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Embedding $S(X)$ into $S(Y)$

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INTRODUCTION

Let A and B be semigroups. When we say that A can be *embedded* in B, we mean it strictly in an algebraic sense. That is, there exists an isomorphism from A into B. If A and B happen to be provided with topologies and there exists an isomorphism from A into B which is a homeomorphism as well, then we will say that A can be *T-embedded* in B. We do not always require in these instances that the topologies will result in topological semigroups. The semigroups we are primarily interested in here are semigroups of continuous functions on topological spaces and it will be assumed, unless something specifically to the contrary is mentioned, that the spaces under consideration are all Hausdorff.

The semigroup, under composition, of all continuous selfmaps of the topological space X will be denoted by $S(X)$. We treat here two problems: that of determining precisely when $S(X)$ can be embedded in $S(Y)$ and, in the event they both have topologies, precisely when $S(X)$ can be T-embedded in $S(Y)$. A large part of Chapter 1 is devoted to getting the machinery we need to approach these two problems. However, we do treat in the first chapter the case where Y is discrete and hence $S(Y)$ is just the full transformation semigroup on Y. We prove, for example, that if $S(X)$ is doubly transitive on X, then $S(X)$ can be embedded in $\mathscr{T}_Y$ (the full transformation semigroup on Y) if and only if $\operatorname{card} X \leqslant \operatorname{card} Y$ and, when both $S(X)$ and $\mathscr{T}_Y$ have the compact-open topologies $S(X)$ can be T-embedded in $\mathscr{T}_Y$ if and only if X is discrete and $\operatorname{card} X \leqslant \operatorname{card} Y$.

It seems appropriate to remark at this point that if we assume the continuum hypothesis, then every semigroup of order less than c (the cardinality of the continuum) can be embedded in the full transformation semigroup $\mathscr{T}_N$ on the natural numbers. In spite of this and the fact that the order of $\mathscr{T}_N$ is c, one would still expect that there are many semigroups of order c which cannot be embedded in $\mathscr{T}_N$. Indeed, J. S. V. Symons has observed [20] that the semigroup of real numbers under the supremum operation cannot be embedded in $\mathscr{T}_N$. One of the results mentioned above supplies us with many additional semigroups of order c which cannot be embedded in $\mathscr{T}_N$. Simply take X to be any separable, completely regular, Hausdorff, arcwise connected space with more than one point (which, of course, will force it to have c points) or any separable 0-dimensional space with c points. Then $S(X)$ will have order c but according to the first result mentioned above, it cannot possibly be embedded in $\mathscr{T}_N$.

In Chapter 2, we treat the problems of embedding and T-embedding $S(X)$ into $S(Y)$ when Y is not necessarily discrete. Indeed, the conditions we place on Y are ofen of such a nature that they will prevent Y from being discrete. We show first in Chapter 2 that if X is a compact absolute retract and Y is normal and both $S(X)$ and $S(Y)$ are equipped with the compact-open topologies, then $S(X)$ can be T-embedded in $S(Y)$ if and only if X can be topologically embedded in Y. We then turn to the problem of embedding $S(X)$ algebraically into $S(Y)$. Here also we find necessary and sufficient conditions, for certain X and Y, for $S(X)$ to be embedded in $S(Y)$. Finally, by restricting both X and Y still further, we are able to give the form that these isomorphisms must take.

SOME RELEVANT HISTORY

First of all, what is known to date seems to accentuate the extremes more than anything else. That is, it is known that there are nontrivial spaces Y such that very few $S(X)$ can be embedded in $S(Y)$ and there are also spaces Y such that $S(Y)$ contains copies of many different $S(X)$. Examples of the former include the closed unit interval I and the space R of real numbers. A fairly extensive class of spaces was given in [15] such that from this entire class, there are precisely three whose semigroups can be embedded in $S(I)$ and precisely five whose semigroups can be embedded in $S(R)$. As for examples of the latter, it is well known that given any collection of semigroups, one can embed each of the semigroups in the full transformation semigroup on a sufficiently large set and of course for discrete Y, $S(Y)$ is just the full transformation semigroup on Y. With a bit more effort, one can produce a compact space whose semigroup contains copies of each semigroup in the collection. Simply take the Stone–Čech compactification βY of the discrete space Y. The mapping which takes a function in $S(Y)$ into its natural extension over βY is an isomorphism from $S(Y)$ into $S(\beta Y)$. Of course, the space βY is badly disconnected but there are even compact connected spaces whose semigroups will contain copies of all the semigroups in the collection. A result in [15] assures us that there is an arcwise connected metric space with the desired property and one simply takes the Stone–Čech compactification of this space.

There is, however, a much nicer result in this direction. It is a theorem of Paalman-de Miranda [18] but before we state it, we should place it in the proper setting by first discussing a rather remarkable result due to J. de Groot. He proved in 1959 [6] that for any group G, there exists a connected metric space Y such that G is isomorphic to $H(Y)$, the group of all homeomorphisms on Y and since $H(Y)$ is isomorphic to $H(\beta Y)$, he also gets the fact that any group can be regarded as the group of all homeomorphisms on a compact connected space. In 1963, a contribution was made to an analogous problem for semigroups when Z. Hedrlin and A. Pultr [7] showed that for each semigroup T with identity, there exists a T_0 space Y such that T is isomorphic to the semigroup of all local homeomorphisms on Y. A local homeomorphism is a selfmap h of X with the property that each point belongs to a neighborhood such that the restriction of h to this neighborhood is a homeomorphism. Hedrlin and Pultr had asked if one could always take the space Y to be Hausdorff and this was answered negatively in the 1966 paper [18] of Paalman-de Miranda. Moreover, in that paper she obtained the interesting result of which we spoke earlier that given any semigroup T with identity, there exists a connected metric space Y such that T is isomorphic to the semigroup of all quasi-local homeomorphisms on Y. A quasi-local homeomorphism on Y is a continuous selfmap h such that for each nonempty open subset G of Y, there exists a nonempty open subset H of G such that the restriction of h to H is a homeomorphism. Furthermore, the mapping which takes a quasi-local homeomorphism on Y into its extension over βY is an isomorphism from the semigroup of all quasi-local homeomorphisms on Y into the semigroup of all quasi-local homeomorphisms on βY. So, not only can any semigroup T with identity be regarded as the semigroup of all quasi-local homeomorphisms on a connected metric space but the space can also be chosen to be compact and connected. Since one can take T to be a full transformation semigroup, it is immediate that the semigroups of these spaces of Paalman-de Miranda can be chosen to contain copies of many different $S(X)$.

The general theory we develop here seems to indicate that the cases we have just discussed are somewhat extreme.

Chapter 1

1. LEFT ZERO SUBSEMIGROUPS OF TOPOLOGICAL SEMIGROUPS

Information concerning the left zero subsemigroups of $S(Y)$ is very helpful in determining if and how $S(X)$ can be embedded in $S(Y)$. We recall that a left zero semigroup is one in which the product of any two elements is the element on the left. In this section, T is any topological semigroup whose subset E of idempotents is nonempty. We define a relation Δ on E by

$$\Delta = \{(v, w) \in E \times E: vw = v \text{ and } wv = w\}.$$

The proof of the following result is straightforward and is omitted.

LEMMA 1.1. *Δ is an equivalence relation on E.*

LEMMA 1.2. *E is closed in T.*

Proof. Let $a \in T - E$. Then $a^2 \neq a$ and there exist disjoint open sets G and H such that $a \in G$ and $a^2 \in H$. There is also an open subset V of T such that $a \in V$ and $V^2 \subset H$. Now take any $b \in G \cap V$. Then $b^2 \in H$, which implies that $b^2 \notin G$. Thus $b^2 \neq b$ and we see that $G \cap V$ is a neighborhood of a which does not intersect E.

PROPOSITION 1.3. *The Δ-classes are the maximal left zero subsemigroups of T and each of these is closed in T.*

Proof. The first assertion follows in a straightforward manner. In order to verify the second, we let L be any one of the Δ-classes of E. In view of the previous lemma, it is sufficient to show that L is closed in E. Take any $a \in E - L$. There are two possibilities. Either (1) $va \neq v$ for some $v \in L$ or (2) $av \neq a$ for some $v \in L$. Suppose (1) holds. Then there exist disjoint open subsets G and H of T containing va and v, respectively, and there also exist open subsets V and W containing v and a, respectively, such that $VW \subset G$. Choose any $b \in W$. Then $vb \in VW \subset G$. Thus $vb \neq v$, which implies that $b \notin L$. Hence $W \cap E$ is a neighborhood of a in E which does not intersect L.

Now suppose that (2) holds. Then there are disjoint open subsets G and H of T containing av and a, respectively. There also exist neighborhoods W and V of a and v, respectively, such that $WV \subset G$. Now take any point $b \in H \cap W$. Then $bv \in WV \subset G$, which implies that $bv \neq b$, since $b \in H$. Thus $H \cap W \cap E$ is a neighborhood of a in E and we conclude that L is closed in E, and hence in T.

Remark. The case for the maximal left zero subsemigroups of T is just the same as for the maximal subgroups with respect to the fact that any two either coincide or are disjoint.

Suppose that T contains left zeros and denote the set of all left zeros of T by K. Then $K \subset E$ and we have

LEMMA 1.4. *K is the kernel of T.*

Proof. It is immediate that K is a right ideal. Moreover, if $v \in K$ and $a, b \in T$, then $(av)b = a(vb) = av$. Thus $av \in K$ for any $a \in T$ and K is a two-sided ideal. Since $KA = K$ for any nonempty subset A of T, it follows that K is contained in every two-sided ideal of T. Thus, K is the kernel of T.

LEMMA 1.5. *Let L be any left zero subsemigroup of T. Then either $L \cap K = \emptyset$ or $L \subset K$.*

Proof. Suppose $a \in L \cap K$; choose any $b \in L$. Then $b = ba$, and for any $c \in T$ we have $bc = (ba)c = b(ac) = ba = b$. Thus b is a left zero of T and we have $L \subset K$.

Next we associate a semigroup to each left zero subsemigroup L of T.

DEFINITION 1.6. Let L be any left zero subsemigroup of T. We denote the *shell* of L by $\mathrm{Sh}(L)$ and define it by

$$\mathrm{Sh}(L) = \{a \in T \colon va = v \text{ for each } v \in L\}.$$

It is immediate that if $J \subset L$, then $\mathrm{Sh}(L) \subset \mathrm{Sh}(J)$. In fact, we shall see that in this case, the two are identical. Furthermore, the two shells may be identical even if L and J do not intersect. The next result gives necessary and sufficient conditions for one shell to be contained in another.

PROPOSITION 1.7. *The following statements are equivalent.*

(1) $\mathrm{Sh}(L) \subset \mathrm{Sh}(J)$.

(2) *$wv = w$, for all $v \in L$ and $w \in J$.*

(3) *There exists $v \in L$ such that $wv = w$ for all $w \in J$.*

Proof. It is immediate that (1) implies (2) and (2) implies (3). We show that (3) implies (1). Choose $v \in L$ which satisfies (3) and let any $a \in \mathrm{Sh}(L)$ be given. Then

$$wa = (wv)a = w(va) = wv = w.$$

Thus $a \in \mathrm{Sh}(L)$ and the verification is complete.

The next three corollaries follow easily from the previous proposition.

COROLLARY 1.8. $\mathrm{Sh}(L) = \mathrm{Sh}(J)$ *if and only if* L *and* J *are contained in the same* Δ*-class.*

COROLLARY 1.9. *Let* D *be the unique* Δ*-class which contains* L. *Then* $\mathrm{Sh}(L) = \mathrm{Sh}(D)$.

COROLLARY 1.10. $\mathrm{Sh}(L)$ *is a proper subsemigroup of* $\mathrm{Sh}(J)$ *if and only if there exists a* $v \in L$ *such that* $wv = w$ *for all* $w \in J$, *but* $vu \neq v$ *for some* $u \in J$.

Remark. We see that $\mathrm{Sh}(L)$ is the maximal subsemigroup of T for which the Δ-class containing L is the kernel.

We mentioned previously that $\mathrm{Sh}(L)$ and $\mathrm{Sh}(J)$ may coincide even if $L \cap J = \emptyset$. This is immediate from Corollary 1.8, for we need only choose two distinct idempotents in the same Δ-class and note that the shells they generate are identical. In order to obtain a case where one shell is a proper subsemigroup of another, we choose two idempotents w and v such that $wv = w$ but $vw \neq v$. Then Corollary 1.10 tells us that $\mathrm{Sh}(\{v\})$ is a proper subsemigroup of $\mathrm{Sh}(\{w\})$. To be a bit more specific, let T be any semigroup with left identity e and left zero z. Then $\mathrm{Sh}(\{e\})$ is a proper subsemigroup of $\mathrm{Sh}(\{z\})$. The former, of course, consists only of the element e, while the latter is all of T.

PROPOSITION 1.11. *In any topological semigroup* T, *each shell in* T *is a closed subsemigroup of* T.

Proof. Suppose $a \in T - \mathrm{Sh}(L)$. Then $va \neq v$ for some $v \in L$, so there exist open subsets V and W such that $v \in V$, $a \in W$, and $VW \subset T - \{v\}$. Suppose there is a point $b \in W \cap \mathrm{Sh}(L)$. Then $vb \in VW \subset T - \{v\}$ which is a contradiction since $b \in \mathrm{Sh}(L)$ implies $vb = v$. Thus, W is a neighborhood of a which does not intersect $\mathrm{Sh}(L)$.

PROPOSITION 1.12. *Suppose that* D *is a* Δ*-class or equivalently, a maximal left zero subsemigroup of* T. *Then* $aD \subset D$ *for each* $a \in \mathrm{Sh}(D)$.

Proof. Let $v \in D$. Then for any $w \in D$, we have $(av)w = a(vw) = av$, and $w(av) = (wa)v = wv = w$. Thus av must belong to D since D is maximal.

We close this section with a few remarks. It is immediate from our previous results that each left zero subsemigroup of a semigroup T is contained in a unique shell $\mathrm{Sh}(D)$ and that we can take D to be maximal among the left zero subsemigroups. Thus it follows that if S is any semigroup which contains left zeros, then any embedding of S into T must actually send S into one of these shells. Since $S(X)$ does have left zeros, this is the motivation for considering shells.

Our next step is to obtain interpretations in the context of semigroups of continuous functions.

2. LEFT ZERO SUBSEMIGROUPS OF $S(Y)$

We first need some convenient notation. Now each function f in $S(Y)$ induces a natural decomposition of X. The sets in the decomposition are the pre-images of the points in the range of f. We denote this family of sets by $\mathscr{D}(f)$. Thus $\mathscr{D}(f) = \{f^{-1}(y): y \in f(Y)\}$.

PROPOSITION 2.1. *Let L be any subset of $S(Y)$. Then L is a left zero subsemigroup of $S(Y)$ if and only if the following two conditions are satisfied.*

(1) *For each f in L and $A \in \mathscr{D}(f)$, there is an $a \in A$ such that $f(x) = a$ for each $x \in A$.*

(2) *$\mathscr{D}(f) = \mathscr{D}(g)$ for all $f, g \in L$.*

Proof. We suppose that L is a left zero subsemigroup of $S(Y)$ and we show first that (1) holds. Now there is an $a \in X$ such that $A = f^{-1}(a)$, which implies that $f(x) = a$ for all $x \in A$. Moreover, f is idempotent and for any such x, we have $f(a) = f(f(x)) = f(x) = a$. This places a in A.

Now we show that (2) holds. Let f and g be any two elements of L and suppose that $B \in \mathscr{D}(g)$. Then since (1) holds, there is a $b \in B$ such that $g(x) = b$ for all x in B. Since L is a left zero semigroup, we have $f \circ g = f$, so that for any $x \in B$, we obtain $f(x) = f(g(x)) = f(b)$. Thus $B \subset f^{-1}(f(b))$. This shows that $\mathscr{D}(g)$ is a refinement of $\mathscr{D}(f)$. In a similar manner, $\mathscr{D}(f)$ is also a refinement of $\mathscr{D}(g)$ and it then follows that $\mathscr{D}(f) = \mathscr{D}(g)$.

Now suppose that both (1) and (2) hold and let $x \in Y$ be given. Then $x \in A$ for some $A \in \mathscr{D}(f) = \mathscr{D}(g)$. By (1), there exist $a, b \in A$ such that $f(y) = a$ and $g(y) = b$ for all $y \in A$. Then $f(g(x)) = f(b) = a = f(x)$. Thus $f \circ g = f$; that is, L is a left zero subsemigroup of $S(Y)$.

Since a function f is constant on X if and only if $\mathscr{D}(f) = \{X\}$, we immediately obtain the following

COROLLARY 2.2. *Let L be any left zero subsemigroup of $S(X)$. Then either all functions in L are constant or no functions in L are constant.*

We remark that the latter corollary is also an immediate consequence of Lemma 1.5 and the fact that a function in $S(X)$ is a left zero of $S(X)$ if and only if it is a constant function.

Now denote the set of idempotents of $S(Y)$ by $E(Y)$ and, as before, let $K(Y)$ denote the kernel of $S(Y)$, which is just the collection of all constant functions. From Proposition 2.1, we can determine the Δ-classes of $E(Y)$, but first we introduce some definitions and some notation.

DEFINITION 2.3. Any decomposition $\mathscr{D}$ of Y is an *eligible* decomposition if $\mathscr{D} = \mathscr{D}(v)$ for some idempotent v in $S(Y)$.

It is immediate that every decomposition of Y is eligible if and only if Y is discrete. So, in particular, there are many decompositions of I

which are not eligible. In fact, there are many functions $f \in S(I)$ such that even $\mathscr{D}(f)$ is not eligible. The following result shows how to obtain some of these.

PROPOSITION 2.4. *Let f be a function in $S(I)$ such that $f(x) \neq f(0)$ for $x \neq 0$, and $f(x) \neq f(1)$ for $x \neq 1$. Suppose further that f is not injective. Then $\mathscr{D}(f)$ is not an eligible decomposition.*

Proof. Suppose, to the contrary, that $\mathscr{D}(f) = \mathscr{D}(v)$ for some idempotent v. Then $\{0\}$ and $\{1\}$ are in $\mathscr{D}(v)$. Since v is an idempotent, $v(v(0)) = v(0)$. Thus $v(0)$ and 0 belong to the same set in the decomposition, which means that $v(0) = 0$. Similarly, $v(1) = 1$. Thus all of I belongs to the range of v, and since v is idempotent, it must be the identity on I. Thus $\mathscr{D}(v)$ consists of singletons, which is a contradiction since f is not injective.

The non-constant everywhere locally recurrent functions in $S(I)$ [4], [17] also provide such examples. An everywhere locally recurrent function is one with the property that for any point $p \in I$ and any neighborhood G of p, there exists an $x \in G$ such that $x \neq p$ and $f(x) = f(p)$. The first example of such a function that we know of was given by J. Gillis in [4]. It follows ([17], p. 283) that for any such function f, each set in $\mathscr{D}(f)$ is a perfect set. One easily verifies that for any nonconstant idempotent, any point in the interior of its range is an isolated point with respect to the set in the decomposition to which it belongs. Thus $\mathscr{D}(f)$ is not eligible if f is any one of these nonconstant everywhere locally recurrent functions.

It is immediate from Proposition 2.1 that every left zero subsemigroup L of $S(Y)$ induces, in a natural way, an eligible decomposition on Y. This decomposition is $\mathscr{D}(v)$, where v is any function chosen from L.

DEFINITION 2.5. For any left zero subsemigroup L of $S(Y)$, we define $\mathscr{D}(L) = \mathscr{D}(v)$, where v is any element in L, and we refer to $\mathscr{D}(L)$ as the *decomposition induced by L.*

The following result is a straightforward consequence of Proposition 2.1 so we omit its proof.

COROLLARY 2.6. *Let L and J be left zero subsemigroups of $S(Y)$. Then $\mathscr{D}(L) = \mathscr{D}(J)$ if and only if L and J are both contained in the same Δ-class.*

DEFINITION 2.7. Let $\mathscr{D}$ be any eligible decomposition of Y. We define:

$$D(Y, \mathscr{D}) = \{f \in S(Y)\colon \text{ for each } A \in \mathscr{D}, \text{ there exists } a \in A \text{ such that } f(x) = a \text{ for all } x \in A\},$$

$$S(Y, \mathscr{D}) = \{f \in S(Y)\colon f(A) \subset A \text{ for each } A \in \mathscr{D}\}.$$

PROPOSITION 2.8. *Let L be any left zero subsemigroup of $S(Y)$. Then $D(Y, \mathscr{D}(L))$ is the unique Δ-class containing L and $S(Y, \mathscr{D}(L))$ is the shell of L and, of course, of $D(Y, \mathscr{D}(L))$ as well.*

Proof. The first assertion follows from Proposition 2.1. We verify that $\mathrm{Sh}(L) = S(Y, \mathscr{D}(L))$. Take any $f \in \mathrm{Sh}(L)$. Then $v \circ f = v$ for all $v \in L$. Then, for any $A \in \mathscr{D}(L) = \mathscr{D}(v)$ and any $x \in A$, we have $v(f(x)) = v(x)$, which implies that $f(x) \in v^{-1}(v(x)) = A$. Thus, f is in $S(Y, \mathscr{D}(L))$.

On the other hand, suppose $f \in S(Y, \mathscr{D}(L))$ and $v \in L$. Then $f(A) \subset A$ for each $A \in \mathscr{D}(L) = \mathscr{D}(v)$. By Definition 2.7 there exists for each $A \in \mathscr{D}(v)$, a point $a \in A$ such that $v(x) = a$ for each $x \in A$. Then $v(f(x)) = a = v(x)$ for each $x \in A$. It follows that $v \circ f = v$, and hence that $f \in \mathrm{Sh}(L)$.

EXAMPLE 2.9. Let Y be any discrete space and let α be any cardinal number which does not exceed $\mathrm{card}\, Y$. Then there exists a maximal left zero subsemigroup of $S(Y)$ with precisely α elements. For, we can choose any subset A of Y such that $\mathrm{card} A = \alpha$ and let $\mathscr{D}$ consist of A and all singletons composed of elements not in A. Then $D(Y, \mathscr{D})$ is a maximal left zero subsemigroup of $S(Y)$ which contains α elements. Furthermore, any shell of $S(Y)$ is the direct product of full transformation semigroups. This fact follows directly from the previous theorem.

EXAMPLE 2.10. Let v be the function from I into I which is defined by $v(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $v(x) = \frac{1}{2}$ for $\frac{1}{2} \leqslant x \leqslant 1$. Then $\mathscr{D}(v) = \{\{x\}:\ 0 \leqslant x \leqslant \frac{1}{2}\} \cup \{[\frac{1}{2}, 1]\}$. Suppose $w \in D(I, \mathscr{D}(v))$. Then w is idempotent and for any x such that $0 \leqslant x \leqslant \frac{1}{2}$, we have $w(w(x)) = w(x)$. Thus x and $w(x)$ belong to the same set in $\mathscr{D}(v)$ which implies that $w(x) = x$ for any x in $[0, \frac{1}{2}]$. Furthermore, w must map all of $[\frac{1}{2}, 1]$ onto a single point in that interval. This forces $w(x) = \frac{1}{2}$ for $\frac{1}{2} \leqslant x \leqslant 1$. In other words, $w = v$ and $\{v\}$ is a maximal left zero subsemigroup of $S(I)$. Moreover, it is straightforward to check that the shell of $\{v\}$ consists of all functions $f \in S(I)$ such that $f(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $f(x) \geqslant \frac{1}{2}$ for $\frac{1}{2} \leqslant x \leqslant 1$.

It was shown in [15] that if L is any uncountable left zero subsemigroup of $S(I)$, then it must consist of constant functions. Thus, the Δ-class which contains it is the kernel of the semigroup.

EXAMPLE 2.11. We consider $S(I)$ again. Define v by $v(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $v(x) = 1-x$ for $\frac{1}{2} \leqslant x \leqslant 1$, and define w by $w(x) = 1-x$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $w(x) = x$ for $\frac{1}{2} \leqslant x \leqslant 1$. Let $L = \{v, w\}$. Then

$$\mathscr{D}(v) = \mathscr{D}(w) = \mathscr{D}(L) = \{\{x, 1-x\}:\ 0 \leqslant x \leqslant \tfrac{1}{2}\},$$

and by Proposition 2.1, L is a left zero subsemigroup of $S(Y)$. Now suppose D is the Δ-class containing L and let $u \in D$. It follows that $u(\frac{1}{2}) = \frac{1}{2}$ and that either (1) $u(0) = 0$ and $u(1) = 0$ *or* (2) $u(0) = 1$ and $u(1) = 1$. This, in turn, implies that u must be either v or w. Thus L is a maximal left zero subsemigroup of $S(I)$. With some calculation, one can also verify that the shell of L consists precisely of v, w, i, and j, where i is the identity and j is defined by $j(x) = 1-x$ for $0 \leqslant x \leqslant 1$. Thus, in this case, both the Δ-class and its shell are finite in contrast with Example 2.10, where the Δ-class was finite but its shell was not.

Example 2.12. Let E^2 denote the Euclidean plane. For each real number r, define a continuous selfmap v_r of E^2 by $v_r(x, y) = (r, y)$. Then $\{v_r\colon r \in R\}$ is a maximal left zero subsemigroup of $S(E^2)$, none of whose elements are constant functions. Moreover, there are c of them and, as we noted in Example 2.10, there is no possibility for this to occur in $S(I)$.

3. SOME NATURAL HOMOMORPHISMS

Let $\mathscr{D}$ be any eligible decomposition of Y. Then for each A in $\mathscr{D}$, there is a natural homomorphism η_A from $S(Y, \mathscr{D})$ into $S(A)$. For any $f \in S(Y, \mathscr{D})$, we just define $\eta_A(f)$ to be the restriction of f to A.

Definition 3.1. The homomorphism described above will be referred to as the *A-sectional homomorphism* from $S(Y, \mathscr{D})$ into $S(A)$.

One can use the A-sectional homomorphisms to define a natural isomorphism μ from $S(Y, \mathscr{D})$ into the direct product of the semigroups $S(A)$, where $A \in \mathscr{D}$. For each f in $S(Y, \mathscr{D})$ and $A \in \mathscr{D}$, we define

$$\big(\mu(f)\big)_A = \eta_A(f).$$

It follows that μ is an isomorphism from $S(Y, \mathscr{D})$ into $\Pi\{S(A)\colon A \in \mathscr{D}\}$. The next result will give a rather intuitive necessary and sufficient condition that μ be an isomorphism from $S(Y, \mathscr{D})$ onto $\Pi\{S(A)\colon A \in \mathscr{D}\}$. The proof is straightforward and is omitted.

Lemma 3.2. *Let $\mathscr{D}$ be any eligible decomposition of Y. Then the mapping μ is an isomorphism from $S(Y, \mathscr{D})$ onto $\Pi\{S(A)\colon A \in \mathscr{D}\}$ if and only if for every selfmap f of Y, $f|_A$ is a continuous selfmap of A for each $A \in \mathscr{D}$ implies f is continuous on Y.*

Corollary 3.3. *Let $\mathscr{D}$ be any decomposition of Y into clopen subsets. Then the mapping μ is an isomorphism from $S(Y, \mathscr{D})$ onto $\Pi\{S(A)\colon A \in \mathscr{D}\}$.*

The condition of Lemma 3.2 can be translated into conditions on the decomposition which are fairly easy to state. We shall do this after the following convenient definition.

Definition 3.4. Let $\mathscr{D}$ be any decomposition of Y and let $\{x_n\colon n \in W\}$ be any net in Y. Then a *companion net* of $\{x_n\colon n \in W\}$ is any net $\{y_n\colon n \in W\}$ with the same directed set W, such that, for each $n \in W$, both x_n and y_n belong to the same set in the decomposition.

Proposition 3.5. *Let Y be any topological space and let $\mathscr{D}$ be any eligible decomposition of Y. Then μ is an isomorphism from $S(Y, \mathscr{D})$ onto $\Pi\{S(A): A \in \mathscr{D}\}$ if and only if the following two conditions are satisfied:*

(1) *every set in $\mathscr{D}$ is either clopen or a singleton;*

(2) *if $\{p\} \in \mathscr{D}$ and some net $\{x_n: n \in W\}$ converges to p, then every companion net of $\{x_n: n \in W\}$ also converges to p.*

Proof. First suppose that conditions (1) and (2) hold and let f be any selfmap of Y with the property that the restriction of f to each $A \in \mathscr{D}$ is a continuous selfmap of A. We show that f is continuous on Y. Let V denote the union of all clopen subsets in $\mathscr{D}$. Then $Y - V$ is closed and the restriction of f to $Y - V$ is continuous since by (1), f is just the identity map there. Now we show that f is continuous on $\mathrm{Cl}(V)$. Since V is the union of clopen sets, it is immediate that f is continuous at any point in V. Let $p \in \mathrm{Cl}(V) - V$. By (1), $\{p\}$ is one of the sets in $\mathscr{D}$. Thus $f(p) = p$. If $\{x_n: n \in W\}$ is any net converging to p, then $\{f(x_n): n \in W\}$ is a companion net and hence also converges to p. This proves that f is continuous at p. Thus, f is continuous on $\mathrm{Cl}(V)$. We have shown that the restriction of f to each of two closed subsets, whose union is all of Y, is continuous. Hence f is continuous on Y and it follows from Lemma 3.2 that μ is an isomorphism onto $\Pi\{S(A): A \in \mathscr{D}\}$.

Now we prove the converse, and here also we rely on Lemma 3.2. We first show by contradiction that (1) holds. If there is a set A in $\mathscr{D}$ which is neither clopen (and hence not open) nor a singleton, we choose any $a \in A - \mathrm{Int}\,A$ and $b \in A - \{a\}$ and we define a selfmap f of Y as follows:

$$f(x) = \begin{cases} b & \text{for} \quad x \in A, \\ x & \text{for} \quad x \in Y - A. \end{cases}$$

Then $f|_B$ is continuous for each $B \in \mathscr{D}$. We show, however, that f is not continuous on Y, and this will contradict Lemma 3.2. Choose two disjoint open subsets G and H containing a and b, respectively. Then $f(a) = b \in H$, but for any neighborhood V of a, there is a point $p \in (V \cap G) - A$, since a is not an interior point. Then $f(p) = p \notin H$. Thus, f is not continuous at a and we have our contradiction.

Now we show, by contradiction, that (2) holds. Suppose that a net $\{x_n: n \in W\}$ converges to p but some companion net $\{y_n: n \in W\}$ does not converge to p. Then there exists an open neighborhood G of p such that $\{y_n: n \in W\}$ is not eventually in G. That is, $\{y_n: n \in W^*\}$ lies outside G for some cofinal subset W^* of W. Define x_n, x_m $(n, m \in W^*)$ to be equivalent if they both belong to the same set in the decomposition. Then for each equivalence class E of $\{x_n: n \in W^*\}$, we have $E \subset A_E$ for some A_E in $\mathscr{D}$. Choose any x_n in E and define $f(x) = y_n$ for all x in A_E. For any $A \in \mathscr{D}$ which does not correspond to one of these equivalence classes define $f(x) = x$ for all x in A. Then $f|_B$ is a continuous selfmap of B for each B in $\mathscr{D}$. However, Lemma 3.2 is again contradicted since $\{x_n: n \in W^*\}$ converges to p while $\{f(x_n): n \in W^*\}$ does not converge to $p = f(p)$.

We give two examples of eligible decompositions where, in the first example, (2) of Proposition 3.5 is satisfied and, in the second example, it is not.

EXAMPLE 3.6. Let A_n denote the closed interval $[1/2n, 1/(2n-1)]$ and let $X = [\bigcup_n A_n] \cup \{0\}$. Let $\mathscr{D}$ consist of all the A_n together with $\{0\}$. It is not difficult to show that $\mathscr{D}$ is an eligible decomposition of X and that both (1) and (2) of Proposition 3.5 are satisfied.

If the decomposition consists of singletons, it is immediate that (2) of Proposition 3.5 is satisfied. There are also nontrivial examples and the following is one.

EXAMPLE 3.7. This example is a subspace of the plane. Let

$$A_n = \left\{(x, y):\ y = \frac{x}{n} \text{ and } \frac{1}{n} \leqslant x \leqslant 1\right\}$$

and let $X = [\bigcup_n A_n] \cup \{(0, 0)\}$. Then let $\mathscr{D}$ consist of the A_n together with $\{(0, 0)\}$. Here, too, $\mathscr{D}$ is an eligible decomposition and (1) of Proposition 3.5 is satisfied but (2) is not. In particular, let $x_n = \left(\frac{1}{n}, \frac{1}{n^2}\right)$ and $y_n = \left(1, \frac{1}{n}\right)$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are companion sequences but the first converges to $(0, 0)$ and the second does not even converge.

We prove one more result about the mapping μ but first we need to recall the definition of the compact-open topology. For any topological space X choose a compact subset K and an open subset G and define

$$\langle K, G\rangle = \{f \in S(X): f(K) \subset G\}.$$

The compact-open topology on $S(X)$ has for a subbasis of closed sets all sets of the form $\langle K, G\rangle$ and it is well known that if X is locally compact, $S(X)$ with this topology is a topological semigroup. The converse is not true in general but is true for a very large class of spaces [19]. We will not ask that our spaces be locally compact.

PROPOSITION (3.8) *Let $\mathscr{D}$ be any decomposition of Y into clopen subsets. Let $S(Y)$ and each $S(A)$, $A \in \mathscr{D}$, have the compact-open topologies. Then μ maps $S(Y, \mathscr{D})$ isomorphically and homeomorphically onto $\Pi\{S(A): A \in \mathscr{D}\}$, where the latter has the product topology.*

Proof. First of all, it is immediate from Corollary 3.3 that μ maps $S(Y, \mathscr{D})$ isomorphically onto $\Pi\{S(A): A \in \mathscr{D}\}$. We show that μ is a homeomorphism as well. Let $\langle K, G\rangle$ be any subbasic open subset of $S(Y)$.

Then, since the sets in the decomposition are open, K is contained in the union of a finite number of them, say $\{A_i\}_{i=1}^n$. Then for any f in $S(Y, \mathscr{D})\cap\langle K, G\rangle$, we have $f(K\cap A_i) \subset G\cap A_i$ for each i. This implies that

$$(\mu(f))_{A_i} \in \langle K\cap A_i, G\cap A_i\rangle, \quad \text{for each } i.$$

Thus,

$$\mu(f) \in \bigcap_{i=1}^{n} P_{A_i}^{-1}\langle K\cap A_i, G\cap A_i\rangle \subset \mu[S(Y, \mathscr{D})\cap\langle K, G\rangle],$$

where P_{A_i} is a projection map. This proves that μ is an open map.

To show that μ is continuous, we observe that $P_A\circ\mu = \eta_A$, the A-sectional homomorphism for each $A \in \mathscr{D}$, and we show that η_A is continuous. Let $\langle K, G\rangle$ be a subbasic open set of $S(A)$. Then, since A is open, G is also an open subset of Y and, of course, K is compact in Y. Let $\langle K, G\rangle_Y$ denote the family of all functions of $S(Y)$ which carry K into G. It is straightforward to verify that

$$\eta_A^{-1}\langle K, G\rangle = S(Y, \mathscr{D})\cap\langle K, G\rangle_Y,$$

and this completes the proof.

Remarks. First of all in this last result, we did not require Y to be locally compact so it is possible that none of the semigroups involved is a topological semigroup. Of course, if Y is locally compact, then each $A \in \mathscr{D}$ is also locally compact and in this case all of the semigroups are topological semigroups, including the product semigroup.

Furthermore, we remark that the same techniques will result in the same conclusion as in Proposition 3.8 if, instead of the compact-open topologies, one uses various smaller topologies, including the point-open topology in particular. With such topologies, however, the semigroups involved will seldom be topological semigroups.

We conclude this section with some remarks about isomorphisms from a fairly arbitrary semigroup T into $S(Y)$.

Let T be any semigroup which contains left zeros and let φ be any isomorphism from T into $S(Y)$. We can associate with φ, in a natural way, a unique eligible decomposition of Y which we shall denote by $\mathscr{D}(\varphi)$. We do this in the following way: we first note that φ must take the kernel (that is, the subsemigroup of left zeros of T) into some unique Δ-class, $D(Y, \mathscr{D})$, where $\mathscr{D}$ is the decomposition of Y which is induced by any one of the functions in $D(Y, \mathscr{D})$. We define $\mathscr{D}(\varphi) = \mathscr{D}$. It follows that φ maps T into the shell of $D(Y, \mathscr{D})$ which is the subsemigroup $S(Y, \mathscr{D})$ of $S(Y)$. In order for φ to be an isomorphism from T into $S(Y)$, it is sufficient that $\eta_A\circ\varphi$ be injective for some $A \in \mathscr{D}$, where η_A is the A-sectional homomorphism from $S(Y, \mathscr{D}(\varphi))$ into $S(A)$ (we remark that $\eta_A\circ\varphi$ may well be injective even though η_A is not).

4. EMBEDDING $S(X)$ INTO A FULL TRANSFORMATION SEMIGROUP

When the space Y is discrete $S(Y)$ is, as we have observed before, simply the full transformation semigroup $\mathcal{T}_Y$ on Y. When we topologize $\mathcal{T}_Y$, it will be with the compact-open topology which, in this case, coincides with the point-open topology. Now $\mathcal{T}_Y$ is a topological semigroup when topologized this way and topological properties of shells and maximal left zero subsemigroups of $\mathcal{T}_Y$ can be completely described by rather simple conditions on the sets of the corresponding decomposition. We shall list a number of such results. After that, we shall look at the problem of embedding $S(X)$ into $\mathcal{T}_Y$ where X is an arbitrary topological space. It is not difficult to see that

(4.1) *if X is any topological space and* $\operatorname{card} X \leqslant \operatorname{card} Y$, *then* $S(X)$ *can be embedded in* $\mathcal{T}_Y$.

Indeed, let h be any injection of X into Y and let v be any mapping of Y into the range H of h such that v restricted to H is the identity. Then the mapping φ from $S(X)$ into $\mathcal{T}_Y$ given by $\varphi(f) = h\circ f\circ h^{-1}\circ v$ is an embedding. Moreover, when Y is infinite, there are many such injections and many idempotents v and each pair gives rise to a different embedding. There are also many embeddings which are not of this form.

Now the converse of (4.1) does not hold. J. de Groot introduced some spaces in [6] all of which have c points and we will eventually see that if X is any one of these spaces, then $S(X)$ can be embedded in $\mathcal{T}_N$ where N is the set of natural numbers. But there are few continuous selfmaps on these spaces and because of this they are traditional counterexamples in the theory of semigroups of continuous functions. So one should really ask if the converse to (4.1) is true when $S(X)$ has a rich supply of continuous functions and the answer here is yes.

We will prove that if $S(X)$ is doubly transitive on X and if $S(X)$ can be embedded in $\mathcal{T}_Y$, then $\operatorname{card} X \leqslant \operatorname{card} Y$. We also solve the related problem where the semigroups are given topologies and the embedding is topological as well as algebraic.

We return to our discussion of shells and maximal left zero subsemigroups of $\mathcal{T}_Y$. We first prove a lemma which is essentially a translation of a previous result.

LEMMA 4.2. *Let $\mathcal{D}$ be any decomposition of Y into mutually disjoint nonempty subsets. Then $S(Y, \mathcal{D})$ is isomorphic to $\Pi\{\mathcal{T}_A\colon A\in\mathcal{D}\}$ and $D(Y, \mathcal{D})$ is isomorphic to $\Pi\{K_A\colon A\in\mathcal{D}\}$, where K_A is the set of constant functions on A; that is, K_A is the kernel of $\mathcal{T}_A$. Furthermore, if all of the full transformation semigroups are given compact-open topologies, then the isomorphisms can be taken to be homeomorphism as well.*

Proof. The result follows from Propositions 2.8 and 3.8 and the observation that the mapping μ of Proposition 3.8 takes $D(Y, \mathscr{D})$ onto the semigroup $\Pi\{K_A: A \in \mathscr{D}\}$.

It is convenient to have still another lemma before we prove the first proposition of this section.

LEMMA 4.3. *Let $\mathscr{T}_Y$ have the compact-open topology. Then the following statements are equivalent:*

(1) *$\mathscr{T}_Y$ is compact;*

(2) *$\mathscr{T}_Y$ is locally compact;*

(3) *Y is finite.*

Proof. The only implication that is not immediately evident is that (2) implies (3). So suppose (3) is false. Then, topologically, $\mathscr{T}_Y$ is the product of card Y copies of Y since the compact-open topology is just the point-open topology in this case. Thus $\mathscr{T}_Y$ is an infinite product of noncompact spaces. But it is well known that a product is locally compact if and only if all of the factor spaces are locally compact and all but a finite number are actually compact. Thus, when (3) fails to hold, so does (2). This completes the proof.

PROPOSITION 4.4. *Let $\mathscr{T}_Y$ have the compact-open topology and let $\mathscr{D}$ be any decomposition of Y. Then the following statements are equivalent.*

(1) *$S(Y, \mathscr{D})$ is compact.*

(2) *$S(Y, \mathscr{D})$ is locally compact.*

(3) *$D(Y, \mathscr{D})$ is compact.*

(4) *Each set in $\mathscr{D}$ is finite.*

Proof. It is immediate from Lemmas 4.2 and 4.3 that each of (1) and (2) is equivalent to (4). Furthermore, since each K_A inherits the discrete topology, it follows from Lemma 4.2 that (3) and (4) are equivalent.

PROPOSITION 4.5. *Let $\mathscr{T}_Y$ have the compact-open topology and let $\mathscr{D}$ be any decomposition of Y. Then $D(Y, \mathscr{D})$ is locally compact if and only if all but a finite number of sets in $\mathscr{D}$ are finite.*

Proof. The proof is deduced from the following: Lemma 4.2; the fact that a product is locally compact if and only if each factor is locally compact and all but a finite number are compact; and the fact that each K_A is discrete and has as many points as A.

Remark. It follows from the previous two results that there are shells $S(Y, \mathscr{D})$ which are not locally compact but whose kernels, $D(Y, \mathscr{D})$, are.

PROPOSITION 4.6. *Let $\mathscr{T}_Y$ have the compact-open topology and let $\mathscr{D}$ be any decomposition of Y. Then $S(Y, \mathscr{D})$ is metrizable if and only if every $A \in \mathscr{D}$ is countable and at most countably many sets in $\mathscr{D}$ have more than one point.*

Proof. It is well known that a product is metrizable if and only if each factor is metrizable and at most a countable number of factors have more than one point. So $\mathcal{T}_A$ is metrizable if and only if A is countable. The proof now follows from Lemma 4.2.

PROPOSITION 4.7. *Let $\mathcal{T}_Y$ have the compact-open topology and let $\mathcal{D}$ be any decomposition of Y. Then $D(Y, \mathcal{D})$ is metrizable if and only if at most countably many of the sets in $\mathcal{D}$ have more than one point.*

Proof. Each K_A is metrizable since it is discrete, so that the proposition on the metrizability of products mentioned in the previous proof and Lemma 4.2 combine to prove this proposition also.

PROPOSITION 4.8. *Let $\mathcal{T}_Y$ have the compact-open topology and let $\mathcal{D}$ be any decomposition of Y. Then $D(Y, \mathcal{D})$ has no isolated points if and only if an infinite number of sets in $\mathcal{D}$ has more than one point.*

Proof. Again we use the fact that $D(Y, \mathcal{D})$ is homeomorphic to $\Pi\{K_A: A \in \mathcal{D}\}$ where each K_A is discrete and $\operatorname{card} K_A = \operatorname{card} A$. One can straightforwardly verify that the product of any collection of discrete spaces has no isolated points if and only if infinitely many of the factor spaces have more than one point, and the proposition follows.

COROLLARY 4.9. *Let $\mathcal{T}_Y$ have the compact-open topology and let $\mathcal{D}$ be any decomposition of A. If $D(Y, \mathcal{D})$ has an isolated point, then it is discrete.*

PROPOSITION 4.10. *Let $\mathcal{T}_Y$ have the compact-open topology and let $\mathcal{D}$ be any decomposition of Y. Then $S(Y, \mathcal{D})$ has no isolated points if and only if at least one of the following two conditions holds.*

(1) *An infinite number of sets in $\mathcal{D}$ have more than one point.*

(2) *At least one set in $\mathcal{D}$ is infinite.*

Proof. Suppose (1) holds. Then infinitely many of the $\mathcal{T}_A$ have more than one point so $\Pi\{\mathcal{T}_A: A \in \mathcal{D})\}$ has no isolated points. Now suppose that (2) holds. Then some $A \in \mathcal{D}$ is infinite, so $\mathcal{T}_A$ has no isolated points. Hence, in this case also $\Pi\{\mathcal{T}_A: A \in \mathcal{D})$ has no isolated points.

Now suppose both (1) and (2) are false. Then there are only finitely many sets in $\mathcal{D}$ with more than one point and none of these is infinite. Thus $\Pi\{\mathcal{T}_A: A \in \mathcal{D}\}$ is finite and hence all of its points are isolated.

COROLLARY 4.11. *Let $\mathcal{T}_Y$ have the compact-open topology and let $\mathcal{D}$ be any decomposition of Y. If $S(Y, \mathcal{D})$ has an isolated point, then it is finite.*

The next two results are concerned with the cardinalities of $D(Y, \mathcal{D})$ and $S(Y, \mathcal{D})$. The proofs are standard applications of set-theoretic arguments to the semigroups $\Pi\{K_A: A \in \mathcal{D}\}$ and $\Pi\{\mathcal{T}_A: A \in \mathcal{D}\}$ and will be omitted. We mention, however, that we do need $2^a > c$ if a is uncountable so we assume the continuum hypothesis.

PROPOSITION 4.12. *Assume the continuum hypothesis and let $\mathcal{D}$ be any* *decomposition of* Y. *Then* $\operatorname{card} D(Y, \mathscr{D}) = c$ *if and only if at least one of the following conditions holds.*

(1) *The collection of the sets of the decomposition having more than one point is at most countable, and at least one set of the decomposition has* c *points.*

(2) *The collection of sets of the decomposition having more than one point is countably infinite, and none of them has more than* c *points.*

PROPOSITION 4.13. *Assume the continuum hypothesis and let* $\mathscr{D}$ *be any decomposition of* Y. *Then* $\operatorname{card} S(Y, \mathscr{D}) = c$ *if and only if at least one of the following conditions is satisfied.*

(1) *The collection of sets of the decomposition having more than one point is at most countable, and at least one of the sets has a countable number of points.*

(2) *The collection of sets of the decomposition having more than one point is countably infinite and none of them has more than a countable number of points.*

We are now in a position to verify a result we mentioned previously. J. de Groot proved [6] the existence of 2^c one-dimensional connected subspaces of the plane with the property that only constant functions map any one of these spaces into another and the only continuous selfmaps of any one of these spaces are the constant maps together with the identity map. Of course, all these spaces have c points.

PROPOSITION 4.14. *Let* X *be any one of de Groot's spaces and let* N *denote the set of natural numbers. Then* $S(X)$ *can be embedded in* $\mathscr{T}_N$.

Proof. Decompose N into a countable collection $\mathscr{D}$ of mutually disjoint subsets, each having more than one point. Then $D(N, \mathscr{D})$ is a left zero subsemigroup of $\mathscr{T}_N$ with c elements. Now $S(X)$ consists only of the constant functions (and there are c of these) and the identity function. Map the constant functions of $S(X)$ injectively into $D(N, \mathscr{D})$ in any manner whatsoever and send the identity of $S(X)$ into the identity of $\mathscr{T}_N$. The result is an embedding of $S(X)$ into $\mathscr{T}_N$.

It is well known that $\mathscr{T}_N$ with the compact-open topology is, topologically, nothing more than the space of irrational numbers ([9], p. 14). For a discrete Y in general, $\mathscr{T}_Y$ is compact if and only if Y is finite, so that one can never hope to get the Cantor discontinuum in this manner. However, as the next result shows, many of the semigroups $D(Y, \mathscr{D})$ and $S(Y, \mathscr{D})$ are homeomorphic to the Cantor discontinuum.

THEOREM 4.15. *Let* $\mathscr{T}_Y$ *have the compact-open topology and let* $\mathscr{D}$ *be any decomposition of* Y. *Then the following statements are equivalent.*

(1) $D(Y, \mathscr{D})$ *is homeomorphic to the Cantor discontinuum.*

(2) *$S(Y, \mathscr{D})$ is homeomorphic to the Cantor discontinuum.*

(3) *All sets in $\mathscr{D}$ are finite and the number of sets with more than one element is countably infinite.*

Proof. It is known that a space X is homeomorphic to the Cantor discontinuum if and only if it is compact, metrizable, totally disconnected, and has no isolated points. Now, any subsemigroup of $\mathscr{T}_Y$ is totally disconnected, and the result follows from Propositions 4.4, 4.6–4.8 and 4.10.

Now we consider the general problem of embedding $S(X)$ into a full transformation semigroup, and the theorem we prove is concerned with spaces X upon which $S(X)$ is doubly transitive. This means that, given a, b, x, y in X with $a \neq b$, there is an $f \in S(X)$ such that $f(a) = x$ and $f(b) = y$. These spaces include all 0-dimensional spaces and all completely regular arcwise connected spaces.

THEOREM 4.16. *Let X be a topological space such that $S(X)$ is doubly transitive on X. Then $S(X)$ can be embedded in $\mathscr{T}_Y$ if and only if* $\operatorname{card} X \leqslant \operatorname{card} Y$.

Proof. We have already observed that the condition is sufficient. Now suppose the condition is false; that is, assume $\operatorname{card} X > \operatorname{card} Y$. Let φ be an isomorphism from $S(X)$ into $\mathscr{T}_Y$. Then by previous considerations, φ maps $S(X)$ into $S(Y, \mathscr{D}(\varphi))$ and the kernel $K(X)$ of $S(X)$ into $D(Y, \mathscr{D}(\varphi))$. For any $x \in X$ we denote $\varphi(\langle x\rangle)$ by v_x where $\langle x\rangle$ denotes the constant function which sends everything into the point x. Then $v_x \in D(Y, \mathscr{D}(\varphi))$ for each $x \in X$ and $v_x \neq v_y$ when $x \neq y$. We choose any two such points x and y, and it follows, for some $A \in \mathscr{D}(\varphi)$, that $v_x(a) \neq v_y(a)$ for each $a \in A$.

Now $\operatorname{card} X > \operatorname{card} Y$ implies $\operatorname{card} X > \operatorname{card} A$, so there are two distinct points w and z of X such that v_w and v_z agree on A (we recall that v_x, v_y, v_w and v_z are all constant on A). Since $S(X)$ is doubly transitive, there exists an $f \in S(X)$ such that $f(w) = x$ and $f(z) = y$. Thus $f \circ \langle w\rangle = \langle x\rangle$ and $f \circ \langle z\rangle = \langle y\rangle$. This implies that

$$v_x = \varphi(\langle x\rangle) = \varphi(f)\circ\varphi(\langle w\rangle) = \varphi(f)\circ v_w$$

and

$$v_y = \varphi(\langle y\rangle) = \varphi(f)\circ\varphi(\langle z\rangle) = \varphi(f)\circ v_z.$$

Now for any $a \in A$, $v_w(a) = v_z(a)$. It follows from this that

$$v_x(a) = \varphi(f)\big(v_w(a)\big) = \varphi(f)\big(v_z(a)\big) = v_y(a)$$

which, of course, is a contradiction.

It is known that a metric space is homeomorphic to a subspace of the rationals if and only if it is countable ([9], p. 287). This, together with the previous theorem, results in

COROLLARY 4.17. *Let X be a metric space such that $S(X)$ is doubly transitive on X, and let N denote the countably infinite discrete space. Then*

$S(X)$ can be embedded in $\mathscr{T}_N$ if and only if X is homeomorphic to a subspace of the rational numbers.

Now, suppose we endow both $S(X)$ and $\mathscr{T}_Y$ with the compact-open topologies (we simply regard Y as a discrete space) and suppose we consider embedding $S(X)$ into $\mathscr{T}_Y$ with a map that is topological as well as algebraic. We will call such a map a *T-embedding map*. One might expect that the condition of Theorem 4.16, that $\operatorname{card} X \leqslant \operatorname{card} Y$, is not sufficient for this. Indeed, one can show that if $S(X)$ can be T-embedded in $\mathscr{T}_Y$, then X must be 0-dimensional. For, $\mathscr{T}_Y$ with the compact-open topology is 0-dimensional and, of course, X is homeomorphic to a subspace of $S(X)$. Thus, if we are looking for those spaces X for which $S(X)$ can be T-embedded in $\mathscr{T}_Y$, we may restrict out attention to 0-dimensional spaces. But even this approximation is not a good one. This will be evident from the next theorem.

THEOREM 4.18. *Let both $S(X)$ and $\mathscr{T}_Y$ have the compact-open topologies. Then the following statements are equivalent.*

(1) *$S(X)$ can be T-embedded as a closed subsemigroup of $\mathscr{T}_Y$.*

(2) *$S(X)$ can be T-embedded in $\mathscr{T}_Y$.*

(3) *X is discrete and* $\operatorname{card} X \leqslant \operatorname{card} Y$.

Proof. It is clear that (1) implies (2). We now verify that (2) implies (3). We first show, by contradiction, that X is discrete. Assume X has a limit point p and let φ be a T-embedding from $S(X)$ into $\mathscr{T}_Y$. Then φ maps $S(X)$ into the shell of $S(Y, \mathscr{D}(\varphi))$ and it sends $K(X)$, the family of all constant functions, into $D(Y, \mathscr{D}(\varphi))$. Now let x_1 and x_2 be distinct from each other and from p. Denote $\varphi(\langle p\rangle)$ by v_p and $\varphi(\langle x_i\rangle)$ by v_i, $i = 1, 2$. Let U be any neighborhood of v_p which does not contain either v_1 or v_2. Since $v_1 \neq v_2$, there exists an $A \in \mathscr{D}$ such that $v_1(a) \neq v_2(a)$ for each $a \in A$. Let $G = U \cap \langle a, v_p(a)\rangle$ for some $a \in A$. This is a neighborhood of v_p which contains neither v_1 nor v_2. Moreover, for any g in $S(Y, \mathscr{D}) \cap G$, we have $g(y) = v_p(a)$ for each $y \in A$. Since X and $K(X)$ are homeomorphic, $\langle p\rangle$ is a limit point of $K(X)$, and since $\langle p\rangle \in \varphi^{-1}(G)$, we can choose $\langle x_3\rangle$ in $\varphi^{-1}(G)$ distinct from $\langle p\rangle$. Now since φ is a topological embedding and $\mathscr{T}_Y$ is 0-dimensional, $S(X)$ must be also and we choose a clopen set H of $S(X)$ containing $\langle p\rangle$ and not containing $\langle x_3\rangle$. Next we choose $\langle x_4\rangle \in H - \{\langle p\rangle\}$ and define a selfmap of X by

$$f(x) = \begin{cases} x_1 & \text{for} \quad \langle x\rangle \in H, \\ x_2 & \text{for} \quad \langle x\rangle \in K(X) - H. \end{cases}$$

Then f is continuous since X and $K(X)$ are homeomorphic, and we have

$$f \circ \langle x_4\rangle = \langle x_1\rangle \quad \text{and} \quad f \circ \langle x_3\rangle = \langle x_2\rangle.$$

Now denote $\varphi(f)$ by g and $\varphi(\langle x_3\rangle)$ and $\varphi(\langle x_4\rangle)$ by v_3 and v_4, respectively, and we have $g\circ v_4 = v_1$ and $g\circ v_3 = v_2$. But v_3, v_4 are in G since $\langle x_3\rangle$, $\langle x_4\rangle$ are in $\varphi^{-1}(G)$, and this implies that $v_3(a) = v_4(a) = v_p(a)$.

Hence

$$v_1(a) = g\big(v_4(a)\big) = g\big(v_3(a)\big) = v_2(a).$$

This is the contradiction we seek. Thus, X is discrete and $\operatorname{card} X \leqslant \operatorname{card} Y$ by Theorem 4.16.

To complete the proof, we need only show that (3) implies (1). So let h be any injection of X into Y. Denote $h(X)$ by A and $Y-h(X)$ by B and let $\mathscr{D} = \{A, B\}$. By Proposition 1.11, $S(Y, \mathscr{D})$ is a closed subsemigroup of $\mathscr{T}_Y$, and by Lemma 4.2, $S(Y, \mathscr{D})$ is topologically isomorphic to $\mathscr{T}_A \times \mathscr{T}_B$, where both $\mathscr{T}_A$ and $\mathscr{T}_B$ have the compact-open topologies. Since X is discrete, $S(X)$ is just $\mathscr{T}_X$ and it is sufficient to exhibit a topological isomorphism from $\mathscr{T}_X$ onto a closed subsemigroup of $\mathscr{T}_A \times \mathscr{T}_B$. Let j denote the identity mapping on B, and for each f in $\mathscr{T}_X$ define

$$\varphi(f) = (h\circ f\circ h^{-1}, j).$$

It is straightforward to show that φ is an isomorphism from $\mathscr{T}_X$ onto $\mathscr{T}_A \times \{j\}$. Moreover, if $\langle K_1, G_1\rangle$ and $\langle K_2, G_2\rangle$ are any two subbasic open sets in $\mathscr{T}_A$ and $\mathscr{T}_B$, respectively, we have

$$\varphi^{-1}(\langle K_1, G_1\rangle \times \langle K_2, G_2\rangle) = \langle h^{-1}(K_1), h^{-1}(G_1)\rangle$$

if $K_2 \subset G_2$; otherwise, it is empty. Hence φ is continuous. Finally, for any subbasic open subset $\langle K, G\rangle$ of $\mathscr{T}_X$, we have

$$\varphi(\langle K, G\rangle) = (\langle h(K), h(G)\rangle \times \mathscr{T}_B) \cap \mathscr{T}_A \times \{j\}).$$

Thus φ is an open map onto its range and the proof is complete.

Chapter 2

We closed the previous chapter with results on embedding and T-embedding $S(X)$ into $S(Y)$ whenever Y is discrete and hence $S(Y)$ is just the full transformation semigroup on Y. In this chapter, we consider the case where Y is not necessarily discrete and, in fact, will often be prevented from being discrete by the conditions placed upon it.

1. A THEOREM ON T-EMBEDDINGS

Before proving our theorem on T-embeddings, we prove a lemma. The result is known in the case of the full transformation semigroup and the proof is essentially the same. We recall that $S(X)$ is said to be *doubly transitive* on X if for each quadruple of points a, b, p, q with $a \neq b$, there exists an f in $S(X)$ such that $f(a) = p$ and $f(b) = q$.

LEMMA 1.1. *Let X be any topological space upon which S is doubly transitive and let φ be any homomorphism from $S(X)$ into a semigroup T. Then if φ is not injective, it takes all the constant functions into one single element of T.*

Proof. Suppose $f \neq g$ but $\varphi(f) = \varphi(g)$. Let $\langle a\rangle$ and $\langle b\rangle$ be any two constant functions in $S(X)$. Since $f \neq g$, there exists a point x in X such that $f(x) \neq g(x)$ and since $S(X)$ is doubly transitive on X, there exists an h in $S(X)$ such that $h(f(x)) = a$ and $h(g(x)) = b$. Then $h \circ f \circ \langle x\rangle = \langle a\rangle$ and $h \circ g \circ \langle x\rangle = \langle b\rangle$ and we have

$$\varphi(\langle a\rangle) = \varphi(h)\circ\varphi(f)\circ\varphi(\langle x\rangle) = \varphi(h)\circ\varphi(g)\circ\varphi(\langle x\rangle) = \varphi(\langle b\rangle).$$

It is appropriate at this point to recall some remarks from a discussion following Proposition 3.8 of Chapter 1. Suppose T is any semigroup which contains left zeros and φ is an isomorphism from T into $S(Y)$ where Y is any topological. The isomorphism φ carries the subsemigroup of left zeros of T into some unique Δ-class $D(Y, \mathscr{D}(\varphi))$ where $\mathscr{D}(\varphi)$ is the decomposition of Y which is induced by any one of the functions in $D(Y, \mathscr{D}(\varphi))$. Furthermore, φ carries T into the shell $S(Y, \mathscr{D}(\varphi))$ of $D(Y, \mathscr{D}(\varphi))$. In order for φ to be injective, it is sufficient that there exist at least one set A in $\mathscr{D}(\varphi)$ with the property that $\eta_A \circ \varphi$ is injective where η_A is the A-sectional homomorphism (Chapter 1, Definition 3.1) from $S(Y, \mathscr{D}(\varphi))$ into $S(A)$. In fact, if T is taken to be some $S(X)$ which is doubly transitive on X, then it turns out that the condition is necessary as well as sufficient. This fact is crucial in a number of proofs which are to follow, including the next one.

THEOREM 1.2. *Let X be a compact absolute retract, let Y be normal, and let both $S(X)$ and $S(Y)$ have the compact-open topologies. Then $S(X)$ can be T-embedded in $S(Y)$ if and only if X can be topologically embedded in Y.*

Proof. First suppose that there exists a homeomorphism h from X into Y. Since $h(X)$ is a compact absolute retract and Y is normal, there exists a continuous map v in $S(X)$ whose range is $h(X)$ and v is such that $v(y) = y$ for all y in $h(X)$. Define a mapping φ from $S(X)$ into $S(Y)$ by

$$\varphi(f) = h \circ f \circ h^{-1} \circ v, \quad \text{for all} \quad f \in S(X).$$

Then φ is an algebraic embedding of $S(X)$ into $S(Y)$. Moreover, for any subbasic open set $\langle K, G\rangle$ of $S(X)$, if H is any open subset of Y such that $h(G) = H \cap h(X)$, it can be verified that

$$\varphi(\langle K, G\rangle) = \langle h(K), H\rangle \cap \varphi(S(X)).$$

In a similar manner, if $\langle K, G\rangle$ is any subbasic open set of $S(Y)$, then we obtain

$$\varphi^{-1}(\langle K, G\rangle) = \langle h^{-1}(v(K)), h^{-1}(G)\rangle.$$

Thus φ is topological as well as algebraic.

Now suppose that φ is any T-embedding from $S(X)$ into $S(Y)$. Then φ actually maps $S(X)$ into $S(Y, \mathscr{D}(\varphi))$ and it takes $K(X)$, the family of all constant functions on X into $D(Y, \mathscr{D}(\varphi))$, where $\mathscr{D}(\varphi)$ is the decomposition induced by φ. We claim that there is at least one A in $\mathscr{D}(\varphi)$ with the property that $\eta_A \circ \varphi$ is an isomorphism from $S(X)$ into $S(A)$, where η_A is the A-sectional homomorphism from $S(Y, \mathscr{D}(\varphi))$ into $S(A)$. If the claim is false, then by Lemma 1.1, each $\eta_A \circ \varphi$ maps all the constant functions of $S(X)$ into one single constant function in $S(A)$ ($S(X)$ is doubly transitive since X is an absolute retract). It follows directly from this that φ sends all constant functions of $S(X)$ into one single idempotent of $S(Y)$, which is, of course, not true. So we take any A such that $\eta_A \circ \varphi$ is injective and we define a maping h from X into Y as follows: choose any $a \in A$ and define

$$h(x) = \varphi(\langle x\rangle)(a).$$

Also define maps $e_a\colon S(Y)\to Y$ and $c\colon X\to S(X)$ by

$$e_a(f) = f(a), \quad \text{and} \quad c(x) = \langle x\rangle.$$

The map e_a is continuous in any topology for $S(X)$ which contains the point-open topology, and of course, c is continuous if $S(X)$ has any set-open topology. Thus h is continuous since $h = e_a\circ\varphi\circ c$. Now let x be any point in X. We obtain

$$h(x) = \varphi(\langle x\rangle)(a) = \varphi(\langle x\rangle)|_A(a) = (\eta_A\circ\varphi)(\langle x\rangle)(a).$$

But $\eta_A\circ\varphi$ maps constant functions of $S(X)$ into constant functions of $S(A)$, and since it is injective, we must have

$$(\eta_A\circ\varphi)(\langle x\rangle) \neq (\eta_A\circ\varphi)(\langle y\rangle) \quad \text{when} \quad x\neq y.$$

Thus

$$(\eta_A\circ\varphi)(\langle x\rangle)(a) \neq (\eta_A\circ\varphi)(\langle y\rangle)(a),$$

which implies that $h(x)\neq h(y)$. Thus h is injective and continuous, and since X is compact, it is a homeomorphism.

2. SOME RESULTS ON ALGEBRAIC EMBEDDINGS

The results in this section are concerned primarily with algebraic embeddings. As we noted in the last section, each isomorphism φ from $S(X)$ into $S(Y)$ induces a decomposition $\mathscr{D}(\varphi)$ on Y and if $S(X)$ is doubly transitive on X, then there is at least one A in $\mathscr{D}(\varphi)$ such that $\eta_A\circ\varphi$ is injective where η_A is the A-sectional homomorphism. We will denote $\eta_A\circ\varphi$ more simply by φ_A. The first result of this section is a fundamental lemma which gives us some information about φ_A.

LEMMA 2.1. *Let X and Y be any two topological spaces and suppose that $S(X)$ is doubly transitive on X. Let φ be any isomorphism from $S(X)$ into $S(Y)$. Then there exists a set A in $\mathscr{D}(\varphi)$ and an injection h of X into A such that the following conditions are satisfied*:

(1) $\big(\varphi_A(f)\big)\circ h = h\circ f$ *for all f in $S(X)$; and*

(2) $h\big(f^{-1}(x)\big) = \big(\varphi_A(f)\big)^{-1}\big(h(x)\big)\cap h(X)$ *for all $f\in S(X)$, $x\in X$.*

Proof. Just as in the proof of the previous theorem, at least one of the homomorphisms $\varphi_A = \eta_A\circ\varphi$ must be injective. We choose any such A and we show that there exists an injection h from X into A satisfying (1) and (2). Let any $x\in X$ be given. Then $\varphi_A = \eta_A\circ\varphi$ sends $\langle x\rangle$ into some $\langle y\rangle$ in $S(A)$. We define $h(x) = y$. The function h must be injective because φ_A is injective. Moreover, we note that $\varphi_A(\langle x\rangle) = \langle h(x)\rangle$. We use this several times in the next string of equalities:

$$\begin{aligned}\langle h\big(f(x)\big)\rangle &= \varphi_A\big(\langle f(x)\rangle\big) = \varphi_A(f\circ\langle x\rangle)\\ &= \varphi_A(f)\circ\varphi_A(\langle x\rangle) = \varphi_A(f)\circ\langle h(x)\rangle = \langle\big(\varphi_A(f)\big)\big(h(x)\big)\rangle.\end{aligned}$$

It follows from this that (1) is valid.

Now let y be any point in $h(f^{-1}(x))$. Then $y=h(a)$ for some $a\in f^{-1}(x)$. Thus $f(a)=x$ or, equivalently, $f\circ\langle a\rangle=\langle x\rangle$. Then we obtain

$$\begin{aligned}\langle h(x)\rangle &= \varphi_A(\langle x\rangle) = \varphi_A(f\circ\langle a\rangle) = \varphi_A(f)\circ\varphi_A(\langle a\rangle)\\ &= \varphi_A(f)\circ\langle h(a)\rangle = \langle(\varphi_A(f))(h(a))\rangle.\end{aligned}$$

Thus $(\varphi_A(f))(h(a)) = h(x)$ and it follows that

$$y = h(a)\in(\varphi_A(f))^{-1}(h(x))\cap h(X).$$

Now suppose that $y\in(\varphi_A(f))^{-1}(h(x))\cap h(X)$. Then $y=h(b)$ for some $b\in X$ and $(\varphi_A(f))(y) = h(x)$. But since (1) holds, this implies that

$$h(x) = (\varphi_A(f))(y) = (\varphi_A(f)\circ h)(b) = h(f(b)),$$

and since h is injective, it follows that $f(b)=x$. Hence $y=h(b)\in h(f^{-1}(x))$. So (2) is valid.

We recall the definition of an S^*-space ([11], p. 295).

DEFINITION 2.2. A space X is an *S^*-space* if for each closed subset H of X and each point $p\in X-H$, there exists a continuous selfmap f of X and a point q in X such that $f(x)=q$ for all $x\in H$ and $f(p)\neq q$.

The above definition differs slightly from the one given in [11], p. 295. There, the spaces were required to be only T_1 while here everything is Hausdorff.

We remark that S^*-spaces include all completely regular spaces containing an arc as well as all 0-dimensional spaces. We also remark that one easily verifies that a space is an S^*-space if and only if the pre-images of points under continuous selfmaps form a basis for the closed subsets.

LEMMA 2.3. *Let X be any S^*-space upon which $S(X)$ is doubly transitive. Then the function h^{-1} of Lemma 2.1 which maps $h(X)$ onto X is continuous.*

Proof. The proof follows immediately from (2) of Lemma 2.1 and the fact that pre-images of points under continuous selfmaps form a basis for the closed subsets of an S^*-space.

We recall two more definitions from [15]. See also [12], p. 327.

DEFINITION 2.4. A topological space is a *strong S^*-space* if for each pair of nonempty mutually disjoint closed subsets A and B of X, there exists a continuous selfmap f of X and distinct points a and b in X such that $f(x)=a$ for $x\in A$ and $f(x)=b$ for $x\in B$.

DEFINITION 2.5. A topological space X is said to be *strongly conformable* if it is a first countable strong S^*-space and for each pair of compact countable subspaces A and B, each having exactly one limit point, there exists a continuous selfmap f of X mapping A into B such that $B-f(A)$ is finite.

By a Lebesgue 0-dimensional space we mean any space in which any finite open cover has a refinement by a partition of that space. This agrees with the definition given in [5], p. 246, when the space is normal. Every Lebesgue 0-dimensional space is 0-dimensional as we have previously defined it here, but 0-dimensional spaces need not be Lebesgue 0-dimensional. However, the two concepts do coincide if the spaces involved are Lindelöf.

We now state, without proof, a result which was first stated in [15].

THEOREM 2.6. *All locally Euclidean normal spaces and all Lebesgue 0-dimensional metric spaces are strongly conformable.*

THEOREM 2.7. *Let X be strongly conformable and suppose $S(X)$ is doubly transitive on X. Let Y be a first countable space with the property that every subspace whose cardinality is equal to that of X has a limit point. Then the function h in Lemma 2.1 is a homeomorphism from X into Y and $h(X)$ is a closed subset of Y.*

Proof. We know from Lemma 2.3 that h^{-1} is continuous. Since X is first countable, we can use sequences to show that h is continuous. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence in X which converges to a point p. We must show that $\lim h(a_n) = h(p)$. Since h is an injection, $h(X)$ has a limit point q and since it is first countable, there is a sequence of distinct points $h(X)$ converging to q. Denote the points of the sequence, together with the limit point q, by B. Then B is a compact countable subset of $h(X)$ with precisely one limit point, and since h^{-1} restricted to B is a homeomorphism, $h^{-1}(B)$ is also a compact countable subset of X with exactly one limit point.

Now denote the points of $\{a_n\}_{n=1}^{\infty}$ together with the limit point p by D. Since X is strongly conformable, there exists a continuous selfmap f of X mapping $h^{-1}(B)$ into D such that $D-f(h^{-1}(B))$ is finite. Thus, there is a positive integer N such that $n > N$ implies that $a_n \in f(h^{-1}(B))$. For each such N, we choose $b_n \in B$ such that

(1) $f(h^{-1}(b_n)) = a_n$.

Since $h^{-1}(q)$ is the unique limit point of $h^{-1}(B)$, we have

(2) $f(h^{-1}(q)) = p$.

Otherwise, f would map all but a finite number of points of $h^{-1}(B)$ into one of the isolated points of D.

It follows from (1) and (2) above and from Lemma 2.1 that

$$h(a_n) = (h\circ f)(h^{-1}(b_n)) = (\varphi_A(f)\circ h)(h^{-1}(b_n)) = \varphi_A(f)(b_n)$$

and

$$h(p) = (h\circ f)(h^{-1}(q)) = (\varphi_A(f)\circ h)(h^{-1}(q)) = \varphi_A(f)(q).$$

Now since $\{b_n\}_{n=1}^{\infty}$ is an infinite sequence of distinct points of B, it must converge to the unique limit point q of B. Thus

$$\lim \varphi_A(f)(b_n) = \varphi_A(f)(q),$$

which implies that

$$\lim h(a_n) = h(p).$$

Thus h is continuous as well as h^{-1}.

We have yet to show that $h(X)$ is closed in Y. Suppose this is false. Then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of distinct points of $h(X)$ converging to a point $t \in Y - h(X)$. Then $\{c_n\}_{n=1}^{\infty}$ is a discrete subset of $h(X)$ and hence $\{h^{-1}(c_n)\}_{n=1}^{\infty}$ is a discrete subset of X. Let

$$V = \{h^{-1}(c_{2n-1})\}_{n=1}^{\infty}$$

and

$$W = \{h^{-1}(c_{2n})\}_{n=1}^{\infty}.$$

The sets V and W are disjoint closed subsets of X, and since X is a strong S^*-space, there exists a continuous selfmap f of X and two distinct points v and w of X such that $f(x) = v$ for $x \in V$ and $f(x) = w$ for $x \in W$. From this we get

$$\varphi_A(f)(c_{2n-1}) = \big(\varphi_A(f)\circ h\big)\big(h^{-1}(c_{2n-1})\big) = (h\circ f)\big(h^{-1}(c_{2n-1})\big) = h(v),$$

and similarly,

$$\varphi_A(f)(c_{2n}) = h(w).$$

This, however, is a contradiction since

$$\lim \varphi_A(f)(c_n) = \varphi_A(f)(t),$$

and the proof is complete.

COROLLARY 2.8. *Let X be a strongly conformable space with the property that $S(X)$ is doubly transitive on X, and suppose that Y is a first countable space with the property that every subspace whose cardinality is equal to that of X has a limit point. If $S(X)$ can be embedded in $S(Y)$, then X is homeomorphic to a closed subspace of Y.*

In general, the converse to the corollary need not hold. For example, let K denote the Cantor discontinuum and let R denote the reals. These spaces satisfy the conditions of the corollary and K is certainly homeomorphic to a closed subspace of R. However, $S(K)$ cannot be embedded in $S(R)$. The space K is strongly conformable and quasi-homogeneous [15], in fact, even homogeneous. But Theorem (5.5) of [15] tells us that if X is a strongly conformable quasi-homogeneous completely regular space, the $S(X)$ can be embedded in $S(R)$ if and only if X is homeomorphic to either R, a half-open interval, the two-point discrete space, the one-point space, or the closed unit interval. Thus, further conditions are needed if we wish the converse also to be true.

THEOREM 2.9. *Let X be an absolute retract and suppose that Y is a first countable normal space with the property that each subspace of Y whose cardinality is equal to that of X has a limit point. Then $S(X)$ can be embedded into $S(Y)$ if and only if X is homeomorphic to a closed subspace of Y.*

Proof. Since X is an absolute retract, it is normal and arcwise connected. Thus it is a strong S^*-space. It also follows from the fact that X is an absolute retract that for any two compact countable subsets A and B; each with exactly one limit point, there exists a continuous selfmap f or X which maps A homeomorphically onto B. Thus, X is strongly conformable and the necessity now follows from Theorem 2.7.

As for the sufficiency, let h be a homeomorphism from X onto a closed subspace V of Y. Since V is an absolute retract and closed in Y, there is an idempotent continuous selfmap v of Y such that the range of v is V. The mapping φ which is defined by

$$\varphi(f) = h \circ f \circ h^{-1} \circ v \quad \text{for each } f \in S(X),$$

is an embedding of $S(X)$ into $S(Y)$.

THEOREM 2.10. *Let X be an absolute retract and let Y be a second countable normal space. Then $S(X)$ can be embedded in $S(Y)$ if and only if X is homeomorphic to a closed subspace of Y.*

Proof. First of all, the conclusion is immediate if X consists of more than one point. If X has more than one point, its cardinality must be at least that of the continuum and the result now follows from Theorem 2.9 and since any uncountable subset of a second countable space has a limit point.

The latter theorem no longer holds if the requirement that Y be second countable is dropped. For, let X be an absolute retract and let Y be any set such that $\operatorname{card} X \leqslant \operatorname{card} Y$. Then Y with the discrete topology is certainly normal and $S(Y)$ is just the full transformation semigroup on Y. The space X is not homeomorphic to any subspace of Y, much less a closed one, and yet $S(X)$ can be embedded in $S(Y)$.

THEOREM 2.11. *Let X and Y be 0-dimensional separable metric spaces and suppose that X is uncountable. Then $S(X)$ can be embedded in $S(Y)$ if and only if X is homeomorphic to a closed subspace of Y.*

Proof. Let h be a homeomorphism from X onto a closed subspace V of Y. Then by Corollary 2 of [9], p. 281, there is an idempotent continuous selfmap v of Y whose range is V and the map φ defined by $\varphi(f) = h \circ f \circ h^{-1} \circ v$ is an embedding of $S(X)$ into $S(Y)$.

On the other hand, suppose $S(X)$ can be embedded in $S(Y)$. It follows from Corollary 3 of [9], p. 281 that X is strongly conformable. Furthermore, since Y is separable and metrizable, every uncountable subset has a limit point. Thus, it now follows from Corollary 2.8 that X is homeomorphic to a closed subspace of Y.

The uncountability of X was used only in the necessity portion of the proof and one cannot prove the necessity without it. The identity map from $S(X)$ into $S(N)$ where X is any countable nondiscrete space and N is the same set with the discrete topology, will serve as a counter-example.

THEOREM 2.12. *Let E^N denote the Euclidean N-space. Then any isomorphism from $S(E^N)$ into $S(E^N)$ is actually on automorphism of $S(E^N)$.*

Proof. Since E^N is an absolute retract, it is strongly conformable and $S(E^N)$ is doubly transitive on E^N. It follows from Theorem 2.7 that the function h whose existence is guaranteed by Lemma 2.1 is a homeomorphism and $h(E^N)$ is closed in E^N. It follows from a well-known theorem of Brouwer ([8], p. 95) that $h(E^N)$ is also open. Thus, h is a homeomorphism from E^N onto E^N. All this implies that the subspace A in Lemma 2.1 is all of E^N and $\varphi_A = \varphi$. Thus, by (1) of Lemma 2.1, we have $\varphi(f) = h \circ f \circ h^{-1}$ for all f in $S(X)$ and this implies that φ is an automorphism of $S(E^N)$.

This latter result was first established for E^1 in [15]. It follows from Corollary 4.2 of [9] that any epimorphism of $S(E^N)$ is actually an automorphism. This, together with the previous result, implies that if an endomorphism of $S(E^N)$ is not an automorphism, then it is neither surjective nor injective. Such endomorphisms do exist. One can, for example, map everything into a single idempotent. To get a less trivial example, choose any idempotent v in $S(E^N)$ which is different from the identity map i and define $\varphi(f) = i$ if f is a unit (that is, a homeomorphism from E^N onto E^N) and let $\varphi(f) = v$ otherwise. Corollary 3.8 of [2] states that $S(E^N)$ is the union of its maximal proper ideal and its group of units and this implies that the product of two elements of $S(E^N)$ can be a unit only if each of the elements is a unit. It follows from this fact that the mapping φ defined previously is indeed an endomorphism which, of course, is far from being an automorphism.

We close this section with two more theorems both of which involve T-embeddings. First, we introduce some notation. Let Q be any topological property which is preserved under taking continuous images. For any space X, $\mathscr{T}_Q$ will denote the set-open topology on $S(X)$ which is obtained by taking all sets of the from

$$\langle K, G\rangle = \{f \in S(X) : f(K) \subset G\}$$

as a subbasis for the *closed* sets of $\mathscr{T}_Q$ where G is an open subset of X and K is a subspace of X with property Q.

THEOREM 2.13. *Let X be an absolute retract, let Y be a second countable normal space and let both $S(X)$ and $S(Y)$ have the $\mathcal{T}_Q$ topologies. Then the following statements are equivalent.*

(1) *$S(X)$ can be T-embedded in $S(Y)$.*

(2) *$S(X)$ can be embedded in $S(Y)$.*

(3) *X is homeomorphic to a closed subspace of Y.*

Proof. Statements (2) and (3) are equivalent by Theorem 2.10 and it is evident that (1) implies (2). We need only show that (3) implies (1). As in the proof of Theorem 2.9, we define a mapping φ by

$$\varphi(f) = h \circ f \circ h^{-1} \circ v \quad \text{for } f \text{ in } S(X).$$

Then φ is an algebraic embedding of $S(X)$ into $S(Y)$ and we need only show that it is topological as well. Let $K \subset X$ have property Q and let G be an open subset of X. Choose any open subset H of Y such that $h(G) = H \cap h(X)$. One can easily show that

$$\varphi(\langle K, G\rangle) = \varphi(S(X)) \cap \langle h(K), H\rangle.$$

On the other hand, if $W \subset Y$ has property Q, then $v(W)$ does also and for any open subset G of Y, one can show that

$$\varphi^{-1}(\langle W, G\rangle) = \langle h^{-1}(v(W)), h^{-1}(G)\rangle.$$

Hence, φ maps $S(X)$ homeomorphically into $S(Y)$.

The proof of the next result will be omitted. It is identical to the proof of the previous theorem except for the fact that one appeals to Theorem 2.11 in place of Theorem 2.10.

THEOREM 2.14. *Let X and Y be 0-dimensional separable metric spaces and suppose that X is uncountable. Let both $S(X)$ and $S(Y)$ have $\mathcal{T}_Q$ topologies. Then the following statements are equivalent.*

(1) *$S(X)$ can be T-embedded in $S(Y)$.*

(2) *$S(X)$ can be embedded in $S(Y)$.*

(3) *X is homeomorphic to a closed subspace or Y.*

3. EMBEDDINGS WHICH ARE INDUCED BY AN IDEMPOTENT AND A HOMEOMORPHISM

In several of the proofs in the preceding section, we constructed embeddings of $S(X)$ into $S(Y)$ by choosing an idempotent continuous selfmap v of Y and a homeomorphism h from X onto the range of v and then defining

$$(*) \qquad \varphi(f) = h \circ f \circ h^{-1} \circ v$$

for each f in $S(X)$. In general, however, not every embedding takes this form. We consider two examples.

EXAMPLE 3.1. Let X be any space and let Y be any discrete space with $\operatorname{card} Y > \operatorname{card} X$. Choose any injection h of X into Y and define

$$\varphi(f)(y) = h\big(f(h^{-1}(y))\big) \quad \text{for } y \in h(X),$$
$$\varphi(f)(y) = y \quad \text{for } y \in Y - h(X).$$

The mapping φ embeds $S(X)$ into $S(Y)$ but it cannot possibly be of the form (*). Among other things, any embedding of the form (*) always takes constant functions into constant functions and, of course, the embedding of this example does not.

EXAMPLE 3.2. E is the space of real numbers and E^2 is the plane. Define an embedding φ of $S(E)$ into $S(E^2)$ by

$$\big(\varphi(f)\big)(x, y) = \big(f(x), y\big).$$

One easily verifies that φ is, indeed, an embedding and since it does not take constant functions into constant functions, it cannot be of the form (*).

The main purpose of this section is to find conditions for the spaces X and Y which will insure that every embedding of $S(X)$ into $S(Y)$ takes the form (*). It was shown in [14] that every α-monomorphism ([14], Definition 4.3, p. 151) from $S(X)$ into $S(Y)$ has the form (*) if X and Y satisfy various conditions. For example, Theorem 5.6 of that paper states that if X is any Hausdorff (it was not assumed that the spaces were necessarily Hausdorff) S^*-space and Y is any compact T_1 space, then each α-monomorphism from $S(X)$ into $S(Y)$ has the form (*). Theorem 5.7 of the same paper states that if X is conformable Y is any first countable T_1 space which is not the union of an infinite number of mutually disjoint nonempty open subsets, then every α-monomorphism from $S(X)$ into $S(Y)$ has the form (*). All this indicates that in a great many instances, α-monomorphisms are given by (*). However, considerably more stringent conditions must be placed on the spaces, particularly on the space Y if one expect every monomorphism from $S(X)$ into $S(Y)$ to have the form (*). Now we need a definition.

DEFINITION 3.3. A topological space X is *quasi-homogeneous* if for each nonempty open subset G of X and each point p in X, there exist continuous selfmaps f and g of X such that $g(p) \in G$ and $f \circ g$ is the identity map on X.

Quasi-homogeneous spaces were introduced in [14]. It was observed that if X is any absolute retract with the property that each nonempty open subset contains a closed copy of X, then X is quasi-homogeneous, so that, in particular, every closed cube I^N in E^N is quasi-homogeneous. Of course, every homogeneous space is quasi-homogeneous, so each E^N is also quasi-homogeneous. It was also shown in [15] that products of quasi-homogeneous spaces are quasi-homogeneous.

THEOREM 3.4. *Let X be strongly conformable and quasi-homogeneous and suppose $S(X)$ is doubly transitive on X. Suppose also that Y is first countable and connected, that every subspace of Y which is homeomorphic to X has nonempty interior, and the every subset of Y with cardinality equal to* card X *has a limit point. Then, for each isomorphism φ from $S(X)$ into $S(Y)$, there exists an idempotent continuous selfmap v of X and a homeomorphism h from X onto the range of such v that $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for each f in $S(X)$.*

Proof. Let $\mathscr{D}(\varphi)$ be the decomposition on Y which is induced by φ. According to Lemma 2.1, there exists an $A\in\mathscr{D}(\varphi)$ and an injection h of X into A which satisfies (1) and (2) of that lemma. It is immediate from Theorem 2.7 that h is actually a homeomorphism from X onto a closed subset of Y. We first show that A is all of Y; that is, the decomposition $\mathscr{D}(\varphi)$ is trivial. Since $h(X)$ is homeomorphic to X, it has nonempty interior. Choose any point $h(a)$ in the interior and let any point b in A be given. We want to show that b is an interior point of A. We recall that for any $x\in X$, the restriction of $\varphi(\langle x\rangle)$ to A is a constant selfmap of A, so that, in particular $\varphi(\langle a\rangle)$ maps all of A into a single point of A, and it follows from (1) of Lemma 2.1 that this point must be $h(a)$. Since $\varphi(\langle a\rangle)(b) = h(a)$ is in the interior of A, there exists an open subset G of Y containing b such that $\varphi(\langle a\rangle)(G)\subset A$. This means that $G\subset A$ since $\varphi(\langle a\rangle)$ maps point of other sets of the decomposition into the same sets. Thus A is open. Since it is also closed and Y is connected, $A = Y$ and the decomposition $\mathscr{D}(\varphi)$ consists of the single set Y.

Now denote $\varphi(i)$ by v, where i is the identity map on X. By Lemma 2.1, we obtain

$$v\circ h = \varphi(i)\circ h = h\circ i = h.$$

Thus, $h(X)\subset V$, where V is the range of v. The subspace V is connected since it is a continuous image of Y. Assume $h(X)\neq V$. Then there exists a point $p\in h(X)\cap \mathrm{Cl}_V\big(V-h(X)\big)$; otherwise V would not be connected. Let G be any subset of $h(X)$ which is open in Y. Since X is $f\circ g = i$ and $g\big(h^{-1}(p)\big)\in h^{-1}(G)$. We use Lemma 2.1 again, to get

$$\varphi(g)(p) = \big(\varphi(g)\circ h\big)\big(h^{-1}(p)\big) = (h\circ g)\big(h^{-1}(p)\big)\in G.$$

Thus, there exists an open subset H of Y containing p such that $\varphi(g)(H)\subset G$.

Next we wish to observe that $\varphi(f)$ maps G into $h(X)$. Let $y\in G$. Since $G\subset h(X)$, we have

$$\varphi(f)(y) = \big(\varphi(f)\circ h\big)\big(h^{-1}(y)\big) = (h\circ f)\big(h^{-1}(y)\big)\in h(X).$$

Now, since $p \in \mathrm{Cl}_V(V-h(X))$, there exists a point $q \in H \cap (V-h(X))$. But for this point q, we have

$$q = v(q) = (\varphi(f)\circ\varphi(g))(q) \in \varphi(f)(G) \subset h(X),$$

which is a contradiction. Thus, $V = h(X)$ and it now readily follows from (1) of Lemma 2.1 that

$$\varphi(f) = h\circ f\circ h^{-1}\circ v, \quad \text{for each } f \in S(X).$$

COROLLARY 3.5. *Let X be any M-dimensional strongly conformable, quasi-homogeneous space such that $S(X)$ is doubly transitive, and let Y be of any connected subspace of E^N the Euclidean N-space, suppose that $M \geqslant N$. Then for each isomorphism φ from $S(X)$ into $S(Y)$, there exists an idempotent v in $S(Y)$ and a homeomorphism h from X onto the range of v such that $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for each f in $S(X)$.*

Proof. Let W be any subspace of Y which is homeomorphic to X. Then $\dim W = N$ and it follows from Theorem IV–3 of [8], p. 44 that W has nonempty interior. Moreover, any subset H of Y with $\mathrm{card} H = \mathrm{card} X$ must have cardinality c, so it has a limit point. Hence, the hypothesis of the previous theorem is satisfied and the conclusion follows. Of course, this means that M and N coincide.

COROLLARY 3.6. *For each isomorphism φ from $S(I^N)$ into $S(I^N)$ there exists an idempotent continuous selfmap v of I^N and a homeomorphism h from I^N onto the range of v such that $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for each f in $S(I^N)$.*

Some remarks. First of all, the requirement in Theorem 3.5 that $M \geqslant N$, is crucial. Example 3.2 is evidence of this. Secondly, we remark that there are many isomorphisms of $S(I^N)$ into $S(I^N)$ which are not automorphisms. This is in contrast to the case for $S(E^N)$, in view of Theorem 2.12. Consequently, $S(I^N)$ has many proper subsemigroups which are isomorphic to it while $S(E^N)$ has none.

It follows from several of the preceding results that there are a number of situations where any embedding must necessarily be a T-embedding. The additional result we need to verify this is the following whose proof we omit since it was essentially verified in the proof of Theorem 2.13.

THEOREM 3.7. *Let Q be any topological property which is closed under taking continuous images. Let X and Y be any two spaces and let $S(X)$ and $S(Y)$ have the $\mathscr{T}_Q$ topologies. Let v be any idempotent in $S(Y)$ and let h be any homeomorphism from X onto the range of v. Then the map φ defined by $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for each f in $S(X)$, is a T-embedding of $S(X)$ into $S(Y)$.*

Now, the conclusion of several of the previous results were to the effect that each embedding φ is given by $\varphi(f) = h\circ f\circ h^{-1}\circ v$, so in view of Theorem 3.7, each embedding is automatically a T-embedding when the semigroups have $\mathscr{T}_Q$ topologies. We state a simple result whose proof follows from the previous observations and Corollary 3.6.

COROLLARY 3.8. *Let Q be any topological property which is closed under taking continuous images and let $S(I^N)$ have the $\mathscr{T}_Q$ topology. Then every embedding of $S(I^N)$ into itself is automatically a T-embedding.*

4. ENTIRE ISOMORPHISMS

In the discussion preceding Theorem 1.2 we noted that an isomorphism φ from a semigroup T (which was assumed to have left zeros) into $S(Y)$ is injective if at least one $\eta_A \circ \varphi$ is injective, where $A \in \mathscr{D}(\varphi)$, the decomposition on Y induced by φ, and η_A is the A-sectional homomorphism. In many cases, a particularly nice situation develops when $\eta_A \circ \varphi$ is injective for each $A \in \mathscr{D}(\varphi)$ so we give these isomorphisms a name.

DEFINITION 4.1. When $\eta_A \circ \varphi$ is injective for each A in $\mathscr{D}$, we refer to φ as an *entire isomorphism.*

The main result of this section is the next one and it is curiously reminiscent of a result of Clifford and Miller on endomorphisms of semigroups of binary relations ([3], p. 310). We shall discuss this in more detail later. Before we state our next result, we need some notation. For any isomorphism φ from $S(X)$ into $S(Y)$, we denote by $\pi(\varphi)$ the equivalence relation $\bigcup \{A \times A\colon\ A \in \mathscr{D}(\varphi)\}$.

THEOREM 4.2. *Let X be an N-dimensional strongly conformable quasi-homogeneous space and let φ be any entire isomorphism from $S(X)$ into $S(Y)$ such that each set in $\mathscr{D}(\varphi)$ is homeomorphic to a connected subspace of E^N. Then there exists an idempotent continuous selfmap v of Y and a continuous function k from the range of v onto X such that*

(1) *the restriction of k to each $v(A)$, $A \in D(\varphi)$, is a homeomorphism onto X, and*

(2) $\varphi(f) = \pi(\varphi) \cap (k^{-1} \circ f \circ k \circ v)$ *for each* $f \in S(X)$.

Proof. Let any $A \in \mathscr{D}(\varphi)$ be given. Then $\eta_A \circ \varphi$ is an isomorphism from $S(X)$ into $S(A)$ and a repetition of the proof of Theorem 3.4 results in the conclusion that there is an idempotent map v_A of $S(A)$ and a homeomorphism h_A from X onto the range of v_A such that

$$(\eta_A \circ \varphi)(f) = h_A \circ f \circ h_A^{-1} \circ v_A \quad \text{for each } f \text{ in } S(X).$$

Now denote $\varphi(i)$ by v, where i is the identity on X and let $y \in A$. Then

$$v(y) = \varphi(i)(y) = \big((\eta_A \circ \varphi)(i)\big)(y) = v_A(y).$$

Thus, v restricted to A is v_A. Denote the range of v by V and define a function k from V onto X by

$$k(y) = h_A^{-1}(y) \quad \text{for} \quad y \in V \cap A.$$

Now we show that k is continuous. Since X is an S^*-space, the preimages of the points of X under continuous selfmaps is a basis for the closed subsets of X, so it is sufficient to show that $k^{-1}(f^{-1}(x))$ is closed for each $x \in X$ and $f \in S(X)$. With some calculation, one shows that

$$k^{-1}(f^{-1}(p)) = (\varphi(f))^{-1}(k^{-1}(p)) \cap V,$$

so in order to conclude that k is continuous, we need only show that $k^{-1}(p)$ is closed for each $p \in X$. First of all, we note that for any $D \in \mathscr{D}(\varphi)$, any $f \in S(X)$, and any $r \in D \cap V$, we have

(3) $\varphi(f)(r) = \varphi_D(f)(r) = h_D\big(f\big(h_D^{-1}(v(r))\big)\big) = h_D\big(f(h_D^{-1}(r))\big) = h_D\big(f(k(r))\big).$

Now suppose $k^{-1}(p)$ is not closed. Since V is closed, this implies that there is a point

(4) $q \in \big[(\mathrm{Cl}\,k^{-1}(p)) - k^{-1}(p)\big] \cap V.$

Then $q \in A$ for some $A \in \mathscr{D}(\varphi)$; and for the identity map i of $S(X)$ and the map $\langle p \rangle$ in $S(X)$, we have from (3) then

(5) $\varphi(i)(q) = h_A(k(q)),$

and

(6) $\varphi(\langle p \rangle)(q) = h_A(p).$

Now $k(q) \neq p$, so $h_A(k(q)) \neq g_A(p)$ and we can find disjoint open sets G_1 and G_2 such that $h_A(k(q)) \in G_1$ and $h_A(p) \in G_2$. Then (5) and (6) imply the existence of open subsets H_1 and H_2, both containing q, such that

(7) $\varphi(i)(H_1) \subset G_1$

and

(8) $\varphi(\langle p \rangle)(H_2) \subset G_2.$

Then since $q \in \mathrm{Cl}(k^{-1}(p))$, there exists a point $t \in H_1 \cap H_2 \cap k^{-1}(p) \cap B$ for some $B \in \mathscr{D}(\varphi)$. So by (3), we have

(9) $\varphi(i)(t) = h_B(k(t)) = h_B(p)$

and also

(10) $\varphi(\langle p \rangle)(t) = h_B(p).$

But (7), (8), (9) and (10) result in a contradiction. Thus $k^{-1}(p)$ is closed, and so k is continuous.

We have previously noted that $v(A) = v_A(A)$ and that h_A is a homeomorphism from X onto $v_A(A)$. Since k restricted to $v_A(A)$ is just h_A^{-1}, condition (1) follows.

Now we verify (2). Suppose $(a, b) \in \varphi(f)$. Then $a \in A$ for some $A \in \mathscr{D}(\varphi)$, and since $\varphi(f)$ maps A into A, we have $b \in A$. Thus $(a, b) \in \pi(\varphi)$. Furthermore,

$$b = \varphi(f)(a) = \varphi_A(f)(a) = h_A\Big(f\big(h_A^{-1}(v_A(a))\big)\Big) = h_A\Big(f\big(k(v(a))\big)\Big).$$

Thus, $\Big(a, f\big(k(v(a))\big)\Big) \in f \circ k \circ v$ and $\Big(f\big(k(v(a))\big), b\Big) \in h_A \subset k^{-1}$. Thus

$$(a, b) \in \pi(\varphi) \cap (k^{-1} \circ f \circ k \circ v).$$

The inclusion in the other direction follows in a similar manner.

EXAMPLE. Define a mapping φ from $S(I)$ into $S(E^2)$ as follows:

$$\varphi(f)(x, y) = \begin{cases} (f(0), y) & \text{if} \quad x \leqslant 0, \\ (f(x), y) & \text{if} \quad 0 \leqslant x \leqslant 1, \\ (f(1), y) & \text{if} \quad 1 \leqslant x \end{cases}$$

for each $(x, y) \in E^2$. Then one can show that φ is an embedding of $S(I)$ into $S(E^2)$. In fact, φ is an entire isomorphism. Let $A_r = \{(x, r)\colon x \in R\}$. Then

$$\mathscr{D}(\varphi) = \{A_r\colon r \in R\}.$$

Since each A_r is homeomorphic to R, the hypothesis of Theorem 5.2 is satisfied. Also,

$$\pi(\varphi) = \bigcup \{A_r \times A_r\colon r \in R\}.$$

The function v is given by

$$v(x, y) = \begin{cases} (0, y) & \text{if} \quad x \leqslant 0, \\ (x, y) & \text{if} \quad 0 \leqslant x \leqslant 1, \\ (1, y) & \text{if} \quad 1 \leqslant x, \end{cases}$$

and its range V is $\{(x, y)\colon 0 \leqslant x \leqslant 1\}$. The function k which maps V onto I is simply the projection map. Finally, for any $r \in I$, $v(A_r) = \{(x, r)\colon 0 \leqslant x \leqslant 1\}$ and it is easy to see that k maps $v(A_r)$ homeomorphically onto I.

The latter theorem has quite a resemblance to a topological version of a theorem of Clifford and Miller on endomorphisms of semigroups of binary relations. Their original result appeared in [3], p. 310. The topological version we state appeared in [16], p. 63 (see also [1] for some related results). We first need to agree on some conventions and notation. For any two binary relations α and β on X, we define

$$\alpha \circ \beta = \{(x, y)\colon (x, z) \in \beta \text{ and } (z, y) \in \alpha \text{ for some } z \in X\}.$$

This is consistent with the manner in which we are composing functions.

DEFINITION 4.3. Let $\mathscr{S}_X$ and $\mathscr{S}_Y$ be any two semigroups of relations on the sets X and Y, respectively. A homomorphism Φ from $\mathscr{S}_X$ into $\mathscr{S}_Y$ is *union preserving* if whenever $a \in \mathscr{S}_X$ and $a = \bigcup \{\beta_a \colon a \in A \text{ and } \beta_a \in \mathscr{S}_X\}$, then $\Phi(a) = \bigcup \{\Phi(\beta_a) \colon a \in A\}$. It is *symmetry preserving* if it takes symmetric relations into symmetric relations. A homomorphism which is both union and symmetry preserving and is nonconstant is referred to as a CM-*homomorphism*. Every CM-homomorphism is, in fact, injective so we might as well use the term CM-*isomorphism.*

Clifford and Miller completely determined the CM-isomorphisms of the full binary relation semigroup on a set ([3], p. 310). Before we state a topological version, we need another definition.

DEFINITION 4.4. A space X is a *C-space* if $a \circ \beta$ is a closed relation on X (i.e. a closed subset of $X \times X$) for any two closed relations a and β on X.

Hence, the family $C[X]$ of all closed relations on a C-space X is a semigroup under composition. It was shown in [13], p. 191 that a first countable space is a C-space if and only if it is either sequentially compact or discrete.

THEOREM 4.5 (Magill and Yamamuro [16]). *Let X and Y be C-spaces. Let π and ζ be any two partial equivalences on Y such that $\pi \neq \emptyset$ and both ζ and $\pi \cup \zeta$ are closed subsets of $Y \times Y$. The domain of π is a subspace of Y which we denote by E and let μ be any continuous function from E onto X which satisfies*

(1) $\mu \circ \pi = E \times X.$

Define a mapping Φ by

(2) $\Phi(a) = \big(\pi \cap (\mu^{-1} \circ a \circ \mu)\big) \cup \zeta.$

Then Φ is a CM-*isomorphism from $C[X]$ into $C[Y]$ and all such isomorphisms are obtained in exactly this manner.*

A homomorphism Φ from $C[X]$ into $C[Y]$ is 0-preserving if it sends the zero of $C[X]$ into the zero of $C[Y]$. The following corollary is an immediate consequence of the previous theorem and the fact that the empty relation is the zero of both $C[X]$ and $C[Y]$.

COROLLARY 4.6. *Let X nad Y be C-spaces. Let π be a closed partial equivalence on Y and let μ be a continuous map from the domain E of π onto X such that*

(1) $\mu \circ \pi = E \times X.$

Define a mapping Φ by

(2) $\Phi(a) = \pi \cap (\mu^{-1} \circ a \circ \mu).$

Then Φ is a 0-preserving CM-*isomorphism from $C[X]$ into $C[Y]$ and all such isomorphisms are obtained in exactly this manner.*

Now we state an immediate corollary of Theorem 4.2.

COROLLARY 4.7. *Let X be an n-dimensional strongly conformable, quasi-homogeneous space and let φ be any entire isomorphism which takes the identity of $S(X)$ into the identity of $S(Y)$ and is such that each set in $\mathscr{D}(\varphi)$ is homeomorphic to a connected subspace of E^N. Then there exists a continuous map k from Y onto X such that*

(1) *k maps each $A \in \mathscr{D}(\varphi)$ homeomorphically onto X; and*

(2) $\varphi(f) = \pi(\varphi) \cap (k^{-1} \circ f \circ k)$ *for each $f \in S(X)$.*

Now we compare the last two results. We did not prove it, but in Corollary 4.7, $\pi(\varphi)$ is a closed equivalence relation on Y. Of course, π in Corollary 4.6 is closed but it may only be a partial equivalence, that is, symmetric and transitive. The function k maps all of Y continuously onto X while μ maps only the domain of π onto X. Both π and $\pi(\varphi)$ are the disjoint union of rectangles. Condition (1) of Corollary 4.6 is equivalent to the requirement that μ maps each projection of each rectangle continuously onto X. The function k does more. It maps each projection of each rectangle homeomorphically onto X.

We consider an example. Define an isomorphism φ from $S(I)$ into $S(I^2)$ by

$$\varphi(f)(x, y) = \big(f(x), y\big).$$

Now $S(I)$ is a subsemigroup of $C[I]$ and $S(I^2)$ is a subsemigroup of $C[I^2]$. It follows from the previous two corollaries that φ can be extended to a 0-preserving CM-isomorphism $\hat{\varphi}$ from $C[I]$ into $C[I^2]$. First, let

$$A_r = \{(x, r)\colon\ 0 \leqslant x \leqslant 1\}.$$

Then,

$$\mathscr{D}(\varphi) = \{A_r\colon\ 0 \leqslant r \leqslant 1\}$$

and

$$\pi(\varphi) = \bigcup \{A_r \times A_r\colon\ 0 \leqslant r \leqslant 1\}.$$

The mapping k whose existence is assured by Corollary 4.7 is just the projection map $k(x, y) = x$ for all $(x, y) \in I^2$. According to that corollary,

$$\varphi(f) = \pi(\varphi) \cap (k^{-1} \circ f \circ k), \quad \text{for all} \quad f \in S(I).$$

We define

$$\hat{\varphi}(a) = \pi(\varphi) \cap (k^{-1} \circ a \circ k)$$

for all $a \in C[I]$, and according to Corollary 4.6 $\hat{\varphi}$ is a 0-preserving CM-isomorphism from $C[I]$ into $C[I^2]$.

This brings up another point. Among other things, we have shown that there exist an isomorphism from $C[I]$ into $C[I^2]$ which carries the subsemigroup $S(I)$ of $C[I]$ into the subsemigroup $S(I^2)$ of $C[I^2]$. Now, it follows from Lemma 5.1 of [16], p. 71, that there exists an isomorphism, in fact, a 0-preserving CM-isomorphism, from $C[I^2]$ into $C[I]$, but in view of Theorem 2.10 it cannot possibly carry $S(I^2)$ into $S(I)$.

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