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Universal rational spaces

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CONTENTS

1. Introduction

All spaces in this paper are separable metric with totally bounded metric. A space is said to be *rational* iff it has a basis of open sets with countable boundaries. Necessarily, rational spaces are one-dimensional. A space is said to be *scattered* iff every nonempty subset has an isolated point. Scattered spaces are classified according to their *topological type* by the countable ordinals. Necessarily, scattered spaces are zero-dimensional. Rational spaces which have a basis with scattered boundaries are called *rim-scattered spaces* and are classified according to their *rim-type* by the countable ordinals. (See Section 2.1 for the definitions of topological type and rim-type.) Rim-scattered spaces which have a basis with finite boundaries are said to be *rim-finite spaces*.

A space X is said to be *universal* for a class $\mathscr{C}$ of spaces iff both of the following conditions are satisfied: (1) $X \in \mathscr{C}$, and (2) for each $Y \in \mathscr{C}$, there is an embedding $e: Y \to X$. If only condition (2) is satisfied, we say that X is a *containing space* for class $\mathscr{C}$.

In this paper we study containing spaces for compact rim-scattered spaces and universal spaces for (possibly non-compact) rational and rim-scattered spaces. In a subsequent paper we will consider the similar spaces that arise when we restrict our attention to subsets of the plane.

1.1. History. The classification of rational compacta by their rim-type forms part of classical topology [K], II, p. 290. Our present study of rim-scattered and rational spaces was motivated by conversations held among the co-authors and Professor S. D. Iliadis at the University of Saskatchewan over a period of several weeks in the Fall and Winter of 1983. Many of the questions we address in this paper were raised at that time by Professor Iliadis. Our current investigations have benefited from these discussions.

Subsequent to these conversations, the co-authors and Professor Iliadis have investigated rational spaces independently, utilizing somewhat different methods. Our methods grew out of an earlier interest in the Menger Universal One-Dimensional Curve [M–O–T].

Investigation of containing and universal spaces for rim-scattered spaces begins with the classical result of Nöbeling [N] that there is no universal element in the class of rim-finite compacta (that is, compacta of rim type 1). Iliadis [I–3] proved that for each countable ordinal $\alpha > 0$, the class of compacta (continua) of rim-type $\leqslant \alpha$ contains no universal element. In [I–2] Iliadis showed that there is a· rim-finite T_1-space (nonmetric) which contains every rim-finite space. For rational spaces, it is well known [K] that the class of rational compacta (continua) contains no universal element.

One is led by the above results to consider three general sorts of questions regarding universal spaces and "minimal" or "best possible" containing spaces:

(1) Are there metric containing compacta of minimal rim-type for compacta of a given rim-type?

(2) In general, is there a (necessarily noncompact) universal space for spaces of a given rim-type (or for rational spaces)?

(3) What can be said in a similar vein about subsets of the plane?

In what follows, we address the first two of the above questions. We reference, where appropriate, specific questions that have since appeared in Professor Iliadis' papers [I–2], [I–3], and [I–4], and which may have been raised in one form or another in our aforementioned conversations.

Let $\mathscr{P}$ be one of the following classes of spaces:

(1) Spaces of rim-type $\leqslant \alpha$.

(2) Rational spaces.

(3) One-dimensional spaces.

For each of the above classes, we will define later a $\mathscr{P}$-*defining sequence* of partitions. The definition of a $\mathscr{P}$-defining sequence is inspired by Anderson's characterization theorem for the Menger curve in [A–1]. Techniques for handling partitions similar to those used herein are developed rather completely in [M–O–T]. The readers of the present paper may find it helpful to read the derivation of the properties of the Menger curve in [M–O–T].

The following theorem includes the well-known theorem that the Menger curve is a containing space (in fact, a universal space) for one-dimensional spaces:

1.2. THEOREM. *If X is a complete space with a $\mathscr{P}$-defining sequence of partitions, then X is a containing space for class $\mathscr{P}$.*

A space with a $\mathscr{P}$-defining sequence of partitions need not itself be in class $\mathscr{P}$. Hence, Theorem 1.2 does not, in general, give a universal space for class $\mathscr{P}$.

We construct geometrically in E^3 spaces M_α and M in the class $\mathscr{P}$ (for $\mathscr{P}$ the class (1) or (2) above, respectively) which admit $\mathscr{P}$-defining sequences. In the metric inherited from E^3, M_α and M are not complete. Though M_α is topologically complete, M is not. However, the proof that each is universal for its respective class is an easy modification of the proof of Theorem 1.2. Our specific universal and containing space theorems are the following:

1.3. RIM-TYPE $\leqslant \alpha$ UNIVERSAL SPACE THEOREM. *For each countable ordinal $\alpha > 0$, there is a universal element M_α in the class of spaces of rim-type $\leqslant \alpha$. Moreover, M_α is topologically complete.*

1.4. UNIVERSAL RATIONAL SPACE THEOREM. *There is a universal element M in the class of rational spaces.*

1.5. RIM-FINITE CONTAINING SPACE THEOREM. *There is a Peano continuum C_2 of rim-type 2 which is a containing space for all rim-finite spaces.*

1.6. RIM-TYPE $< \alpha$ CONTAINING SPACE THEOREM. *For each countable ordinal $\alpha > 0$, there is a Peano continuum C_α of rim-type α which is a containing space for all compacta of rim-type $< \alpha$.*

In [T], Tymchatyn has shown that the spaces which admit rational compactifications are exactly the rim-scattered spaces. Extending this result, it is the main theorem of Iliadis and Tymchatyn in [I–T] that a space of rim-type $\leqslant \alpha + n$, for α a limit ordinal (possibly 0) and n a nonnegative integer, has a compactification of rim-type $\leqslant \alpha + 2n + \min\{\alpha, 1\}$, and that this is the best result possible. Thus the hypothesis of compactness cannot be removed from Theorem 1.6.

In Section 2, we introduce the concepts of topological type, rim-type, decompositions and partitions of a given rim-type, prove certain embedding theorems for spaces of a given topological type, and prove the Rim Decomposition Theorem 2.5.2. In Section 3, we define defining sequence of α-partitions, and we develop partition-matching techniques for constructing embeddings. These techniques are largely combinatorial in nature. In Section 4, we apply these techniques to prove first Theorem 4.2, which is a special case of Theorem 1.2. Theorem 4.2 is not used anywhere in this paper, but its proof is the model for all subsequent proofs in the paper. In Section 5, we construct the required universal and containing spaces to prove Theorems 1.3 through 1.6. The reader may find it helpful to read the construction of the universal space of rim-type $\leqslant \alpha$ in Section 5 before attempting the details of the proofs in Sections 3 and 4.

At about the same time that we obtained the results in this paper, Iliadis announced the existence for each α of a universal space of rim-type $\leqslant \alpha$, and the existence of a universal rational space. His methods involve decomposition spaces of subsets of the Cantor set. We wish to thank Professor Iliadis for introducing us to the study of rim-scattered spaces with the object of discovering containing and universal spaces. We also wish to thank him for pointing out an error in an earlier version of this paper.

All spaces in this paper are separable metric spaces with totally bounded metric. A *compactum* is a compact metric space; a *continuum* is a connected compactum; a *Peano continuum* is a locally connected continuum. A space

has *Property S* iff for every $\varepsilon > 0$, it can be represented as the union of finitely many connected sets of diameter $< \varepsilon$. A space is *uniformly locally connected* iff for every $\varepsilon > 0$, there is a $\delta > 0$, such that if two points are within δ of each other, then there is a connected open set of diameter $< \varepsilon$ containing them.

2. Rim-type and decompositions

The proofs of Theorems 1.3–1.6 in Section 4 involve first, constructing a containing (or universal) space for the given class of spaces, and second, showing how to embed any member of the given class in the space constructed. In this section, we construct universal spaces of type α and obtain some elementary properties of spaces of rim-type α.

2.1. Type and rim-type. Let A be a nonempty subset of a metric space (X, d). For $\varepsilon > 0$, we let $S(A, \varepsilon) = \{x \in X|\ d(x, A) < \varepsilon\}$. By $\mathrm{Bd}(A)$ and $\mathrm{Cl}(A)$ we denote the boundary and closure of A, respectively. By $\mathrm{Int}(A)$ we denote the interior of A, namely $A - \mathrm{Bd}(A)$. By $\mathrm{card}(A)$ we denote the cardinality of A. If $\{A_i\}_{i=1}^{\infty}$ is a sequence of subsets of X and $\lim \mathrm{diam}(A_i) = 0$, we say that, $\{A_i\}_{i=1}^{\infty}$ is a *null sequence*. A basis $\mathscr{B}$ of X is a *null basis* iff $\mathscr{B}$ is a null sequence.

2.1.1. Topological type. Let A be a subset of X. Let $A^{(0)} = A$, let $A^{(1)}$ be the set of limit points of A, and for each countable ordinal $\alpha \geqslant 0$, define $A^{(\alpha+1)} = (A^{(\alpha)})^{(1)}$. If α is a limit ordinal, let $A^{(\alpha)} = \bigcap\{A^{(\beta)}|\ \beta < \alpha\}$. We call $A^{(\alpha)}$ the *derivative of A of order* α or the α-*derivative* of A.

If A is scattered, then for some countable ordinal $\alpha \geqslant 0$, $A^{(\alpha)} = \varnothing$. If $A^{(\alpha)} = \varnothing$, we say that A is of *type* $\leqslant \alpha$, and write $\mathrm{type}(A) \leqslant \alpha$. If α is the least such ordinal, we say that A is of *type* α, and we write $\mathrm{type}(A) = \alpha$. If $B \subset A$, then $\mathrm{type}(B) \leqslant \mathrm{type}(A)$. If A and B are closed sets of type $\leqslant \alpha$, then $A \cup B$ is of type $\leqslant \alpha$. A compactum is scattered iff it is countable. A scattered compactum has type α for some isolated ordinal α. There exist compacta of type α for every *isolated* countable ordinal α. A compactum has type 1 iff it is finite and nonempty. (See [K, II] for proofs of some of the statements above.)

2.1.2. Rim-type. A space X is said to be of *rim-type* $\leqslant \alpha$ iff X has a basis of open sets whose boundaries each have type $\leqslant \alpha$, and we write rim-type $(X) \leqslant \alpha$. If α is the least such ordinal, then we say that X is of *rim-type* α, and we write rim-type$(X) = \alpha$. If $B \subset A$, then rim-type$(B) \leqslant$ rim-type(A). A space has rim-type 0 iff it is zero-dimensional. A space has rim-type 1 iff it has a basis of open sets with discrete boundaries. A compactum has rim-type 1 iff it has a basis of open sets with finite boundaries. Hence, a compactum of rim-type $\leqslant 1$ is said to be *rim-finite*. There exist continua of rim-type α for each countable ordinal $\alpha > 0$.

2.2. Decompositions. Let X be a space. Let $\mathscr{U}$ be a finite collection of closed subsets of X such that

(1) $\mathscr{U}$ covers X,

(2) for each $U \in \mathscr{U}$, $\mathrm{Int}(U)$ is dense in U and $\dim(\mathrm{Bd}(U)) \leqslant 0$, and

(3) for all $U \neq V \in \mathscr{U}$, $\mathrm{Int}(U) \cap \mathrm{Int}(V) = \varnothing$.

Then we say that $\mathscr{U}$ is a *decomposition* of X. We say that $\mathscr{U}$ is of *order* $\leqslant n$ iff the intersection of any $n+1$ distinct elements of $\mathscr{U}$ is empty. For the remainder of this paper we shall assume that all decompositions are of order $\leqslant 2$.

2.2.1. Decreasing sequences of decompositions. A decomposition $\mathscr{U}$ is said to *refine* a decomposition $\mathscr{V}$ iff every element of $\mathscr{U}$ is contained in some element of $\mathscr{V}$. For any decomposition $\mathscr{W}$, we define

$$\mathrm{mesh}(\mathscr{W}) = \max\{\mathrm{diam}(W)| \ W \in \mathscr{W}\}.$$

A sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of decompositions is said to be a *decreasing sequence* of decompositions iff, for all $i > 0$, $\mathscr{U}_{i+1}$ refines $\mathscr{U}_i$, and $\mathrm{mesh}(\mathscr{U}_i)$ approaches 0 as i approaches ∞.

2.2.2. Notation. Let $\mathscr{U}$ refining $\mathscr{V}$ be decompositions with $Y \subset \bigcup \mathscr{V}$ and $\mathscr{W} \subset \mathscr{U}$. We adopt the following notational conveniences:

(1) $\mathscr{W}(Y) = \{U \in \mathscr{W}| \ U \subset Y\}$.

(2) $\mathscr{U}|Y = \{U \cap Y| \ U \in \mathscr{U}\}$.

(3) $\partial \mathscr{W} = \bigcup\{W_1 \cap W_2| \ W_1 \neq W_2 \in \mathscr{W}\}$.

(4) The *star of Y in $\mathscr{W}$* is

$$\mathrm{Star}(Y, \mathscr{W}) = \{W \in \mathscr{W}| \ W \cap Y \neq \varnothing\}.$$

(5) The *boundary of Y in $\mathscr{W}$* is

$$\mathrm{Bound}(Y, \mathscr{W}) = \{W \in \mathscr{W}(Y)| \ \text{for some} \ V \in \mathscr{W} - \mathscr{W}(Y), V \cap W \neq \varnothing\}.$$

(6) The *core of Y in $\mathscr{W}$* is

$$\mathrm{Core}(Y, \mathscr{W}) = \mathscr{W}(Y) - \mathrm{Bound}(Y, \mathscr{W}).$$

2.2.3. Strongly decreasing sequences of decompositions. A decomposition $\mathscr{U}$ is said to *strongly refine* a decomposition $\mathscr{V}$ iff $\mathscr{U}$ refines $\mathscr{V}$ and, for all $U_1, U_2, V_1, V_2 \in \mathscr{V}$ such that $U_i \neq V_i$ ($i = 1, 2$) and $\{U_1, V_1\} \neq \{U_2, V_2\}$,

$$\mathrm{Star}(U_1 \cap V_1, \mathscr{U}) \cap \mathrm{Star}(U_2 \cap V_2, \mathscr{U}) = \varnothing.$$

A decreasing sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ is said to be a *strongly* decreasing sequence iff $\mathscr{U}_{i+1}$ strongly refines $\mathscr{U}_i$, for each i.

2.2.4. Isomorphisms of decompositions. Let $\mathscr{V}_1$ and $\mathscr{U}_1$ be decompositions of a space X with $\varphi_1: \mathscr{V}_1 \to \mathscr{U}_1$ a one-to-one, onto function such that for all $U, V \in \mathscr{V}_1$,

$$U \cap V \neq \varnothing \quad \text{iff} \quad \varphi_1(U) \cap \varphi_1(V) \neq \varnothing.$$

Then we say that φ_1 is an *isomorphism*. Suppose futher that $\mathscr{V}_2$ and $\mathscr{U}_2$ are decompositions of order $\leqslant 2$ refining $\mathscr{V}_1$ and $\mathscr{U}_1$, respectively. Let φ_2: $\mathscr{V}_2 \to \mathscr{U}_2$ be an isomorphism such that for all $V_2 \in \mathscr{V}_2$, for all $V_1 \in \mathscr{V}_1$,

$$V_2 \subset V_1 \quad \text{iff} \quad \varphi_2(V_2) \subset \varphi_1(V_1).$$

We then say φ_2 is an isomorphism *with respect to* φ_1.

The following theorem is a generalized version of a well-known theorem for inducing homeomorphisms by matching covers of compacta.

2.2.5. DECOMPOSITION MATCHING THEOREM. *Let X and Y be spaces with* $\{\mathscr{U}_i\}_{i=1}^\infty$ *and* $\{\mathscr{V}_i\}_{i=1}^\infty$ *decreasing sequences of decompositions of X and Y, respectively. Suppose that for all $i > 0$.*

(1) φ_i: $\mathscr{V}_i \to \mathscr{U}_i$ *is an isomorphism,*

(2) φ_{i+1} *is an isomorphism with respect to φ_i, and*

(3) *for all* $y \in Y$, $\bigcap_{i=1}^\infty \bigcup \varphi_i(\text{Star}(y, \mathscr{V}_i)) \neq \varnothing$.

Then there is an embedding φ: $Y \to X$ *such that for all* $y \in Y$,

$$\varphi(y) = \bigcap_{i=1}^\infty \bigcup \varphi_i(\text{Star}(y, V_i)).$$

2.3. Containing and universal spaces for rim-scattered and rational spaces. In order to show the existence of a containing (universal) space of a given rim-type, it is first necessary to show the existence of a containing (universal) space of a given type. These spaces will form the boundary sets or "rims" of the decomposition elements used subsequently to construct containing (or universal) spaces for rim-scattered and rational spaces.

The following theorem is well-known and easy to prove:

2.3.1. UNIVERSAL COUNTABLE SPACE THEOREM. *The space Q of rational numbers is a universal space for all countable spaces. Moreover, if $p \in Q$, and U is neighborhood of p, then any countable space embeds in U.*

2.3.2. UNIVERSAL SPACE THEOREM FOR SPACES OF TYPE $\leqslant \alpha$. *For each countable ordinal* $\alpha \geqslant 0$, *there exists a universal space D_α of type α.*

Proof. The containing space D_α constructed below has been constructed previously in [I–T]. The empty set is the universal space D_0 of type 0. The set of positive integers is a universal space of type 1, since a space of type 1 is discrete.

Suppose for each ordinal number $\beta < \alpha_0$, there exists a universal space D_β of type β as in the theorem.

If $\alpha_0 > 1$ is a limit ordinal, let $2 = \alpha(1) < \alpha(2) < \ldots$ be an infinite sequence of ordinals coveringing to α_0. We prove that the free union

$$D_{\alpha_0} = \bigcup_{n=1}^\infty D_{\alpha(n)}$$

is a universal space of type α_0. It is clear from the definition of type that for $\gamma < \alpha_0$,

$$D_{\alpha_0}^{(\gamma)} \subset \bigcup \{ D_{\alpha(n)}^{(\gamma)} \mid \gamma < \alpha(n) < \alpha_0 \}.$$

So, $D_{\alpha_0}^{(\alpha_0)} = \varnothing$ and D_{α_0} is of type α_0.

Let A be a space of type α_0 and let $U_0 = A$. We may write $A = \{x_1, x_2, \ldots\}$, where $x_n \in A - A^{(\alpha(n))}$, for each n. Let U_1 be an open and closed set in A such that

$$A^{(\alpha(1))} \subset U_1 \subset A - \{x_1\}.$$

If U_n is defined to be an open and closed set in A such that

$$A^{(\alpha(n))} \subset U_n \subset (A - \{x_n\}) \cap U_{n-1},$$

let U_{n+1} be an open and closed set in A such that

$$A^{(\alpha(n+1))} \subset U_{n+1} \subset U_n - \{x_{n+1}\}.$$

By induction, $\{U_n\}_{n=0}^{\infty}$ is defined and $\bigcap_{n=1}^{\infty} U_n = \varnothing$. Then A embeds in D_{α_0} so that $U_n - U_{n+1} \subset D_{\alpha(n)}$, for each $n > 0$ and $U_0 - U_1 \subset D_{\alpha(1)}$.

Now suppose that $\alpha_0 > 1$ is an isolated ordinal. Let $\{x_0, x_1, \ldots\}$ be an infinite discrete space. Let

$$D_{\alpha_0} = \{x_0, x_1, \ldots\} \cup \bigcup_{i,j=0}^{\infty} T_{i,j}.$$

be the disjoint union of spaces where each $T_{i,j}$ is homemorphic to D_{α_0-1}. Let D_{α_0} be topologized so that U is open in D_{α_0} iff

(1) $U \cap T_{i,j}$ is open in $T_{i,j}$ for each (i, j), and

(2) if $x_i \in U$, then $T_{i,j} \subset U$ for all but finitely many values of j.

Let A be a space of type α_0. Then $A^{(\alpha_0 - 1)}$ is a discrete space. We may suppose that $A^{(\alpha_0 - 1)} = \{y_1, y_2, \ldots\}$. Let $\{U_i\}_{i=1}^{\infty}$ be a null sequence of pairwise disjoint, open and closed sets in A such that $y_i \in U_i$.

Let

$$U_0 = A - \bigcup_{i=1}^{\infty} U_i.$$

Then U_0 is closed in A. If $x \in U_0$, then there is a neighborhood U of x such that $\operatorname{diam}(U) < \frac{1}{2} d(A^{(\alpha_0 - 1)}, x)$. Since $\{U_i\}_{i=1}^{\infty}$ is a null sequence, each member of which meets $A^{(\alpha_0 - 1)}$, at most finitely many of the closed sets $\{U_i\}_{i=1}^{\infty}$ meet U. Hence, $U - \bigcup_{i=1}^{\infty} U_i$ is a neighborhood of x. Thus, U_0 is also open in A.

For each positive integer i, let $\{U_{i,j}\}_{j=0}^{\infty}$ be a null sequence of open and closed neighborhoods of y_i such that $U_{i,0} = U_i$ and $U_{i,j+1} \subset U_{i,j}$. For each

(i, j) with $i > 0$, let $h_{i,j}$: $U_{i,j} - U_{i,j+1} \to T_{i,j}$ be an embedding, and let h_0: $U_0 \to T_{0,0}$ be an embedding. Define h: $A \to D_{\alpha_0}$ by

$$h(x) = \begin{cases} h_0(x), & \text{if } x \in U_0, \\ h_{i,j}(x), & \text{if } x \in U_{i,j} - U_{i,j+1}, \\ x_i, & \text{if } x = y_i. \end{cases}$$

Then h is an embedding of X into D_{α_0}. It is shown in [I–T] that if $\alpha = \gamma + n$, γ a limit ordinal (possibly 0) and $n \geqslant 0$, then the compactification K of least type of D_α is of type $\gamma + 2n + \min\{\gamma, 1\}$. ∎

2.3.3. *Doubling decompositions of type α.* The definition below is motivated by the following elementary observation about the Cantor set C and the rationals Q: there is a decomposition of C (respectively, Q) into at least two mutually disjoint closed sets, each of which is homeomorphic to C (respectively, Q). It is possible to decompose the universal space D_α in a similar fashion.

If α is a countable limit ordinal and $D = \bigcup \{D_{\alpha(i)}\}_{i=1}^\infty$ where $\alpha(i) < \alpha(i+1) < \alpha$ is a universal space of type α (as constructed in Section 2.3.2), then for some j, let $\mathscr{D}$ be a decomposition containing the elements

$$D_{\alpha(j)}, \quad \bigcup_{i=1}^\infty D_{\alpha(j+2i)}, \quad \bigcup_{i=1}^\infty D_{\alpha(j+2i+1)}.$$

If α is a countable isolated ordinal and

$$D = \{x_1, x_2, \ldots\} \cup \bigcup_{i,j=1}^\infty T_{i,j}$$

is a universal space of type α (as constructed in Section 2.3.2), where each $T_{i,j}$ is a copy of $D_{\alpha-1}$, then for some k, let $\mathscr{D}$ be a decomposition containing the elements

$$\{x_k\} \cup \bigcup_{j=1}^\infty T_{k,j}, \quad \{x_{k+2i}\}_{i=1}^\infty \cup \bigcup_{i,j=1}^\infty T_{k+2i,j}, \quad \{x_{k+2i+1}\}_{i=1}^\infty \cup \bigcup_{i,j=1}^\infty T_{k+2i+1,j}.$$

If α is a countable isolated ordinal and $D = \{x\} \cup \bigcup_{i=1}^\infty T_i$, where each T_i is a copy of $D_{\alpha-1}$, for $i \neq j$, $T_i \cap (T_j \cup \{x\}) = \varnothing$, and $\lim T_i = \{x\}$, then for some j, let $\mathscr{D}$ be a decomposition containing the elements T_j and $\{x\} \cup \bigcup_{i=j+1}^\infty T_i$.

In each of the above cases we call the decomposition $\mathscr{D}$ a *doubling decomposition* of D. If D is as in one of the above cases and $\{\mathscr{D}_i\}_{i=1}^\infty$ is a decreasing sequence of decompositions of D such that $\mathscr{D}_1 = \{D\}$ and for each $B \in \mathscr{D}_i$, $\mathscr{D}_{i+1}(B)$ is a doubling decomposition of B, then $\{\mathscr{D}_i\}_{i=1}^\infty$ is said to be a *sequence of doubling decompositions* of D.

2.3.4. *Strongly-refining embedding.* Let $\{\mathscr{V}_i\}_{i=1}^\infty$ and $\{\mathscr{U}_j\}_{j=1}^\infty$ be decreasing sequences of decompositions of Y and X, respectively. Let $A \subset Y$, and let h: $A \to X$ be an embedding. We say that h is *strongly refining with respect to*

$\{\mathcal{V}_i\}_{i=1}^{\infty}$ and $\{\mathcal{U}_j\}_{j=1}^{\infty}$ iff for all $i \geqslant 1$, for all $V, V' \in \mathcal{V}_i$ with $V \cap V' = \emptyset$, there exists $j \geqslant 1$ such that

$$\mathrm{Star}\big(h(A \cap V), \mathcal{U}_j\big) \cap \mathrm{Star}\big(h(A \cap V'), \mathcal{U}_j\big) = \emptyset,$$

and if $i = 1$, then $j = 1$.

2.3.5. EMBEDDING THEOREM FOR SPACES OF TYPE α. *Let α be a countable ordinal, D_α a universal space of type α (as constructed in Section 2.3.2), and $\{\mathcal{D}_j\}_{j=1}^{\infty}$ a sequence of doubling decompositions of D_α. Let X be a space of type $\leqslant \alpha$, and let $\{\mathcal{W}_i\}_{i=1}^{\infty}$ be a decreasing sequence of order 1 decompositions of X. Then there exists an integer j_0 and an embedding $h: X \to D_\alpha$ which is strongly refining with respect to $\{\mathcal{W}_i\}_{i=1}^{\infty}$ and $\{\mathcal{D}_j\}_{j=j_0}^{\infty}$.*

Proof. We prove the theorem by induction on α. Suppose $\alpha = 1$. Then D_1 is a countably infinite discrete space and X is countable and discrete. Thus, $\mathcal{W}_1 = \{U_{1,1}, \dots, U_{1,k}, V_{1,1}, \dots, V_{1,m}\}$ where $U_{1,i}$ is a singleton for each i and $V_{1,i}$ is not a singleton for each i. Using a sequence of doubling decompositions of D_1, choose $j(1) > 1$ such that $\mathcal{D}_{j(1)}$ has at least k singleton elements and at least m infinite elements. Then there is a function $\varphi_1: \mathcal{W}_1 \to \mathcal{D}_{j(1)}$ such that $\varphi_1(U_{1,i})$ is a singleton element of $\mathcal{D}_{j(1)}$ for each i, $\varphi_1(V_{1,i})$ is an infinite element of $\mathcal{D}_{j(1)}$ for each i, and φ_1 is an isomorphism onto its image.

Similarly, there exists $j(2) > j(1)$ and $\varphi_2: \mathcal{W}_2 \to \mathcal{D}_{j(2)}$ such that φ_2 is an isomorphism with respect to φ_1 onto its image. Inductively, there exist $j(i) > j(i-1)$ and $\varphi_i: \mathcal{W}_i \to \mathcal{D}_{j(i)}$ such that φ_i is an isomorphism with respect to φ_{i-1} onto its image. Let $j_0 = j(1)$. Since every point $x \in X$ is in a singleton element of some $\mathcal{W}_i$, $\{\varphi_i\}_{i=1}^{\infty}$ induces an embedding $h: X \to D_1$, by Theorem 2.2.5, which is strongly refining with respect to $\{\mathcal{W}_i\}_{i=1}^{\infty}$ and $\{\mathcal{D}_j\}_{j=j_0}^{\infty}$.

As inductive hypothesis, assume the theorem is true for all ordinals $\alpha_0 < \alpha$, where α is a countable ordinal.

First suppose that $\alpha > 1$ is an isolated ordinal. Then

$$D_\alpha = \{x_1, x_2, \dots\} \cup \bigcup_{i,j=1}^{\infty} T_{i,j}$$

where $T_{i,j}$ is a universal space $D_{\alpha-1}$ of type $\alpha - 1$, and X is of type β for some $\beta \leqslant \alpha$. If $\beta < \alpha$, then by the induction hypothesis, there exists an integer j_α and an embedding $h: X \to D_{\alpha-1} = T_{1,1} \subset D_\alpha$ which is strongly refining with respect to $\{\mathcal{W}_i\}_{i=1}^{\infty}$ and $\{\mathcal{D}_j\}_{j=j_\alpha}^{\infty}$. So we may assume $\mathrm{type}(X) = \alpha$.

Then $X^{(\alpha-1)}$ is a nonempty countable discrete space, possibly infinite, and thus

$$\mathcal{W}_1 = \{U_{1,1}, \dots, U_{1,k}, V_{1,1}, \dots, V_{1,m}, W_{1,1}, \dots, W_{1,n}\}$$

where $\mathrm{type}\,(U_{1,i}) < \alpha$ for each i, $(V_{1,i})^{(\alpha-1)}$ is a singleton for each i, and $\mathrm{card}((W_{1,i})^{(\alpha-1)}) > 1$ for each i. Choose $j(1) \geqslant j_\alpha$ such that the decomposition $\mathcal{D}_{j(1)}$ of D_α has at least k elements of type $\alpha - 1$, at least m elements M such that $M^{(\alpha-1)}$ is a singleton, and at least n elements M such that $M^{(\alpha-1)}$ is infinite.

Let $\varphi_1\colon \mathscr{W}_1 \to \mathscr{D}_{j(1)}$ be an isomorphism onto its image such that $\varphi_1(U_{1,i})$ has type $\alpha - 1$ for each i, $(\varphi_1(V_{1,i}))^{(\alpha - 1)}$ is a singleton for each i, and $(\varphi_1(W_{1,i}))^{(\alpha - 1)}$ is infinite for each i.

Similarly, choose $j(2) > j(1)$ and define $\varphi_2\colon \mathscr{W}_2 \to \mathscr{D}_{j(2)}$ an isomorphism with respect to φ_1 onto its image such that if $V_{2,s} \subset V_{1,t} \in \mathscr{W}_1$, then

$$(\varphi_2(V_{2,s}))^{(\alpha - 1)} \neq \varnothing \quad \text{iff} \quad (V_{2,s})^{(\alpha - 1)} \neq \varnothing.$$

The φ_i's for $i > 2$ are constructed inductively as above. Since each point $x \in X^{(\alpha - 1)}$ is the unique point in $W^{(\alpha - 1)}$ for some W in some $\mathscr{W}_i$, the induced embedding $h\colon X \to D_\alpha$ is well-defined on $X^{(\alpha - 1)}$, and, hence, on X. Moreover, for $j_0 = j(1)$, h is strongly-refining with respect to $\{\mathscr{W}_i\}_{i=1}^\infty$ and $\{\mathscr{D}_j\}_{j=j_0}^\infty$.

Now suppose that $\alpha > 0$ is a limit ordinal. Then $D_\alpha = \bigcup_{i=1}^\infty D_{\alpha(i)}$ where $\{\alpha_i\}_{i=1}^\infty$ is a sequence of ordinals increasing to α, and for each i, $D_{\alpha(i)}$ is a universal space of type $\alpha(i)$. If type$(X) < \alpha$, then there are integers i_0 and j_α and an embedding of X in $D_{\alpha(i_0)}$, which is strongly refining with respect to $\{\mathscr{W}_i\}_{i=1}^\infty$ and $\{\mathscr{D}_j\}_{j=j_\alpha}^\infty$, by the induction hypothesis. So we may assume type$(X) = \alpha$.

Then $\mathscr{W}_1 = \{U_{1,1}, \ldots, U_{1,k}, V_{1,1}, \ldots, V_{1,m}\}$ where type$(U_{1,i}) < \alpha$ for each i and type$(V_{1,j}) = \alpha$ for each i. Choose $j_0 = j(1) \geqslant j_\alpha$ such that the decomposition $\mathscr{D}_{j(1)}$ of D_α contains k elements M_i such that type$(M_i) \geqslant$ type$(U_{1,i})$ and at least m other elements of type α. Let $\varphi_1\colon \mathscr{W}_1 \to \mathscr{D}_{j(1)}$ be an isomorphism onto its image such that $\varphi_1(U_{1,i}) = M_i$ for each i, and $\varphi_1(V_{1,i})$ is an element of type α for each i.

We continue as before to define the φ_i's inductively, and obtain an induced embedding $h\colon X \to D_\alpha$ which is strongly refining with respect to $\{\mathscr{W}_i\}_{i=1}^\infty$ and $\{\mathscr{D}_j\}_{j=j_0}^\infty$. For each $x \in X$, there is an i such that $x \notin X^{(\alpha(i))}$ (which set is closed in X). Hence, there is some $\mathscr{W}_j$ such that $x \in W \in \mathscr{W}_j$ and type$(W) < \alpha(i)$. Therefore, h is well defined on W by induction. ∎

2.3.6. Containing Space Theorem for Compacta of Type α. *Each compactum $B_{\alpha+1}$ of type $\alpha + 1$ is a containing space for compacta of type $\leqslant \alpha$. Moreover, if $\beta \leqslant \alpha$, $p \in (B_{\alpha+1})^{(\beta)}$, U is a neighborhood of p, and A is a compactum of type $\leqslant \beta$, then there is an embedding $e\colon A \to U$ such that $e(A) \cap (B_{\alpha+1})^{(\beta)} = \varnothing$.*

Proof. Note that compacta of type $\alpha + 1$ exist. The type of a scattered compactum and the cardinality of its highest nonempty derivative together characterize it, by a theorem of Mazurkiewicz and Sierpiński [M–S]. The theorem is an easy corollary to this. ∎

2.4. Decompositions and partitions. Let X be a connected, locally connected space and $\mathscr{U}$ a decomposition of X such that for all $U \in \mathscr{U}$, Int(U) is connected and has Property S. Then we say that $\mathscr{U}$ is a *partition of X*. If it is also the case that for all U, $V \in \mathscr{U}$, Int $(U \cup V)$ is uniformly locally connected, then we say that $\mathscr{U}$ is a *brick partition* of X.

An order 2 decomposition (respectively, partition, brick partition) $\mathscr{U}$ is said to be *rational* iff for all $U \in \mathscr{U}$, $\mathrm{Bd}(U)$ is countable. An order 2 decomposition (respectively, partition, brick partition) $\mathscr{U}$ is said to be an *α-decomposition* (respectively, *α-partition*, *brick α-partition*), for a countable ordinal $\alpha > 0$, iff for all $U \in \mathscr{U}$, $(\mathrm{Bd}(U))^{(\alpha)} = \varnothing$. We call a 1-decomposition whose elements all have finite boundaries a *rim-finite* decomposition.

A decomposition $\mathscr{V}$ is said to be an *amalgam* of a decomposition $\mathscr{U}$ iff each element of $\mathscr{V}$ is the union of a subcollection of $\mathscr{U}$ and $\bigcup \mathscr{V} = \bigcup \mathscr{U}$. We usually denote an amalgam of $\mathscr{U}$ by $\hat{\mathscr{U}}$. Note that an amalgam of a decomposition is again a decomposition. A subcollection $\bar{\mathscr{U}}$ of an amalgam $\hat{\mathscr{U}}$ of a decomposition $\mathscr{U}$ is said to be a *partial amalgam* of $\mathscr{U}$.

2.4.1. THEOREM. *Let X be a rim-scattered space. Then the following are equivalent*:

(1) *X is of rim-type $\leqslant \alpha$.*

(2) *Every pair of disjoint closed subsets of X can be separated by a closed set of type $\leqslant \alpha$.*

(3) *For each $\varepsilon > 0$ and each α-decomposition $\mathscr{U}$ of X, there is an α-decomposition $\mathscr{V}$ of X such that $\mathscr{V}$ strongly refines $\mathscr{U}$ and $\mathrm{mesh}(\mathscr{V}) < \varepsilon$.*

(4) *X has a strongly decreasing sequence of α-decompositions.*

PROOF. That (1) implies (2) is proved in Theorem 6 of [I–T]. To see that (2) implies (3), let $\mathscr{U}$ be an α-decomposition of X and let $\varepsilon > 0$. Fix $U \in \mathscr{U}$ and let $U_1, \ldots, U_n$ be the elements of $\mathscr{U} - \{U\}$ which meet U. Inductively, for each $i = 1, \ldots, n$, let V_i be a regular closed set in U such that $U_i \cap U \subset \mathrm{Int}_U(V_i)$ and $\mathrm{Bd}(V_i)$ has type $\leqslant \alpha$. Let $V_U = \mathrm{Cl}(U - \bigcup \{V_1, \ldots, V_n\})$. Let $\mathscr{W}(U) = \{V_1, \ldots, V_n, V_U\}$, and let $\mathscr{W} = \bigcup \{\mathscr{W}(U) \mid U \in \mathscr{U}\}$. Using (2), by standard decomposition arguments as in [M–O–T], one can find an α-decomposition $\mathscr{V}$ of X refining $\mathscr{W}$ such that $\mathrm{mesh}(\mathscr{V}) < \varepsilon$. Then $\mathscr{V}$ is the required strong refinement of $\mathscr{U}$. That (3) implies (4) and that (4) implies (1) is clear. ∎

2.4.2. THEOREM. *Let X be a space. Then the following are equivalent*:

(1) *X is rational.*

(2) *Every pair of disjoint closed subsets of X can be separated by a countable set.*

(3) *For each $\varepsilon > 0$ and each rational decomposition $\mathscr{U}$ of X, there is a rational decomposition $\mathscr{V}$ of X such that $\mathscr{V}$ strongly refines $\mathscr{U}$ and $\mathrm{mesh}(\mathscr{V}) < \varepsilon$.*

(4) *X has a strongly decreasing sequence of rational decompositions.*

Proof. The proof is as in Theorem 2.4.1, except that we use Theorem 5 of [I–T]. ∎

2.4.3. Chains, coherent collections, and core refinement. A subcollection $\{U_1, \ldots, U_n\}$ of a decomposition $\mathscr{U}$ is called a *chain* provided that $U_i \cap U_j \neq \varnothing$ iff $|i-j| \leqslant 1$, and it is called a *circular chain* provided that $U_i \cap U_j \neq \varnothing$ iff

$|i-j| \leqslant 1$ or $\{i, j\} = \{1, n\}$. A subcollection $\mathscr{V}$ of a decomposition $\mathscr{U}$ is said to be *coherent* iff for any pair of elements $U, V \in \mathscr{V}$, there is a chain in $\mathscr{V}$ with U as the first element and V as the last element. A subcollection $\mathscr{V}$ of $\mathscr{U}$ is called a *graph-chain* provided that $\mathscr{V}$ is coherent, and it is called a *tree-chain* provided that in addition $\mathscr{V}$ contains no circular chain.

If $\mathscr{V}$ is a graph-chain, we call the elements of $\mathscr{V}$ *links*, and we call those elements of $\mathscr{V}$ which meet at most one other element of $\mathscr{V}$ *endlinks*.

Let $\mathscr{U}$ refining $\mathscr{V}$ be partitions. We say that $\mathscr{U}$ *core refines* $\mathscr{V}$ iff for every $V \in \mathscr{V}$, $\mathrm{Core}(V, \mathscr{U})$ is a coherent nonempty subcollection of $\mathscr{U}(V)$ such that for every $U \in \mathrm{Bound}(V, \mathscr{U})$, $U \cap (\bigcup \mathrm{Core}(V, \mathscr{U})) \neq \varnothing$.

2.4.4. EXAMPLE. We give an example to show that Theorem 2.4.1 cannot be improved to obtain a decreasing sequence of α-partitions even when X is a Peano continuum. It is a special case of Bing's Partitioning Theorems [B] that every one-dimensional Peano continuum has a decreasing sequence of core-refining brick partitions. Though a rim-finite Peano continuum has arbitrarily small brick partitions by Bing's theorem, and has arbitrarily small rim-finite decompositions by Theorem 2.4.1, it cannot be expected that arbitrarily small, rim-finite brick partitions exist. For example, if we attach to each vertex of each triangle of the Sierpiński Triangular Curve [K, II, page 276], a null sequence of arcs, then the resulting rim-finite Peano continuum has *no* rim-finite brick partitions of small mesh.

2.4.5. THEOREM. [I–1, Theorem 3]. *Each compactum of rim-type $\leqslant \alpha$ can be embedded in a Peano continuum of rim-type $\leqslant \alpha$.*

Proof. Using a decreasing sequence of α-decompositions obtained *via* Theorem 2.4.1, and then carefully adding a null sequence of arcs between elements of the decompositions, one can show that each compactum of rim-type $\leqslant \alpha$ embeds in a Peano continuum of rim-type $\leqslant \alpha$. In fact, the construction can be carried out in such a way that the resulting Peano continuum *does* have a decreasing sequence of brick α-partitions. ∎

2.5. Incidence matrices and decompositions. In the proofs of Theorem 1.2–1.6 we will need to obtain, for an arbitrary space of rim-type $\leqslant \alpha$ or a rational space, a nice decomposition which matches a given decomposition of another space.

2.5.1. Incidence matrices. An $n \times n$ matrix $A = (a_{i,j})_{n \times n}$ whose entries are zeroes and ones is said to be an *incidence matrix* iff $a_{i,i} = 1$ and $a_{i,j} = a_{j,i}$,

and it is said to be a *coherent incidence matrix* iff for each $i \neq j$, there exists a set $\{i = i(0), i(1), \ldots, i(m) = j\} \subset \{1, \ldots, n\}$ such that for $k \in \{1, \ldots, m\}$, $a_{i(k-1), i(k)} = 1$. An incidence matrix $A = (a_{i,j})_{n \times n}$ is said to be the *incidence matrix of* the decomposition $\mathcal{U} = \{U_1, \ldots, U_n\}$ provided that $U_i \cap U_j \neq \emptyset$ iff $a_{i,j} = 1$.

2.5.2. Rim Decomposition Theorem. *Let X be a space of rim-type $\leqslant \alpha$ (respectively, rational space). Let $\{\mathcal{W}_i\}_{i=1}^{\infty}$ be a strongly decreasing sequence of α-decompositions (respectively, rational decompositions) of X. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a collection of pairwise disjoint closed subsets of X such that for $i \neq j$, there exists k such that $\mathrm{Star}(B_i, \mathcal{W}_k) \cap \mathrm{Star}(B_j, \mathcal{W}_k) = \emptyset$. Let $A = (a_{i,j})_{n \times n}$ be a coherent incidence matrix. Then there is an α-decomposition (respectively, rational decomposition) $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X such that*
(1) $\partial \mathcal{U} \cap (\bigcup \mathcal{B}) = \emptyset$,
(2) $B_j \subset U_j$,
(3) *if* $U_i \cap U_j \neq \emptyset$, *then* $a_{i,j} = 1$,
(4) $\mathcal{U} = \mathcal{W}_i$, *for some* i.

Proof. In the statement and proof of the theorem, note the following: some of the B_i's may be empty; since we may have simultaneously $a_{i,j} = 1$ and $U_i \cap U_j \neq \emptyset$, A need not be the incidence matrix of $\mathcal{U}$; the hypothesis that A is coherent cannot be omitted.

The proof is by induction on n. There are three cases: $n = 1$, $n = 2$, and $n > 2$. The first case is the base case, the second case is needed as a lemma for the third, inductive, case.

If $n = 1$, then $A = (1)$, and we take $\mathcal{U} = \{X\}$. Hence $\partial \mathcal{U} = \emptyset$, and the theorem follows trivially.

If $n = 2$, then $B = \{B_1, B_2\}$ and $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Choose i such that $\mathrm{Star}(B_1, \mathcal{W}_i) \cap \mathrm{Star}(B_2, \mathcal{W}_i) = \emptyset$. Without loss of generality, if $B_1 \neq \emptyset$, let $U_1 = \bigcup \mathrm{Star}(B_1, \mathcal{W}_i)$ and $U_2 = \bigcup(\mathcal{W}_i - \mathrm{Star}(B_1, \mathcal{W}_i))$. If $B_1 = \emptyset = B_2$, then let U_1 be any link of $\mathcal{W}_i$ and let $U_2 = \bigcup(\mathcal{W}_i - \{U_1\})$.

Suppose that $n > 2$ and that the theorem holds for $n-1$. Since A is coherent, we may assume without loss of generality that $a_{1,2} = 1$. Define the coherent incidence matrix $B = (b_{i,j})_{(n-1) \times (n-1)}$ by

$$
b_{i,j} = \begin{cases} a_{i+1,j+1}, & \text{if } i, j \geqslant 2, \\ \min\{1, a_{1,j+1} + a_{2,j+1}\}, & \text{if } i = 1, \\ \min\{1, a_{i+1,1} + a_{i+1,2}\}, & \text{if } j = 1. \end{cases}
$$

In a sense, B "amalgamates" two rows (respectively, columns) of A into a single row (respectively, column).

For $j \in \{2, \ldots, n-1\}$, let $B'_j = B_{j+1}$, let $B'_1 = B_1 \cup B_2$, and let $\mathscr{B}' = \{B'_1, \ldots$
$\ldots, B'_{n-1}\}$. By induction, there is an α-decomposition (respectively, rational decomposition) $\mathscr{U}' = \{U'_1, \ldots, U'_{n-1}\}$ of X such that

    (i) $\partial \mathscr{U}' \cap (\bigcup \mathscr{B}') = \varnothing$,

    (ii) for all $j \in \{1, \ldots, n-1\}$, $B'_j \subset U'_j$,

    (iii) for all $i \neq j \in \{1, \ldots, n-1\}$ $U'_i \cap U'_j \neq \varnothing$ implies $b_{i,j} = 1$,

    (iv) $\mathscr{U}' = \mathscr{W}_{i'}$ for some i'.

Define D_1 and D_2 to be disjoint closed sets in U'_1 such that

    (v) $D_1 = B_1 \cup (\bigcup \{U'_1 \cap U'_j | \; j > 1 \text{ and } a_{1,j+1} = 1\})$,

    (vi) $D_2 = B_2 \cup (\bigcup \{U'_1 \cap U'_j | \; j > 1 \text{ and } a_{1,j+1} = 0\})$.

Note that if $b_{1,j} = 1$ $(j > 1)$ and $a_{1,j+1} = 0$, then $a_{2,j+1} = 1$. Note that for some i, $\text{Star}(D_1, \mathscr{W}_i) \cap \text{Star}(D_2, \mathscr{W}_i) = \varnothing$. We apply the case $n = 2$ with U'_1 replacing X and $\{D_1, D_2\}$ replacing $\mathscr{B}$. We obtain a decomposition $\{U_1, U_2\}$ of U'_1 satisfying conditions (1)–(4).

For $j \in \{3, \ldots, n\}$, let $U_j = U'_{j-1}$, and let $\mathscr{U} = \{U_1, U_2, U_3, \ldots, U_n\}$. Then $\mathscr{U}$ is a decomposition of X satisfying the theorem. ∎
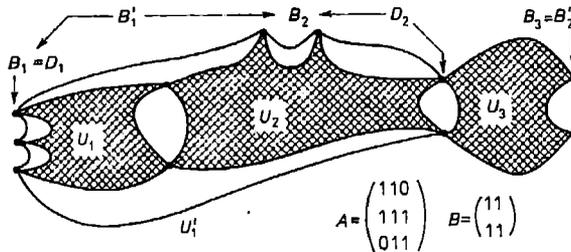

Fig. 2.1. Illustrates the rim-finite case for $n = 3$

3. Defining sequences and isomorphisms

We will define our containing (universal) spaces in terms of decreasing sequences of brick partitions of subsets of E^3 satisfying certain combinatorial properties. Chief among these are the properties of α-*connectedness*, α-*interlacing*, and α-*splitting*. The later two notions are closely related to similar notions used by Anderson in [A-1,2].

3.1. DEFINITIONS. Let $\mathscr{U}$ refining $\mathscr{V}$ be brick α-partitions of a space. Let $\mathscr{W}$ denote a subcollection of $\mathscr{U}$.

3.1.1. α-*Connected collections.* We say that $\mathscr{W}$ is α-*connected* iff $\mathscr{W}$ is coherent and for all $U \neq V \in \mathscr{W}$, if $U \cap V \neq \varnothing$, then $U \cap V$ contains a universal space of type α. If $\mathscr{W}$ is a chain (respectively, circular chain, graph-chain, or tree-chain) which is α-connected, then we say that $\mathscr{W}$ is an α-*chain* (respectively, α-*circular-chain*, α-*graph-chain*, or α-*tree-chain*).

3.1.2. *α-Interlacing.* We say that $\mathcal{U}$ is *α-interlaced* in $\mathcal{V}$ iff for every $V \in \mathcal{V}$, $\mathcal{U}(V)$ contains two α-tree-chains $\mathcal{U}_1$ and $\mathcal{U}_2$ such that the elements of $\mathcal{U}_1 \cap \mathcal{U}_2$ = Bound$(V, \mathcal{U})$ are endlinks of both $\mathcal{U}_1$ and $\mathcal{U}_2$, and $(\mathcal{U}_1 \cup \mathcal{U}_2)$-Bound$(V, \mathcal{U})$ is α-connected.

3.1.3. *α-Splitting.* We say that $\mathcal{U}$ *α-splits* $\mathcal{V}$ iff for all $U \neq V \in \mathcal{V}$,

(1) $\mathcal{U} | U \cap V$ is a doubling decomposition of $U \cap V$,

(2) for each $W_1 \in \mathcal{U}(U)$ such that $W_1 \cap V \neq \varnothing$, there is a unique $W_2 \in \mathcal{U}(V)$ such that $W_1 \cap V = W_2 \cap U = W_1 \cap W_2$.

3.2. α-Defining sequences of partitions. Let X be a space. We say that a strongly decreasing sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of decompositions of X is a *defining sequence of α-partitions* iff for all $i > 0$,

(1) $\mathcal{U}_i$ is a brick α-partition,

(2) $\mathcal{U}_{i+1}$ core refines $\mathcal{U}_i$,

(3) for all $U \in \mathcal{U}_i$, for all $V \neq W \in$ Bound$(U, \mathcal{U}_{i+1})$, $V \cap W = \varnothing$,

(4) $\mathcal{U}_1$ is α-connected,

(5) for all $U \in \mathcal{U}_i$, $\mathcal{U}_{i+1}(U)$ is α-connected,

(6) $\mathcal{U}_{i+1}$ α-splits $\mathcal{U}_i$,

(7) $\mathcal{U}_{i+1}$ is α-interlaced in $\mathcal{U}_i$.

Conditions (1)–(7) above are not entirely independent. Any decreasing sequence of brick α-partitions of a space wherein one partition is α-interlaced in its predecessor has a subsequence which very nearly satisfies the conditions of Definition 3.2. However, making the conditions explicit somewhat simplifies the constructions to follow.

The reader should note that in this paper a *defining sequence* is a much more restrictive notion than a *decreasing sequence.* The notion of a defining sequence of α-partitions gives rise to a universal space of rim-type α, while the notion of a decreasing sequence of α-decompositions gives rise merely to a space of rim-type $\leqslant \alpha$.

3.3. α-Amalgams. Let $\bar{\mathcal{U}}$ be a partial amalgam of a brick α-partition $\mathcal{U}$. We say that $\bar{\mathcal{U}}$ is an *α-amalgam* of $\mathcal{U}$ iff for all $U \in \bar{\mathcal{U}}$, $\mathcal{U}(U)$ is α-connected.

3.4. α-Interlacing Lemma. *Let X be a space with $\{\mathcal{U}_i\}_{i=1}^{\infty}$ a defining sequence of α-partitions of X. Then for all $i > 0$, for all $j > i$, $\mathcal{U}_j$ is α-interlaced in $\mathcal{U}_i$.*

Proof. We prove the lemma by induction on $j - i$. Since $\mathcal{U}_{i+1}$ is α-interlaced in $\mathcal{U}_i$, we have the base case. So assume that $\mathcal{U}_j$ is α-interlaced in $\mathcal{U}_i$ for $1 \leqslant j - i < n$, and suppose that $j - i = n$. The various parts of Figure 3.1 illustrate rather completely the argument which follows.

Fix $U \in \mathcal{U}_i$. Since $\mathcal{U}_{j-1}$ is α-interlaced in $\mathcal{U}_i$, there are α-tree-chains $\mathcal{W}_1$ and $\mathcal{W}_2$ in $\mathcal{U}_{j-1}(U)$ such that the elements of

$$\mathcal{W}_1 \cap \mathcal{W}_2 = \text{Bound}(U, \mathcal{U}_{j-1})$$

are endlinks of each of $\mathcal{W}_1$ and $\mathcal{W}_2$, and $(\mathcal{W}_1 \cup \mathcal{W}_2)$-$\text{Bound}(U, \mathcal{U}_{j-1})$ is α-connected. See Figure 3.1(a).
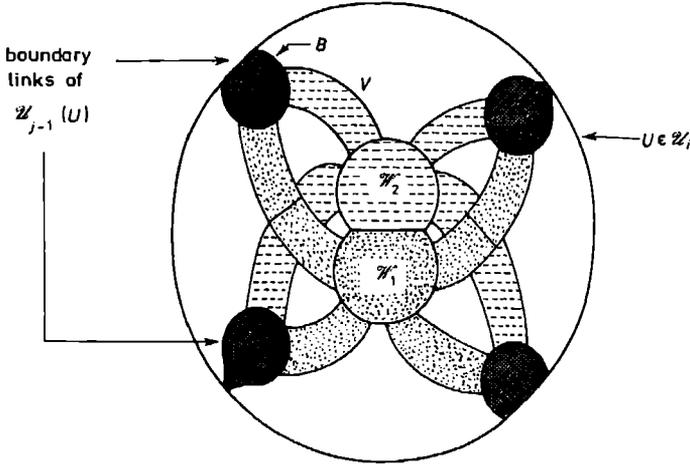

Fig. 3.1(a). α-Tree-chains $\mathcal{W}_1$ and $\mathcal{W}_2$ in $\mathcal{U}_{j-1}(U)$

We will construct α-tree-chains $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\mathcal{U}_j((\bigcup \mathcal{W}_1) \cup (\bigcup \mathcal{W}_2))$ by working on one link of $\mathcal{W}_k$, for $k \in \{1, 2\}$, at a time. Let $B \in \text{Bound}(U, \mathcal{U}_{j-1})$. Since $\mathcal{U}_j$ is α-interlaced in $\mathcal{U}_{j-1}$, there are α-tree-chains $\mathcal{W}_{B,1}$ and $\mathcal{W}_{B,2}$ in $\mathcal{U}_j(B)$ such that

$$\mathcal{W}_{B,1} \cap \mathcal{W}_{B,2} = \text{Bound}(B, \mathcal{U}_j)$$

is in the set of endlinks of each of $\mathcal{W}_{B,1}$ and $\mathcal{W}_{B,2}$. See Figure 3.1(b)
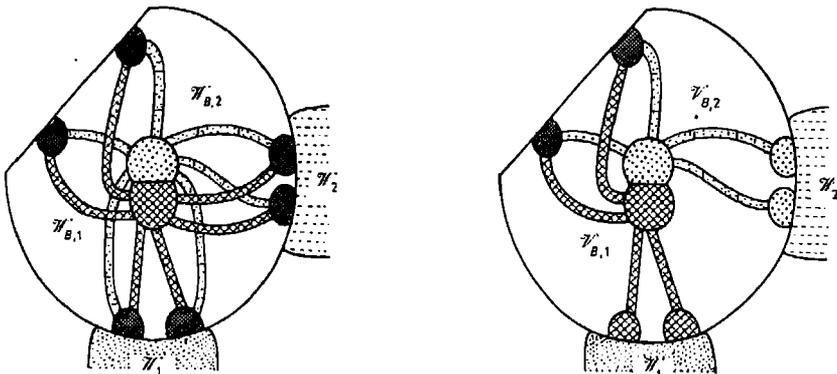

Fig. 3.1(b). Magnified view of boundary link $B \in \text{Bound}(U, \mathcal{U}_{j-1})$

For $k \in \{1, 2\}$, let $\mathscr{V}_{B,k}$ be an α-tree-chain in $\mathscr{W}_{B,k}$ minimal with respect to containing $\mathrm{Bound}(U, \mathscr{U}_j) \cap \mathscr{U}_j(B)$ and covering $B \cap K$ for each $K \in \mathscr{W}_k - \{B\}$ such that $B \cap K \neq \varnothing$. Thus, $\mathscr{V}_{B,k}$ reaches *out* to the boundary of U and *in* to the link of $\mathscr{W}_k - \{B\}$ which meets B. Since $\mathscr{U}_{j-1}$ is order two,

$$B \cap \left(\bigcup (\mathscr{W}_1 - \{B\})\right) \cap \left(\bigcup (\mathscr{W}_2 - \{B\})\right) = \varnothing.$$

It follows that

$$\mathscr{V}_{B,1} \cap \mathscr{V}_{B,2} = \mathrm{Bound}(U, \mathscr{U}_j) \cap \mathscr{U}_j(B) \subset \mathrm{Bound}(U, \mathscr{U}_j).$$

Since $(\mathscr{W}_1 \cup \mathscr{W}_2) - \mathrm{Bound}(U, \mathscr{U}_{j-1})$ is α-connected, there are links $G \in \mathscr{W}_1 - \mathrm{Bound}(U, \mathscr{U}_{j-1})$ and $H \in \mathscr{W}_2 - \mathrm{Bound}(U, \mathscr{U}_{j-1})$ such that $G \cap H$ contains a universal space of type α. Since $\mathscr{U}_j$ is α-interlaced in $\mathscr{U}_{j-1}$, there is an α-tree-chain $\mathscr{V}_{G,1} \subset \mathscr{U}_j(G)$ minimal with respect to covering $G \cap H$ and with respect to covering $G \cap K$ for each $K \in \mathscr{W}_1 - \{G\}$ such that $G \cap K \neq \varnothing$. Similarly, there is an α-tree-chain $\mathscr{V}_{H,1} \subset \mathscr{U}_j(H)$ minimal with respect to covering $H \cap G$ and with respect to covering $H \cap K$ for each $K \in \mathscr{W}_2 - \{H\}$ such that $H \cap K \neq \varnothing$.

Let $V \in \mathscr{W}_1 - \left(\mathrm{Bound}(U, \mathscr{U}_{j-1}) \cup \{G\}\right)$. There is an α-tree-chain $\mathscr{V}_{V,1} \subset \mathscr{U}_j(V)$ minimal with respect to covering $V \cap K$ for each $K \in \mathscr{W}_1 - \{V\}$ such that $V \cap K \neq \varnothing$. Similarly, for $V \in \mathscr{W}_2 - \left(\mathrm{Bound}(U, \mathscr{U}_{j-1}) \cup \{H\}\right)$, there is an α-tree-chain $\mathscr{V}_{V,2} \subset \mathscr{U}_j(V)$ minimal with respect to covering $V \cap K$ for each $K \in \mathscr{W}_2 - \{V\}$ such that $V \cap K \neq \varnothing$. See Figure 3.1(c).
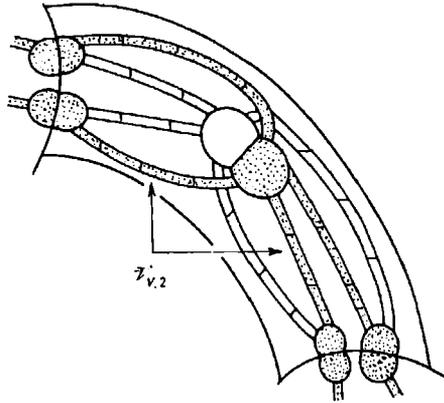

Fig. 3.1(c). Magnified view of link $V \in \mathscr{W}_2$

Within each link V of $\mathscr{W}_k$ $(k = 1, 2)$ we now have an α-tree chain. Moreover, by construction, $\mathscr{V}'_1 = \bigcup \{\mathscr{V}_{V,1} \mid V \in \mathscr{W}_1\}$ and $\mathscr{V}'_2 = \bigcup \{\mathscr{V}_{V,2} \mid V \in \mathscr{W}_2\}$ are graph-chains. For $k \in \{1, 2\}$, although $\mathscr{V}'_k$ need not be an α-graph-chain $\mathscr{V}'_k$ contains an α-tree-chain $\mathscr{V}_k$, minimal with respect to containing $\mathrm{Bound}(U, \mathscr{U}_j)$ and covering a universal space of type α in $G \cap H$. This follows because $\mathscr{U}_j$ α-splits $\mathscr{U}_{j-1}$. Then $\mathscr{V}_1$ and $\mathscr{V}_2$ are the desired α-tree-chains in $\mathscr{U}_j(\bigcup \mathscr{W}_1)$ (respectively, $\mathscr{U}_j(\bigcup \mathscr{W}_2)$) which show that $\mathscr{U}_j$ is α-interlaced in $\mathscr{U}_i$. See Figure 3.1(d). ∎

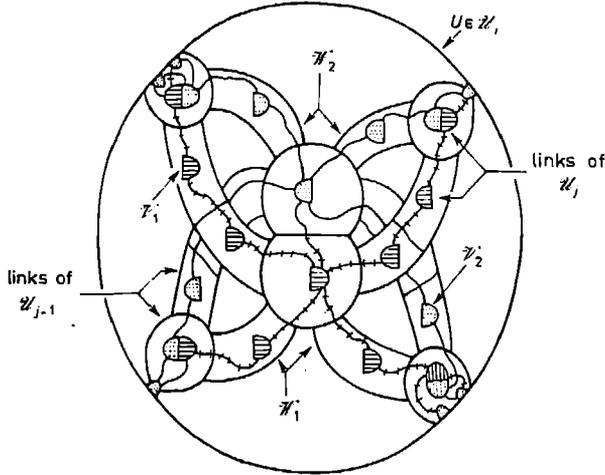Fig. 3.1(d). α-Tree-chains $\mathscr{V}_1$ and $\mathscr{V}_2$ each containing all boundary links of $\mathscr{U}_j(U)$

3.4.1. COROLLARY. *Let X be a space with $\{\mathscr{U}_i\}_{i=1}^{\infty}$ a defining sequence of α-partitions of X. For all $i > 0$, for all partial amalgams $\bar{\mathscr{U}}_i$ of $\mathscr{U}_i$, for all $U \in \bar{\mathscr{U}}_i$ such that $\mathscr{U}_i(U)$ is α-connected, for all $j > i$, there is an α-tree-chain $\mathscr{V}$ in $\mathscr{U}_j(U)$ such that $\mathscr{V}$ contains the elements of $\mathrm{Bound}(U, \mathscr{U}_j)$ as endlinks.*

3.5. Matching decompositions by isomorphisms. The construction, in subsequent sections, of the embedding of a member of a given class of spaces into one of our containing or universal spaces involves constructing two sequences of matched decompositions, one a sequence of decompositions of an arbitrary member of the given class, and the other a sequence of α-amalgams of a subsequence of a defining sequence of α-partitions of the containing or universal space under consideration. We have previously defined isomorphisms of decompositions and stated the Decomposition Matching Theorem 2.2.5.

3.5.1. Boundary embeddings. Let $\mathscr{V}_1$ and $\mathscr{U}_1$ be decompositions with φ_1: $\mathscr{V}_1 \to \mathscr{U}_1$ an isomorphism. Suppose that $\partial \varphi_1 : \partial \mathscr{V}_1 \to \partial \mathscr{U}_1$ is an embedding such that for all $V \in \mathscr{V}_1$,

$$\partial \varphi_1(\mathrm{Bd}(V)) \subset \mathrm{Bd}(\varphi_1(V)),$$

and for all $V \neq W \in \mathscr{V}_1$.

$$\partial \varphi_1(V \cap W) \subset \varphi_1(V) \cap \varphi_1(W).$$

Then we say that $\partial \varphi_1$ is a *boundary embedding corresponding to* φ_1.

Together, the following lemma and the Rim Decomposition Theorem 2.5.2 are used inductively in the proofs of Theorems 1.2 to 1.6 in Sections 4 and 5 to

establish the existence of the sequences of matched decompositions required to induce an embedding.

3.6. BRICK PULVERIZING LEMMA. *Let α be a countable ordinal. Let Y be a space of rim-type $\leqslant \alpha$ and $\{\mathscr{V}_i\}_{i=1}^{\infty}$ a strongly decreasing sequence of α-decompositions of Y. Let $\vec{\mathscr{V}}_1 = \{V_1, V_2\}$ be an α-decomposition of $V = V_1 \cup V_2 \subset Y$. Let X be a space with $\{\mathscr{U}_j\}_{j=1}^{\infty}$ a defining sequence of α-partitions of X. Let $\vec{\mathscr{U}}_1$ be an α-amalgam of $\mathscr{U}_1$, and let $U \in \vec{\mathscr{U}}_1$. Suppose that h: $\mathrm{Bd}(V) \to \mathrm{Bd}(U)$ is an embedding which is strongly refining with repect to $\{\mathscr{V}_i\}_{i=1}^{\infty}$ and $\{\mathscr{U}_j\}_{j=1}^{\infty}$. Then there exists an integer $i > 1$, an α-decomposition $\vec{\mathscr{V}}_i(V)$ refining $\vec{\mathscr{V}}_1$, an integer $j_0 > 1$, an α-amalgam $\vec{\mathscr{U}}_{j_0}(U)$ of $\mathscr{U}_{j_0}(U)$ refining $\mathscr{U}_1(U)$, and an isomorphism $\varphi: \vec{\mathscr{V}}_i(V) \to \vec{\mathscr{U}}_{j_0}(U)$ with a corresponding boundary embedding $\partial\varphi: \partial\vec{\mathscr{V}}_i(V) \to \partial\vec{\mathscr{U}}_{j_0}(U)$ such that*

(1) $\varphi(W) \cap \mathrm{Bd}(U) \supset h(\mathrm{Bd}(V) \cap W)$, *for all $W \in \vec{\mathscr{V}}_i(V)$,*

(2) $\partial\varphi \cup h: \partial\vec{\mathscr{V}}_i(V) \cup \mathrm{Bd}(V) \to \partial\vec{\mathscr{U}}_{j_0}(U) \cup \mathrm{Bd}(U)$ *is an embedding which is strongly refining with repect to $\{\mathscr{V}_i\}_{i=1}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j_0}^{\infty}$.*

Proof. For $k \in \{1, 2\}$, let

$$\mathscr{B}_k = \mathrm{Star}\big(h(\mathrm{Bd}(V) \cap V_k), \mathscr{U}_1(U)\big) \subset \mathrm{Bound}(U, \mathscr{U}_1).$$

Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$. Now $V_1 \cap V_2 \cap \mathrm{Bd}(V) = \varnothing$, and $\{\mathscr{V}_i\}_{i=1}^{\infty}$ is strongly decreasing, so we have

$$\mathrm{Star}\big(\mathrm{Bd}(V) \cap V_1, \mathscr{V}_2\big) \cap \mathrm{Star}\big(\mathrm{Bd}(V) \cap V_2, \mathscr{V}_2\big) = \varnothing.$$

Since h is strongly refining with respect to $\{\mathscr{V}_i\}_{i=1}^{\infty}$ and $\{\mathscr{U}_j\}_{j=1}^{\infty}$, we may assume without loss of generality that $\mathscr{B}_1 \cap \mathscr{B}_2 = \varnothing$. Since $\vec{\mathscr{U}}_1$ is an α-amalgam, $\mathscr{U}_1(U)$ is α-connected. By Corollary 3.4.1 we may assume that $\mathscr{U}_1(U)$ is a nonempty α-tree-chain whose set of endlinks is $\mathscr{B}$.

Let G_0 be any link in $\mathscr{U}_1(U) - \mathscr{B}$. For $k = 1, 2$, let $\mathscr{G}_k$ be the minimal α-tree-chain in $\mathscr{U}_1(U)$ that contains $\mathscr{B}_k \cup \{G_0\}$. Then $\mathscr{G}_1$ and $\mathscr{G}_2$ have at least one element, namely G_0, in common. (However, they may have more than one element in common.)

For $k \in \{1, 2\}$, let $\mathscr{G}_k = \{G_{k,1}, \ldots, G_{k,n(k)}\}$, where $G_{k,1} = G_0$. Note that an element of $\mathscr{U}_1(U)$ may now have two names. We now apply the Rim Decomposition Theorem 2.5.2 twice, independently to $\mathscr{G}_1$ and $\mathscr{G}_2$. Since $\mathscr{G}_k$ has a coherent incidence matrix, and

$$\big\{h^{-1}\big(h(\mathrm{Bd}(V)) \cap G_{k,r}\big)\big| \ r = 2, 3, \ldots, n(k)\big\} \cup \{V_1 \cap V_2\}$$

satisfies the hypotheses of the Rim Decomposition Theorem, we can pull back the partition $\mathscr{G}_k$ to an α-decomposition $\mathscr{W}_k = \{W_{k,1}, \ldots, W_{k,n(k)}\}$ of V_k such that

(a) $\mathscr{W}_k = \vec{\mathscr{V}}_i(V_k)$, for some $i > 1$,

(b) for all $W_{k,r}$ and $W_{k,s}$ in $\mathscr{W}_k$, $W_{k,r} \cap W_{k,s} \neq \varnothing$ implies $G_{k,r} \cap G_{k,s} \neq \varnothing$,

(c) for all $r \in \{1, \ldots, n(k)\}$, $h^{-1}\big(h(\mathrm{Bd}(V)) \cap G_{k,r}\big) \subset W_{k,r}$,

(d) $V_1 \cap V_2 \subset W_{k,1}$.

For $k \in \{1, 2\}$, define the one-to-one, onto function $\varphi_k \colon \mathscr{W}_k \to \mathscr{G}_k$ by $\varphi_k(W_{k,r}) = G_{k,r}$, for $r \in \{1, \ldots, n(k)\}$. Note that $\varphi_1 \cup \varphi_2$ is a function, but *not* one-to-one. See Figure 3.2(a). It remains to modify the $G_{k,r}$, allowing us to replace $\varphi_1 \cup \varphi_2$ by a one-to-one function φ.
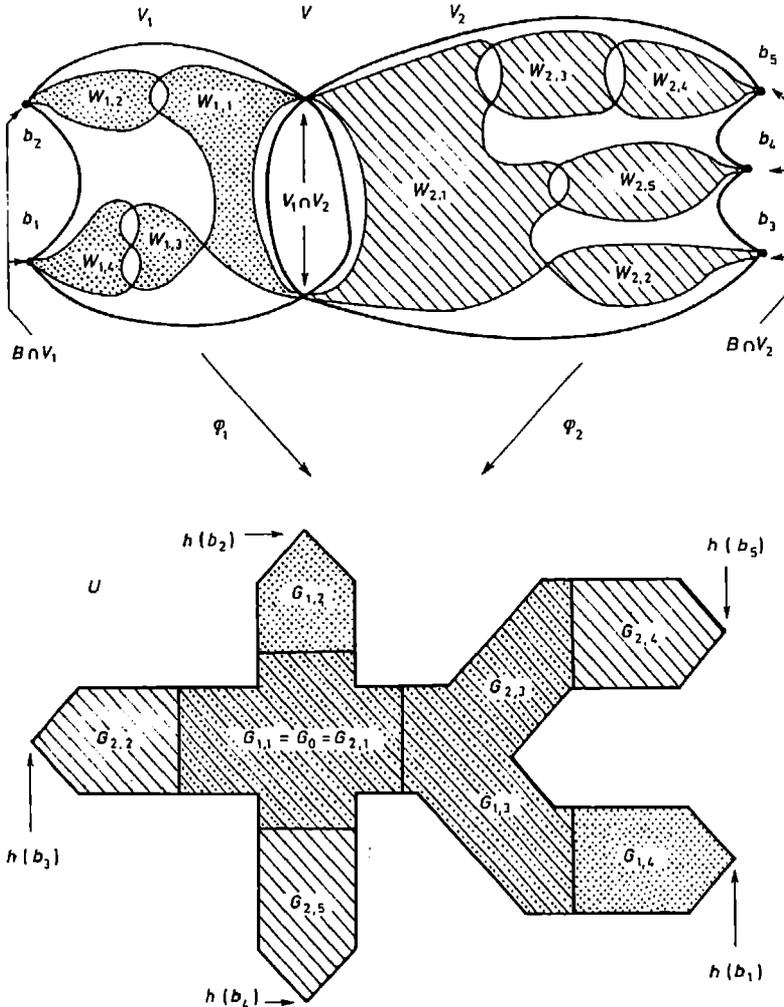

Fig. 3.2(a). Boundaries are shown as finite for simplicity

Since type $(\partial \mathscr{W}_1 \cup \partial \mathscr{W}_2) \leqslant \alpha$, we apply α-splitting and the Embedding Theorem 2.3.5 to obtain an embedding $g \colon \partial \mathscr{W}_1 \cup \partial \mathscr{W}_2 \to \partial \mathscr{G}_1 \cup \partial \mathscr{G}_2$ such that

(e) $g(W \cap W') \subset \varphi_k(W) \cap \varphi_k(W')$, for all $W \neq W' \in \mathscr{W}_k$.

Because $\{\mathcal{U}_j\}_{j=1}^{\infty}$ is strongly decreasing, there exists an integer $j_0 > 1$ such that g is strongly refining with respect to $\{\mathcal{V}_i\}_{i=i_0}^{\infty}$ and $\{\mathcal{U}_j\}_{j=j_0}^{\infty}$. Moreover, we may assume that

$$\text{Star}\big(g(\partial \mathcal{W}_1 \cup \partial \mathcal{W}_2), \mathcal{U}_{j_0}\big) \cap \text{Star}\big(h(\text{Bd}(V)), \mathcal{U}_{j_0}\big) = \varnothing.$$

Thus, we have

(f) $h \cup g$ is strongly refining with respect to $\{\mathcal{V}_i\}_{i=i_0}^{\infty}$ and $\{\mathcal{U}_j\}_{j=j_0}^{\infty}$.

For $k \in \{1, 2\}$ and $r \in \{1, \ldots, n(k)\}$, define the closed set $D_{k,r}$ of type $\leqslant \alpha$ by

(g) $D_{k,r} = g\big(\partial \mathcal{W}_k(W_{k,r})\big) \cup h\big(\text{Bd}(V) \cap G_{k,r}\big)$.

By (f), let $\mathcal{N}_{k,r}$ be a minimal nonempty α-tree-chain in $\mathcal{U}_{j_0}(G_{k,r})$ that contains $\text{Star}\big(D_{k,r}, \mathcal{U}_{j_0}(G_{k,r})\big) \subset \text{Bound}(G_{k,r}, \mathcal{U}_{j_0})$, as guaranteed by the α-Interlacing Lemma 3.4, subject to the following additional conditions:

(h) if $D_{k,r} = \varnothing$, then $\mathcal{N}_{k,r}$ is a single nonboundary link of $\mathcal{U}_{j_0}(G_{k,r})$,

(i) $(\bigcup \mathcal{N}_{1,1}) \cap (\bigcup \mathcal{N}_{2,1})$ is a universal space of type α,

(j) $(\bigcup \mathcal{N}_{1,1}) \cap (\bigcup \mathcal{N}_{2,1})$ is contained in a single link of each of $\mathcal{N}_{1,1}$ and $\mathcal{N}_{2,1}$ (recall $\mathcal{N}_{1,1} \cup \mathcal{N}_{2,1} \subset G_0 = G_{1,1} = G_{2,1}$),

(k) $(\bigcup \mathcal{N}_{1,r}) \cap (\bigcup \mathcal{N}_{2,s}) = \varnothing$, if $\{r, s\} \neq \{1\}$.

Since $\text{type}(V_1 \cap V_2) \leqslant \alpha$, it follows from (i), α-splitting, and the Embedding Theorem 2.3.5 that there is an embedding

$$f \colon V_1 \cap V_2 \to \big((\bigcup \mathcal{N}_{1,1}) \cap (\bigcup \mathcal{N}_{2,1})\big),$$

such that $h \cup g \cup f$ is strongly refining with respect to $\{\mathcal{V}_i\}_{i=i_0}^{\infty}$ and $\{\mathcal{U}_j\}_{j=j_0}^{\infty}$, as in (f) above.

For $k \in \{1, 2\}$ and $r \in \{1, \ldots, n(k)\}$, let $U_{k,r} = \bigcup \mathcal{N}_{k,r}$, and let

$$\overline{\mathcal{U}}_j(U) = \{U_{k,r} \mid k \in \{1, 2\} \text{ and } r \in \{1, \ldots, n(k)\}\}.$$

It follows from (g) and the definition of the embedding f that

(m) $g\big(\partial \mathcal{W}_k(W_{k,r})\big) \subset D_{k,r} \subset \text{Bd}(U_{k,r})$,

(n) $f(V_1 \cap V_2) \subset U_{1,1} \cap U_{2,1}$.

Clearly, $\overline{\mathcal{U}}_j(U)$ is an α-amalgam of $\mathcal{U}_j(U)$ refining $\mathcal{U}_1(U)$. See Figure 3.2(b). Let $\hat{\mathcal{V}}_i(V) = \mathcal{W}_1 \cup \mathcal{W}_2$. Then $\hat{\mathcal{V}}_i(V)$ refines $\mathcal{V}_1$.

Define an isomorphism $\varphi \colon \hat{\mathcal{V}}_i(V) \to \overline{\mathcal{U}}_j(U)$ by $\varphi(W_{k,r}) = U_{k,r}$. By (k), the ranges of $\varphi | \mathcal{W}_1$ and $\varphi | \mathcal{W}_2$ have no common elements. Condition (1) of the lemma is satisfied because of (g).

Define the corresponding boundary embedding $\partial \varphi \colon \partial \hat{\mathcal{V}}_i(V) \to \partial \overline{\mathcal{U}}_j(U)$ by $\partial \varphi = g \cup f$. Then $\partial \varphi$ is well defined and an embedding, because g and f have pairwise disjoint domains and ranges. Thus, condition (2) of the lemma is satisfied. The boundary embedding $\partial \varphi$ corresponds of φ because of (e), (m), and (n). ∎
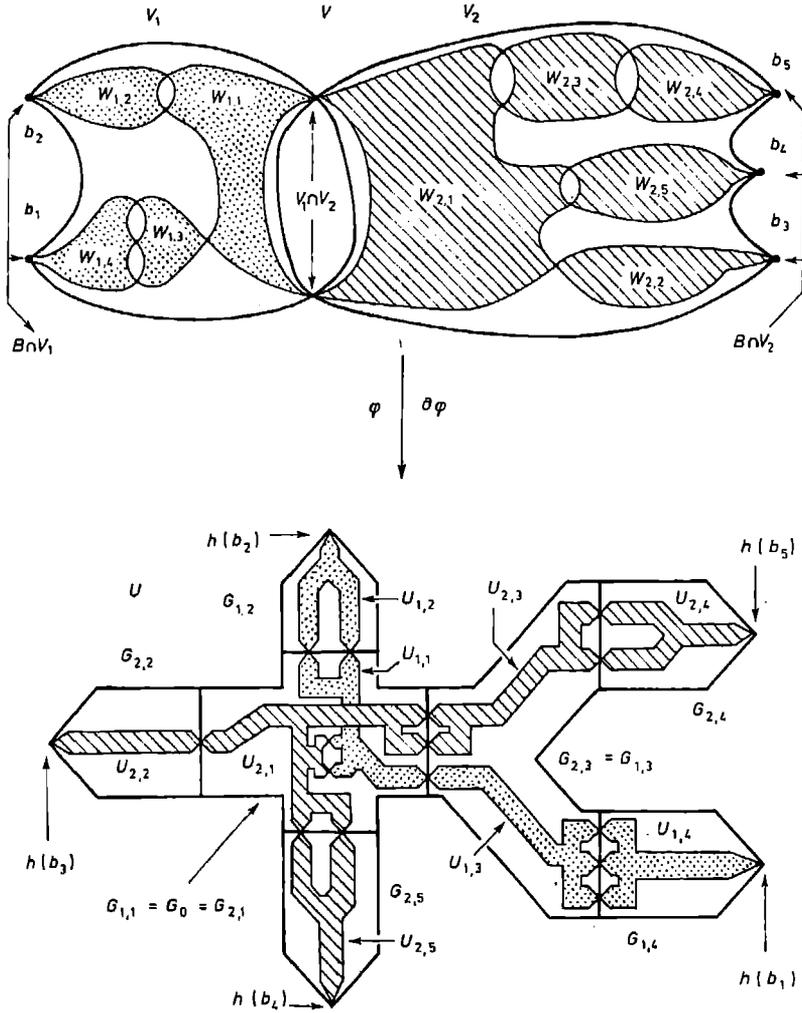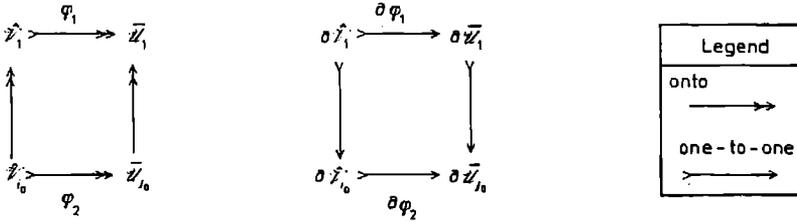
Fig. 3.2(b)

4. Embedding theorem

We first prove Theorem 4.2: that *a complete space with a defining sequence of α-partitions is a universal space for all spaces of rim-type* $\leqslant \alpha$. The statement of Theorem 4.2 is not used anywhere else in this paper. We include it because the structure of its proof makes transparent all of the proofs that follow it. We then indicate how to modify the above definitions so as to define $\mathcal{P}$-defining sequences of partitions for $\mathcal{P}$ one of the classes: spaces of rim-type $\leqslant \alpha$, rational spaces, one-dimensional spaces. We conclude the section by indicating how to prove Theorem 1.2 in its full generality.

4.1. LEMMA. *Let $\{\mathcal{V}_i\}_{i=1}^{\infty}$ be a strongly decreasing sequence of α-decompositions of a space Y. Let $\{\mathcal{U}_j\}_{j=1}^{\infty}$ be a defining sequence of α-partitions of a space X. Suppose that $\varphi_1\colon \hat{\mathcal{V}}_1 \to \bar{\mathcal{U}}_1$ is an isomorphism with a corresponding boundary embedding $\partial\varphi_1\colon \partial\hat{\mathcal{V}}_1 \to \partial\bar{\mathcal{U}}_1$ which is strongly refining with respect to $\{\mathcal{V}_i\}_{i=1}^{\infty}$ and $\{\mathcal{U}_j\}_{j=1}^{\infty}$. Then there exist $i_0, j_0 > 1$ such that $\hat{\mathcal{V}}_{i_0}$ refines $\mathcal{V}_1$, $\mathcal{U}_{j_0}$ refines $\mathcal{U}_1$, and $\bigcup \bar{\mathcal{U}}_{j_0} \subset \bigcup \bar{\mathcal{U}}_1$; moreover, there exists an isomorphism $\varphi_2\colon \hat{\mathcal{V}}_{i_0} \to \bar{\mathcal{U}}_{j_0}$ with respect to φ_1 with a corresponding boundary embedding $\partial\varphi_2\colon \partial\hat{\mathcal{V}}_{i_0} \to \partial\bar{\mathcal{U}}_{j_0}$ which is strongly refining with respect to $\{\mathcal{V}_i\}_{i=i_0}^{\infty}$ and $\{\mathcal{U}_j\}_{j=j_0}^{\infty}$, and such that $\partial\varphi_2|\partial\hat{\mathcal{V}}_1 = \partial\varphi_1$.*

Idea of proof. Consider the following diagrams:

We are given the top row with $\partial\varphi_1$ corresponding to φ_1; we must find the bottom row so that both diagrams commute and $\partial\varphi_2$ corresponds to φ_2. At the same time, the meshes of the amalgams of decompositions of Y and X must decrease.

Proof. We define $\hat{\mathcal{V}}_{i_0}$ by working on one element of $\hat{\mathcal{V}}_1$ at a time. Let $V \in \hat{\mathcal{V}}_1$, and let $U = \varphi_1(V) \in \bar{\mathcal{U}}_1$. We induct on the number of elements of $\mathcal{V}_1(V)$. By passing, if necessary, to finer covers in the sequences $\{\mathcal{V}_i\}_{i=1}^{\infty}$ and $\{\mathcal{U}_j\}_{j=1}^{\infty}$, we may assume that $\operatorname{card}(\mathcal{V}_1(V)) \geq 2$. There are two cases to consider.

Case 1. Suppose that $\mathcal{V}_1(V) = \{V_1, V_2\}$. We apply the Brick Pulverizing Lemma 3.6 to V and U with $\partial\varphi_1|\mathrm{Bd}(V)$ as h. We obtain an integer $i(V) > 1$, a decomposition $\hat{\mathcal{V}}_{i(V)}(V)$ refining $\mathcal{V}_1(V)$, an integer $j(V) > 1$, an α-amalgam $\bar{\mathcal{U}}_{j(V)}(U)$ refining $\mathcal{U}_1(U)$, and an isomorphism $\varphi_V\colon \hat{\mathcal{V}}_{i(V)}(V) \to \bar{\mathcal{U}}_{j(V)}(U)$ with a corresponding boundary embedding $\partial\varphi_V\colon \partial\hat{\mathcal{V}}_{i(V)}(V) \to \partial\bar{\mathcal{U}}_{j(V)}(U)$ so that the following conditions are satisfied:

(1) $\varphi_V(W) \cap \mathrm{Bd}(U) \supset \partial\varphi_1(W \cap \mathrm{Bd}(V))$, for all $W \in \hat{\mathcal{V}}_{i(V)}(V)$.

(2) $\partial\varphi_V \cup (\partial\varphi_1|\mathrm{Bd}(V))\colon \partial\hat{\mathcal{V}}_{i(V)}(V) \cup \mathrm{Bd}(V) \to \partial\bar{\mathcal{U}}_{j(V)}(U) \cup \mathrm{Bd}(U)$ is strongly refining with respect to $\{\mathcal{V}_i\}_{i=i(V)}^{\infty}$ and $\{\mathcal{U}_j\}_{j=j(V)}^{\infty}$.

Case 2. Suppose $\mathcal{V}_1(V) = \{V_1, V_2, \ldots, V_n\}$, for $n > 2$. We prove case 2 by induction on n. Let $\mathcal{V}'(V) = \{V_1 \cup V_2, V_3, \ldots, V_n\}$ which has $n-1$ elements. We suppose that we can find an integer $i'(V) > 1$, a decomposition $\hat{\mathcal{V}}_{i'(V)}(V)$ refining $\mathcal{V}''(V)$, an integer $j'(V) > 1$, and α-amalgam $\bar{\mathcal{U}}_{j'(V)}(U)$ refining $\mathcal{U}_1(U)$, and an isomorphism $\varphi'_V\colon \hat{\mathcal{V}}_{i'(V)}(V) \to \bar{\mathcal{U}}_{j'(V)}(U)$ with a corresponding boundary

embedding $\partial\varphi'_V\colon \partial\hat{\mathscr{V}}_{i'(V)}(V) \to \partial\bar{\mathscr{U}}_{j'(V)}(U)$ so that the following conditions are satisfied:

(3) $\varphi'_V(W) \cap \mathrm{Bd}(U) \supset \partial\varphi_1(W \cap \mathrm{Bd}(V))$, for all $W \in \hat{\mathscr{V}}_{i'(V)}(V)$.

(4) $\partial\varphi'_V \cup (\partial\varphi_1|\mathrm{Bd}(V))\colon \partial\hat{\mathscr{V}}_{i'(V)}(V) \cup \mathrm{Bd}(V) \to \partial\bar{\mathscr{U}}_{j'(V)}(U) \cup \mathrm{Bd}(U)$ is strongly refining with respect to $\{\mathscr{V}_i\}_{i=i'(V)}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j'(V)}^{\infty}$.

We would be done if $\hat{\mathscr{V}}_{i'(V)}(V)$ refined $\mathscr{V}_1(V)$ instead of $\mathscr{V}'(V)$. However, we need only modify $\hat{\mathscr{V}}_{i'(V)}(V)$ on $\hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)$. This will necessitate modifying the α-amalgam $\bar{\mathscr{U}}_{j'(V)}(U)$ and the isomorphism φ'_V, and we will have to extend the embedding $\partial\varphi'_V$ in correspondence with our modifications.

Since there are only finitely many elements in $\hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)$, we do our modifications of $\hat{\mathscr{V}}_{i'(V)}(V)$ on one element of $\hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)$ at a time. We do not alter $\partial\hat{\mathscr{V}}_{i'(V)}(V)$, nor do we change $\partial\varphi'_V$; rather, we obtain a finer decomposition and extend the boundary embedding to the added boundary points. The difficult part of what follows is keeping track of the notation, because there are several steps, and at each step we may need to go to finer decompositions in the decreasing sequences.

Suppose $W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)$. It is possible that $W \subset V_1$ or that $W \subset V_2$. If either $W \subset V_1$ or $W \subset V_2$, let $i''(W) = i'(V)$, $\hat{\mathscr{V}}_{i''(W)}(W) = \{W\}$, $j''(W) = j'(V)$, $\varphi''_W = \varphi'_V|\hat{\mathscr{V}}_{i''(W)}(W)$, and $\bar{\mathscr{U}}_{j''(W)}(\varphi'_V(W)) = \{\varphi''_W(W)\}$. Since $\hat{\mathscr{V}}_{i''(W)}(W)$ is a singleton, $\partial\hat{\mathscr{V}}_{i''(W)}(W) = \varnothing$, and therefore, $\partial\varphi''_W$ is the empty map. Since $W \subset V_1 \in \mathscr{V}_1(V)$ or $W \subset V_2 \in \mathscr{V}_1(V)$, $\hat{\mathscr{V}}_{i''(W)}(W)$ refines $\mathscr{V}_1(V)$, and conditions (5) and (6), stated below, are trivially satisfied.

Suppose that $W \not\subset V_1$ and $W \not\subset V_2$. Note that $\{W \cap V_1, W \cap V_2\}$ is an amalgam of $\hat{\mathscr{V}}_{i'(V)}(W)$. We apply the Brick Pulverizing Lemma 3.6 to W and $\varphi'_V(W) = U' \in \bar{\mathscr{U}}_{j'(V)}(U)$ with $(\partial\varphi'_V \cup \partial\varphi_1)|\mathrm{Bd}(W)$ as h. We obtain an $i''(W) > i'(V)$, a decomposition $\hat{\mathscr{V}}_{i''(W)}(W)$ refining $\{W \cap V_1, W \cap V_2\}$, which itself refines $\mathscr{V}_1(V)$, an integer $j''(W) > j'(V)$, an α-amalgam $\bar{\mathscr{U}}_{j''(W)}(U')$ refining $\bar{\mathscr{U}}_{j'(V)}(U')$, and an isomorphism $\varphi''_W\colon \hat{\mathscr{V}}_{i''(W)}(W) \to \bar{\mathscr{U}}_{j''(W)}(U')$ with a corresponding boundary embedding $\partial\varphi''_W\colon \partial\hat{\mathscr{V}}_{i''(W)}(W) \to \partial\bar{\mathscr{U}}_{j''(W)}(U')$ satisfying the following conditions:

(5) $\varphi''_W(G) \cap \mathrm{Bd}(U') \supset (\partial\varphi'_V \cup \partial\varphi_1)(G \cap \mathrm{Bd}(W))$, for all $G \in \hat{\mathscr{V}}_{i''(W)}(W)$.

(6) $\partial\varphi''_W \cup ((\partial\varphi'_V \cup \partial\varphi_1)|\mathrm{Bd}(W))$ is strongly refining with respect to $\{\mathscr{V}_i\}_{i=i''(W)}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j''(W)}^{\infty}$.

One should note that conditions (5) and (6) take account of the fact that W may meet $\mathrm{Bd}(V)$ because either V_1 or V_2 may do so, and the fact that $\partial\varphi_1$ is already defined on any points of $W \cap \mathrm{Bd}(V)$.

We carry out the procedure in the above paragraphs for each $W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)$. Let

$$i(V) = \max\{i''(W)\mid W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)\} \geqslant i'(V) > 1.$$

$$j(V) = \max\{j''(W)\mid W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)\} \geqslant j'(V) > 1,$$

$$\mathscr{V}_{i(V)}(V) = \hat{\mathscr{V}}_{i'(V)}(V_3 \cup \ldots \cup V_n) \cup \left(\bigcup\{\hat{\mathscr{V}}_{i''(W)}(W)\mid W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)\}\right).$$

Note that $\hat{\mathscr{V}}_{i(V)}(V)$ differs from $\hat{\mathscr{V}}_{i'(V)}(V)$ only on $V_1 \cup V_2$, and that $\hat{\mathscr{V}}_{i(V)}(V)$ refines $\mathscr{V}_1(V)$.

It follows from Corollary 3.4.1 to the α-interlacing Lemma that if $1 < b \leqslant c$ are integers, then we can replace each element W_b of an α-amalgam $\bar{\mathscr{U}}_b(U)$ by a subset W_c which is an element of an α-amalgam $\bar{\mathscr{U}}_c(U)$, and so that for all $W_b' \in \bar{\mathscr{U}}_b(U)$ and corresponding element $W_c' \in \bar{\mathscr{U}}_c(U)$, $W_b \cap W_b' = W_c \cap W_c'$ and $W_b \cap \mathrm{Bd}(U) = W_c \cap \mathrm{Bd}(U)$.

There is, therefore, an α-amalgam $\bar{\mathscr{U}}_{j(V)}(U)$ of $\mathscr{U}_{j(V)}(U)$ constructed in accordance with the above paragraph, and an isomorphism $\varphi_V: \hat{\mathscr{V}}_{i(V)}(V) \to \bar{\mathscr{U}}_{j(V)}(U)$ which has the following properties:

(7) For each $W \in \hat{\mathscr{V}}_{i(V)}(V_3 \cup \ldots \cup V_n)$, $\varphi_V(W)$ is that element of $\bar{\mathscr{U}}_{j(V)}(U)$ by which we replaced $\varphi_V'(W)$ in the above paragraph, with $b = j'(V)$ and $c = j(V)$.

(8) For each $G \in \hat{\mathscr{V}}_{i(V)}(V_1 \cup V_2)$ with $G \subset W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2)$, $\varphi_V(G)$ is that element of $\bar{\mathscr{U}}_{j(V)}(U)$ by which we replaced $\varphi_W''(G)$.

We define $\partial \varphi_V$ as the extension of $\partial \varphi_V'$ by the equality

$$\partial \varphi_V = \partial \varphi_V' \cup \left(\bigcup \left(\{ \partial \varphi_W'' | \ W \in \hat{\mathscr{V}}_{i'(V)}(V_1 \cup V_2) \} \right) \right).$$

Then $\partial \varphi_V$ is well defined and is an embedding because the domains and ranges of the constituent decomposition-respecting embeddings are pairwise disjoint. It follows from conditions (1)–(8) that $\partial \varphi_V$ is a boundary embedding corresponding to φ_V which is strongly refining with respect to $\{\mathscr{V}_i\}_{i=i(V)}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j(V)}^{\infty}$. Thus, conditions (1) and (2) (as stated in case 1, above) are satisfied. This concludes the proof of case 2.

We now return to the main induction. *Via* either case 1 or case 2, for each $V \in \hat{\mathscr{V}}_1$, we define φ_V on $\hat{\mathscr{V}}_{i(V)}(V)$ refining $\mathscr{V}_1(V)$, and we define $\partial \varphi_V$ on $\partial \hat{\mathscr{V}}_{i(V)}(V)$ so that φ_V and $\partial \varphi_V \cup (\partial \varphi_1 | \mathrm{Bd}(V))$ satisfy conditions (1) and (2). We also obtain for each such V, an α-amalgam $\bar{\mathscr{U}}_{j(V)}(\varphi_1(V))$ as the image of φ_V, and a collection of boundary points in $\partial \bar{\mathscr{U}}_{j(V)}(\varphi_1(V))$ as the image of the corresponding boundary embedding $\partial \varphi_V$, respectively.

Let $i_0 = \max \{ i(V) | \ V \in \hat{\mathscr{V}}_1 \} > 1$, $j_0 = \max \{ j(V) | \ V \in \hat{\mathscr{V}}_1 \} > 1$, and $\hat{\mathscr{V}}_{i_0} = \bigcup \{ \hat{\mathscr{V}}_{i(V)}(V) | \ V \in \hat{\mathscr{V}}_1 \}$. As in the proof of case 2, above, there is an α-amalgam $\bar{\mathscr{U}}_{j_0} = \{ \bigcup \{ \bar{\mathscr{U}}_{j_0}(\varphi_1(V)) | \ V \in \hat{\mathscr{V}}_1 \}$, and there is an isomorphism $\varphi_2: \hat{\mathscr{V}}_{i_0} \to \bar{\mathscr{U}}_{j_0}$ with respect to φ_1. It follows from conditions (1) and (2) that $\partial \varphi_2 = \partial \varphi_1 \cup (\bigcup \{ \partial \varphi_V | \ V \in \hat{\mathscr{V}}_1 \})$ is a boundary embedding corresponding to φ_2 which is strongly refining with respect to $\{\mathscr{V}_i\}_{i=i_0}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j_0}^{\infty}$. ∎

4.2. Theorem. *If X is a complete space with a defining sequence of α-partitions, then X is universal for spaces of rim-type $\leqslant \alpha$.*

Proof. Let Y be a space of rim-type $\leqslant \alpha$, and let $\{\mathscr{V}_i\}_{i=1}^{\infty}$ be a strongly decreasing sequence of α-decompositions of Y. Let $\{\mathscr{U}_j\}_{j=1}^{\infty}$ be a defining sequence of α-partitions of X. We construct inductively a decreasing sequence $\{\hat{\mathscr{V}}_{i(k)}\}_{k=1}^{\infty}$ of amalgams of a subsequence of the given decompositions of Y, a decreasing sequence $\{\bar{\mathscr{U}}_{j(k)}\}_{k=1}^{\infty}$ of α-amalgams of a subsequence of the given

defining sequence of α-partitions of X. Further, we construct isomorphisms φ_k: $\hat{\mathscr{V}}_{i(k)} \to \bar{\mathscr{U}}_{j(k)}$, and boundary embeddings $\partial\varphi_k$: $\partial\hat{\mathscr{V}}_{i(k)} \to \partial\bar{\mathscr{U}}_{j(k)}$ which are strongly refining with respect to $\{\mathscr{V}_i\}_{i=i(k)}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j(k)}^{\infty}$ such that φ_{k+1} respects φ_k and $\partial\varphi_k$ corresponds to φ_k, so as to satisfy the conditions of the Decomposition Matching Theorem 2.2.5.

To begin the induction, let $i(1) = j(1) = 1$, $\bar{\mathscr{U}}_{j(1)} = \{X\}$, and $\hat{\mathscr{V}}_{i(1)} = \{Y\}$; let φ_1: $\hat{\mathscr{V}}_{i(1)} \to \bar{\mathscr{U}}_{j(1)}$ be the obvious isomorphism, and let $\partial\varphi_1$ be the empty map.

Suppose that for all k with $1 \leqslant k < m$, we have obtained integers $i(k) > i(k-1)$ and $j(k) > j(k-1)$, a decomposition $\hat{\mathscr{V}}_{i(k)}$ of Y refining $\mathscr{V}_{i(k-1)}$, an α-amalgam $\bar{\mathscr{U}}_{j(k)}$ of $\mathscr{U}_{j(k)}$ such that $\bar{\mathscr{U}}_{j(k)}$ refines $\mathscr{U}_{j(k-1)}$ with $\bigcup \bar{\mathscr{U}}_{j(k)} \subset \bigcup \bar{\mathscr{U}}_{j(k-1)}$, and an isomorphism φ_k: $\hat{\mathscr{V}}_{i(k)} \to \bar{\mathscr{U}}_{j(k)}$ with respect to φ_{k-1} with a corresponding boundary embedding $\partial\varphi_k$: $\partial\hat{\mathscr{V}}_{i(k)} \to \partial\bar{\mathscr{U}}_{j(k)}$ which is strongly refining with respect to $\{\mathscr{V}_i\}_{i=i(k)}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j(k)}^{\infty}$, and such that $\partial\varphi_k|\partial\hat{\mathscr{V}}_{i(k-1)} = \partial\varphi_{k-1}$.

By Lemma 4.1, there is an $i(m) > i(m-1)$, a decomposition $\hat{\mathscr{V}}_{i(m)}$ of Y refining $\mathscr{V}_{i(m-1)}$, an integer $j(m) > j(m-1)$, an α-amalgam $\bar{\mathscr{U}}_{j(m)}$ refining $\mathscr{U}_{j(m-1)}$ with $\bigcup \bar{\mathscr{U}}_{j(m)} \subset \bigcup \bar{\mathscr{U}}_{j(m-1)}$, and an isomorphism φ_m: $\hat{\mathscr{V}}_{i(m)} \to \bar{\mathscr{U}}_{j(m)}$ with respect to φ_{m-1} with a corresponding boundary embedding $\partial\varphi_m$: $\partial\hat{\mathscr{V}}_{i(m)} \to \partial\bar{\mathscr{U}}_{j(m)}$ which is strongly refining with respect to $\{\mathscr{V}_i\}_{i=i(m)}^{\infty}$ and $\{\mathscr{U}_j\}_{j=j(m)}^{\infty}$, and such that $\partial\varphi_m|\partial\hat{\mathscr{V}}_{i(m-1)} = \partial\varphi_{m-1}$.

By induction, $\{\hat{\mathscr{V}}_{i(k)}\}_{k=1}^{\infty}$, $\{\bar{\mathscr{U}}_{j(k)}\}_{k=1}^{\infty}$, $\{\varphi_k$: $\hat{\mathscr{V}}_{i(k)} \to \bar{\mathscr{U}}_{j(k)}\}_{k=1}^{\infty}$, and $\{\partial\varphi_k$: $\partial\hat{\mathscr{V}}_{i(k)} \to \partial\bar{\mathscr{U}}_{j(k)}\}_{k=1}^{\infty}$ are defined. Since $\hat{\mathscr{V}}_{i(k)}$ refines $\mathscr{V}_{i(k-1)}$ and $\bar{\mathscr{U}}_{j(k)}$ refines $\mathscr{U}_{j(k-1)}$, $\mathrm{mesh}(\hat{\mathscr{V}}_{i(k)}) \to 0$ and $\mathrm{mesh}(\bar{\mathscr{U}}_{j(k)}) \to 0$.

Since X is complete, the conditions of the Decomposition Matching Theorem 2.2.5 are satisfied. Hence, there exists an embedding $\varphi\colon Y \to X$ such that for all $y \in Y$, $\varphi(y) = \bigcap\{\bigcup \varphi_k(\mathrm{Star}(y, \hat{\mathscr{V}}_{i(k)}))\}$, and if $y \in \partial\hat{\mathscr{V}}_{i(k)}$, for some k, then $\varphi(y) = \partial\varphi_k(y)$. ∎

4.3. Containing spaces for spaces of rim-type $\leqslant \alpha$.

Let X be a space, and let $\mathscr{U} = \{U_1, \ldots, U_n\}$ be a brick partition of X. Let $A_{i,j} \subset U_i \cap U_j$ for $i \neq j \in \{1, \ldots, n\}$ be a set of type $\leqslant \alpha$ such that $A_{i,j}$ is homeomorphic to one of the three sets defined in Section 2.3.3 (elements of a doubling decomposition of some D_α), and such that $A_{i,j} = A_{j,i}$. Note that $A_{i,j}$ may be empty. We call the pair $(\mathscr{U}, \{A_{i,j}\}_{i,j=1}^n)$ an α-collection. We say the α-collection $(\mathscr{U}, \{A_{i,j}\}_{i,j=1}^n)$ refines the α-collection $(\mathscr{V}, \{B_{r,s}\}_{r,s=1}^l)$ iff $\mathscr{U}$ refines $\mathscr{V}$ and

$$\mathscr{U}|\mathrm{Br}_{r,s} = \{A_{i,j} \,|\, U_i \in \mathscr{U}(V_r),\ U_j \in \mathscr{U}(V_s),\ \text{and}\ V_r \neq V_s \in \mathscr{V}\}.$$

Let the α-collection $(\mathscr{U}, \{A_{i,j}\}_{i,j=1}^n)$ refine the α-collection $(\mathscr{V}, \{B_{r,s}\}_{r,s=1}^l)$. Let the α-collection $(\mathscr{W}, \{C_{a,b}\}_{a,b=1}^c)$ be a subcollection of $(\mathscr{U}, \{A_{i,j}\}_{i,j=1}^n)$.

4.3.1. α-Connected collections. We say that $(\mathscr{W}, \{C_{a,b}\}_{a,b=1}^c)$ is α-connected iff $\mathscr{W}$ is coherent and for all $W_a \neq W_b \in \mathscr{W}$ with $W_a \cap W_b \neq \varnothing$, $C_{a,b}$ is a universal space of type α. If $\mathscr{W}$ is a chain (respectively, circular chain, graph-chain,

or tree-chain) such that $(\mathscr{W}, \{C_{a,b}\}_{a,b=1}^c)$ is α-connected, then we say that $(\mathscr{W}, \{C_{a,b}\}_{a,b=1}^c)$ is an α-chain (respectively, α-*circular-chain*, α-*graph-chain*, or α-*tree-chain*).

4.3.2. α-Interlacing. We say that $(\mathscr{U}, \{A_{i,j}\}_{i,j=1}^n)$ is α-*interlaced* in $(\mathscr{V}, \{B_{r,s}\}_{r,s=1}^t)$ iff for every $V_r \in \mathscr{V}$, $\mathscr{U}(V_r)$ contains two α-tree-chains $(\mathscr{U}_1, \{A_{i_1,j_1}\}_{i_1,j_1=1}^{m_1})$ and $(\mathscr{U}_2, \{A_{i_2,j_2}\}_{i_2,j_2=1}^{m_2})$ such that the elements of $\mathscr{U}_1 \cap \mathscr{U}_2 = \text{Bound}(V, \mathscr{U})$ are the endlinks of both $\mathscr{U}_1$ and $\mathscr{U}_2$, and there exist $U_i \in \mathscr{U}_1 - \text{Bound}(V, \mathscr{U})$ and $U_j \in \mathscr{U}_2 - \text{Bound}(V, \mathscr{U})$ such that $\text{type}(A_{i,j}) = \alpha$.

4.3.3. α-Splitting. We say that $(\mathscr{U}, \{A_{i,j}\}_{i,j=1}^n)$ α-*splits* $(\mathscr{V}, \{B_{r,s}\}_{r,s=1}^t)$ iff for all $V_r \neq V_s \in \mathscr{V}$,

(1) $\mathscr{U}|B_{r,s}$ is a doubling decomposition of $B_{r,s}$ of $\text{type}(B_{r,s})$,

(2) for each $U_i \in \mathscr{U}(V_r)$ with $U_i \cap V_s \neq \emptyset$, there is a unique $U_j \in \mathscr{U}(V_s)$ such that $U_i \cap V_s = U_j \cap V_r = U_i \cap U_j$.

4.3.4. $\mathscr{P}$-Defining sequences of partitions. Let X be a space. Let $\mathscr{P}$ be the class of spaces of rim-type $\leqslant \alpha$. We say that a sequence $\{(\mathscr{U}_i, \{A_{i,j,k}\}_{j,k=1}^n)\}_{i=1}^\infty$ of α-collections is a $\mathscr{P}$-*defining sequence* for X iff for all $i > 0$, the following seven conditions hold:

(1) $\{\mathscr{U}_i\}_{i=1}^\infty$ is a strongly decreasing sequence of brick partitions of X.

(2)–(7) are the corresponding conditions of Definition 3.2 (2)–(7) with only the obvious modifications.

The notion of α-amalgam is extended to α-collections in the obvious way. The appropriate versions of Lemmas 3.4, 3.6, 4.1, and Theorem 4.2 hold for α-collections. Hence, we have the case of Theorem 1.2 for $\mathscr{P}$ the class of spaces of rim-type $\leqslant \alpha$.

4.4 Containing spaces for rational spaces. By Theorem 2.3.1, the set Q of rational numbers is the universal space for all countable spaces. Any decomposition of Q into disjoint, nonempty, closed sets can be taken as a *doubling decomposition* of Q. We can modify the definitions in Section 4.3 in the obvious way to obtain the definition, for $\mathscr{P}$ the class of rational spaces, of a $\mathscr{P}$-*defining sequence* for a space X, and a proof of Theorem 1.2 for $\mathscr{P}$ the class of rational spaces.

4.5. Containing spaces for one-dimensional spaces. The Cantor ternary set C is a universal space for all zero-dimensional spaces. For a *doubling decomposition* of C take any decomposition of C into disjoint, nonempty, closed sets. Each element of such a decomposition is a copy of C. We can modify the definitions in Section 4.3 in the obvious way to obtain, for $\mathscr{P}$ the class of one-dimensional spaces, the definition of a $\mathscr{P}$-*defining sequence* for a space X, and a proof of Theorem 1.2 for $\mathscr{P}$ the class of one-dimensional spaces. This, of course, is one direction of the Anderson Characterization Theorem in terms of covers for the Menger Curve in [A-1].

5. Construction of universal and containing spaces

In this section we construct our universal and containing spaces and prove Theorem 1.3, that *for each countable ordinal $\alpha > 0$, there is a universal space M_α of rim-type α* and Theorem 1.4, that *there is a universal rational space M.*

We first construct in E^3 the claimed universal space M_α of rim-type $\leqslant \alpha$. Theorem 1.3 follows immediately from the Construction (5.1) and a slight modification (5.2) of the proof of Theorem 4.2 (required because in our construction M_α is not complete). We then indicate the necessary further modifications needed to obtain a universal rational space M, thus proving Theorem 1.4. We then indicate how to obtain a compactum of rim-type α which contains each compactum of rim type $< \alpha$, proving Theorems 1.5 and 1.6.

5.1. Rim-type $\leqslant \alpha$ universal space. A *sticky-3-ball* is a subset U of E^3 such that $\mathrm{Cl}(U)$ in E^3 is a closed 3-ball with a 2-sphere boundary, $\mathrm{Int}(U)$ is an open 3-ball, and $U - \mathrm{Int}(U)$ is a countable set with countable closure in E^3. One can take $U - \mathrm{Int}(U)$ to be any scattered set desired because a scattered set has a countable compactification [I–T].

We will construct a decreasing sequence $\{\mathcal{U}_i\}_{i=1}^\infty$ of collections of sticky 3-balls in E^3 so that for all $U \neq V \in \mathcal{U}_i$, $\mathrm{type}(U \cap V) \leqslant \alpha$. It will follow from the construction that $M_\alpha = \bigcap_{i=1}^\infty \bigcup \mathcal{U}_i$ is a space of rim-type α with $\{\{U \cap M_\alpha \mid U \in U_i\}\}_{i=1}^\infty$ a defining sequence of α-partitions of M_α. We will use the notation defined in Sections 2 and 3. The various parts of Figure 5.1 illustrate the construction for the case $\alpha = 1$.

Let D_α be the universal space of type α constructed in the proof of Theorem 2.3.2. Let $\mathcal{U}_1 = \{U_1, U_2\}$ be two sticky-3-balls in E^3 of diameter $< 2^{-1}$ such that $U_1 \cap U_2 \subset \mathrm{Bd}(U_1) \cap \mathrm{Bd}(U_2)$ is the homeomorphic image of D_α with a countable closure. See Figure 5.1(a).

Fig. 5.1(a). $\mathcal{U}_1 = \{U_1, U_2\}$ illustrated

We may assume that $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) = \mathrm{Cl}(U_1 \cap U_2)$. We identify $U_1 \cap U_2$ with D_α. We call an embedding of D_α satisfying these conditions a *standard copy of D_α*.

We will construct

$$\mathscr{U}_2 = \{U_{i,j}| \ i = 1, 2 \ \text{and} \ j = 1, 2, \ldots, n_i\} = \mathscr{U}_2(U_1) \cup \mathscr{U}_2(U_2)$$

as a collection of sticky-3-balls such that $\bigcup \mathscr{U}_2(U_i) \subset U_i$ and

$$\left(\bigcup \mathscr{U}_2(U_1)\right) \cap \mathrm{Bd}(U_2) = \left(\bigcup \mathscr{U}_2(U_2)\right) \cap \mathrm{Bd}(U_1) = U_1 \cap U_2.$$

For $i = 1, 2$, let $U_{i,1}, \ldots, U_{i,m}$ be disjoint sticky-3-balls in U_i such that $\{U_{i,1} \cap \mathrm{Bd}(U_i), \ldots, U_{i,m} \cap \mathrm{Bd}(U_i)\}$ is a doubling decomposition of $U_1 \cap U_2$. We may suppose that $\mathrm{diam}(U_{i,j}) < 2^{-2}$ for $i = 1, 2$ and $j = 1, \ldots, m$. We may assume that

$$\mathrm{Cl}(U_{i,j}) \cap \mathrm{Bd}(U_i) \subset \mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2),$$

and for $j \neq k$,

$$\mathrm{Cl}(U_{i,j}) \cap \mathrm{Cl}(U_{i,k}) \subset \mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) - (U_1 \cap U_2).$$

We may also assume that $U_1 \cap U_{2,j} = U_2 \cap U_{1,j}$.

Let $\mathrm{Bound}(U_i, \mathscr{U}_2) = \{U_{i,1}, \ldots, U_{i,m}\}$, for $i = 1, 2$. See Figure 5.1(b). Note that in general some two elements of $\mathrm{Bound}(U_i, \mathscr{U}_2)$ will not have disjoint
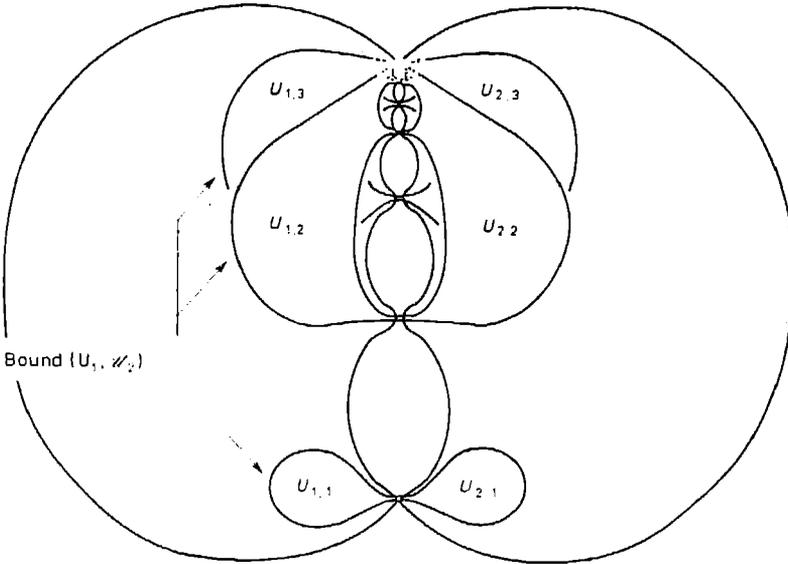

Fig. 5.1(b)

closures when $\{U_{i,1} \cap \mathrm{Bd}(U_i), \ldots, U_{i,m} \cap \mathrm{Bd}(U_i)\}$ is a doubling decomposition of $U_1 \cap U_2$.

We now construct those sticky-3-balls of $\mathscr{U}_2$ which will become $\mathrm{Core}(U_i, \mathscr{U}_2)$, for $i = 1, 2$. Let $V_{i,1}$ and $V_{i,2}$ be two sticky-3-balls inside U_i of diameter $< 2^{-2}$ at a positive distance from $\bigcup \mathrm{Bound}(U_i, \mathscr{U}_2) \cup \mathrm{Bd}(U_i)$, and such that $V_{i,1} \cap V_{i,2}$ is a standard copy of D_2. See Figure 5.1(c).
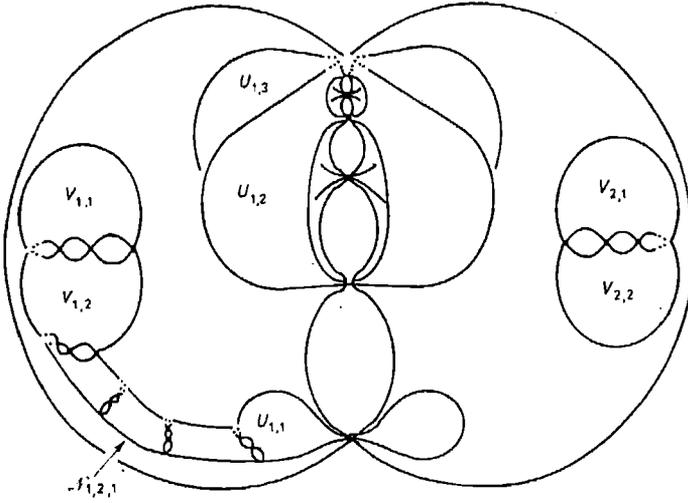
Fig. 5.1(c). Chain $\mathcal{N}_{1,2,1}$ from $V_{1,2}$ to $U_{1,1} \in \mathrm{Bound}(U_1, \mathcal{U}_2)$ illustrated

For each $i, j \in \{1, 2\}$ and $k \in \{1, \ldots, m\}$, let $\mathcal{N}_{i,j,k}$ be a chain of sticky 3-balls inside U_i from $V_{i,j}$ to $U_{i,k}$, such that $\mathrm{mesh}(\mathcal{N}_{i,j,k}) < 2^{-2}$. For fixed i, j, k, we require that $\{\mathrm{Cl}(W)|\ W \in \mathcal{N}_{i,j,k}\}$ be a chain, that any two different elements of $\mathcal{N}_{i,j,k}$ which meet do so in a standard copy of D_α, that one endlink of $\mathcal{N}_{i,j,k}$ meets $V_{i,j}$ and that the other endlink meets $U_{i,k}$, each in a standard copy of D_α, and that for each (i, j, k), $\bigcup \mathcal{N}_{i,j,k}$ is at a positive distance from each of the following:

(1) $\mathrm{Bd}(U_i)$,
(2) $U_{i,p}$ for $p \neq k$,
(3) $V_{i,q}$ for $q \neq j$,
(4) $\bigcup \mathcal{N}_{i,p,q}$ for either $p \neq j$ or $q \neq k$.
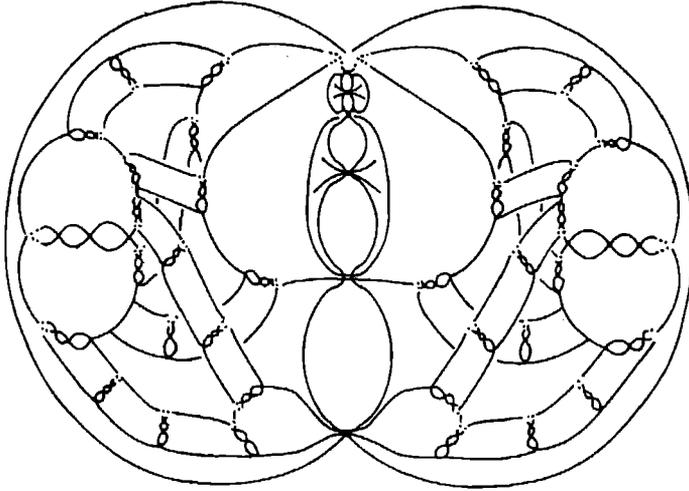
Then, for $i = 1, 2$, we let

$$\mathrm{Core}(U_i, \mathcal{U}_2) = \{V_{i,1}, V_{i,2}\} \cup \left(\bigcup \{\mathcal{N}_{i,j,k}|\ j \in \{1, 2\}\ \text{and}\ k \in \{1, \ldots, m\}\right),$$

and we let

$$\mathcal{U}_2(U_i) = \mathrm{Bound}(U_i, \mathcal{U}_2) \cup \mathrm{Core}(U_i, \mathcal{U}_2).$$

We have now defined $\mathcal{U}_2$. See Figure 5.1(d). It is clear from the construction that $\mathcal{U}_1$ is α-connected, $\mathcal{U}_2$ is α-interlaced in $\mathcal{U}_1$ and α-splits $\mathcal{U}_1$, and that for each $U \in \mathcal{U}_1$, $\mathcal{U}_2(U)$ is α-connected.

If $\mathcal{U}_i$ is constructed, we construct $\mathcal{U}_{i+1}$ inductively to be a collection of sticky-3-balls of diameter $< 2^{-(i+1)}$ refining $\mathcal{U}_i$ by working within one link of $\mathcal{U}_i$ at a time, exactly as we did above in constructing that part of $\mathcal{U}_2$ that lay in each element of $\mathcal{U}_1$. Let $M_\alpha = \bigcap_{i=1}^{\prime} \bigcup \mathcal{U}_i$. Note that $\{\{U \cap M_\alpha|\ U \in \mathcal{U}_i\}\}_{i=1}^{\infty}$ is a defining sequence of α-partitions of M_α. ∎

Fig. 5.1(d). $\mathscr{U}_2$ refining $\mathscr{U}_1$ illustrated

5.1.1. Remark. Observe that M_α has the property that for all $p \neq q \in M_\alpha$, there are infinitely many independent arcs from p to q in M_α. However, there is not a Cantor set of independent arcs between any pair of points in M_α.

5.2. Proof that M_α is topologically complete and universal. To see that M_α is topologically complete, note that

$$M_\alpha \subset \bigcap_{j=1}^{\infty} \mathrm{Cl}(\bigcup \mathscr{U}_j).$$

Moreover, by construction we have that for each j

$$\mathrm{Cl}(\bigcup \mathscr{U}_{j+1}) \subset (\bigcup \mathscr{U}_j) \cup \mathrm{Cl}(\partial \mathscr{U}_j).$$

Since $\mathscr{U}_j$ is finite and each two elements of $\mathscr{U}_j$ which meet do so in a scattered set with countable closure, we have $\mathrm{Cl}(\partial \mathscr{U}_j)$ is a countable set. It follows that

$$\mathrm{Cl}(M_\alpha) = M_\alpha \cup \bigcup_{j=1}^{\infty} \mathrm{Cl}(\partial \mathscr{U}_j).$$

Since $\mathrm{Cl}(M_\alpha)$ is the union of M_α and a countable set, M_α is a G_δ in E^3. Consequently, M_α is topologically complete [K], I. p. 430.

Let Y be a space of rim-type $\leqslant \alpha$. To prove that Y embeds in M_α, we may follow the proof of Theorem 4.2 with M_α in place of X up to the point where the completeness of the space X with a defining sequence of α-partitions is used to show that the embedding $\varphi: Y \to X$, induced by the decomposition-matching isomorphisms $\varphi_k: \mathscr{V}_{i(k)} \to \mathscr{U}_{j(k)}$ is well defined. Since M_α is not complete in the metric inherited from E^3, we must provide an alternative proof that $\varphi: Y \to M_\alpha$ is well defined.

Let $y \in Y$. By definition of M_α, it is clear that

$$\bigcap_{k=1}^{\infty} \left(\bigcup \varphi_k(\text{Star}(y, \hat{\mathcal{V}}_{i(k)})) \right) \subset M_\alpha.$$

By the Decomposition Matching Theorem 2.2.5, it suffices to prove that

$$\bigcap_{k=1}^{\alpha_i} \left(\bigcup \varphi_k(\text{Star}(y, \hat{\mathcal{V}}_{i(k)})) \right) \neq \varnothing.$$

Suppose that $y \in \partial \hat{\mathcal{V}}_{i(n)}$ for some n. Note that $\partial \hat{\mathcal{V}}_{i(k)} \supset \partial \hat{\mathcal{V}}_{i(k)} \supset \partial \hat{\mathcal{V}}_{i(k-1)}$ for all $k > 1$. Hence,

$$\bigcup_{i=1}^{\infty} \partial \hat{\mathcal{V}}_i = \bigcup_{k=1}^{\infty} \partial \hat{\mathcal{V}}_{i(k)}.$$

Since the $\partial \varphi_k$'s are corresponding boundary embeddings, it follows that $\partial \varphi_m(y) = \partial \varphi_n(y)$ for all $m > n$, and that

$$\bigcap_{k=1}^{\alpha_i} \left(\bigcup \varphi_k(\text{Star}(y, \hat{\mathcal{V}}_{i(k)})) \right) = \partial \varphi_n(y).$$

Now suppose that $y \in Y - \bigcup_{k=1}^{\infty} \partial \hat{\mathcal{V}}_{i(k)}$. By construction of the amalgams of the $\mathcal{V}_i$'s, for each r, there is an $s > r$ such that

$$\text{Cl}\left(\bigcup \text{Star}(y, \hat{\mathcal{V}}_{i(s)}) \right) \cap \partial \hat{\mathcal{V}}_{i(r)} = \varnothing.$$

Hence, by construction of φ_r, φ_s, and $\bar{\mathcal{U}}_{j(r)}$, we have

$$\text{Cl}\left(\bigcup \varphi_s(\text{Star}(y, \hat{\mathcal{V}}_{i(s)})) \right) \cap \text{Cl}(\partial \mathcal{U}_{j(r-1)}) = \varnothing.$$

Thus, we have

(*) $$\text{Cl}\left(\bigcup \varphi_s(\text{Star}(y, \hat{\mathcal{V}}_{i(s)})) \right) \subset \bigcup \varphi_{r-1}(\text{Star}(y, \hat{\mathcal{V}}_{i(r-1)})).$$

It follows from (*) that

(**) $$\bigcap_{k=1}^{\infty} \left(\bigcup \varphi_k(\text{Star}(y, \hat{\mathcal{V}}_{i(k)})) \right) = \bigcap_{k=1}^{\infty} \text{Cl}\left(\bigcup \varphi_k(\text{Star}(y, \hat{\mathcal{V}}_{i(k)})) \right).$$

The intersection on the right in (**) is nonempty because it is the intersection of a tower of nonempty compact sets. Since the conditions (in particular, condition (3)) of the Decomposition Matching Theorem 2.2.5 is thereby satisfied, there exists an embedding $\varphi : Y \to M_\alpha$.

5.2.1. *Question.* Is the space M_α unique among absolute G_δ-spaces which are universal for the class of spaces of rim-type $\leq \alpha$ and have a defining sequence of α-partitions?

5.3. Universal rational space. Iliadis has recently proved the existence of a universal rational space using techniques different from ours. Our construction in Section 5.1 is easily modified to yield a space M, universal for all rational spaces.

We construct a decreasing sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of sticky-3-balls for M as in Section 5.1, except that each stricky-3-ball in $\mathcal{U}_i$ is an open 3-ball plus a copy of Q which has zero-dimensional closure in E^3 in its boundary. (Note that Q can be embedded densely in the Cantor ternary set C.) A *doubling decomposition* of Q is any decomposition of Q into nonempty disjoint closed subsets. Each element of such a decomposition is a copy of Q. Then $M = \bigcap_{i=1}^{\infty} \bigcup \mathcal{U}_i$ is the required universal rational space, and $\{\{U \cap M \mid U \in \mathcal{U}_i\}\}_{i=1}^{\infty}$ is a *rational defining sequence of partitions* of M.

The proof of the universality of M_α in Section 5.2 can now be easily adapted to this situation to show that M is a universal rational space.

5.3.1. R e m a r k. The space M above is not complete because it is not closed as a subset of E^3. Furthermore, no universal rational space can be topologically complete. To see this, let $Y = [0, 1] \times Q$, a rational space. Assume that X is a universal rational space and that $y \in Y \subset X$. Let U be a neighborhood of y in X such that $\mathrm{Bd}(U)$ is countable and $\mathrm{Bd}(U) \cap Y$ is homeomorphic to Q. Let $Z = \mathrm{Cl}(\mathrm{Bd}(U) \cap Y) \subset \mathrm{Bd}(U)$. Observe that Z is a countable closed subset of X and that $\mathrm{Bd}(U) \cap Y$ is a G_δ-subset of Z. Hence, $\mathrm{Bd}(U) \cap Y$ is a G_δ-subset of X. If X were topologically complete, then $\mathrm{Bd}(U) \cap Y$ would be topologically complete as well. But this is a contradiction, since $\mathrm{Bd}(U) \cap Y$ is homeomorphic to Q. (We are indebted to the referee for this remark.)

5.4. Containing spaces for compacta of rim-type $< \alpha$. We now prove Theorem 1.6: *for each countable ordinal α, there is a Peano continuum C_α of rim-type α which is a containing space for all compacta of rim-type $< \alpha$.* Theorem 1.5, a special case of 1.6, follows immediately.

5.4.1. Proof of Theorem 1.6. First suppose α is a countable isolated ordinal. If A is a compactum of type α, then $A^{(\alpha-1)}$ is a non-empty finite set. We consider A to be embedded in the Cantor ternary set C. A *doubling decomposition* of A is the restriction to A of any decomposition of C into nonempty disjoint closed sets. If B and D are compacta of type $< \alpha$, then B and D embed disjointly in $A - A^{(\alpha-1)}$ by Theorem 2.3.6.

We indicate how to modify the construction of Section 5.1 to construct a Peano continuum C_α of rim-type α such that C_α is a containing space for all compacta of rim-type $< \alpha$. Let A be a compactum of type α with $A^{(\alpha-1)}$ a singleton. We construct a decreasing sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of 3-balls in E^3 as in Section 5.1, except that each link of $\mathcal{U}_i$ is taken to be a closed 3-ball instead of a sticky-3-ball, and if two elements U and V of $\mathcal{U}_i$ meet, then

$U \cap V = \mathrm{Bd}(U) \cap \mathrm{Bd}(V)$ is homeomorphic to a member of a doubling decomposition of A. Let $C_\alpha = \bigcap_{i=1}^{\infty} \bigcup \mathcal{U}_i$. Then C_α is a Peano continuum of rim-type α, and $\{\{U \cap C_\alpha | \ U \in \mathcal{U}_i\}\}_{i=1}^{\infty}$ is an α-*defining sequence of partitions* of C_α.

If K is a compactum of rim-type $< \alpha$, then, by the proof of Theorem 4.2 and the first paragraph of this section, there exists an embedding

$$h: K \to C_\alpha - \bigcup_{i=1}^{\infty} \partial \mathcal{U}_i^{(\alpha - 1)} = N_{\alpha - 1}.$$

Hence, $N_{\alpha-1}$ is a (non-compact) space of rim-type $\alpha - 1$ which contains every compactum of rim-type $< \alpha$, and $N_{\alpha-1}$ has a compactification, namely C_α, of rim-type α.

Now suppose α is a countable limit ordinal. Let $\{\beta_i\}_{i=1}^{\infty}$ be a monotone sequence of isolated ordinals converging up to α. For each i, let C_{β_i} be the Peano continuum constructed above. Let $p \in E^3$. We may suppose that for each i, $C_{\beta_i} \subset E^3$, $C_{\beta_i} \cap C_{\beta_j} = \{p\}$ for all $i \neq j$, and $\lim C_{\beta_i} = \{p\}$. Then $C_\alpha = \bigcup_{i=1}^{\infty} C_{\beta_i}$ is a Peano continuum of rim-type α which is a containing space for all compacta of rim-type $< \alpha$. For suppose K is a compactum of rim-type $< \alpha$. Then for some i, $\alpha \leq \beta_i$. Hence, K embeds in $C_{\beta_i} \subset C_\alpha$. This concludes the proof of Theorem 1.6.

5.4.2. Remark. In fact, for α a limit ordinal, the space C_α constructed above is a containing space for all spaces (not just compacta) of rim-type $< \alpha$. For if Y is a space of rim-type $\gamma + n < \alpha$, where γ is a limit ordinal and $n \geq 0$ is an integer, then by Theorem 8 of [I–T] Y has a compactification K of rim-type $\leq \gamma + 2n + \min\{\gamma, 1\}$. Choose i such that $\gamma + 2n + \min\{\gamma, 1\} < \beta_i$. Then K, and hence Y, embeds in $C_{\beta_i} \subset C_\alpha$.

5.4.3. Proof of Theorem 1.5. It is a theorem of Nöbeling [N] that each rim-finite space has a rim-finite compactification. Hence, Theorem 1.5 follows from Theorem 1.6 with $\alpha = 2$.

6. References

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