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CHARACTERIZING PLANE ANR'S BY EXISTENCE OF LOCAL MEANS

BY

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The concept of a mean on a topological space is a generalization of the concept of the arithmetic mean for real numbers. Precisely, a *mean* on a topological space X is a map $m: X\times X\to X$ such that $m(x, y) = m(y, x)$ and $m(x, x) = x$ for all x, y in X. Means have been studied, among others, by Eckmann, Bacon and Sigmon; see references.

In the following definition we localize this concept by requiring that the map m be defined only on a symmetric neighbourhood of the diagonal ΔX in $X\times X$:

Definition. A *local mean* on a topological space X is a pair (U, m), where U is a symmetric neighbourhood of the diagonal ΔX in $X\times X$ and $m: U\to X$ is a map satisfying $m(x, y) = m(y, x)$ for all (x, y) in U and $m(x, x) = x$ for all x in X.

What we have called a local mean may more properly be called a local 2-mean. We can define the notion of a local n-mean for any integer $n > 1$ in a similar fashion by considering the n-fold product X^n instead of $X\times X$ and requiring that U and m be invariant under the action of S_n, the group of permutations of n symbols, on X^n. The results proved for local means in this paper carry over for local n-means without difficulty.

A slightly different (but essentially identical) way of looking at local means and means is as follows. Given a space X consider the quotient space Y obtained from $X\times X$ by identifying (x, y) with (y, x) for all x, y in X. Let $p: X\times X\to Y$ be the identification map and let $Z = p(\Delta X)$. Clearly, X and Z are homeomorphic, and a local mean on X corresponds uniquely to a retraction of a neighbourhood of Z in Y onto Z, while a mean on X corresponds to a retraction of Y onto Z. In particular, it follows that every compact metric absolute neighbourhood retract admits a local mean on itself while every compact metric absolute retract admits a mean. There are, of course, other examples. For example, the solenoid admits a mean and, more generally, the n-solenoid admits an n-mean for all $n > 1$ (see [12]).

Not all spaces admit a local mean. Topological consequences of the existence of a mean on a space have been studied in [1], [3], [7], [11] and [12]. Bacon [1] proved that a plane compactum having a mean on it cannot disconnect the plane and an analogous result for higher dimensional Euclidean spaces was proved by Sigmon [12]. We shall show that among plane Peano continua, absolute neighbourhood retracts are the only ones which admit a local mean. For this we shall use the result of Bacon, just mentioned, as well as Borsuk's characterizations of plane absolute retracts and plane absolute neighbourhood retracts [4].

We begin with a few facts about locally connected continua. We consider only metric spaces. By a *continuum* we mean a compact, connected metric space. It is well known that a continuum is locally connected if and only if it is a Peano continuum, i.e. can be obtained as a continuous image of the unit interval.

LEMMA 1. *Given a locally connected continuum X, a point x in X and a neighbourhood V of x in X, there exists a neighbourhood N of x in X such that $N \subset V$ and N itself is a locally connected continuum.*

For a proof of this lemma, see [8], p. 219. The next result is proved in [10], p. 232.

LEMMA 2. *Suppose X is locally connected, and $X = C \cup D$, where C, D are closed subsets of X. If $C \cap D$ is locally connected, then both C and D are locally connected.*

We shall also need the following non-trivial fact about plane Peano continua:

THEOREM 1. *Let X be a compact, connected, locally connected subset of the plane. Then given any $\varepsilon > 0$, there are only finitely many components of the complement of X whose diameters exceed ε.*

A proof is given in [5], p. 232, or can also be constructed from a theorem of Schoenflies (see [10], p. 515).

Borsuk's characterization of plane absolute retracts (see [4], p. 137) states that if X is a non-empty Peano continuum in the plane E^2, which does not disconnect E^2, then there exists a retraction r_0 of E^2 onto X such that $r_0(E^2 - X) \subset \partial X$, where ∂X denotes the boundary of X in E^2.

We use this along with Lemmas 1 and 2 to prove

THEOREM 2. *Let X be a plane Peano continuum, $x \in X$, and V a neighbourhood of x in X. Then there exists a neighbourhood N of x, contained in V, and a retraction $r: X \to N$.*

Proof. Let $\varepsilon > 0$ be such that if C is the closed disk of a radius ε and center x, then $C \cap X \subset V$. $C \cap X$ is also a neighbourhood of x in X and so, by Lemma 1, there is a neighbourhood N' of x in X such that $N' \subset C \cap X$ and N' is a Peano continuum. Let M be the union of N' and all bounded components of $E^2 - N'$ and M' be the union of N' and of the unbounded component of $E^2 - N'$. Let $N = M \cap X$. Then, $N' \subset N$ and $N \subset C \cap X$ because the unbounded component of $E^2 - N'$ contains $E^2 - C$ and, consequently, $M \subset C$. Moreover, the boundary of M in E^2 is contained in N'.

Now, M and M' are closed subsets of E^2 such that $M \cup M' = E^2$ and $M \cap M' = N'$. Since E^2 and N' are locally connected, it follows from Lemma 2 that M (and also M') is locally connected. Obviously, M is also compact, connected and does not disconnect the plane. Hence, by Borsuk's theorem, there is a retraction r_0 of E^2 onto M such that $r_0(E^2 - M) \subset \partial M$, where ∂M is the boundary of M in E^2. We claim that r_0 maps X onto $N = M \cap X$. Clearly, $r_0(M \cap X) = M \cap X$. If $y \in X$ but $y \notin M$, then $r_0(y) \in \partial M$. But $\partial M \subset N' \subset N$. Thus r_0 defines a retraction $r: X \to N$. Since $N' \subset N$, N is a neighbourhood of x in X. Finally, $N \subset V$, and so the proof is complete.

We are now ready for the main result of this paper.

Theorem 3. *A plane Peano continuum X admits a local mean on itself if and only if it is an absolute neighbourhood retract.*

Proof. Clearly, only necessity of the condition needs to be proved. For this, in view of Borsuk's characterization of plane ANR's (see [4], p. 138), it suffices to show that $E^2 - X$ has only finitely many components. Suppose this is not the case. Then, we can write the components of $E^2 - X$ in an infinite sequence $U_0, U_1, U_2, \ldots$, where U_0 is the unbounded component. Pick any points x_i in U_i for all $i \geqslant 1$. The sequence $\{x_i\}$ is clearly bounded and, hence, has a convergent subsequence. We denote this subsequence by $\{y_n\}$, and the corresponding U's by V's. Thus $y_n \in V_n$ for each $n \geqslant 1$, and $\{y_n\}$ converges to a point, say, y^*. Clearly, y^* is in X. We claim that there is a compact neighbourhood N of y^* in X which admits a mean.

We are given a symmetric neighbourhood U of ΔX in $X \times X$ and a map $m: U \to X$ satisfying $m(x, y) = m(y, x)$ for all $(x, y) \in U$ and $m(x, x) = x$ for all $x \in X$. Since $(y^*, y^*) \in \Delta X$, there is a neighbourhood V of y^* in X such that $V \times V \subset U$. By Theorem 2, there is a neighbourhood N of y^* in X such that $N \subset V$ and a retraction $r: X \to N$. Define $f: N \times N \to N$ by $f(x, y) = r\big(m(x, y)\big)$ for x, y in N. Then f is a mean on N. Hence, by Bacon's result [1], N cannot disconnect E^2. But, in view of Theorem 1, we can see that this is not the case. Indeed, let $t > 0$ be such that if C is the closed disk of radius t and center y^*, then $C \cap X \subset N$. Choose k_1 such that, for $n \geqslant k_1$, $d(y_n, y^*) < t/4$ (by d we mean the usual metric on E^2). By Theorem 1, there also exists k_2 such that, for $n \geqslant k_2$, the diameter of V_n is less than $t/4$. Let $k = \max\{k_1, k_2\}$. Since $y_n \in V_n$ for all n, it follows that, for $n \geqslant k$, $V_n \subset C$. But the boundary of each V_n is contained in X.

Hence, for all $n \geqslant k$, the boundary of V_n is contained in $C \cap X$ and, therefore, in N. Thus, for $n, n' \geqslant k$, $n \neq n'$, y_n and $y_{n'}$ are in distinct components of $E^2 - N$. In particular, N disconnects E^2. This contradiction completes the proof of the theorem.

It should be noted that the hypothesis that X is locally connected cannot be dropped from the statement of the preceding theorem so as to contend that existence of a local mean on a plane continuum X implies that $E^2 - X$ has only finitely many components. The problem of deciding exactly which plane continua admit a local mean remains unsolved (**P 851**).

We give here an example of a plane continuum X with a local mean on it such that $E^2 - X$ has infinitely many components. Let Z be the subspace of the real line consisting of the point 0 and the points $1/n$ for each positive integer n. Let X be the non-reduced suspension on Z. A local mean on X can be constructed as follows:

Let $Y = Z \times I$, where I is the unit interval and let $p\colon Y \to X$ be the identification map which identifies $Z \times \{0\}$ to a single point of X and $Z \times \{1\}$ to another point of X. Define closed subsets A, B and C of Y as follows:

$$A = \{(z, t) \in Y \mid 0 \leqslant t \leqslant 1/2\},$$

$$B = \{(z, t) \in Y \mid 1/2 \leqslant t \leqslant 1\},$$

$$C = \{(z, t) \in Y \mid 1/4 \leqslant t \leqslant 3/4\}.$$

Let $V = (A \times A) \cup (B \times B) \cup (C \times C)$. Then V is a symmetric neighbourhood of the diagonal ΔY in $Y \times Y$. Let q be the map

$$q = p \times p\colon Y \times Y \to X \times X$$

and let

$$U = q(V).$$

Then U is a symmetric neighbourhood of the diagonal ΔX in $X \times X$. By $q'\colon V \to U$ we denote the map defined by q. Since V is compact, q' is an identification map.

To define a local mean $m\colon U \to X$ we first define a local mean $\mu\colon V \to Y$ as follows:

$$\mu\big((z_1, t_1), (z_2, t_2)\big) = (\min\{z_1, z_2\}, \min\{t_1, t_2\}) \text{ if } (z_i, t_i) \in A \text{ for } i = 1, 2.$$

$$\mu\big((z_1, t_1), (z_2, t_2)\big) = (\min\{z_1, z_2\}, \max\{t_1, t_2\}) \text{ if } (z_i, t_i) \in B \text{ for } i = 1, 2.$$

It remains to define μ for points of $C \times C$ in such a way that the definition will agree with the previous definitions of μ for poins of $(A \cap C) \times \times (A \cap C)$ and of $(B \cap C) \times (B \cap C)$. To do this, let W be the square $J \times J$, where J is the interval $[1/4, 3/4]$. Using the fact that J is an absolute extensor, we can easily get a map $r\colon W \to J$ which is symmetric (i.e., $r(t_1, t_2) = r(t_2, t_1)$ for t_1, t_2 in J) and such that

$$r(t_1, t_2) = \begin{cases} \min\{t_1, t_2\} & \text{if } 1/4 \leqslant t_1,\ t_2 \leqslant 1/2, \\ \max\{t_1, t_2\} & \text{if } 1/2 \leqslant t_1, t_2 \leqslant 3/4. \end{cases}$$

Now, if $(z_i, t_i) \in C$ for $i = 1, 2$, write

$$\mu\big((z_1, t_1), (z_2, t_2)\big) = \big(\min\{z_1, z_2\}, r(t_1, t_2)\big).$$

Putting these definitions of μ together, we get a well-defined map $\mu\colon V \to Y$ which is symmetric and such that $\mu(y, y) = y$ for all y in Y. The composition $p \circ \mu\colon V \to X$ induces a unique function $m\colon U \to X$ such that $m \circ q' = p \circ \mu$. Continuity of m follows from the fact that q' is an identification map. Moreover, m is symmetric and $m(x, x) = x$ for all x in X because μ enjoyes the corresponding properties. Thus the space X has a local mean. Clearly, X can also be thought of as a plane continuum for which $E^2 - X$ has infinitely many components.

Combining Theorem 3 with the following propositions, it is possible to answer a question raised by Bacon [2] about the existence of an acyclic Peano continuum which admits no mean:

PROPOSITION 1. *Suppose that X, Y are metric spaces, $X \subset Y$ and that X is a retract of some neighbourhood V in Y. Then if Y has a local mean, so does X.*

Proof. Let (U, m) be a local mean on Y. Write

$$U' = U \cap m^{-1}(V) \cap (X \times X).$$

Clearly, U' is a neighbourhood of the diagonal ΔX in $X \times X$. Also U' is symmetric since U and m are symmetric. Define $m'\colon U' \to X$ by $m'(x, y) = r\big(m(x, y)\big)$ for $(x, y) \in U'$, where $r\colon V \to X$ is a retraction. Then (U', m') is a local mean on X.

PROPOSITION 2. *Let X be a compact metric space and let CX be the cone over X. Then if CX has a mean, X has a local mean.*

Proof. This follows from Proposition 1 because X is, clearly, a neighbourhood retract of CX.

Now, let X be a plane Peano continuum which is not an ANR. Then Theorem 3 and Proposition 2 imply that CX admits no mean although CX is a contractible (and, hence, acyclic) Peano continuum. After these results were obtained another paper by Bacon [3] appeared, in which he gives an example where

$$X = \bigcup_{n=1}^{\infty} X_n,$$

each X_n being the circle of radius $1/n$ with center at $(1/n, 0)$ in the plane. In this particular example, the non-existence of a local mean on X can be proved by a very easy and direct argument, without invoking Theorem 3.

This proof of the fact that CX cannot have a mean seems simpler than that given in [3].

We conclude with a few remarks.

Remark 1. Using the well-known Dowker's lemma (see [9], p. 116), it is not hard to show that a contractible space has a local mean if and only if it has a mean. In view of this, we have actually given examples of acyclic Peano continua which do not even admit local means. The latter is also obvious from Proposition 1.

Remark 2. The proof of Theorem 3 relies heavily on Borsuk's characterizations of plane absolute retracts and plane absolute neighbourhood retracts, as well as Theorem 1, none of which holds in higher dimensional Euclidean spaces. The question whether Theorem 3 itself holds in higher dimensional Euclidean spaces remains open (**P 852**).

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REFERENCES

[1] P. Bacon, *Compact means in the plane*, Proceedings of the American Mathematical Society 22 (1969), p. 242-246.

[2] — *An acyclic continuum that admits no mean*, Fundamenta Mathematicae 67 (1970), p. 11-13.

[3] — *Unicoherence in means*, Colloquium Mathematicum 21 (1970), p. 211-215.

[4] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa 1967.

[5] E. Čech, *Point sets*, Prague 1969.

[6] B. Eckmann, *Räume mit Mittelbildungen*, Commentarii Mathematici Helvetici 28 (1954), p. 329-340.

[7] — T. Ganea and P. Hilton, *Generalized means* from *Studies in analysis and related topics*, Stanford 1962.

[8] D. Hall and G. Spencer II, *Elementary topology*, New York 1955.

[9] W. T. Hu, *Theory of retracts*, Detroit 1965.

[10] K. Kuratowski, *Topology*, Vol. 2, Warszawa 1968.

[11] K. Sigmon, *A note on means in Peano continua*, Aequationes Mathematicae 1 (1968), p. 85-86.

[12] — *Acyclicity of compact means*, Michigan Mathematical Journal 16 (1969), p. 111-115.

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