sum_{k=0}^{n-1} z_k R^k e + z_n\Big)E(\lambda g) = \Big(\sum_{k=0}^{n-1} z_k R^{k+1} e + z_n\Big)E(\lambda g) \\ &= \Big(\sum_{j=1}^{n} z_j R^j e + z_n\Big)E(\lambda g) = \Big(\sum_{j=0}^{n} z_j R^j e\Big)E(\lambda g). \end{aligned}$$

Clearly, $\sum_{j=0}^{n-1} R^j z_j \in \ker D^n$. Moreover, since X is a Leibniz D-algebra, we have $R^j z = zR^j e$ for $j = 0, \ldots, n-1$ and $z \in \ker D$. We therefore conclude that x is of the required form. ■

THEOREM 5.6. *Let* $\mathcal{F}$ *be algebraically closed. Let*

$$P(t) = \prod_{j=1}^{n} (t - t_j)^{r_j}, \quad r_j \geq 1,\ r_1 + \ldots + r_n = N, \qquad t_j \in \mathcal{F},\ t_j \neq t_k \text{ if } j \neq k\ (j, k = 1, \ldots, n). \tag{5.20}$$

Suppose that Condition $[\mathbf{L}]$ *holds*, $(L, E) \in G[\Omega]$ *and there is an* $R \in \mathcal{R}_D$ *such that* $t_j g = t_j Re \in \operatorname{dom} \Omega^{-1} (j = 1, \ldots, n)$. *Let* $x_0 \in I(X) \cap \operatorname{dom} \Omega$. *Then* $x_0 \in \ker P(D)$

if and only if

$$x_0 = \sum_{j=0}^{n} \Big(\sum_{k=0}^{r_j - 1} z_{jk} R^k e \Big) E(t_j g), \tag{5.21}$$

where $z_{jk} \in \ker D$ *are arbitrary* $(k = 0, \ldots, r_j - 1; j = 1, \ldots, n)$.

Proof. The operator D is an algebraic operator on the space $X_0 = \ker P(D)$ and its characteristic polynomial is $P(t)$. We therefore conclude that $X_0 = X_1 \oplus \ldots \oplus X_n$, where $X_j = \ker(D - t_j I)^{r_j}$ $(j = 1, \ldots, n)$. Proposition 5.6 now implies that x_0 is of the form (5.9). ∎

Proposition 5.6. *Suppose that all assumptions of Proposition* 5.5 *are satisfied. Then the equation*

$$(D - \lambda I)^n x = y, \qquad y \in \operatorname{dom} \Omega, \tag{5.22}$$

has the general solution of the form

$$x = x_1 + x_0, \tag{5.23}$$

where

$$x_1 = E(\lambda g) R^n [E(-\lambda g) y],$$

and x_0 *is given by* (5.18).

In order to prove this proposition, we shall prove

Lemma 5.1. *Suppose that Condition* **[L]** *holds*, $u, v \in \operatorname{dom} D^n$ *for some* $n \in \mathbb{N}$ *and* $\lambda \in \mathcal{F}$. *Then*

$$(D - \lambda I)^n (uv) = \sum_{j=0}^{n} \binom{n}{j} [(D - \lambda I)^{n-j} u] D^j v, \qquad \lambda \in \mathcal{F}. \tag{5.24}$$

Proof. By the Leibniz formula for $u, v \in \operatorname{dom} D^n$ and $\lambda \in \mathcal{F}$ we have

$$\begin{aligned}
(D - \lambda I)^n (uv) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \lambda^{n-k} D^k (uv) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \lambda^{n-k} \sum_{j=0}^{k} \binom{k}{j} (D^{k-j} u) D^j v \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} \frac{n! k!}{k!(n-k)! j!(k-j)!} (-1)^{n-k} \lambda^{n-k} (D^{k-j} u) D^j v \\
&= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (D^j v) \sum_{k=j}^{n} \frac{(n-j)!}{(n-k)!(k-j)!} (-1)^{n-k} \lambda^{n-k} D^{k-j} u \\
&= \sum_{j=0}^{n} \binom{n}{j} (D^j v) \sum_{m=0}^{n-j} \frac{(n-j)!}{(n-j-m)! m!} (-1)^{n-j-m} \lambda^{n-j-m} D^m u
\end{aligned}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (D^j v) \sum_{m=0}^{n-j} \binom{n-j}{m} (-1)^{n-j-m} \lambda^{n-j-m} D^m u$$
$$= \sum_{j=0}^{n} \binom{n}{j} [(D-\lambda I)^{n-j} u] D^j v. \ \blacksquare$$

Proof of Proposition 5.6. By Corollary 5.3, $E(\lambda g) \in \ker(D - \lambda I)$. Then, by Lemma 5.1, for every $n \in \mathbb{N}$ we get

$$\begin{aligned}(D-\lambda I)^n x_1 &= (D-\lambda I)^n \{E(\lambda g) R^n [E(-\lambda g) y]\} \\ &= \sum_{j=0}^{n} \binom{n}{j} [(D-\lambda I)^{n-j} E(\lambda g)] D^j R^n [E(-\lambda g) y] \\ &= E(\lambda g) D^n R^n [E(-\lambda g) y] \\ &\quad + \sum_{j=0}^{n-1} \binom{n}{j} (D-\lambda I)^{n-1-j} [(D-\lambda I) E(\lambda g)] R^{n-j} [E(-\lambda g) y] \\ &= E(\lambda g) E(-\lambda g) y = y. \ \blacksquare\end{aligned}$$

THEOREM 5.7. *Suppose that all assumptions of Theorem 5.6 are satisfied. Then the equation*

$$P(D)x = y, \quad y \in \operatorname{dom} \Omega, \tag{5.25}$$

has the general solution of the form

$$x = x_1 + x_0, \tag{5.26}$$

where $x_0 \in \ker P(D)$ and

$$\begin{aligned} x_1 &= E(t_n g) R^{r_n} [E(-t_n g) y_{n-1}], \quad y_0 = y, \\ y_j &= E(t_j g) R^{r_j} [E(-t_j g) y_{j-1}] \quad \text{for } j = 1, \ldots, n-1 \ (n \geq 2). \end{aligned} \tag{5.27}$$

Proof. By Proposition 5.3, we have

$$\begin{aligned} P(D) x_1 &= \prod_{j=1}^{n} (D - t_j I)^{r_j} x_1 \\ &= \Big[\prod_{j=1}^{n-1} (D-\lambda I)^{r_j}\Big] (D-\lambda I)^{r_n} \{E(t_n g) R^{r_n} [E(-t_n g) y]\} \\ &= \Big[\prod_{j=1}^{n-1} (D - t_j I)^{r_j}\Big] y_{n-1} \\ &= \Big[\prod_{j=1}^{n-2} (D - t_j I)^{r_j}\Big] (D - t_j I)^{r_{n-1}} y_{n-1} \end{aligned}$$

$$= \Big[\prod_{j=1}^{n-2} (D - t_j I)^{r_j} \Big] y_{n-2} = \ldots = (D - t_1 I)^{r_1} y_1 = y_0 = y. \ \blacksquare$$

Note. If $\mathcal{F} = \mathbb{R}$, there may appear pairs of conjugate imaginary roots: $i\lambda$, $-i\lambda$, where $\lambda \in \mathbb{R} \setminus \{0\}$. To every such pair there corresponds the operator $D^2 + \lambda^2 I$. This operator, the particular properties of logarithms and antilogarithms when $\mathcal{F} = \mathbb{C}$ and their relation to the real case will be considered in the next sections. We point out that the properties of right invertible operators in the complex case have recently been studied in a series of papers by Binderman (cf. for instance, Binderman [1], [2]).

EXAMPLE 5.1. Suppose that Condition [**L**] holds and $(L, E) \in G_{R,1}[\Omega]$ for an $R \in \mathcal{R}_D$. We recall that the *Wrońskian* of the system $\{x_1, \ldots, x_N\} \subset \operatorname{dom} D^{N-1}$ is the determinant

$$W_D = W_D(x_1, \ldots, x_N) = \det(D^{j-1} x_k)_{j,k=1,\ldots,N}$$

and that this determinant has the same properties as in the classical case, i.e. we can prove the Wroński theorems for linear equations in D with coefficients belonging to X (cf. Przeworska-Rolewicz [1], Section 6.3). Let

$$a(D) = \sum_{k=0}^{N} a_k D^k, \quad \text{where } a_0, \ldots, a_N \in X, \ a_N = e.$$

Then the *Liouville formula* holds:

$$(5.28) \qquad DW_D = (-1)^N a_{N-1} W_D.$$

A system $\{x_1, \ldots, x_N\} \in \operatorname{dom} D^{N-1}$ is said to be *fundamental* if $W_D(x_1, \ldots, x_N) \in I(X)$ (cf. Przeworska-Rolewicz [1], Definition 6.3.3). If $\{x_1, \ldots, x_N\}$ is a fundamental system of solutions to the linear equation $a(D)x = 0$ then (5.28) implies that $DLW_D = (-1)^N a_{N-1}$. Hence we get

$$x = LW_D = (-1)^N R a_{N-1} \in \operatorname{dom} \Omega^{-1} \quad \text{and} \quad W_D = Ex.$$

Let $x_1 = Dx$. Suppose that either $x_1 \in V_R^l(X)$ or $x_1 \in V_R^r(X)$. Then Propositions 1.7 and 1.8, respectively, imply $Ex = FEx$, where F is an initial operator for D corresponding to R. We therefore conclude that

$$(5.29) \qquad W_D = 0 \quad \text{if and only if} \quad FW_D = 0$$

(cf. also Przeworska-Rolewicz [1], Theorem 6.3.4, with a different proof).

6. Exponential, power and polylogarithmic functions

DEFINITION 6.1. Let $X \in \mathbf{Lg}(D)$. Write

$$(6.1) \qquad E_\lambda u = E(\lambda L u) \quad \text{for } (L, E) \in G[\Omega], \ u \in \operatorname{dom} \Omega, \ \lambda \in \mathcal{F}.$$

Then E_λ is said to be of *power type with exponent* λ.

Proposition 6.1. *Suppose that* $X \in \mathbf{Lg}(D)$, $(L,E) \in G[\Omega]$ *and the mapping* E_λ *is defined by* (6.1). *Then for all* $\lambda, \mu \in \mathcal{F}$,

(i) $E_\lambda(\operatorname{dom}\Omega) \subset \operatorname{dom}\Omega$;
(ii) $LE_\lambda = \lambda L$;
(iii) $E_\lambda E_\mu = E_{\lambda\mu}$.

Proof. Let $u \in \operatorname{dom}\Omega$ and $\lambda \in \mathcal{F}$. By definitions,

$$DE_\lambda u = DE(\lambda Lu) = E(\lambda Lu)D(\lambda Lu) = (E_\lambda u)D(\lambda Lu).$$

Hence $E_\lambda u \in \operatorname{dom}\Omega$ and $LE_\lambda u = \lambda Lu$. Again by definition, we get

$$E_\lambda E_\mu = E\{\lambda L[E(\mu L)]\} = E[\lambda LE(\mu L)] = E(\lambda\mu L) = E_{\lambda\mu}. \ \blacksquare$$

Proposition 6.2. *Suppose that Condition* **[L]** *holds and* $(L,E) \in G[\Omega]$. *Then for all* $\lambda, \mu \in \mathcal{F}$ *and* $u \in I(X) \cap \operatorname{dom}\Omega$,

(i) $E_\lambda \in M(X)$;
(ii) $DE_\lambda u = \lambda(E_{\lambda-1}u)Du$;
(iii) $(E_\lambda u)(E_\mu u) = E_{\lambda+\mu}u$; *in particular*, $(E_\lambda u)(E_{-\lambda}u) = e$;
(iv) $E_\lambda u \in I(X)$ *and* $(E_\lambda u)^{-1} = E_{-\lambda}u$.

Proof. By Theorem 1.3, L is of exponential type. Let $u, v \in \operatorname{dom}\Omega$ and $\lambda \in \mathcal{F}$. Then

$$\begin{aligned}(E_\lambda u)(E_\lambda v) &= [E(\lambda Lu)][E(\lambda Lv)] = E(\lambda Lu + \lambda Lv)\\ &= E[\lambda(Lu + Lv)] = E[\lambda L(uv)] = E_\lambda(uv),\end{aligned}$$

i.e. $E_\lambda \in M(X)$.

We have already shown that $DE_\lambda u = \lambda(E_\lambda u)DLu$ (cf. the proof of Proposition 6.1). On the other hand, if $u \in I(X) \cap \operatorname{dom}\Omega$ and $\lambda, \mu \in \mathcal{F}$, then

$$\begin{aligned}E_{\lambda-1}u &= E[(\lambda-1)Lu] = E(\lambda Lu - Lu) = E(\lambda Lu)E(-Lu)\\ &= (ELu)^{-1}E_\lambda u = u^{-1}E_\lambda u.\end{aligned}$$

Hence $DE_\lambda u = \lambda(E_\lambda u)u^{-1}Du = \lambda(E_{\lambda-1}u)Du$. Moreover,

$$(E_\lambda u)(E_\mu u) = E(\lambda Lu)E(\mu Lu) = E(\lambda Lu + \mu Lu) = E[(\lambda+\mu)Lu] = E_{\lambda+\mu}u.$$

If $\mu = -\lambda$ then $(E_\lambda u)(E_{-\lambda}u) = E(0 \cdot Lu) = E(0) = e$. This implies (iv). $\blacksquare$

In general, we have

Proposition 6.3. *Let* $X \in \mathbf{Lg}(D)$ *and* $(L,E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *If* $\lambda \in \mathcal{F}$ *and* $u, v \in \operatorname{dom}\Omega$, $E_\lambda u, E_\lambda v \in I(X)$ *then*

$$(E_\lambda u)(E_\lambda v) = E\{c_D\lambda(Lu + Lv) + R[(E_\lambda u)^{-1}(E_\lambda v)^{-1}f_D(E_\lambda u, E_\lambda v)]\}.$$

This follows immediately from Definition 6.1 and Theorem 1.2.

Corollary 6.1. *Suppose that all assumptions of Proposition* 6.3 *are satisfied and* $c_D = 0$. *Then the mapping* E_λ *is not defined for* $\lambda \neq 1$. *If* $\lambda = 1$ *then* $E_1 = I|_{\operatorname{dom}\Omega}$.

P r o o f. Let $u \in \operatorname{dom} \Omega$. Then $x = Lu \in \operatorname{dom} \Omega^{-1}$. Let $\lambda \in \mathcal{F}$. By Proposition 4.3, if $\lambda x \in \operatorname{dom} \Omega^{-1}$ then $\lambda = 1$. Hence E_λ is not defined for $\lambda \neq 1$. If $\lambda = 1$ then $E_\lambda(u) = E(\lambda Lu) = ELu = u$. ■

Corollary 6.1 implies that for multiplicative D the mapping E_λ is not defined.

DEFINITION 6.2. Suppose that X is a D-algebra with unit e and a complete linear metric space, and $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$. Write $x^0 = e$ and

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } x \in X$$

whenever this series is convergent. The function e^x is said to be an *exponential function*.

DEFINITION 6.3. Suppose that $X \in \mathbf{Lg}(D)$ with unit $e \in \operatorname{dom} \Omega^{-1}$ is a complete linear metric space and $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$. Write

$$\mathcal{E}_D(X) = \Big\{ x \in \operatorname{dom} \Omega^{-1} : \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is convergent} \Big\}, \tag{6.2}$$

$$\mathcal{E}'_D(X) = \{ u \in \operatorname{dom} \Omega : \lambda Lu \in \mathcal{E}_D(X) \text{ for } (L, E) \in G[\Omega],\ \lambda \in \mathcal{F} \}, \tag{6.3}$$

$$u^\lambda = \mathrm{e}^{\lambda Lu} \quad \text{for } u \in \mathcal{E}'_D(X),\ \lambda \in \mathcal{F}. \tag{6.4}$$

The function u^λ is said to be a *power function*.

Note that (as in the classical case)

$$\mathrm{e}^x \mathrm{e}^y = \mathrm{e}^{x+y} \quad \text{whenever} \quad x, y, x+y \in \mathcal{E}_D(X). \tag{6.5}$$

PROPOSITION 6.4. *Suppose that X is a complete linear metric space, $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, Condition* [**L**] *holds, X has unit $e \in \operatorname{dom} \Omega^{-1}$, $(L, E) \in G[\Omega]$ and D is closed. Then*

(i) *if $x \in \mathcal{E}_D(X)$ then $\mathrm{e}^x \in \operatorname{dom} \Omega$, $Ex = \mathrm{e}^x$ and $L\mathrm{e}^x = x$;*

(ii) *if $u \in \mathcal{E}'_D(X)$ and $\lambda \in \mathcal{F}$ then $\mathrm{e}^{\lambda Lu} \in \operatorname{dom} \Omega$, $\mathrm{e}^{\lambda Lu} = E_\lambda u = u^\lambda$ and $Lu^\lambda = \lambda Lu$;*

(iii) *if $u \in I(X) \cap \mathcal{E}'_D(X)$ then*

$$Du^\lambda = \lambda u^{\lambda - 1} Du. \tag{6.6}$$

P r o o f. (i) Since X is a Leibniz D-algebra, we have $Dx^n = nx^{n-1}Dx$ for all $x \in \operatorname{dom} D$ and $n \in \mathbb{N}$ (cf. Przeworska-Rolewicz [1], Section 6.1). Since D is closed and

$$\begin{aligned} \sum_{n=0}^{\infty} D \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \frac{1}{n!} Dx^n = \sum_{n=1}^{\infty} \frac{1}{n!} n (Dx^{n-1}) Dx \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} Dx = \Big(\sum_{j=0}^{\infty} \frac{x^j}{j!} \Big) Dx = \mathrm{e}^x Dx, \end{aligned}$$

we get $D\mathrm{e}^x = \mathrm{e}^x Dx$. Hence $\mathrm{e}^x = Ex \in \operatorname{dom} \Omega$ and $L\mathrm{e}^x = LEx = x$.

(ii) Suppose that $\lambda u \in \mathcal{E}_D(X)$ for a $\lambda \in \mathcal{F}$ and $u \in \operatorname{dom} \Omega$. Then, by Proposition 6.2, $u^\lambda = E_\lambda = E(\lambda L u) = \mathrm{e}^{\lambda L u}$ and $Du^\lambda = D\mathrm{e}^{\lambda L u} = \mathrm{e}^{\lambda L u} D(\lambda L u) = u^\lambda D(\lambda L u)$. Hence $u^\lambda = \mathrm{e}^{\lambda L u} \in \operatorname{dom} \Omega$ and $Lu^\lambda = \lambda L u$.

(iii) If $u \in I(X) \cap \mathcal{E}'_D E(X)$ then we have $Du^\lambda = DE_\lambda u = \lambda(E_{\lambda-1} u) Du = \lambda u^{\lambda-1} Du$. ■

COROLLARY 6.2. *Suppose that all assumptions of Proposition* 6.2 *are satisfied. Then* $(L', E') \in G[\Omega]$, *where*

$$E'x = z\mathrm{e}^x, \ L'(z\mathrm{e}^x) = x \quad \text{for } x \in \mathcal{E}_D(X), \ z \in \ker D$$

(cf. Proposition 1.15).

Proof. Let $x \in \mathcal{E}_D(X)$ and $z \in \ker D$. By Proposition 6.2(i), $D(z\mathrm{e}^x) = (Dz)\mathrm{e}^x + zD\mathrm{e}^x = zD\mathrm{e}^x = z\mathrm{e}^x Dx$. Hence $x = L'E'x = L'(z\mathrm{e}^x)$. ■

PROPOSITION 6.5. *Suppose that X is a complete linear metric space, either* $\mathcal{F} = \mathbb{R}$ *or* $\mathcal{F} = \mathbb{C}$, *Condition* [**L**] *holds, X has a unit* $e \in \operatorname{dom} \Omega^{-1}$, $(L, E) \in G[\Omega], F \in ML(X)$ *is a continuous initial operator for D corresponding to an* $R \in \mathcal{R}_D$ *and* $g = Re$. *Then for all* $\lambda \in \mathcal{F}$,

(i) $\lambda g \in \mathcal{E}_D(X)$, $E(\lambda g) = \mathrm{e}^{\lambda g}$, $F\mathrm{e}^{\lambda g} = e$;
(ii) $D^n \mathrm{e}^{\lambda g} = \lambda^n \mathrm{e}^{\lambda g}$ *and* $FD^n \mathrm{e}^{\lambda g} = \lambda^n e$ *for* $n \in \mathbb{N}$.

Proof. (i) By (5.2), in Leibniz D-algebras with unit e the series

$$\sum_{n=0}^{\infty} \frac{\lambda^n g^n}{n!}$$

is convergent for every $\lambda \in \mathbb{R}$. Hence $\lambda g \in \mathcal{E}_D(X)$. By Proposition 6.2, $E(\lambda g) = \mathrm{e}^{\lambda g}$. Since for all $n \geq 1$ we have $Fg^n = (Fg)^n = (FRe)^n = 0$ and $Fe = e$, we find that

$$F\mathrm{e}^{\lambda g} = F \sum_{n=0}^{\infty} \frac{\lambda^n g^n}{n!} = F \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} F g^n = e.$$

(ii) is proved by an easy induction. ■

EXAMPLE 6.1. Let $X = C(\mathbb{R})$ with pointwise multiplication and with the topology induced by uniform convergence on compact intervals. Let $D = d/dt$ and $x \in X$ with $x(t) > 0$ for $t \in \mathbb{R}$. Then $x^\lambda(t) = \mathrm{e}^{\lambda \ln x(t)}$, $Dx^\lambda = \lambda x^{\lambda-1} Dx$.

DEFINITION 6.4. Suppose that X is a D-algebra with unit e and a complete linear metric space over $\mathbb{R}$ or $\mathbb{C}$. Write

$$\mathrm{Li}_p x = \sum_{n=1}^{\infty} \frac{x^n}{n^p} \quad \text{for } x \in X \ (p \in \mathbb{N}_0) \tag{6.7}$$

whenever this series is convergent. If $p = 0$ then $\mathrm{Li}_0 x$ is a *geometric series* and

$$\mathrm{Li}_0 x = (e - x)^{-1} - e = -x(e - x)^{-1}.$$

If $p \geq 1$ then Li_p is called a *polylogarithmic function*.

EXAMPLE 6.2. Let X and D be as in Example 6.1. Then

$$\mathrm{Li}_p(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^p} \quad \text{for } |t| < 1\ (p \in \mathbb{N}_0),$$
$$\mathrm{Li}_1(t) = -\ln(1-t) \quad (|t| < 1)$$

and

$$\mathrm{Li}_p(t) = \int_0^t \mathrm{Li}_{p-1}(s)s^{-1}\,ds \quad \text{for } |t| < 1\ (p \geq 2) \tag{6.8}$$

(cf. Zagier [1]).

DEFINITION 6.5. Suppose that $X \in \mathbf{Lg}(D)$ is a complete linear metric space over $\mathbb{R}$ or $\mathbb{C}$. Let $p \in \mathbb{N}_0$. Write

$$\mathcal{L}_D^{(p)}(X) = \left\{u \in \operatorname{dom}\Omega : \sum_{n=1}^{\infty} \frac{u^j}{n^p} \text{ are convergent } (j = 0, 1, \ldots, p)\right\}. \tag{6.9}$$

PROPOSITION 6.6. *Suppose that X is a complete linear metric space over $\mathbb{R}$ or $\mathbb{C}$, Condition* **[L]** *holds*, $(L, E) \in G[\Omega]$, *D is closed and $p \in \mathbb{N}$. If $u \in \mathcal{L}_D^{(p)}(X)$ then*

$$\mathrm{Li}_1 u = -L(e-u) \quad \textit{and} \quad u = e - E(-\mathrm{Li}_1 u) \in \operatorname{dom}\Omega. \tag{6.10}$$

If, in addition, $u \in I(X)$ then

$$D\mathrm{Li}_p u = (\mathrm{Li}_{p-1}u)u^{-1}Du = (\mathrm{Li}_{p-1}u)DLu \quad \textit{for } p \geq 2. \tag{6.11}$$

Proof. Since X is a Leibniz D-algebra, we have $e \in \ker D$. Let $u \in \mathcal{L}_D^{(p)}(X)$ and $x = \mathrm{Li}_1 u$. Then

$$\sum_{n=1}^{\infty} \frac{Du^n}{n} = \sum_{n=1}^{\infty} \frac{nu^{n-1}}{n} Du = u^{-1}\Big(\sum_{n=1}^{\infty} u^n\Big)Du = (e-u)^{-1}Du.$$

Since D is closed, we conclude that

$$Dx = \sum_{n=1}^{\infty} \frac{Du^n}{n} = (e-u)^{-1}Du = -(e-u)D(e-u) = -DL(u-e),$$

which implies (6.10). Since $\mathrm{Li}_1 u = x = -L(e-u)$, we find $e - u = E(-x)$ and $u = e - E(-x)$.

Similarly, for $p \geq 2$ we get

$$\begin{aligned} D\mathrm{Li}_p u &= \sum_{n=1}^{\infty} \frac{Du^n}{n^p} = \sum_{n=1}^{\infty} \frac{nu^{n-1}}{n^p} \\ &= \sum_{n=1}^{\infty} \frac{u^{n-1}}{n^{p-1}} Du = u^{-1}\Big(\sum_{n=1}^{\infty} \frac{u^n}{n^{p-1}}\Big) \\ &= (\mathrm{Li}_{p-1}u)u^{-1}Du = (\mathrm{Li}_{p-1}u)DLu. \ \blacksquare \end{aligned}$$

An immediate consequence of the last proposition is

COROLLARY 6.3. *Suppose that all assumptions of Proposition* 6.6 *are satisfied and* $g = Re \in I(X) \cap \mathcal{L}_D^{(p)}(X)$ *for an* $R \in \mathcal{R}_D$. *Let* $p \in \mathbb{N}$. *Then*

$$\mathrm{Li}_p g = \sum_{n=1}^{\infty} \frac{g^n}{n^p} \quad \text{and} \quad \mathrm{Li}_p g = R[(g^{-1}\mathrm{Li}_p g)]. \tag{6.12}$$

N o t e. Observe that both (6.11) and the second formula of (6.12) are generalizations of (6.8).

PROPOSITION 6.7. *Suppose that* X *is a complete linear metric space over* $\mathbb{R}$ *or* $\mathbb{C}$, *Condition* **[M]** *holds*, $(L, E) \in G[\Omega]$ *and* D *is closed. Then*

$$D\mathrm{Li}_p u = \mathrm{Li}_p(Du) \quad \text{for } u \in \mathcal{L}_D^{(p)}(X) \ (p \in \mathbb{N}_0). \tag{6.13}$$

P r o o f. Let $p \in \mathbb{N}_0$. Since, by our assumption, $D \in ML(X)$, for $u \in \mathcal{L}_D^{(p)}(X)$ we have

$$\sum_{n=1}^{\infty} \frac{D(u^n)}{n^p} = \sum_{n=1}^{\infty} \frac{(Du)^n}{n^p} = \mathrm{Li}_p(Du).$$

Since D is closed, we get (6.13). ∎

COROLLARY 6.4. *Suppose that all assumptions of Proposition* 6.7 *are satisfied. Then*

$$D\mathrm{Li}_p g = e\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{where } g = Re,\ R \in \mathcal{R}_D \ (p \in \mathbb{N}_0). \tag{6.14}$$

If $F \in ML(X)$ *is a continuous initial operator for* D *corresponding to* R *then*

$$\mathrm{Li}_p g = g\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{where } g = Re,\ R \in \mathcal{R}_D \ (p \in \mathbb{N}_0). \tag{6.15}$$

P r o o f. By definition, $Dg = e$. Proposition 6.7 now immediately implies (6.14). Suppose that $F \in \mathcal{F}_D \cap ML(X)$ corresponds to R. Then $Fg^n = (Fg)^n = (FRe)^n = 0$ for every $n \in \mathbb{N}$. Hence

$$F\mathrm{Li}_p g = F\sum_{n=1}^{\infty} \frac{g^n}{n^p} = \sum_{n=1}^{\infty} \frac{Fg^n}{n^p} = 0,$$

and (6.14) shows that

$$\mathrm{Li}_p g = R\Big[e\sum_{n=1}^{\infty} \frac{1}{n^p}\Big] + F\mathrm{Li}_p g = (Re)\sum_{n=1}^{\infty} \frac{1}{n^p} = g\sum_{n=1}^{\infty} \frac{1}{n^p}. \ ∎$$

The functions Li_p have been introduced and studied from the very beginning of Mathematical Analysis, since they have several interesting properties, and also some important applications to physics and engineering.

7. Complex case

In this section we make use of the following condition:

[**C**] $\mathcal{F} = \mathbb{C}$, $\operatorname{dom} \Omega^{-1}$ *is symmetric, i.e.* $-x \in \operatorname{dom} \Omega^{-1}$ *whenever* $x \in \operatorname{dom} \Omega^{-1}$, $X \in \mathbf{Lg}(D)$ *and* $c_D \neq 0$.

As a matter of fact, our previous considerations show that for a multiplicative D we have $c_D = 0$ and $\operatorname{dom} \Omega^{-1}$ is not symmetric (cf. Proposition 4.4). However, we have

OPEN QUESTION. *Does the symmetry of* $\operatorname{dom} \Omega^{-1}$ *imply* $c_D \neq 0$*?*

By Corollary 4.1, in the multiplicative case $ix \notin \operatorname{dom} \Omega^{-1}$ for $x \in \operatorname{dom} \Omega^{-1}$. This also justifies the assumption $c_D \neq 0$.

DEFINITION 7.1. Suppose that Condition [**C**] holds and $(L, E) \in G[\Omega]$. For $ix \in \operatorname{dom} \Omega^{-1}$ we write

$$Cx = \frac{1}{2}[E(ix) + E(-ix)], \quad Sx = \frac{1}{2i}[E(ix) - E(-ix)]. \tag{7.1}$$

The mappings C and S are said to be *cosine* and *sine mappings*, respectively, or *trigonometric mappings*. The elements Cx and Sx are said to be *cosine* and *sine elements*, or *trigonometric elements*.

PROPOSITION 7.1. *Suppose that Condition* [**C**] *holds. Let* $(L, E) \in G[\Omega]$. *Then the trigonometric mappings* C *and* S *are well-defined for all* $ix \in \operatorname{dom} \Omega^{-1}$ *and*

(i) *the* De Moivre formula *holds*:

$$E(ix) = Cx + iSx \quad \text{for } ix \in \operatorname{dom} \Omega^{-1}; \tag{7.2}$$

(ii) C *and* S *are even and odd functions*, *respectively*, *i.e.*

$$C(-x) = Cx, \quad S(-x) = -Sx \quad \text{for } ix \in \operatorname{dom} \Omega^{-1}; \tag{7.3}$$

(iii) $C(0) = z \in \ker D \setminus \{0\}$ *and* $S(0) = 0$;

(iv) *for all* $ix \in \operatorname{dom} \Omega^{-1}$ *we have*

$$(Cx)^2 + (Sx)^2 = E(ix)E(-ix), \tag{7.4}$$

$$DCx = -(Sx)Dx, \quad DSx = (Cx)Dx, \tag{7.5}$$

$$(DCx)^2 + (DSx)^2 = [(Cx)^2 + (Sx)^2](Dx)^2 \tag{7.6}$$

and

$$D^2Cx = -c_D[(Cx)(Dx)^2 + (Sx)D^2x] + f_D(Sx, Dx), \tag{7.7}$$

$$D^2Sx = -c_D[(Sx)(Dx)^2 - (Cx)D^2x] + f_D(Cx, Dx). \tag{7.8}$$

P r o o f. Let $ix \in \operatorname{dom} \Omega^{-1}$. By Condition [**C**], $-ix \in \operatorname{dom} \Omega^{-1}$. Thus the mappings C and S are well-defined.

(i) By definition,

$$Cx + iSx = \frac{1}{2}[E(ix) + E(-ix)] + i\frac{1}{2i}[E(ix) - E(-ix)] = E(ix).$$

(ii) immediately follows from Definition 7.1. Note that

$$Cx - iSx = C(-x) + iS(-x) = E(-ix) \quad \text{for } ix \in \operatorname{dom} \Omega^{-1}.$$

(iii) By Proposition 1.5, $E(0) = z \in \ker D$ and $z \neq 0$. Then, by definitions, $C(0) = \frac{1}{2}(z + z) = z$ and $S(0) = \frac{1}{2i}(z - z) = 0$.

(iv) Again by definitions, we have

$$\begin{aligned}
(Cx)^2 + (Sx)^2 &= \frac{1}{4}[E(ix) + E(-ix)]^2 + \frac{1}{4i^2}[E(ix) - E(-ix)]^2 \\
&= \frac{1}{4}[2E(ix)E(-ix) - (-2)E(ix)E(-ix)] = E(ix)E(-ix),
\end{aligned}$$

$$\begin{aligned}
DCx &= \frac{1}{2}[DE(ix) + DE(-ix)] = \frac{1}{2}[E(ix)D(ix) + E(-ix)D(-ix)] \\
&= \frac{1}{2}i[E(ix) - E(-ix)]Dx = -i^2\frac{1}{2i}[E(ix) - E(-ix)]Dx = -(Sx)Dx,
\end{aligned}$$

$$\begin{aligned}
DSx &= \frac{1}{2i}[DE(ix) - DE(-ix)] = \frac{1}{2i}[E(ix)D(ix) - E(-ix)D(-ix)] \\
&= i\frac{1}{2i}[E(ix) + E(-ix)]Dx = \frac{1}{2}[E(ix) + E(-ix)]Dx = (Cx)Dx,
\end{aligned}$$

$$(DCx)^2 + (DSx)^2 = [(-Sx)Dx]^2 + [(Cx)Dx]^2 = [(Cx)^2 + (Sx)^2](Dx)^2,$$

$$\begin{aligned}
D^2Cx &= D[-(Sx)Dx] = -c_D[(DSx)Dx + (Sx)D^2x] - f_D(Sx, Dx) \\
&= -c_D[(Cx)(Dx)^2 + (Sx)D^2x] - f_D(Sx, Dx),
\end{aligned}$$

$$\begin{aligned}
D^2Sx &= D[(Cx)Dx] = c_D[-(Sx)(Dx)^2 + (Cx)D^2x] + f_D(Cx, Dx) \\
&= -c_D[(Sx)(Dx)^2 - (Cx)D^2x] + f_D(Cx, Dx). \ \blacksquare
\end{aligned}$$

COROLLARY 7.1. *Suppose that all assumptions of Proposition* 7.1 *are satisfied. Then the mappings* C' *and* S' *defined by*

$$C'x = C(x + z), \quad S'x = S(x + z) \quad \textit{for } ix \in \operatorname{dom} \Omega^{-1},\ z \in \ker D$$

also satisfy assertions (i)–(iv) *of that proposition.*

P r o o f. Let $x' = x + z$. Then $ix' \in \operatorname{dom} \Omega^{-1}$ and $Dx' = Dx$, $D^2x' = D^2x$. Now, Proposition 1.12 implies the conclusion. ■

COROLLARY 7.2. *Suppose that Condition* **[C]** *holds*, $(L, E) \in G[\Omega]$ *and* X *is a Leibniz* D*-algebra* (*i.e. Condition* **[L]** *holds*). *Then the* Trigonometric Identity *holds*, *i.e.*

(7.9) $$(Cx)^2 + (Sx)^2 = e \quad \textit{whenever } ix \in \operatorname{dom} \Omega^{-1}.$$

P r o o f. By our assumptions and Proposition 5.1(i), $E(ix)E(-ix) = e$. ■

PROPOSITION 7.2. *Suppose that Condition* **[C]** *holds*, $(L, E) \in G[\Omega]$ *and the trigonometric identity* (7.9) *holds. Then*

(i) $E(ix)$, $E(-ix) \in I(X)$ *and* $E(-ix) = [E(ix)]^{-1}$ *for all* $ix \in \operatorname{dom} \Omega^{-1}$;

(ii) $E(0) = e$, *hence* $Le = 0$;

(iii) $e \in \ker D$;

(iv) *if* X *is an* almost Leibniz D-*algebra*, *i.e. if* $f_D(x,z) = 0$ *for* $x \in \operatorname{dom} D$ *and* $z \in \ker D$, *then* $c_D = 1$;

(v) *if* $c_D = 1$ *then* $f_D(u,e) = 0$ *for all* $u \in I(X) \cap \operatorname{dom} \Omega$, *i.e.* $g_D = 0$.

Proof. (i) Suppose that the trigonometric identity (7.9) holds. By (7.5), for all $ix \in \operatorname{dom} \Omega^{-1}$,

$$E(ix)E(-ix) = (Cx)^2 + (Sx)^2 = e.$$

Hence $E(ix)$, $E(-ix) \in I(X)$ and $E(-ix) = [E(ix)]^{-1}$.

(ii) Put $x = 0$ in (7.9). By Proposition 1.5, $E(0) = z \in \ker D$ and $z \neq 0$. We find that

$$z^2 = E(0)E(0) = [C(0)]^2 + [S(0)]^2 = e.$$

Hence $E(0) = z = e$. Indeed, suppose that $z = E(0) = -e$. Then $L(-e) = LE(0) = 0$ and

$$Le = L[(-e)(-e)] = 2c_D L(-e) + R[(-e)^{-2} f_D(-e,-e)] = Rf_D(e,e).$$

Thus

$$Le = Le^2 = 2c_D Le + Rf_D(e,e) = 2c_D Le + Le,$$

which implies $2c_D Le = 0$. Since, by our assumption, $c_D \neq 0$, we get $Le = 0$, which implies $e = ELe = E(0) = -e$, a contradiction. Thus $z = e$ and $Le = LE(0) = 0$. This also proves (iii), since $e = z \in \ker D$.

(iv) Suppose that X is an almost Leibniz algebra. We have already proved that $e \in \ker D$. This means that $g_D(u) = u^{-1} f_D(u,e) = 0$ for all $u \in I(X) \cap \operatorname{dom} \Omega$, i.e. the mapping g_D is constant. Since, by our assumption, $X \in \mathbf{Lg}(D)$, Theorem 1.4 implies that $c_D = 1$.

(v) Suppose that $c_D = 1$. Since $Le = 0$, for all $u \in I(X) \cap \operatorname{dom} \Omega$ we have

$$\begin{aligned} f_D(u,e) &= uDRu^{-1} f_D(u,e) = uDRg_D(u) = uD(1-c_D)Lu \\ &= (1-c_D)uDLu = (1-c_D)Du. \end{aligned}$$

This implies that $g_D = 0$. Note that, by Theorem 1.4, the mapping g_D is constant, but not necessarily zero. ■

Open Question. *Do there exist non-Leibniz D-algebras with the trigonometric identity* (7.9)*?*

Corollary 7.3. *Suppose that Condition* **[C]** *holds*, $(L,E) \in G[\Omega]$, $R \in \mathcal{R}_D$, $g = Re$ *and* $\lambda \in \mathbb{C}$. *If* $\lambda i g \in \operatorname{dom} \Omega^{-1}$ *then*

$$\begin{aligned} (D^2 + c_D\lambda^2 I)C(\lambda g) &= \lambda[S(\lambda g)De + f_D(S(\lambda g),e)], \\ (D^2 + c_D\lambda^2 I)S(\lambda g) &= -\lambda[C(\lambda g)De - f_D(C(\lambda g),e)]. \end{aligned} \tag{7.10}$$

P r o o f. Let $x=\lambda g$. By definition, $Dx=\lambda Dg = \lambda DRe = \lambda e$ and $D^2x = \lambda De$. Now, (7.5) and (7.6) imply that

$$D^2C(\lambda g) = -c_D[\lambda^2 C(\lambda g) + \lambda S(\lambda g)De] + f_D(S(\lambda g), e),$$
$$D^2S(\lambda g) = -c_D[\lambda^2 S(\lambda g) - \lambda C(\lambda g)De] + f_D(C(\lambda g), e). \ \blacksquare$$

Corollary 7.4. *Suppose that all assumptions of Corollary* 7.3 *are satisfied and X is a Leibniz D-algebra. If $\lambda ig \in \operatorname{dom} \Omega^{-1}$ then*

$$C(\lambda g), S(\lambda g) \in \ker(D^2 + \lambda^2 I).$$

P r o o f. Here $c_D = 1$ and $f_D = 0$. Corollary 7.3 yields the conclusion. ■

Proposition 7.3. *Suppose that Condition* **[C]** *holds*, $(L, E) \in G[\Omega]$ *and* $ix \in \operatorname{dom} \Omega^{-1}$. *Then*

$$D^2E(ix) = -c_D E(ix)[(Dx)^2 - iD^2x] + if_D(E(ix), Dx). \tag{7.11}$$

P r o o f. Let $ix \in \operatorname{dom}\Omega^{-1}$. Then, by definition,

$$\begin{aligned} DE(ix) &= E(ix)D(ix) = iE(ix)Dx, \\ D^2E(ix) &= D[iE(ix)Dx] = c_D i\{[DE(ix)]Dx + E(ix)D^2x\} + if_D(E(ix), Dx) \\ &= c_D i[iE(ix)(Dx)^2 + E(ix)D^2x] + if_D(E(ix), Dx) \\ &= -c_D E(ix)[(Dx)^2 - iD^2x] + if_D(E(ix), Dx). \ \blacksquare \end{aligned}$$

Corollary 7.5. *Suppose that all assumptions of Corollary* 7.3 *are satisfied and X is a Leibniz D-algebra. If $\lambda ig \in \operatorname{dom} \Omega^{-1}$ then*

$$E(\lambda ig), E(-\lambda ig) \in \ker(D^2 + \lambda^2 I).$$

P r o o f. By our assumptions, $c_D = 1$, $f_D = 0$ and $x = \lambda ig \in \operatorname{dom} \Omega^{-1}$. Since $Dx = \lambda Dg = \lambda DRe = \lambda e$ and $D^2x = \lambda De = 0$, (7.11) implies that $E(\lambda ig) \in \ker(D^2 + \lambda^2 I)$. The same arguments are used for $-x$. ■

Proposition 7.4. *Suppose that Condition* **[C]** *holds*, *X is a Leibniz D-algebra*, $\lambda \in \mathbb{C}$, $R \in \mathcal{R}_D$, $g = Re$ *and* $\lambda ig \in \operatorname{dom}\Omega^{-1}$. *Then* $u \in \ker(D^2 + \lambda^2 I)$ *if and only if there is an* $(L_2, E_2) \in G[\Omega_2]$ *such that*

$$u = zE_2\left(\frac{\lambda^2 g^2}{2}\right), \quad \text{where } z = z'g + z'',\ z', z'' \in \ker D \text{ are arbitrary.}$$

P r o o f. *Necessity*. Suppose that $u \in \ker(D^2 + \lambda^2 I)$. Then $D^2u = -\lambda^2 u$. We are looking for x such that $D^2x = -\lambda^2 e$. Clearly,

$$x = -\lambda^2 R^2 e + Rz_1 + z_0, \quad \text{where } z_0, z_1 \text{ are arbitrary.}$$

Since X is a Leibniz D-algebra and $g = Re$, we find that $R^2e = (Re)^2/2 = g^2/2$ and

$$E_2(\lambda^2 R^2 e + Rz_1 + z_0) = E_2(Rz_1 + z_0)E_2(\lambda^2 g^2/2).$$

Since $Rz_1 + z_0 \in \ker D^2$, we conclude that there is a $z \in \ker D^2$ such that

$$E_2(Rz_1 + z_0) = z = Rz'' + z' = z''g + z', \quad \text{where } z', z'' \in \ker D.$$

Sufficiency is proved by checking. ∎

COROLLARY 7.6. *Suppose that Condition* **[C]** *holds*, X *is a Leibniz* D*-algebra*, $\lambda \in \mathbb{C}$, $R \in \mathcal{R}_D$, $g = Re$ *and* $\lambda ig \in \operatorname{dom} \Omega^{-1}$. *If* $(L, E) \in G[\Omega]$ (*we write* $L_1 = L$, $E_1 = E$, $\Omega_1 = \Omega$) *and* $(L_2, E_2) \in G[\Omega_2]$ *then*

$$\begin{aligned} \ker(D^2 + \lambda^2 I) &= \{z_1 E(\lambda ig) + z_2 E(-\lambda ig) : z_1, z_2 \in \ker D\} \\ &= \{zC(\lambda g) + \widetilde{z} S(\lambda g) : z, \widetilde{z} \in \ker D\} \\ &= \{(z'' g + z') E_2(\lambda^2 g^2/2) : z', z'' \in \ker D\}. \end{aligned} \tag{7.12}$$

P r o o f. Corollaries 7.1 and 7.5 and arguments similar to those used in the proof of Proposition 7.4 imply the first and second equality in (7.12). The third one follows from Proposition 7.4 (cf. also Proposition 5.5). ∎

COROLLARY 7.7. *Suppose that all assumptions of Corollary* 7.6 *are satisfied. If* $-\lambda^2 g^2/2 \in \operatorname{dom} \Omega_2^{-1}$ *then all solutions of the equation*

$$(D^2 + \lambda^2 I)x = y, \quad y \in X, \tag{7.13}$$

are of the form

$$x = x_1 + x_0, \tag{7.14}$$

where

$$x_1 = E_2(-\lambda^2 g^2/2) R^2 [E_2(\lambda^2 g^2/2) y], \quad x_0 \in \ker(D^2 + \lambda^2 I),$$

and x_0 *is determined by Corollary* 7.6.

P r o o f. Put $m = 2$ and $a = -\lambda^2 e$ in Corollary 5.6. Since X is a Leibniz D-algebra, we get $R^2 a = -\lambda^2 g^2/2$. ∎

N o t e. The assumption $\lambda i, -\lambda i \in v_{\mathbb{C}} R$ ensures that $\lambda ig, -\lambda ig \in \operatorname{dom} \Omega^{-1}$ (cf. Propositions 1.10 and 5.5). In this case, $-\lambda^2 \in v_{\mathbb{C}} R^2$.

EXAMPLE 7.1. We shall show connections of the trigonometric mappings and elements with the trigonometric operators and elements induced by a right inverse of a $D \in R(X)$ (cf. Przeworska-Rolewicz [1], Sections 2.3 and 6.2). Suppose that Condition **[C]** holds, $R \in \mathcal{R}_D$ and $\lambda \in v_{\mathbb{C}} R$. Let $g = Re$. By Proposition 1.10, there are $(L, E) \in G[\Omega]$ such that

$$E(\lambda g) = e_\lambda z \in \ker(D - \lambda I), \quad \text{where } z \in \ker D \text{ is arbitrary.}$$

Recall that elements of this form are called *exponentials* for D since they are eigenvectors for D corresponding to the eigenvalue λ.

Suppose now that $\lambda i, -\lambda i \in v_{\mathbb{C}} R$. The operators

$$c_\lambda = I + \lambda^2 R^2, \quad s_\lambda = \lambda R(I + \lambda^2 R^2)$$

are called *cosine* and *sine operators*, respectively. Let $z \in \ker D$. Then $c_\lambda z$ and $s_\lambda z$ are called *cosine* and *sine elements*, respectively. It is not difficult to verify that

$$c_\lambda = \frac{1}{2}(e_{\lambda i} + e_{-\lambda i}), \quad s_\lambda = \frac{1}{2i}(e_{\lambda i} - e_{-\lambda i}).$$

Thus

$$c_\lambda z = C(\lambda g), \quad s_\lambda z = S(\lambda g).$$

We therefore conclude that c_λ and s_λ have all the properties listed in Proposition 7.1. Clearly, the proofs in the author's book [1] are different, since they follow just from definitions. Also the assumption made there was much stronger. Namely, we assumed that R is a *Volterra operator*, i.e. $I - \lambda R$ is invertible for all scalars λ.

DEFINITION 7.2. Suppose that Condition **[C]** is satisfied and the trigonometric identity holds. Let $w = u + iv \in \operatorname{dom} \Omega$. Write $w^* = u - iv$ and

$$\mathbb{C}(X) = \{w = u + iv \in \operatorname{dom} \Omega : ww^* \in I_2(\operatorname{dom} \Omega)\}.$$

By definition, $w \in I(X)$. Clearly, $(w^*)^* = w$ and $(ie)^* = -ie$. We have $ww^* = (u + iv)(u - iv) = u^2 + v^2$. Write $\lfloor w \rfloor = (ww^*)^{1/2}$. Let

$$\mathbb{R}(X) = \{w \in \mathbb{C}(X) : w^* = w\}.$$

It is easy to verify that $\lfloor w \rfloor \in \mathbb{R}(X)$ and

$$u = \frac{w + w^*}{2} \in \mathbb{R}(X), \quad v = \frac{w - w^*}{2i} \in \mathbb{R}(X).$$

Note. We could define $0^* = 0$ and $\lfloor 0 \rfloor = 0$, which would require some little modification in the definitions of $\mathbb{C}(X)$ and $\mathbb{R}(X)$. However, since $0 \notin \operatorname{dom} \Omega$, this modification is useless.

PROPOSITION 7.5. *Suppose that all assumptions of Definition* 7.2 *are satisfied and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *If* $w \in \mathbb{C}(X)$ *then there is an* x *such that* $ix \in \operatorname{dom} \Omega$ *and*

$$Lw = c_D(L\lfloor w \rfloor + ix) + R[\lfloor w \rfloor^{-1} E(-ix) f_D(\lfloor w \rfloor, E(ix))], \tag{7.15}$$

namely,

$$x = -iL[u' + (e - (u')^2)^{1/2}], \quad \textit{where} \quad u' = \tfrac{1}{2}(w + w^*)\lfloor w \rfloor^{-1}. \tag{7.16}$$

In particular, *if* X *is a Leibniz* D-*algebra then*

$$Lw = L\lfloor w \rfloor + ix. \tag{7.17}$$

Proof. Let $u' = u\lfloor w \rfloor^{-1}$ and $v' = v\lfloor w \rfloor^{-1}$. Clearly, $(u')^2 + (v')^2 = e$ and $w' = u' + iv' \in \operatorname{dom} \Omega$. Let $x = -iL(u' + iv')$, i.e. x is defined as in (7.16). By definition,

$$w' = u' + iv' = E[i(-i)L(u' + v')] = E(ix) = Cx + iSx.$$

By Proposition 7.2, $[E(ix)]^{-1} = E(-ix)$. We therefore conclude that

$$\begin{aligned} Lw = L[\lfloor w \rfloor E(ix)] &= c_D[L\lfloor w \rfloor + LE(ix)] + R[\lfloor w \rfloor^{-1} E(ix)^{-1}] \\ &= c_D(L\lfloor w \rfloor + ix) + R[\lfloor w \rfloor E(-ix)]. \end{aligned}$$

Putting $c_D = 1$ and $f_D = 0$, we get (7.17) with x defined by (7.16). ∎

EXAMPLE 7.2. Suppose that all assumptions of Definition 7.2 are satisfied. Let X be a D-algebra over $\mathbb{R}$ with unit e. Write

$$Y = X + iX = \{a + ib : a, b \in X\}.$$

Clearly, Y is a D-algebra over $\mathbb{C}$ with unit e with addition, multiplication by scalars and multiplication of elements defined as follows: for $a, b, c, d \in X$ and $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}(a + ib) + (c + id) &= (a + c) + i(b + d);\\ (\lambda + i\mu)(a + ib) &= (\lambda a - \mu b) + i(\lambda b + \mu a);\\ (a + ib)(c + id) &= (ac - bd) + i(ad + bc).\end{aligned}$$

The embedding $a \to a+i0$ permits us to consider X as a subspace of Y. For every $A \in L(X)$ we write $A(a + ib) = Aa + iAb$ for $a + ib \in Y$. It is easy to verify that $\mathbb{R}(Y) = \mathbb{R}(X) = I_2(\operatorname{dom} \Omega) \cap X$. The space Y is said to be the *complexification* of the space X (cf. Example 2.3).

DEFINITION 7.3. Suppose that all assumptions of Definition 7.2 are satisfied. The element x defined by (7.16) is called an *argument* of w and we write $x = \arg w$. The element $\lfloor w \rfloor$ is said to be the *$*$-modulus* of w.

PROPOSITION 7.6. *Suppose that all assumptions of Definition* 7.2 *are satisfied*, *X is a complete linear metric space* (*over* $\mathbb{C}$), $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$, $w \in \mathbb{C}(X)$ *and* $x = \arg w$. *If* $x \in \mathcal{E}_D(X)$ *then also*

$$x' = x + 2\pi ike = \arg w \quad \textit{for } k \in \mathbb{Z}.$$

Proof. For all $\lambda \in \mathbb{C}$ we have

$$e^{\lambda e} = \sum_{n=0}^{\infty} \frac{(\lambda e)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right] e = e^{\lambda} e.$$

Now, Proposition 6.4(i) implies that $E(\lambda e) = e^{\lambda} e$. Let $\omega = 2\pi i$ and let $k \in \mathbf{Z}$. Suppose that $x = \arg w \in \mathcal{E}_D(X)$. Again, by Proposition 6.4(i),

$$E(ix') = E(ix + 2\pi ik) = e^{x+\omega e} = e^{\omega e} e^{x} = e^{\omega} e e^{x} = e^{2\pi ik} e^{x} = e^{x} = E(ix). \ \blacksquare$$

An immediate consequence of this proposition is

COROLLARY 7.8. *Suppose that all assumptions of Definition* 7.2 *are satisfied*, *X is a complete linear metric space* (*over* $\mathbb{C}$) *and* $(L, E) \in G_{R,1}[\Omega]$ *for an* $R \in \mathcal{R}_D$. *Then for all* $w \in \mathbb{C}(X)$ *and* $x = \arg w \in \mathcal{E}_D(X)$ *the mapping Ei and the trigonometric mappings are* $2\pi e$-*periodic*, *i.e.*

$$E[i(x + 2\pi e)] = E(ix), \quad C(x + 2\pi e) = Cx, \quad S(x + 2\pi e) = Sx.$$

Note. The periodicity of the elements $e_\lambda, c_\lambda, s_\lambda$ which appear in Example 7.1 has been proved in another way, by means of the properties of shifts induced by a right invertible operator and its right inverse (cf. also Binderman [1], [2] for generalizations of the mentioned shifts).

8. Smooth logarithms and antilogarithms

Let X be a linear space over a field $\mathcal{F}$ of scalars. Let $D \in R(X)$ and let F be an initial operator for D corresponding to a right inverse R of D. Recall that the space of all *smooth elements* induced by D is

$$D_\infty = \bigcap_{k \in \mathbb{N}_0} D_k,$$

where $D_k = \operatorname{dom} D^k$ for $k \in \mathbb{N}$, $D_0 = X$, and the space of *singular elements* is

$$Q_R(D) = \{x \in D_\infty : \forall_{n \in \mathbb{N}_0} FD^n x = 0\} = \{x \in D_\infty : \forall_{n \in \mathbb{N}} R^n D^n x = x\}.$$

Let now X be a complete linear metric space over $\mathbb{R}$. We shall consider the following subspaces of D_∞:

• the space of D-*analytic elements*:

$$A_R(D) = \Big\{x \in D_\infty : x = \sum_{n=0}^{\infty} R^n F D^n x\Big\} = \{x \in D_\infty : \lim_{n \to \infty} R^n D^n x = 0\};$$

• the space of D-*paraanalytic elements*:

$$PA_R(D) = \{x \in D_\infty : \lim_{n \to \infty} R^n F D^n x = 0\} \supset A_R(D) \oplus Q_R(D)$$

(cf. Przeworska-Rolewicz [2]).

Theorem 8.1. *Suppose that $X \in \mathbf{Lg}(D)$ is a complete linear metric space over $\mathbb{R}$, $(L, E) \in G[\Omega]$, there is an $R \in \mathcal{R}_D$ such that the series $\sum_{n=0}^{\infty} \lambda^n R^n z$ is convergent for all $z \in \ker D$ and $\lambda \in v_{\mathcal{F}} R$, and $g = Re$. If $\lambda g \in \operatorname{dom} \Omega$ for $\lambda \in v_{\mathcal{F}} R$ then*

(i) *$E(\lambda g) = (I - \lambda R)^{-1} z = \sum_{n=0}^{\infty} \lambda^n R^n z$, where $z = FE(\lambda g)$ and F is an initial operator for D corresponding to R;*

(ii) *$E(\lambda g) \in A_R(D)$.*

Proof. Let $\lambda \in v_{\mathcal{F}} R$. Observe that $E(\lambda g) \in D_\infty$. Indeed, by definition,

$$DE(\lambda g) = E(\lambda g) D(\lambda g) = \lambda E(\lambda g) DRe = \lambda E(\lambda g).$$

By an easy induction we find that $D^n E(\lambda g) = \lambda^n E(\lambda g)$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}(I - \lambda R) E(\lambda g) &= E(\lambda g) - \lambda R E(\lambda g) \\ &= (I - RD) E(\lambda g) = FE(\lambda g) = z \in \ker(D - \lambda I).\end{aligned}$$

Since $\lambda \in v_{\mathcal{F}} R$, we conclude that $E(\lambda g) = (I - \lambda R)^{-1} z$. This implies that

$$\begin{aligned}\sum_{n=0}^{\infty} R^n F D^n E(\lambda g) &= \sum_{n=0}^{\infty} R^n F \lambda^n E(\lambda g) = \Big(\sum_{n=0}^{\infty} \lambda^n R^n\Big) FE(\lambda g) \\ &= (I - \lambda R)^{-1} z = E(\lambda g).\end{aligned}$$

Hence $E(\lambda g) \in A_R(D)$. ∎

N o t e. According to the author's paper [5], if $R \in \mathcal{R}_D$ and

$$\lim_{n\to\infty} \lambda^n R^n z = 0 \quad \text{for } z \in \ker D,\ \lambda \in v_{\mathcal{F}} R,$$

then R is said to be *almost quasinilpotent on* $\ker D$. Clearly, a right inverse R satisfying the assumptions of Theorem 8.1 is almost quasinilpotent on $\ker D$, since the convergence of the series $\sum_{n=0}^{\infty} \lambda^n R^n z$ for all $z \in \ker D$ and $\lambda \in v_{\mathcal{F}} R$ implies the convergence of its nth term to zero.

THEOREM 8.2. *Suppose that X is a complete linear metric space over $\mathbb{R}$, Condition* [**L**] *holds and $(L, E) \in G_{R,1}[\Omega]$ for a continuous $R \in \mathcal{R}_D$. If $u \in I(X) \cap A_R(D)$ and $u^{-1} \in A_R(D)$ then $Lu \in A_R(D)$.*

P r o o f. If $u, v \in A_R(D)$ then $Du, Ru, uv \in A_R(D)$ (cf. Przeworska-Rolewicz [2]). Let $u \in I(X) \cap A_R(D)$ and $u^{-1} \in A_R(D)$. Then $DLu = u^{-1}Du \in A_R(D)$. Let F be an initial operator for D corresponding to R. By our assumption, $FLu = 0$. This implies

$$Lu = (I - F)Lu = RDLu = R(u^{-1}Du) \in A_R(D). \ ■$$

PROPOSITION 8.1. *Suppose that X is a complete linear metric space over $\mathbb{R}$, Condition* [**L**] *holds, $(L, G) \in G[\Omega]$ and $F \in ML(X)$ is a continuous initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $g = Re$. Then $\mathrm{e}^{\lambda g} \in A_R(D)$ for all $\lambda \in \mathbb{R}$.*

P r o o f. (i) Let $\lambda \in \mathbb{R}$. By Proposition 6.5(i),(ii), $\lambda g \in \mathcal{E}_D(X)$ and for all $n \in \mathbb{N}$ we have

$$R^n F D^n \mathrm{e}^{\lambda g} = R^n(\lambda^n e) = \frac{\lambda^n g^n}{n!}.$$

Hence the series $\sum_{n=0}^{\infty} \lambda^n g^n/n!$ is convergent. Thus $\mathrm{e}^{\lambda g} \in A_R(D)$. ■

PROPOSITION 8.2. *Suppose that X is a complete linear metric space over $\mathbb{R}$, Condition* [**L**] *holds, $(L, G) \in G[\Omega]$, D is closed, $F \in ML(X)$ is a continuous initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $g = Re$. Then*

(i) *$D\mathrm{e}^x = \mathrm{e}^x Dx$ for $x \in \mathcal{E}_D(X)$;*

(ii) *$\mathrm{e}^x - e \in Q_R(D)$ for $x \in Q_R(D) \cap \mathcal{E}_D(X)$;*

(iii) *$\mathrm{e}^{\lambda g + x} \in A_R(D) \oplus Q_R(D)$ for $x \in Q_R(D) \cap \mathcal{E}_D(X)$ and $\lambda \in \mathbb{R}$.*

P r o o f. (i) Let $x \in \mathcal{E}_D(X)$. Since $X \in \mathcal{A}_D(0, e, 0)$, we have $Dx^n = nx^{n-1}Dx$ for all $n \in \mathbb{N}$. Thus

$$\sum_{n=1}^{\infty} \frac{Dx^n}{n!} = \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!}Dx = \sum_{k=0}^{\infty} \frac{x^k}{k!}Dx = \mathrm{e}^x Dx.$$

Since D is closed, we conclude that $D\mathrm{e}^x = \mathrm{e}^x Dx$.

(ii) Let $x \in Q_R(D) \cap \mathcal{E}_D(X)$. Then the series $\sum_{n=0}^{\infty} x^n/n!$ is convergent and $FD^n x = 0$ for all $n \in \mathbb{N}_0$. Since $F \in ML(X)$ and $Fx = 0$, we have $Fx^n =$

$(Fx)^n = 0$ for $n \geq 1$, which implies that

$$Fe^x = F\sum_{n=0}^{\infty}\frac{x^n}{n!} = Fe + \sum_{n=1}^{\infty}\frac{1}{n!}Fx^n = e. \tag{8.1}$$

We shall prove by induction that $FD^n e^x = 0$ for $n \in \mathbb{N}$. Indeed, by (i) and (8.1) we get $FDe^x = F(e^x Dx) = (Fe^x)(FDx) = eFDx = 0$. Suppose that $FD^n e^x = 0$ for some $n \geq 1$. Then, by the Leibniz formula,

$$FD^{n+1}e^x = FD^n(De^x) = FD^n(e^x Dx) = F\sum_{k=0}^{n}\binom{n}{k}(D^{n-k}e^x)D^{k+1}x$$
$$= \sum_{k=0}^{n}[F(D^{n-k}e^x)]FD^{k+1}x = 0.$$

Hence $y = e^x - e \in Q_R(D)$, because $FD^n y = 0$ for $n \in \mathbb{N}_0$.

(iii) Let $\lambda \in \mathbb{R}$. By Proposition 8.1, $e^{\lambda g} \in A_R(D)$. Let $x \in \mathcal{E}_D(X)$. By (ii), $e^x - e \in Q_R(D)$. Hence

$$e^{\lambda g + x} = e^{\lambda g}e^x = e^{\lambda g}(e^x - e) + e^{\lambda g} \in A_R(D) \oplus Q_R(D),$$

since the product of an element belonging to D_∞ by an element of $Q(R_D)$ again belongs to $Q_R(D)$ (cf. Przeworska-Rolewicz [2], Theorem 7.2(i)). ∎

PROPOSITION 8.3. *Suppose that all assumptions of Proposition* 8.1 *are satisfied*, $F_1 \neq F$ *is an initial operator for* D *corresponding to an* $R_1 \in \mathcal{R}_D$, $F_1 \in ML(X)$, $g = Re \in I(X)$ *and* $(L, E) \in G[\Omega]$. *Then*

(i) *if*

$$\lim_{n\to\infty}\frac{1}{n}(FR_1e)^{-n}(R_1e)^n = 0 \tag{8.2}$$

then $Lg \in PA_{R_1}(D)$;

(ii) *if* R_1 *is continuous and* (8.2) *holds then* $g^{-1} \in PA_{R_1}(D)$;

(iii) *if the series*

$$\sum_{n=0}^{\infty}\frac{1}{n}(FR_1e)^{-n}(R_1e)^n \tag{8.3}$$

is convergent then $Lg \in A_{R_1}(D)$;

(iv) *if* R_1 *is continuous and* (8.3) *holds then* $g^{-n} \in A_{R_1}(D)$ *for all* $n \in \mathbb{N}$.

Proof. By our assumptions, $R_1 \neq R$. Since $F_1 \in ML(X)$, we obtain for R_1 the same property as for R:

$$R_1 z = z\frac{(R_1e)^n}{n!} \quad \text{for } z \in \ker D,\ n \in \mathbb{N}$$

(cf. (6.6)). Since $g \in I(X)$, we conclude that $F_1 g \in I(X)$. Indeed, by our assumptions, $e = F_1 e = F_1(gg^{-1}) = (F_1g)(F_1g^{-1})$. By definition, $F_1R = -FR_1$(cf. Przeworska-Rolewicz [1], Section 2.2). Hence $FR_1e = -F_1Re = -F_1g \in I(X)$.

(i) By Proposition 5.3(ii), for all $n \in \mathbb{N}_0$ we have

$$\begin{aligned} R_1^n F_1 D^n Lg &= (F_1 D^n Lg)\frac{(R_1 e)^n}{n!} = F_1[(-1)^{n-1}(n-1)!g^{-n}]\frac{(R_1 e)^n}{n!} \\ &= (-1)^{n-1}(n-1)!(F_1 g^{-n})\frac{(R_1 e)^n}{n!} = \frac{(-1)^{n-1}}{n}(F_1 g)^{-n}(R_1 e)^n \\ &= \frac{(-1)^{n-1}}{n}(F_1 Re)^{-n}(R_1 e)^n = \frac{(-1)^{n-1}}{n}(-FR_1 e)^{-n}(R_1 e)^n \\ &= \frac{(-1)^{n-1-n}}{n}(FR_1 e)^{-n}(R_1 e)^n = -\frac{1}{n}(FR_1 e)^{-n}(R_1 e)^n. \end{aligned}$$

Now, (8.2) shows that $Lg \in PA_{R_1}(D)$.

(ii) By (i), since R_1 is continuous and (8.2) holds, we get

$$\lim_{n\to\infty} R_1^n F_1 D^n g^{-1} = DR_1 \lim_{n\to\infty} D^n Lg = D \lim_{n\to\infty} R_1^{n+1} F_1 D^{n+1} Lg = 0.$$

Hence $g^{-1} \in PA_{R_1} D$.

(iii) By (i) and (8.3), the series $\sum_{n=0}^{\infty} R_1^n F_1 D^n Lg$ is convergent. Thus $Lg \in A_{R_1}(D)$.

(iv) Since R_1 is continuous, the space $A_{R_1}(D)$ is *D-invariant*, i.e.

$$D[A_{R_1}(D)] \subset A_{R_1}(D).$$

Hence, by (iii), $g^{-1} = DLg \in A_{R_1}(D)$. Since the product of two functions belonging to $A_{R_1}(D)$ again belongs to $A_{R_1}(D)$ (cf. Przeworska-Rolewicz [2], Theorem 8.3(ii)), by an easy induction we prove that $g^{-n} = (g^{-1})^n \in A_{R_1}(D)$. ∎

Immediate consequences of Proposition 8.3 are:

COROLLARY 8.1. *Suppose that all assumptions of Proposition* 8.3 *are satisfied*, R_1 *is continuous*, $P(t) = \sum_{k=0}^{N} p_k t^k \in \mathcal{F}[t]$ *and Condition* (8.3) *holds. Then* $P(g^{-1}) = g^{-N} P_1(g) \in A_{R_1}(D)$, *where* $N = \deg P(t)$ *and* $P_1(t) = \sum_{k=0}^{N} p_{N-k} t^k$.

COROLLARY 8.2. *Suppose that all assumptions of Proposition* 8.3 *are satisfied and* $P(t) \in \mathcal{F}[t]$. *Then* $P(Lg) \in A_{R_1}(D)$.

Again we shall see that in the multiplicative case we have a different situation.

PROPOSITION 8.4. *Suppose that* X *is a complete linear metric space over* $\mathbb{R}$, *Condition* [**M**] *holds*, $(L, E) \in G[\Omega]$ *and* D *is closed. Then* $D\mathrm{e}^x = \mathrm{e}^{Dx}$ *whenever* $x, Dx \in \mathcal{E}_D(X)$.

Proof. Let $x, Dx \in \mathcal{E}_D(X)$. Since $D \in ML(X)$ and is closed, we get (as in the proof of Proposition 8.2)

$$\mathrm{e}^{Dx} = \sum_{n=0}^{\infty} \frac{(Dx)^n}{n!} = \sum_{n=0}^{\infty} D\frac{x^n}{n!} = D\sum_{n=0}^{\infty} \frac{x^n}{n!} = D\mathrm{e}^x. \ ∎$$

PROPOSITION 8.5. *Suppose that all assumptions of Proposition* 8.4 *are satisfied and* L *is continuous. Then*

(i) $Lu \in \mathcal{E}_D(X)$ *and* $\mathrm{e}^{Lu} = L\mathrm{e}^u = u$ *for* $u \in \mathcal{E}_D(X) \cap \operatorname{dom} \Omega$;

(ii) $Du = \mathrm{e}^{Du-u}$ *if* $u, Du \in \mathcal{E}_D(X)$.

Proof. By Proposition 3.5, L and E are multiplicative. By definition, if $u \in \mathcal{E}_D(X) \cap \operatorname{dom} \Omega$ then $x = Lu \in \operatorname{dom} \Omega^{-1}$ and

$$u = Ex = \mathrm{e}^x = \mathrm{e}^{Lu} = \sum_{n=0}^{\infty} \frac{(Lu)^n}{n!} = \sum_{n=0}^{\infty} \frac{Lu^n}{n!} = L \sum_{n=0}^{\infty} \frac{u^n}{n!} = L\mathrm{e}^u.$$

By (i), $Du = DL\mathrm{e}^u$. On the other hand, by definition, $D\mathrm{e}^u = \mathrm{e}^u DL\mathrm{e}^u$. By (6.5), we have $\mathrm{e}^{-u} = (\mathrm{e}^u)^{-1}$. By Proposition 8.4,

$$Du = DL\mathrm{e}^u = (\mathrm{e}^u)^{-1} D\mathrm{e}^u = (\mathrm{e}^u)^{-1}\mathrm{e}^{Du} = \mathrm{e}^{-u}\mathrm{e}^{Du} = \mathrm{e}^{Du-u}. \ \blacksquare$$

EXAMPLE 8.1. Suppose that all assumptions of Proposition 8.4 are satisfied, F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and $z \in \ker D \cap \mathcal{E}_D(X)$. Since $D\mathrm{e}^z = \mathrm{e}^{Dz} = 0$, we have $\mathrm{e}^z \in \ker D$. Furthermore,

$$D\mathrm{e}^{Rz} = \mathrm{e}^{DRz} = \mathrm{e}^z.$$

Hence

$$\mathrm{e}^{Rz} = R\mathrm{e}^z + F\mathrm{e}^z = R\mathrm{e}^z + \mathrm{e}^z = (R+I)\mathrm{e}^z, \quad F\mathrm{e}^{Rz} = F(R+I)\mathrm{e}^z = \mathrm{e}^z.$$

By an easy induction we get

$$\mathrm{e}^{R^n z} = \Big(\sum_{k=0}^{n} R^k\Big)\mathrm{e}^z, \quad F(\mathrm{e}^{R^n z}) = \mathrm{e}^z \quad \text{for } z \in \ker D,\ n \in \mathbb{N}. \tag{8.4}$$

DEFINITION 8.1. Suppose that X is a D-algebra with unit $e \in \operatorname{dom} D$, $A \in L(X)$ and $\operatorname{dom} A \supset \operatorname{dom} D$. We write

$$\Gamma_n^A(u) = \prod_{j=1}^{n} A^j u \quad \text{for } u \in \operatorname{dom} A,\ n \in \mathbb{N}. \tag{8.5}$$

PROPOSITION 8.6. *Suppose that Condition* [**M**] *holds and* $(L, E) \in G[\Omega]$. *Then*

(i) $D^n Ex = (Ex)\Gamma_n^D(x)$ *for all* $x \in \operatorname{dom} D^n \cap \operatorname{dom} \Omega^{-1}$ *and* $n \in \mathbb{N}$;

(ii) $Ex \in D_\infty$ *for* $x \in D_\infty \cap \operatorname{dom} \Omega^{-1}$.

Proof. (i) For $n = 1$, by definition, $DEx = (Ex)Dx = (Ex)\Gamma_1^D(x)$. Observe that $\Gamma_n^D(x)$ is well-defined for $x \in \operatorname{dom} D^n$. Suppose (i) to be true for a fixed $n \in \mathbb{N}$. Let $x \in \operatorname{dom} D^{n+1} \cap \operatorname{dom} \Omega^{-1}$. Then

$$D^{n+1}Ex = D[(Ex)\Gamma_n^D(x)] = (DEx)[D\Gamma_n^D(x)] = (Ex)(Dx)D\prod_{j=1}^{n} D^j x$$

$$= (Ex)(Dx)\prod_{j=1}^{n} D^{j+1}x = (Ex)\prod_{k=1}^{n+1} D^k x = (Ex)\Gamma_{n+1}^D(x).$$

If $x \in D_\infty \cap \operatorname{dom} \Omega^{-1}$ then (i) implies (ii). $\blacksquare$

PROPOSITION 8.7. *Suppose that Condition* **[M]** *holds*, $(L,E)\in G[\Omega]$, $F\in ML(X)$ *is an initial operator for* D *corresponding to an* $R\in\mathcal{R}_D$ *and* $x\in D_\infty\cap\operatorname{dom}\Omega^{-1}$. *Then* $Ex\in Q_R(D)$ *if and only if* $FEx=0$.

Proof. Since $F\in ML(X)$ and $FEx=0$, by Proposition 8.6(i) we get

$$FD^nEx=F[(Ex)\Gamma_n^D(x)]=(FEx)[F\Gamma_n^D(x)]=0\quad\text{ for } n\in\mathbb{N}. \ ■$$

COROLLARY 8.3. *Suppose that all assumptions of Proposition* 8.7 *are satisfied and* X *is a complete linear metric space over* $\mathbb{R}$. *If* $FEx=0$ *then* $Ex\notin A_R(D)$.

Indeed, $A_R(D)\cap Q_R(D)=\{0\}$ and $Ex\neq 0$ by definition.

PROPOSITION 8.8. *Suppose that Condition* **[M]** *holds*, $(L,E)\in G[\Omega]$, $F\in ML(X)$ *is an initial operator for* D *corresponding to an* $R\in\mathcal{R}_D$ *and* $u\in D_\infty\cap\operatorname{dom}\Omega$. *Suppose that* $FLu=Fu=0$. *Then* $Lu\in Q_R(D)$ *if and only if* $u\in Q_R(D)$.

Proof. Let $u\in D_\infty\cap\operatorname{dom}\Omega$ and let $x=Lu$. Then $u=Ex$. Since $D,F\in ML(X)$ and $Du=uDLu$, we get for all $n\in\mathbb{N}$,

$$\begin{aligned}(8.6)\qquad FD^nuF[(Du)^n]&=(FDu)^n=[F(uDLu)]^n\\ &=(Fu)^n(FDLu)^n=(Fu)^nF(DLu)^n=(Fu)^nFD^nLu.\end{aligned}$$

If $Lu\in Q_R(D)$ then $FD^nLu=0$ for all $n\in\mathbb{N}$. Hence $FD^nu=0$ for all $n\in\mathbb{N}$, which implies $u\in Q_R(D)$. Conversely, if $u\in Q_R(D)$ then, by (8.6),

$$(Fu)^n(FD^nLu)=FD^nu=0\quad\text{ for all } n\in\mathbb{N}.$$

Since, by our assumption, $\ker D$ has no zero divisors, we find $FD^nLu=0$ for all $n\in\mathbb{N}$, which implies $Lu\in Q_R(D)$. ■

PROPOSITION 8.9. *Suppose that* X *is a complete linear metric space over* $\mathbb{R}$, *Condition* **[M]** *holds*, $F,F_1\in ML(X)$ *are initial operators for* D *corresponding to* $R,R_1\in\mathcal{R}_D$, $F_1\neq F$, $g=Re\in I(X)$ *and* $(L,E)\in G[\Omega]$. *If*

$$(8.7)\qquad \lim_{n\to\infty}R_1^ne=0,$$

then

(i) $Lg\in A_{R_1}(D)$;
(ii) $g^{-1}\in A_{R_1}(D)$, *provided that* R_1 *is continuous*.

Proof. (i) Observe that $Dg^n=Dg^{-n}=e$ for $n\in\mathbb{N}$. Indeed, for $n\in\mathbb{N}$ we have $Dg^n=(Dg)^n=e^n=e$ and $e=De=D(g^ng^{-n})=(Dg^n)(Dg^{-n})=Dg^{-n}$. Hence for $n\geq 2$ we have $D^nLg=D^{n-1}g^{-1}=e$. We therefore obtain

$$R_1F_1D^nLg=\begin{cases}F_1g^{-1} & \text{if } n=0,\\ R_1F_1g^{-1} & \text{if } n=1,\\ R_1^nF_1e & \text{otherwise,}\end{cases}$$

and

$$F_1e=e-R_1D_1e=e-R_1e,\quad F_1g^{-1}=g^{-1}-R_1D_1g^{-1}=g^{-1}-R_1e,$$

$$R_1F_1g^{-1} = R_1g^{-1} - R_1^2e, \quad R_1^nF_1e = R_1^ne - R_1^{n+1}e \quad \text{for } n \ge 2.$$

Hence for all $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{N} R_1^nF_1D^nLg &= R_1F_1g^{-1} + R_1F_1e + \ldots + R_1^NF_1e \\ &= R_1g^{-1} - R_1^2e + R_1^2e - R_1^3e + \ldots + R_1^Ne - R_1^{N+1}e \\ &= R_1g^{-1} - R_1^{N+1}e.\end{aligned}$$

Condition (8.7) implies that the series $\sum_{n=0}^{\infty} R_1^NF_1D^nLg$ is convergent and

$$Lg = \sum_{n=0}^{\infty} R_n^nF_1D^nLg = F_1Lg + R_1g^{-1} = F_1g^{-1} + R_1g^{-1} \in A_{R_1}(D).$$

(ii) follows from (i) just as in the proof of Proposition 8.3(iv)⇒(iii). ■

EXAMPLE 8.2. Let X, e, D, R be as in Example 3.2. Then $D \in ML(X)$ and Condition [**M**] holds. Let $u = \{u_n\}$. If $u_n \ne 0$ for $n \in \mathbb{N}$ then $L\{u_n\} = \{x_n\}$, where $x_1 = 1$ and $x_{n+1} = u_{n+1}/u_n$. Observe that all $x_n \ne 0$. It is easy to verify that $E\{x_n\} = \{u_n\} = u_1\{\gamma_n(x)\}$, where $x = \{x_n\}$ and

$$\gamma_n(x) = \begin{cases} 1 & \text{for } n = 1, \\ \prod_{j=2}^{n} x_j & \text{otherwise.} \end{cases}$$

If $x_n = \lambda$ for all $n \in \mathbb{N}$ then $x = \lambda e$, $\gamma_n(\lambda e) = \lambda^{n-1}$ and $Ex = u_1\{\lambda^{n-1}\}$.

As a matter of fact, all results obtained in this section are examples only which are intended show how to examine the properties of smooth logarithms and antilogarithms. In order to obtain more general results we have to know the properties of non-Leibniz components in any particular case.

9. Logarithmic and antilogarithmic mappings induced by left invertible and invertible operators

We still assume that the multifunction Ω is as in Section 1. As in that section, we get

PROPOSITION 9.1. *Suppose that* $D \in \mathsf{A}(X) \cap \Lambda(X)$. *Then*

$$\operatorname{dom} \Omega \supset \{x = S(u^{-1}Du) : u \in I(X) \cap \operatorname{dom} \Omega;\ S \in \mathcal{L}_D\} \ne \emptyset.$$

Note. By definition, if $D \in \mathsf{A}(X)$ then X is a commutative algebra and the domain of D is a subalgebra of X. On the other hand, since D is left invertible, we have $\operatorname{dom} D = X$. Hence the assumption that the domain of D is a subalgebra of X is trivially satisfied.

PROPOSITION 9.2 (cf. Propositions 1.3 and 1.4). *Let* $D \in \mathsf{A}(X) \cap \Lambda(X)$ *and* $u \in I(X) \cap \operatorname{dom} \Omega$. *If* $x \in \Omega u$ *and* $y \in \Omega u$ *then* $x = y$.

In other words: *If* $D \in \mathsf{A}(X) \cap \Lambda(X)$ *then the multifunction* Ω *is single-valued.*

P r o o f. By our assumptions, $Dx = u^{-1}Du = Dy$. Since $\ker D = \{0\}$, we get $x = y$. ∎

In view of Proposition 9.2 we may write

$$L = \Omega, \quad E = \Omega^{-1} \quad \text{for } (L, E) \in G[\Omega], \quad \text{i.e.} \quad L \in \mathcal{I}(X). \tag{9.1}$$

For left invertible and invertible operators some pathologies may appear, as is shown by

THEOREM 9.1. *Let* $A \in \mathsf{A}(X)$. *Consider the following conditions*:

(i) X *is a Leibniz algebra*, *i.e.* $c_A = 0$ *and* $f_A = 0$;
(ii) X *has a unit* e;
(iii) $\ker A = \{0\}$, *i.e.* A *is left invertible.*

Each pair of these conditions excludes the remaining condition, *i.e.*:

(a) *if* $\ker A = \{0\}$ *then either* X *is not a Leibniz algebra or* X *has no unit*;
(b) *if* X *is a Leibniz algebra then either* X *has no unit or* $\ker A \neq \{0\}$.

P r o o f. (a) Suppose that X is a Leibniz algebra with unit e and $\ker A = \{0\}$. Then $Ae = Ae^2 = 2eAe = 2Ae$, which implies $Ae = 0$. Thus $e = 0$, a contradiction.

(b) Suppose that X is a Leibniz algebra. Let X have unit e. Arguing as before, we conclude that $Ae = 0$, i.e. $\ker A \neq \{0\}$.

Suppose now that $\ker A = \{0\}$. Hence $x = 0$ is a unique element satisfying $Ax = 0$. Suppose, moreover, that there is an $e \in X \setminus \{0\}$ such that $xe = ex = x$ for every $x \in X$. Then $Ax = xAe + eAx = xAe + Ax$. Thus $xAe = 0$ for all x, which implies $Ae = 0$, i.e. $e \in \ker A = \{0\}$. Hence $e = 0$, a contradiction. ∎

EXAMPLE 9.1. Let $X = C[0, T]$ and

$$D = \frac{d}{dt} \quad \text{with} \quad \operatorname{dom} D = \{x \in C^1[0, T] : x(0) = 0\}.$$

It is easy to verify that X (with respect to the usual addition, multiplication by scalars and multiplication of elements) is a Leibniz algebra without unit and that D is invertible with inverse $D^{-1} = \int_0^t$.

This example can be generalized in the following manner:

EXAMPLE 9.2. Suppose that X is a Leibniz D-algebra with unit e and F is an initial operator for D corresponding to an $R \in \mathcal{R}_D$. Let $D' = D$ with

$$\operatorname{dom} D' = \{x \in \operatorname{dom} D : Fx = 0\}.$$

By our assumption, $e \in \ker D$. Hence $Fe = e \neq 0$. We therefore conclude that X is a Leibniz algebra without unit, D' is invertible and $D'^{-1} = R$.

An example of an algebra with a multiplicative invertible operator is shown in Theorem 4.1.

N o t e. Recall that $\mathcal{L}_D = \mathcal{R}_D = \{D^{-1}\}$ for $D \in \mathcal{I}(X)$.

An immediate consequence of Propositions 9.1 and 9.2 is

PROPOSITION 9.3. *Let* $D \in \mathsf{A}(X) \cap \Lambda(X)$ *and* $\Omega = L \in \mathcal{I}(X)$. *Then*

$$Lu = S(u^{-1}u), \qquad [I - S(Dx)]Ex = 0,$$

where $S \in \mathcal{L}_D, (u, x) \in \operatorname{graph} L,\ u \in I(X)$.

An immediate consequence of this proposition is

COROLLARY 9.1. *Let* $D \in \mathsf{A}(X) \cap \Lambda(X)$ *and* $\Omega = L \in \mathcal{I}(X)$. *Write*

$$\Theta(S) = \{x \in \operatorname{dom} E : \ker[I - S(Dx)] \neq \{0\}\} \quad \text{for } S \in \mathcal{L}_D. \tag{9.2}$$

If $\Theta(S) = \emptyset$ *then* $Ex = 0$. *Hence* $Lu = 0$ *for all* $(u, x) \in \operatorname{graph} L$, *i.e. logarithmic and antilogarithmic mappings do not exist.*

DEFINITION 9.1. We denote by $\mathbf{Lg}_{\#}(D)$ the class of commutative algebras X with an operator $D \in \Lambda(X)$ such that $D \in \mathsf{A}(X)$ and $\Omega = L$ is invertible, and by $\mathbf{Lg}^0_{\#}(D)$ the class of algebras belonging to $\mathbf{Lg}_{\#}(D)$ with unit e such that $e \in \operatorname{dom} L$.

Clearly, by this definition and Corollary 9.1, if $X \in \mathbf{Lg}_{\#}(D)$ then there is an $S \in \mathcal{L}_D$ such that the set $\Theta(S)$ defined by (9.2) is not empty. Then we get

$$Lu = S(u^{-1}u), \quad Ex = u \quad \text{for } u \in I(X) \cap \operatorname{dom} L.$$

In this case the operator $I - S(Dx)$ is not left invertible (invertible). However, if $I - S(Dx)$ is right invertible then $Ex = Rz$, where $R \in \mathcal{R}_{I-S(Dx)}$ and $z \in \ker[I - S(Dx)]$ (cf. Propositions 1.7 and 1.8).

An immediate consequence of the fact that Ω is single-valued (cf. Proposition 9.2) is

COROLLARY 9.2 (cf. Propositions 1.11, 1.12 and Corollary 1.3). *Let* $X \in \mathbf{Lg}_{\#}(D)$. *Then logarithms and antilogarithms are uniquely determined.*

PROPOSITION 9.4. *Let* $X \in \mathbf{Lg}_{\#}(D)$. *Then the logarithm and antilogarithm of zero are not defined.*

Proof. Let $L \in \mathcal{I}(X)$. If $x = L(0)$ then x should satisfy $0 = D0 = 0Dx$. Thus $L(0)$ is undetermined. By definition, we have $DE(0) = E(0)D(0) = 0$. Since $\ker D = \{0\}$, we conclude that $E(0) = 0$. This implies that $L(0) = LE(0) = 0$, a contradiction (cf. Corollary 9.2). ∎

PROPOSITION 9.5 (cf. Theorem 1.2 and Corollaries 1.1, 1.2). *Let* $X \in \mathbf{Lg}^0_{\#}(D)$, $S \in \mathcal{L}_D$, $e \in DX$ *and* $g = Se$. *Then* $g \in \operatorname{dom} L$ *if and only if* $g \in I(X)$. *In that case*, *g is not a zero divisor.*

Proof. Since $e \in DX$ and $Se = g = SDg$, we get $Dg = e$. The basic equation implies that $e = Dg = gDLg$. Clearly, g is not a zero divisor. ∎

THEOREM 9.2. *If* $X \in \mathbf{Lg}_{\#}(D)$ *and* $(L, E) \in G[L]$ *then for* $u, v \in I(X) \cap \operatorname{dom} L$ *and* $x = Lu$, $y = Lv$ *we have*, *independently of the choice of an* $S \in \mathcal{L}_D$,

$$L(uv) = c_D(Lu + Lv) + S[u^{-1}v^{-1}f_D(u, v)], \tag{9.3}$$

(9.4) $$(Ex)(Ey) = E\{c_D(x+y) + S[(Ex)^{-1}(Ey)^{-1}f_D(Ex,Ey)]\}.$$

Proof. Let $u, v \in I(X) \cap \operatorname{dom} L$. If $S \in \mathcal{L}_D$ then

$$\begin{aligned} L(uv) &= SDL(uv) = SD\{c_D(Lu+Lv) + S[u^{-1}v^{-1}f_D(u,v)]\} \\ &= c_D(Lu+Lv) + S[u^{-1}v^{-1}f_D(u,v)] \end{aligned}$$

(cf. the proof of Theorem 1.2). By Proposition 9.4, formula (1.9) is independent of the choice of a left inverse S, and (1.10) is an immediate consequence of the definition and (1.9). ∎

Corollary 9.3 (cf. Corollary 1.4). *Suppose that $c_D \neq 0$, $X \in \mathbf{Lg}_{\#}(D)$ and $(L, E) \in G[L]$. Then $u^2 \in I(X) \cap \operatorname{dom} L$ if and only if $u \in I(X) \cap \operatorname{dom} L$. In that case*,

$$Lu^2 = 2c_D Lu + S[u^{-2}f_D(u,u)] \quad (S \in \mathcal{L}_D).$$

Corollary 9.4. *Suppose that $c_D \neq 0$, $X \in \mathbf{Lg}_{\#}(D)$ and $(L, E) \in G[L]$. Then L is of exponential type if and only if X is a Leibniz algebra* (cf. the proof of Theorem 1.3 where, as a matter of fact, we did not use the right invertibility of D).

Corollary 9.5 (cf. Proposition 1.14). *Suppose that $c_D \neq 0$, $X \in \mathbf{Lg}_{\#}(D)$ and $(L, E) \in G[L]$. If $D \in ML(X)$ then L and E are multiplicative.*

Theorem 9.3. *Let $X \in \mathbf{Lg}^0_{\#}(D)$ and $(L, E) \in G[L]$. Let g_D be defined by* (1.11). *Then*

(9.5) $$(1 - c_D)Lu = c_D Le + Sg_D(u) \quad \textit{for } u \in I(X) \cap \operatorname{dom} L.$$

Moreover, *if $c_D \neq 0$ then the mapping g_D is not constant.*

Proof. (9.11) follows immediately from our assumptions and Theorem 9.2 (cf. Proposition 1.16). The remaining part of the proof is similar to that of Theorem 1.4 (cf. also Theorem 9.1). ∎

Theorems 9.1, 9.3 and Proposition 9.4 imply the following proposition (replacing Proposition 1.17):

Proposition 9.6. *Let $X \in \mathbf{Lg}^0_{\#}(D)$ and $(L, E) \in G[L]$. Then*

(i) $(1 - 2c_D)Le = Sf_D(e, e)$, *where $S \in \mathcal{L}_D$*;
(ii) $(1 - 2c_D)DLe = (1 - 2c_D)De = f_D(e, e)$;
(iii) $c_D = 1/2$ *implies* $f_D(e, e) = 0$;
(iv) $c_D \neq 1/2$ *implies* $Le = e$.

Similarly to Theorems 1.5, 1.6 and Corollaries 1.5–1.8 we obtain

Proposition 9.7. *Suppose that $X \in \mathbf{Lg}_{\#}(D)$ and $c_D \neq 0$. Let $\alpha \in \mathcal{F} \setminus \{0\}$. Let $u \in \operatorname{dom} L$. If $\alpha \neq 1$ then $\alpha u \notin \operatorname{dom} L$. In particular*, *if $X \in \mathbf{Lg}^0_{\#}(D)$ then $-e \notin \operatorname{dom} L$.*

PROPOSITION 9.8. *Suppose that* $X \in \mathbf{Lg}^0_{\#}(D)$ *and* $c_D \neq 0$. *Let* $\alpha \in \mathcal{F} \setminus \{0\}$ *and* $u \in \operatorname{dom} L$. *Then* $\alpha u \in \operatorname{dom} L$ *and*

$$L(\alpha u) = c_D[Lu + L(\alpha u)] + Sg_D(u) = Lu - Le + L(\alpha e) \quad (S \in \mathcal{L}_D).$$

PROPOSITION 9.9. *Suppose that* $D \in \mathsf{A}(X) \cap \Lambda(X)$, X *has unit* e *and* $c_D \neq 1$. *Let the mapping* g_D *defined by* (1.11) *be not constant. Let the mapping* A *with* $\operatorname{dom} A \supset \operatorname{dom} \Omega$ *be defined by*

$$Au = \frac{1}{1-c_D}[c_D e + Sg_D(u)] \quad \text{for } u \in I(X) \cap \operatorname{dom} A, \text{ where } S \in \mathcal{L}_D.$$

If A *is invertible then* $(A, A^{-1}) \in G[\Omega]$, *i.e.* $X \in \mathbf{Lg}(D)$.

Proof. By definitions, for all $u \in I(X) \cap \operatorname{dom} A$ we have

$$SDA = Au = \frac{1}{1-c_D}[c_D SDe + Sg_D(u)] = S\frac{1}{1-c_D}[c_D e + g_D(u)],$$

which implies

$$DAu = \frac{1}{1-c_D}[De + g_D(u)]$$

(up to an $s \in \ker S$ which can be assumed to be zero). Further arguments are the same as in the proof of Proposition 1.18. ■

In the same way as in Section 2, we introduce logarithms and antilogarithms of higher order. Instead of Theorem 2.1 and Corollaries 2.1, 2.3 we have the following

THEOREM 9.4. *For* $n \in \mathbb{N}$ *define the multifunction* $\Omega_n : \operatorname{dom} D^n \to 2^{\operatorname{dom} D^n}$ *by*

$$\Omega_n u = \{x \in \operatorname{dom} D^n : D^n u = uD^n x\} \quad \text{for } x \in \operatorname{dom} D^n.$$

For $(L, E) \in G[L]$ *write* $L_1 = L = \Omega$, *and* $E_1 = E = L^{-1}$. *Then* $\operatorname{dom} L_{n+1} \subset \operatorname{dom} L_n$ $(n \in \mathbb{N})$. *If* $X \in \mathbf{Lg}_{\#}(D)$ *then* $X \in \mathbf{Lg}_{\#}(D^n)$ *for all* $n \in \mathbb{N}$. *Moreover, if* $(L_1, E_1) \in G[L_1]$ *then* $(L_n, E_n) \in G[L_n]$ *for every* $n \in \mathbb{N}$, *and for* $(u, x), (v, y) \in \operatorname{graph} L_n$,

$$L_n u = c_D L_{n-1} u + S^n[c_D(DL_1 u)D^{n-1}L_{n-1}u + u^{-1} f_D(u, D^{n-1}L_{n-1}u)] \quad (n \geq 2),$$

while for all $n \in \mathbb{N}$,

$$L_n(uv) = c^n_D(L_n u + L_n v) + R^n[u^{-1}v^{-1} f^{(n)}_D(u, v)],$$

$$(E_n x)(E_n y) = E_n\{c_D(x + y) + S^n[(Ex)^{-1}(Ey)^{-1} f^{(n)}_D(Ex, Ey)]\},$$

independently of the choice of $S \in \mathcal{L}_D$, *i.e.* L *and* E *are uniquely determined. If* $X \in \mathbf{Lg}^0_{\#}(D)$ *then*

$$(1 - 2c^n_D)L_n e = S^n f^{(n)}_D(e, e), \quad \text{i.e.} \quad (1 - 2c^n_D)D^n e = Sf^{(n)}_D(e, e).$$

LEMMA 9.1 (cf. Lemma 4.2). *Let* $D \in \mathsf{A}(X) \cap \Lambda(X)$. *If* $D \in ML(X)$ *and* $S \in \mathcal{L}_D$ *then* $S \in ML(X)$.

P r o o f. Let $D \in ML(X)$. Let $x, y \in X$, $S \in \mathcal{L}_D$ and $u = Dx$, $v = Dy$. Then $x = Su$, $y = Sv$ and $S(uv) = S[(Dx)(Dy)] = SD(xy) = xy = (Su)(Sv)$. ■

By similar arguments to those used in the proof of Theorem 2.2, applying Lemma 9.1 we obtain

THEOREM 9.5. *Let $X \in \mathbf{Lg}_{\#}(D)$ and $D \in ML(X)$. Then $X \in \mathbf{Lg}_{\#}(D^n)$ for all $n \in \mathbb{N}$, $(L_n, E_n) \in G[L_n] \cap ML(X)$ and for $(u, x) \in \operatorname{graph} L$ and $n \in \mathbb{N}$,*

$$L_n u = \prod_{j=0}^{n-1} S^j L_1 u,$$

$$\prod_{j=0}^{n-1} E_n(S^j x) = E_n\Big(\prod_{j=0}^{n-1} S^j x\Big) = E_1 x$$

independently of the choice of $S \in \mathcal{L}_D$.

THEOREM 9.6 (cf. Theorem 2.4). *Let $n \in \mathbb{N}$, $X \in \mathbf{Lg}_{\#}(D^n)$, $(L_n, E_n) \in G[L_n]$, $a \in X$ and $S^n a \in \operatorname{dom} E_n$ for an $S \in \mathcal{L}_D$. Then*

(i) $S^n a \in \operatorname{dom} E_n$;
(ii) $x_0 \in \ker(D^n - a)$ *if and only if* $x_0 = E_n(R^n a)$;
(iii) *if $y \in \operatorname{dom} L_n$ and there is a $v \in \operatorname{dom} L_n$ such that*

$$[c_D^n D^n v + (c_D^n - 1)av]x_0 + f_D^{(n)}(x_0, v) = y, \tag{9.6}$$

then $x = vx_0$ is a unique solution of the equation

$$D^n x = ax + y, \qquad y \in \operatorname{dom} L_n \tag{9.7}$$

(*independently of the choice of $S \in \mathcal{L}_D$*).

P r o o f. *Sufficiency* is proved by checking.

Necessity. Suppose that $x_0 \in \ker(D^n - a)$. Then $D^n x_0 = a x_0$. Hence $D^n L_n x_0 = a$. If $S \in \mathcal{L}_D$ then $L_n x_0 = S^n D^n L_n x_0 = S^n a$ implies that $x_0 = E_n(S^n a)$.

Having thus proved (i) and (ii), we are looking for a $v \in \operatorname{dom} D$ such that $x = vx_0$ is a solution of (9.6). By (ii), $D^n x_0 = a x_0$ and

$$\begin{aligned} D^n x - ax = D^n(vx_0) - avx_0 &= c_D^n(vD^n x_0 + x_0 D^v) + f_D^{(n)}(x_0, v) - avx_0 \\ &= c_D^n D^n(avx_0 + x_0 D^n v) - avx_0 + f_D^{(n)}(x_0, v) \\ &= [c_D^n D^n v + (c_D^n - 1)v]x_0 + f_D^{(n)}(x_0, v). \end{aligned}$$

Clearly, if there is a v satisfying (9.6), then x is a solution of (9.7). ■

COROLLARY 9.6. *Suppose that all assumptions of Theorem* 9.6 *are satisfied, $n = 1$ and X is a Leibniz algebra. If $x_0 = E_1(Sa) \in I(X)$ then $x = x_0 S(yx_0^{-1})$ is a unique solution of* (9.7) (*independently of the choice of $S \in \mathcal{L}_D$*).

P r o o f. Since X is a Leibniz algebra, we have $c_D = 1$ and $f_D = 0$. Hence (9.6) reduces to $x_0 Dv = y$. By our assumption, x_0 is invertible. Thus $v = SDv = S(x_0^{-1} y)$, which implies that x is of the required form. ■

Note. In Corollary 9.6, X is a Leibniz algebra and $D \in \Lambda(X)$. We therefore conclude that X has no unit (cf. Theorem 9.1). This is the reason why we have to assume that $x_0 = E_1(Sa)$ is invertible. For D-algebras the invertibility of elements $E(Ra)$ where $R \in \mathcal{R}_D$ follows from Proposition 5.1(i).

Corollary 9.7. *Suppose that all assumptions of Theorem* 9.6 *are satisfied and* $D \in ML(X)$. *If* $a, x_0 = E_n(S^n a) \in I(X)$ *and* $1 \in v_{\mathcal{F}} S^n$ *then*

$$x = x_0(I - S^n)^{-1}[(S^n a^{-1})(S^n x_0^{-1})(S^n y)]$$

is a unique solution of (9.7) (*independently of the choice of* $S \in \mathcal{L}_D$).

Proof. Since $D \in ML(X)$, we have $D^n \in L(X)$, $c_D^n = 0$ and $f_D^{(n)}(x_0, v) = (D^n x_0) D^n v = a x_0 D v$. Thus (9.6) reduces to $a x_0 D^n v - a v x_0 = y$. By our assumption, $v - S^n v = S^n D^n v - S^n v = S^n(a^{-1} x_0^{-1} y)$. By Lemma 9.1, $S \in ML(X)$, hence $S^n \in ML(X)$. By our assumption, the operator $I - S^n$ is invertible. We therefore conclude (as in the previous cases) that x is of the required form. ∎

Definition 9.2. Denote by $\mathcal{B}_D(a, c, d)$ the class of commutative algebras X with $D \in \mathsf{A}(X)$ such that the product rule is given by the same formula as in Definition 3.1:

$$D(xy) = c(xDy + yDx) + d(Dx)Dy + axy \quad \text{for } x, y \in \operatorname{dom} D = X,$$

where $a, c, d \in X$ depend on D only and do not vanish simultaneously. We denote by $\mathcal{B}_D^0(a, c, d)$ the class of algebras with unit e belonging to $\mathcal{B}_D(a, c, d)$.

Taking into account Theorem 3.1 and Proposition 3.6, we conclude that Theorem 9.1 implies

Theorem 9.7. *If* $X \in \mathcal{B}_D(0, c, 0)$ *then* $X \notin \mathbf{Lg}_{\#}(D)$.

Proposition 9.10 (cf. the proof of Proposition 3.1). *Let* $X \in \mathcal{B}_D(a, c, d)$. *Suppose that* $X \in \mathbf{Lg}_{\#}(D)$ *and* $(L, E) \in G[L]$. *If* $Hu = dDLu + c$ *then*

$$H(uv) = (Hu)Hv + h \quad \text{for } u, v \in I(X) \cap \operatorname{dom} L, \text{ where } h = ad - c^2 + c.$$

Proposition 9.11 (cf. Corollary 3.1). *Let* $X \in \mathcal{B}_D^0(a, c, d)$. *Suppose that* $X \in \mathbf{Lg}_{\#}(D)$ *and* $(L, E) \in G[L]$. *If* $Le = 0$ *then* $ad = 0$.

Proposition 9.12. *Let* $X \in \mathcal{B}_D^0(a, c, d)$, $d \in I(X)$, $D^0 = dD + cI$ *and* $a = cd^{-1}(c - e)$. *Then*

(i) $D^0 \in ML(X)$, *i.e.* $X \in \mathcal{B}_{D^0}^0(0, 0, e)$;

(ii) *if the operator* $dI + Sc$ *is invertible for an* $S \in \mathcal{L}_D$ *then* $D^0 \in \Lambda(X)$ *and* $S^0 = d(dI + Sc)^{-1} S d^{-1} \in \mathcal{L}_{D^0}$;

(iii) *in particular, if* $c = \alpha e$ *and* $d = \beta e$, *where* $\alpha, \beta \in \mathcal{F} \setminus \{0\}$ *and* $-\alpha/\beta \in v_{\mathcal{F}} S$, *then* $D^0 \in \Lambda(X) \cap ML(X)$ *and* $S^0 = (\alpha I + \beta S)^{-1} S \in \mathcal{L}_{D^0}$.

P r o o f (cf. the proof of Proposition 3.2, Corollary 3.2 and Proposition 3.3). We only have to check that $S^0 \in \mathcal{L}_{D^0}$. Indeed,

$$\begin{aligned} S^0 D^0 &= d(dI+Sc)^{-1}Sd^{-1}(dD+cI) = d(dI+Sc)^{-1}(Sd^{-1}dD+Sd^{-1}cI) \\ &= d(dI+Sc)^{-1}(I+Sd^{-1}c) = d(dI+Sc)^{-1}(dI+Sc)d^{-1} = I. \ \blacksquare \end{aligned}$$

PROPOSITION 9.13 (cf. Proposition 3.5). *Suppose that all assumptions of Proposition* 9.12 *are satisfied. Let* $D^0 = dD + cI$,

$$\Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0 u = u D^0 y\} \quad \textit{for } u \in \operatorname{dom} D^0.$$

If $(L^0, E^0) \in G[L^0]$ *then* $X \in \mathbf{Lg}_{\#}(D)$ *and* $(L, E) \in G[L]$, *where*

$$Lu = R(D^0 L^0 u - d^{-1}c) \quad \textit{for } (u, x) \in \operatorname{graph} L^0.$$

THEOREM 9.8. *Let* $X \in \mathbf{Lg}_{\#}(D)$, $(L_n, E_n) \in G[L_n]$ *and let*

$$P(D) = \sum_{k=0}^{N} p_k D^k, \quad p_0, \ldots, p_N \in X, p_N = e.$$

Set

$$D^0 = P(D), \quad \Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D^N \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0 u = u D^0 y\} \quad \textit{for } u \in \operatorname{dom} D^0.$$

Let Ω_k *be as in Theorem* 2.1 *and let* $(L_k, E_k) \in G[L_k]$ *for* $k = 1, \ldots, N$. *Suppose that the operator*

$$P(I, S) = \sum_{k=0}^{N} p_k R^{N-k}$$

is invertible for an $S \in \mathcal{L}_D$ *and*

$$L^0 u = S^N [P(I,S)]^{-1} \Big(p_0 u^{-1} + \sum_{k=1}^{N} p_k D L_k u\Big) \quad \textit{for } u \in \operatorname{dom} L_N \cap I(X).$$

If the mapping L^0 *is invertible then* $X \in \mathbf{Lg}_{\#}(D^0)$ *and* $(L^0, E^0) \in G[L^0]$, *where* $E^0 = (L^0)^{-1}$.

P r o o f. By our assumptions, $D^0 = P(D) \in \Lambda(X)$ and $S^0 = S^N [P(I,S)]^{-1} \in \mathcal{L}_{D^0}$. Indeed,

$$P(D) = \sum_{k=0}^{N} p_k S^{N-k} D^{N-k} D^k = \Big(\sum_{k=0}^{N} p_k S^{N-k}\Big) D^N = P(I,S) D^N.$$

Further considerations are similar to those in the proof of Theorem 3.2, if we take into account the fact that $\ker D = \{0\}$. $\blacksquare$

THEOREM 9.9 (cf. Theorems 3.3 and 9.8). *Let* $X \in \mathbf{Lg}_{\#}(D)$, $(L_n, E_n) \in G[L_n]$ *and let*

$$Q(D) = \sum_{k=0}^{N} Q_k D^k, \qquad Q_k \in L(X)\ (k = 0, 1, \ldots, N-1),\ Q_N = I.$$

Set

$$D^0 = Q(D), \qquad \Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D^N \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0 u = u D^0 y\} \qquad \text{for } u \in \operatorname{dom} D^0.$$

Let Ω_k *be as in Theorem* 2.1 *and let* $(L_k, E_k) \in G[L_k]$ *for* $k = 1, \ldots, N$. *Suppose that the operator*

$$Q(I, S) = \sum_{k=0}^{N} Q_k S^{N-k}$$

is invertible for an $S \in \mathcal{L}_D$, $(L_N, E_N) \in G[L_N]$ *and*

$$L^0 u = S^N [Q(I,S)]^{-1} \{u^{-1} [Q(I,S)]^{-1} (u D^N L_N u) \qquad \text{for } u \in \operatorname{dom} L^0 \cap I(X).$$

If the mapping L^0 *is invertible then* $X \in \mathbf{Lg}_{\#}(D^0)$ *and* $(L^0, E^0) \in G[L^0]$, *where* $E^0 = (L^0)^{-1}$.

Similarly, we have

THEOREM 9.10 (cf. Theorem 3.4). *Let* $X \in \mathbf{Lg}_{\#}(D)$, $(L_n, E_n) \in G[L_n]$ *and let*

$$Q\langle D\rangle = \sum_{k=0}^{N} D^k Q_k, \tag{3.7}$$
$$Q_k \in L(X),\ Q_k X \in \operatorname{dom} D^N\ (k = 0, 1, \ldots, N-1),\ Q_N = I.$$

Set

$$D^0 = Q\langle D\rangle, \qquad \Omega^0 : \operatorname{dom} D^0 = \operatorname{dom} D^N \to 2^{\operatorname{dom} D^0},$$
$$\Omega^0 u = \{y \in \operatorname{dom} D^0 : D^0 u = u D^0 y\} \qquad \text{for } u \in \operatorname{dom} D^0.$$

Let Ω_k *be as in Theorem* 2.1 *and let* $(L_k, E_k) \in G[L_k]$ *for* $k = 1, \ldots, N$. *Suppose that the operator*

$$Q\langle I, S\rangle = \sum_{k=0}^{N} S^{N-k} Q_k$$

is invertible for an $S \in \mathcal{L}_D$, $(L_N, E_N) \in G[L_N]$ *and*

$$L^0 u = [Q\langle I,S\rangle]^{-1} S^N [u^{-1} (Q\langle I,S\rangle u) D^N L_N (Q\langle I,S\rangle u)] \qquad \text{for } u \in \operatorname{dom} L^0 \cap I(X).$$

If the mapping L^0 *is invertible then* $X \in \mathbf{Lg}_{\#}(D^0)$ *and* $(L^0, E^0) \in G[L^0]$, *where* $E^0 = (L^0)^{-1}$.

N o t e. Similar results can be obtained for operators of the form

$$D^M Q(D), \quad Q(D)D^M, \quad D^M Q\langle D\rangle, \quad Q\langle D\rangle D^M \quad (M \in \mathbb{N}),$$

since all these operators are left invertible under the assumptions of Theorems 9.9 and 9.10, respectively.

Consider now the multiplicative case.

PROPOSITION 9.14 (cf. Proposition 4.1). *Suppose that* X *is a commutative algebra with unit* $e \in \operatorname{dom} D$, $D \in ML(X) \cap \mathsf{A}(X) \cap \Lambda(X)$ *and* $\operatorname{dom} D$ *has no zero divisors. Then*

(i) $De = e$;
(ii) $Dx \in I(X)$ *and* $Dx^{-1} = (Dx)^{-1}$ *for* $x \in I(X) \cap \operatorname{dom} D$;
(iii) $Dx^n = (Dx)^{-1}$ *for* $x \in \operatorname{dom} D$ *and* $n \in \mathbb{N}$;
(iv) $Dx \in I_2(\operatorname{dom} D)$ *and* $Dx^{1/2} = (Dx)^{1/2}$ *for* $x \in I_2(\operatorname{dom} D)$.

We will make use of the following condition:

$[\mathbf{M}]_\#$ $\quad X \in \mathbf{Lg}^0_\#(D)$, $D \in ML(X)$ *and* $\operatorname{dom} D$ *has no zero divisors.*

In particular, if Condition $[\mathbf{M}]_\#$ is satisfied, then $X \in \mathcal{B}^0_D(0,0,e) \cap \mathbf{Lg}_\#(D)$ (cf. Condition $[\mathbf{M}]$ defined in Section 4).

PROPOSITION 9.15. *Suppose that Condition* $[\mathbf{M}]_\#$ *holds*, $(L,E) \in G[L]$ *and* $e \in \operatorname{dom} L^{-1}$. *Then*

(i) $e \in \operatorname{dom} L \cap \operatorname{dom} L^{-1}$ *and* $Ee = Le = e$;
(ii) $Ex \in I(X)$ *and* $Ex^{-1} = (Ex)^{-1}$ *for all* $x \in I(X) \cap \operatorname{dom} L^{-1}$;
(iii) $Ex^n = (Ex)^n$ *for all* $x \in \operatorname{dom} L^{-1}$ *and* $n \in \mathbb{N}$.

DEFINITION 9.3 (cf. Definition 4.1). Suppose that $\mathcal{F} = \mathbb{C}$ or $\mathbb{R}$, Condition $[\mathbf{M}]_\#$ holds and $(L,E) \in G[L]$. We say that E is *multiplicative with exponent* λ if there is a $\lambda \in \mathbb{R}_+ \setminus \{0\}$ such that

$$E(\alpha e) = \alpha^\lambda e \quad \text{for } \alpha \in \mathbb{R}_+ \setminus \{0\}.$$

PROPOSITION 9.16 (cf. Proposition 4.3). *Suppose that all assumptions of Definition* 9.3 *are satisfied*, E *is multiplicative with exponent* $\lambda > 0$, $x \in \operatorname{dom} L^{-1}$ *and* $\alpha \in \mathbb{R}_+ \setminus \{0\}$. *If* $\alpha x \in \operatorname{dom} L^{-1}$ *then* $\alpha = 1$.

PROPOSITION 9.17 (cf. Proposition 4.4). *Suppose that Condition* $[\mathbf{M}]_\#$ *holds and* $(L,E) \in G[L]$. *Then* $E(-e)$ *and* $L(-e)$ *do not exist*, *i.e.* $-e \notin \operatorname{dom} L \cap \operatorname{dom} L^{-1}$.

PROPOSITION 9.18 (cf. Corollary 4.1). *Suppose that* $\mathcal{F} = \mathbb{C}$, *Condition* $[\mathbf{M}]_\#$ *holds and* $(L,E) \in G[L]$. *Then* $E(ie)$ *and* $L(ie)$ *do not exist*, *i.e.* $ie \notin \operatorname{dom} L \cap \operatorname{dom} L^{-1}$.

THEOREM 9.11 (cf. Corollary 4.4). *Suppose that Condition* $[\mathbf{M}]_\#$ *holds*, $(L_n, E_n) \in G[L_n]$ *for* $n \in \mathbb{N}$, $a \in I(X)$ *and there is an* $S \in \mathcal{L}_D$ *such that*

$S^n e \in \operatorname{dom} L_n^{-1}$, $S^n a \in I(X) \cap \operatorname{dom} L_n^{-1}$ *and* $1 \in v_{\mathcal{F}} S^n$. *Then the equation*

$$D^n x = ax + y, \qquad y \in X,$$

has a unique solution $x = x_0 + x_1$, *where*

$$x_0 = E_n(S^n a) E_n(S^n e) \in \ker(D^n - a),$$
$$x_1 = E_n(S^n a)(I - S^n)^{-1} S^n y^{\#}_{a,n}, \quad \text{where} \quad y^{\#}_{a,n} = a^{-1}[E_n(S^n a)]^{-1} y.$$

We shall now consider the Leibniz case. Theorem 9.1 shows that we cannot expect too many results, since in this section we consider operators with trivial kernels. We introduce here the following condition:

$[\mathbf{L}]_{\#}$ $\qquad$ $X \in \mathbf{Lg}_{\#}(D)$ *is a Leibniz algebra.*

Proposition 9.19 (cf. Proposition 5.1). *Suppose that Condition* $[\mathbf{L}]_{\#}$ *holds and* $(L, E) \in G[L]$. *Then*

(i) $E(nx) = (Ex)^n$ *for* $x \in \operatorname{dom} L^{-1}$ *and* $n \in \mathbb{N}$;
(ii) $Ex \in I_n(\operatorname{dom} L^{-1})$ *and* $(Ex)^{1/n} = E\big(\frac{1}{n}x\big)$ *for* $x \in \operatorname{dom} L^{-1}$ *and* $n \in \mathbb{N}$;
(iii) $(Ex)^q = E(qx)$ *for* $x \in \operatorname{dom} L^{-1}$ *and* $q \in \mathbb{Q}$;
(iv) $Lu^q = qLu$ *for* $u \in \operatorname{dom} L$ *and* $q \in \mathbb{Q}$.

Theorem 9.12 (cf. Theorem 5.1). *Suppose that Condition* $[\mathbf{L}]_{\#}$ *holds and* X *is a complete linear metric space over* $\mathbb{R}$, $(L, E) \in G[L]$ *and* L *is continuous on* $\operatorname{dom} L$ *and* E *is continuous on* $\operatorname{dom} L^{-1}$. *Then*

$$(Ex)^r = E(rx), \quad Lu^r = rLu \quad \text{for } (u, x) \in \operatorname{graph} L,\ r \in \mathbb{R} \setminus \{0\}.$$

Theorem 9.13 (cf. Corollary 5.6). *Suppose that Condition* $[\mathbf{L}]_{\#}$ *holds. Let* $m \in \mathbb{N}$. *Suppose that* $(L_m, E_m) \in G[L_m]$, $a_1, \ldots, a_n \in X$ *and there is an* $S \in \mathcal{L}_D$ *such that* $S^m a_j \in \operatorname{dom} L_m^{-1}$ $(j = 1, \ldots, n)$. *Then the unique solution of the equation*

$$\Big[\prod_{j=1}^{n} (D^m - a_j)\Big] x = y, \qquad y \in \operatorname{dom} L_m,$$

is of the form

$$x = E_m(S^m a_n) S^m [E_m(L_m y^{\#}_{n-1} - S^m a_n)],$$
$$y^{\#}_0 = y, \quad y^{\#}_j = E_m(S^m a_j) S^m [E_m(L_m y^{\#}_{j-1} - S^m a_j)] \quad (j = 1, \ldots, n).$$

Consider now the properties of the mapping E_λ introduced by Definition 6.1. Clearly, we can extend this definition to the class $\mathbf{Lg}_{\#}(D)$. We get

Proposition 9.20 (cf. Proposition 6.1). *Suppose that* $X \in \mathbf{Lg}_{\#}(D), (L, E) \in G[L]$ *and the mapping* E_λ *is defined by* (6.1). *Then for all* $\lambda \in \mathcal{F}$,

(i) $E_\lambda(\operatorname{dom} L) \subset \operatorname{dom} L$;
(ii) $LE_\lambda = \lambda L$.

Proposition 9.21 (cf. Proposition 6.2). *Suppose that Condition* $[\mathbf{L}]_{\#}$ *holds and* $(L, E) \in G[L]$. *Then for all* $\lambda \in \mathcal{F}$,

(i) $E_\lambda \in M(X)$;
(ii) $DE_\lambda u = \lambda(E_{\lambda-1}u)Du$ *for* $u \in I(X) \cap \operatorname{dom} L$.

In general, we have

Proposition 9.22 (cf. Proposition 6.3). *Let* $X \in \mathbf{Lg}_\#(D)$ *and* $(L,E) \in G[L]$. *If* $\lambda \in \mathcal{F}$ *and* $u, v \in \operatorname{dom} L$, $E_\lambda u, E_\lambda v \in I(X)$ *then for some* $S \in \mathcal{L}_D$,

$$(E_\lambda u)(E_\lambda v) = E\{c_D\lambda(Lu + Lv) + S[(E_\lambda u)^{-1}(E_\lambda v)^{-1} f_D(E_\lambda u, E_\lambda v)]\}.$$

Corollary 9.8 (cf. Corollary 6.1). *Suppose that all assumptions of Proposition* 9.22 *are satisfied and* $c_D = 0$. *Then the mapping* E_λ *is not defined for* $\lambda \neq 1$. *If* $\lambda = 1$ *then* $E_1 = I|_{\operatorname{dom} \Omega}$.

Corollary 6.1 implies that for multiplicative D the mapping E_λ is not defined.

Definition 9.4. Suppose that $D \in \mathsf{A}(X) \cap \Lambda(X)$, X has unit e, X is a complete linear metric space and $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$. Write $\mathrm{e}^0 = e$ and

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } x \in X$$

whenever this series is convergent. The function e^x is said to be an *exponential function* (cf. Definition 6.2). If $X \in \mathbf{Lg}^0_\#(D)$, $(L,E) \in G[L]$, X has a unit $e \in \operatorname{dom} L^{-1}$, X is a complete linear metric space and either $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$ then we shall write

$$\mathcal{E}_D(X) = \left\{x \in \operatorname{dom} L^{-1} : \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is convergent}\right\};$$
$$\mathcal{E}'_D(X) = \{u \in \operatorname{dom} L : \lambda Lu \in \mathcal{E}_D(X) \text{ for } (L,E) \in G[L],\ \lambda \in \mathcal{F}\};$$
$$u^\lambda = \mathrm{e}^{\lambda Lu} \quad \text{for } u \in \mathcal{E}'_D(X),\ \lambda \in \mathcal{F}.$$

The function u^λ is said to be a *power function* (cf. Definition 6.3). Here also we have $\mathrm{e}^x \mathrm{e}^y = \mathrm{e}^{x+y}$ whenever $x, y, x+y \in \mathcal{E}_D(X)$.

Proposition 9.23 (cf. Proposition 8.4). *Suppose that* X *is a complete linear metric space over* $\mathbb{R}$, *Condition* $[\mathbf{M}]_\#$ *holds*, $(L,E) \in G[L]$ *and* D *is closed. Then* $D\mathrm{e}^x = \mathrm{e}^{Dx}$ *whenever* $x, Dx \in \mathcal{E}_D(X)$.

Proposition 9.24 (cf. Proposition 8.6). *Suppose that Condition* $[\mathbf{M}]_\#$ *holds and* $(L,E) \in G[L]$. *Let* Γ^D_n *be defined by* (8.5). *Then*

(i) $D^n Ex = (Ex)\Gamma^D_n(x)$ *for all* $x \in \operatorname{dom} D^n \cap \operatorname{dom} L^{-1}$ *and* $n \in \mathbb{N}$;
(ii) $Ex \in D_\infty$ *for* $x \in D_\infty \cap \operatorname{dom} L^{-1}$.

We now modify Condition $[\mathbf{C}]$ introduced in the complex case (Section 7) in the following way:

$[\mathbf{C}]_\#$ $\qquad \mathcal{F} = \mathbb{C}$, $\operatorname{dom} \Omega^{-1}$ *is symmetric*, $X \in \mathbf{Lg}_\#(D)$ *and* $c_D \neq 0$.

Clearly, Definition 7.1 works with $[\mathbf{C}]_\#$. Assertions (i)–(iii) of Proposition 7.1 hold under Condition $[\mathbf{C}]_\#$. Assertion (iii) of that proposition in our case is $C(0) =$

$S(0) = 0$. By Theorem 9.1, the trigonometric identity (7.9) does not hold in the Leibniz case (since X has no unit). Similarly, Definition 7.2 holds with Condition $[\mathbf{C}]_{\#}$ replacing $[\mathbf{C}]$.

PROPOSITION 9.25 (cf. Proposition 7.6). *Suppose that Condition $[\mathbf{C}]_{\#}$ is satisfied, X has a unit e, the trigonometric identity holds and $(L, E) \in G[L]$. If $w \in \mathbb{C}(X)$ then there is an x such that $ix \in \operatorname{dom} L$ and for some $S \in \mathcal{L}_D$,*

$$Lw = c_D(L\lfloor w\rfloor + ix) + S[\lfloor w\rfloor^{-1}E(-ix)f_D(\lfloor w\rfloor, E(ix))],$$

namely (by Definition 7.2),

$$x = -iL[u' + (e - (u')^2)^{1/2}], \quad \textit{where} \quad u' = \tfrac{1}{2}(w + w^*)\lfloor w\rfloor^{-1}.$$

PROPOSITION 9.26 (cf. Proposition 7.7). *Suppose that all assumptions of Proposition 9.25 are satisfied, X is a complete linear metric space (over $\mathbb{C}$), $w \in \mathbb{C}(X)$ and $x = \arg w$. If $x \in \mathcal{E}_D(X)$ then also*

$$x' = x + 2\pi ike = \arg w \quad \textit{for } k \in \mathbb{Z}.$$

COROLLARY 9.9 (cf. Corollary 7.8). *Suppose that all assumptions of Proposition 9.25 are satisfied and X is a complete linear metric space (over $\mathbb{C}$). Then for all $w \in \mathbb{C}(X)$ and $x = \arg w \in \mathcal{E}_D(X)$ the mapping Ei and the trigonometric mappings are $2\pi e$-periodic, i.e.*

$$E[i(x + 2\pi e)] = E(ix), \quad C(x + 2\pi e) = Cx, \quad S(x + 2\pi e) = Sx.$$

10. Other generalizations

Denote by $\mathcal{K}(X)$ the ideal of all finite-dimensional operators belonging to $L_0(X)$. It is well-known (cf. for instance, Przeworska-Rolewicz and Rolewicz [1]) that an operator $A \in L(X)$ has a finite *nullity* (resp. *deficiency*) if and only if there is a $K \in \mathcal{K}(X)$ such that

$$A = \Delta + K, \quad \text{where } \Delta \in \Lambda(X) \quad (\text{resp. } \Delta \in R(X)).$$

THEOREM 10.1. *Let $X \in \mathbf{Lg}(D)$, $(L, E) \in G[\Omega]$ and $K \in \mathcal{K}(X)$. Write*

$$D' = D + K, \qquad \Omega' : \operatorname{dom} D' \to 2^{\operatorname{dom} D'},$$
$$\Omega' u = \{x \in \operatorname{dom} D' : D'u = uD'x\}, \qquad u \in \operatorname{dom} D' = \operatorname{dom} D.$$

For $u \in I(X) \cap \operatorname{dom}\Omega$ and $R \in \mathcal{R}_D$, set

$$L'u = Lu + R[u^{-1}Ku - KLu].$$

Then the mapping L' is a selector of Ω'. If L' is invertible and $E' = L'^{-1}$ then $(L', E') \in G[\Omega']$.

Proof. Let $D' = D + K$, $u \in I(X) \cap \mathrm{dom}\,\Omega$, $x = Lu$ and $x' = x + R[u^{-1}Ku - Kx]$. Then

$$D'u = Du + Ku = uDx + Ku = u(D'x - Kx) + Ku$$
$$= uD'[x + R(u^{-1}Ku - Kx)] = uD'x'.$$

Hence $Dx' = DL'u$, which proves that L' is a selector of Ω'. ∎

In the same manner we obtain

THEOREM 10.2. *Let $X \in \mathbf{Lg}_{\#}(D)$, $(L, E) \in G[\Omega]$ and $K \in \mathcal{K}(X)$. Let Ω' be as in Theorem* 10.1. *For $u \in I(X) \cap \mathrm{dom}\,\Omega$ and $S \in \mathcal{L}_D$, set*

$$L'u = Lu + S[u^{-1}Ku - KLu].$$

Then the mapping L' is a selector of Ω'. If L' is invertible and $E' = L'^{-1}$ then $(L', E') \in G[\Omega']$.

LEMMA 10.1. *Suppose that all assumptions of Theorem* 10.1 (*respectively*, *of Theorem* 10.2) *are satisfied. Let*

$$Kx = \sum_{k=1}^{n} \varphi_k(x) x_k \quad \text{for } x \in X,$$

where $\varphi_1, \ldots, \varphi_n$ belong to a conjugate space X' (i.e. a total space of linear functionals over X) and $x_1, \ldots, x_n \in X$ are linearly independent. Then the operator $I - RK$ (respectively, $I - SK$) is invertible if and only if

$$\Delta(R) = \det(\delta_{jk} - \varphi_j(Rx_k))_{j,k=1,\ldots,n} \neq 0$$

(*respectively*, $\Delta(S) \neq 0$). *In that case*,

$$v = (I - RK)^{-1} w = w + \sum_{k=1}^{n} v_k R x_k,$$

where $(v_1, \ldots, v_n)$ is the unique solution of the Cramer system

$$\sum_{k=1}^{n} [\delta_{jk} - \varphi_j(Rx_k)] v_k = \varphi_j(w) \quad (j = 1, \ldots, n),$$

and $v_k = \varphi_k(v)$, i.e. $v_k = \Delta(R)^{-1} v_k(w)$, where $v_k(w)$ is the determinant $\Delta(R)$ with the k-th column replaced by $(\varphi_1(w), \ldots, \varphi_n(w))$ (a corresponding formula holds for $(I - SK)^{-1}$).

Proof. We obtain the Cramer system of algebraic equations by standard techniques for finite-dimensional operators. This system has a unique solution if and only if the corresponding determinant is different from zero. In that case, the operator $I - RK$ (respectively, $I - SK$) is invertible. ∎

An immediate consequence of Theorem 10.1 (respectively, Theorem 10.2) and Lemma 10.1 is

Corollary 10.1. *Suppose that all assumptions of Theorem* 10.1 (*respectively*, *of Theorem* 10.2) *are satisfied. Let*

$$Kx = \sum_{k=1}^{n} \varphi_k(x)x_k \quad \textit{for } x \in X,$$

where $\varphi_1, \ldots, \varphi_n$ *belong to a conjugate space* X' (*i.e. a total space of linear functionals over* X) *and* $x_1, \ldots, x_n \in X$ *are linearly independent. If there is an* $R \in \mathcal{R}_D$ (*respectively*, *an* $S \in \mathcal{L}_D$) *such that*

$$\Delta(R) = \det\left(\delta_{jk} - \varphi_j(Rx_k)\right)_{j,k=1,\ldots,n} \neq 0$$

(*respectively*, $\Delta(S) \neq 0$) *then there exists an invertible selector of* Ω'.

Proof. By our assumptions, $x = Lu$, i.e. $u = Ex$, $x' = Lu$ and

$$x' = L'u = Lu + R[u^{-1}Ku - KLu] = x + R(u^{-1}Ku - Kx).$$

This implies

$$(I - RK)x = x' - R(u^{-1}Ku) = x' - R[(Ex)^{-1}KEx] = w.$$

Putting $v = x$ in Lemma 10.1 we conclude that for every x' there exists a unique u such that $L'u = x'$ (x is already fixed). This means that L' is an invertible selector of Ω'. The same arguments work for left invertible operators. ∎

Further considerations for operators having either finite nullity or finite deficiency, i.e. *finite-dimensional perturbations* of right and left invertible operators, are similar to those given in the preceding sections. Observe that the operator RKL (respectively, SKL) which appears in Theorem 10.1 (respectively, in Theorem 10.2) is finite-dimensional. We obtain similar statements and formulae up to a finite-dimensional operator. Another way of obtaining such results is to consider a *paraalgebra* $\mathcal{P}(X;D)$ induced by an operator $D \in \mathsf{A}(X)$ and the quotient paraalgebra $\mathcal{P}(X;D)/\mathcal{K}(X)$ (cf. Przeworska-Rolewicz and Rolewicz [1], Lausch and Przeworska-Rolewicz [1]). This method, however, should be the subject of a separate paper, since it requires other complicated notions.

Now we shall give some remarks concerning the non-commutative case. Denote by $\mathcal{N}_D(a,b,c,d)$ the class of non-commutative algebras X over a field $\mathcal{F}$ of scalars with an operator $D \in L(X)$ such that the domain of D is a subalgebra of X and the product rule for D is the following:

$$D(xy) = cxDy + b(Dx)y + d(Dx)Dy + axy, \qquad x, y \in \operatorname{dom} D, \tag{10.1}$$

where $a, b, c, d \in X$. Non-Leibniz components of algebras belonging to $\mathcal{N}_D(a,b,c,d)$ have been considered by Kornacki [1].

Proposition 10.1. *Let* $X \in \mathcal{N}_D(a,b,c,d)$. *Then*

$$(e - c - b)De = d(De)^2 + a. \tag{10.2}$$

Proof. By (10.1), we have

$$De = De^2 = ceDe + b(De)e + ae^2 = (c+b)De + d(De)^2 + a,$$

which implies (10.2). ∎

PROPOSITION 10.2. *Let* $X \in \mathcal{N}_D(a,b,c,d)$. *Then*

(i) *if* $e \in \ker D$ *then* $a = 0$;
(ii) *if* $e \in \ker D$ *then* $(c-e)Dx = (b-e)Dx = 0$;
(iii) *if* $d = a = 0$ *and* $e - c - b$ *is not a zero divisor then* $e \in \ker D$;
(iv) *if* $b - e$ *and* $c - e$ *are not zero divisors then* $\operatorname{dom} D = \ker D$.

Proof. (i) and (iii) are immediate consequences of (10.2).

(ii) Let $x \in \operatorname{dom} D$. Then, by (10.1),

$$Dx = D(xe) = cxDe + b(Dx)e + d(Dx)De = bDx,$$
$$Dx = D(ex) = ceDx + b(De)x + d(De)Dx = cDx.$$

(iv) is an immediate consequence of our assumptions and (ii). ■

Let

$$\mathcal{N}_D(a,c,d) = \{X \in \mathcal{N}_D(a,b,c,d) : b = c\}. \tag{10.3}$$

This means that the product rule for an $X \in \mathcal{N}_D(a,c,d)$ is

$$D(xy) = c[xDy + (Dx)y] + d(Dx)Dy + axy \quad \text{if } x,y \in \operatorname{dom} D. \tag{10.4}$$

Denote by E^{c} the *commutant* of a subalgebra $E \subset X \in \mathcal{N}_D(a,c,d)$, i.e.

$$E^{\mathrm{c}} = \{y \in E : \forall_{x\in E}\ xy = yx\}.$$

PROPOSITION 10.3. *Let* $X \in \mathcal{N}_D(a,c,d)$ *and* $c,d \in X^{\mathrm{c}}$. *If* $D^0 = dD + cI$ *then*

$$D^0(xy) = (D^0x)D^0y + hxy \quad \textit{for } x,y \in \operatorname{dom} D, \ \textit{where } h = ad + c - c^2.$$

Proof. If $D^0 = dD + cI$ then $dD = D^0 - cI$. Since $c,d \in X^{\mathrm{c}}$, for all $x,y \in \operatorname{dom} D^0 = \operatorname{dom} D$ we have

$$\begin{aligned}
D^0(xy) &= dD(xy) + cxy = dc[xDy + (Dx)y] + d^2(Dx)Dy + daxy + cxy\\
&= c[xdDy + (dDx)y] + (dDx)(dDy) + adxy + cxy\\
&= c[x(D^0y - cy) + (D^0x - cy)] + (D^0x - cx)(D^0y - cy) + (ad + c)xy\\
&= cxD^0y - c^2xy + c(D^0x)y - c^2xy + (D^0x)D^0y - cxD^y\\
&\quad - c(D^0x)y + c^2xy + (ad + c)xy\\
&= (ad - c + c^2)xy + (D^0x)D^0y = (D^0x)D^0y + hxy. \ \blacksquare
\end{aligned}$$

An immediate consequence of this proposition is

COROLLARY 10.2. *Suppose that all assumptions of Proposition* 10.1 *are satisfied and* $h = ad + c - c^2 = 0$. *Then* $D^0 = dD + cI \in ML(X)$, *i.e.* $X \in \mathcal{N}_{D^0}(0,0,e)$.

EXAMPLE 10.1. Let $X \in \mathcal{N}_D(0,e,0)$, i.e. $a = d = 0$ and $c = e$. Hence

$$D(xy) = xDy + (Dx)y \quad \text{for } x,y \in \operatorname{dom} D.$$

Then X is said to be a *non-commutative Leibniz algebra*, briefly, an *ncL-algebra*. The most important example is

$$Dx = [x,\delta] \quad \text{where } [x,\delta] = x\delta - \delta x \text{ for a fixed } \delta \in X \text{ and all } x \in X.$$

Define the multifunction $\Omega : \operatorname{dom} D \to 2^{\operatorname{dom} D}$ by

$$\Omega u = \{x \in (DX)^{\mathrm{c}} \cap \operatorname{dom} D : Du = uDx\} \quad \text{for } u \in \operatorname{dom} D. \tag{10.5}$$

Clearly, if $x \in (DX)^{\mathrm{c}}$ then $uDx = (Dx)u$ for all $u \in \operatorname{dom} D$ and conversely.

Having defined the multifunction Ω by (10.5), we can consider, for right invertible and left invertible operators D, invertible selectors of Ω, as in the previous sections.

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