TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

SOME NEW RESULTS ON NUMBERS OF GENERATORS OF IDEALS IN LOCAL RINGS

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This paper gives an introduction to J. Watanabe's work on numbers of generators of ideals in Artinian local rings based on his inequality $\mu(I) \leqslant l(A/xA)$, and an extension of the inequality to the one-dimensional case.

1

In 1983 Junzo Watanabe started to develop an interesting new theory concerning local rings, especially Artinian local rings ([2], [3]). In particular, he found a remarkable parallelism between the theory of Artinian local rings and the theory of finite posets (poset = partially ordered set), and applied some techniques and theorems of combinatorics to Artinian local rings.

Recently I found that one of his results can be generalized to the one-dimensional case, but our knowledge in this case is still very limited.

A local ring will mean a Noetherian local ring. When (A, m) is a local ring and M is a finitely generated A-module, $l(M)$ and $\mu(M)$ will denote, respectively, the length and the minimum number of generators of M, so that $\mu(M) = l(M/mM)$. If N is a submodule of M and x is an element of m, we will write $(N:x)_M$ for the submodule $\{\xi \in M \mid x\xi \in N\}$.

2

Let (A, m) be an Artinian local ring (i.e. a zero-dimensional local ring). Watanabe found the following interesting theorem: for any ideal I of A and for any element x of m, we have

$$\mu(I) \leqslant l(A/xA).$$

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Consequently,

$$\operatorname{Max}\{\mu(I) \mid I \text{ is an ideal of } A\} \leqslant \operatorname{Min}\{l(A/xA) \mid x \in m\}.$$

He calls the left-hand side the Dilworth number of A, and the right-hand side the Rees number of A, and he denotes them by $d(A)$ and $r(A)$ respectively. He has proved that these two invariants coincide in many cases.

First we give a very short proof of Watanabe's inequality in a slightly sharper form.

THEOREM 1. *Let (A, m) be an Artinian local ring, M a finitely generated A-module and N a submodule. Let $x \in m$. Then $\mu(N) \leqslant l(M/xM)$, and the equality holds iff $(0:x)_M \subset N$ and $mN = xN$.*

Proof. Consider the exact sequences

$$0 \to (0:x)_M \to M \xrightarrow{x} M \to M/xM \to 0$$

and

$$0 \to (0:x)_N \to N \xrightarrow{x} N \to N/xN \to 0.$$

Since all the modules appearing here are of finite length, we get

$$l(M/xM) = l((0:x)_M) \geqslant l((0:x)_N) = l(N/xN) \geqslant l(N/mN) = \mu(N).$$

All the assertions of the theorem are contained in these relations.

Next we generalize the theorem to the one-dimensional case.

THEOREM 2. *Let (A, m) be a local ring and M be a finitely generated A-module with $\dim M \leqslant 1$. Let N be a submodule of M and x be an element of m. Then $\mu(N) \leqslant l(M/xM)$, and the equality holds iff the following conditions are satisfied:*

$$l(M/xM) < \infty, \quad \dim M/N = 0, \quad N \supset (0:x)_M, \quad mN = xN.$$

Proof. Consider the exact sequences

$$0 \to \frac{xM \cap N}{xN} \to \frac{N}{xN} \to \frac{M}{xM} \to \frac{M}{xM+N} \to 0,$$

$$0 \to \frac{xM \cap N}{xN} \to \frac{xM}{xN} \to \frac{M}{N} \to \frac{M}{xM+N} \to 0.$$

We may assume $l(M/xM) < \infty$, since otherwise $\mu(N) < l(M/xM)$ is trivial. We have to distinguish two cases.

Case 1. $l(M/N) < \infty$, or equivalently, $\dim M/N = 0$. Since $xM/xN \simeq M/(xN:x)_M$ is isomorphic to a quotient of M/N, we have $l(xM/xN) < \infty$, hence also $l((xM \cap N)/xN) < \infty$, and since $l(M/xM) < \infty$ by assumption we see that all the modules appearing in the exact sequences are of finite length. Therefore

$$\begin{aligned} l(M/xM)-l(N/xN) &= l(M/(xM+N))-l((xM\cap N)/xN) \\ &= l(M/N)-l(xM/xN) \\ &= l(M/N)-l(M/(xN:x)_M) \\ &= l((xN:x)_M/N). \end{aligned}$$

Hence

$$l(M/xM) = l(N/xN)+l((xN:x)_M/N) = \mu(N)+l(mN/xN)+l((xN:x)_M/N),$$

and it is easy to see that $(xN:x)_M = N$ if and only if $N \supset (0:x)_M$. Thus our assertions hold in this case.

Case 2. $l(M/N) = \infty$. We can reduce this case to Case 1 by the same argument as in [1, p. 50]. Namely, by the Artin–Rees theorem there exists $c > 0$ such that $N \cap m^v M \subset mN$ for $v > c$. Since $l(M/(N+m^v M)) < \infty$, we have by case 1

$$\begin{aligned} l(M/xM) \geqslant \mu(N+m^v M) &= l((N+m^v M)/m(N+m^v M)) \\ &= l((N+m^v M)/(N+m^{v+1} M))+l((N+m^{v+1} M)/(mN+m^{v+1} M)). \end{aligned}$$

Here, the first term $l((N+m^v M)/(N+m^{v+1} M))$ is equal to $l(m^v(M/N)/m^{v+1}(M/N))$ and since $\dim(M/N) = 1$ in this case, this last is equal to the multiplicity $e(M/N)$ of M/N if v is sufficiently large, and hence it is strictly positive. The second term $l((N+m^{v+1} M)/(mN+m^{v+1} M))$ is equal to $l(N/(mN+(N\cap m^{v+1} M)))$, which is $l(N/mN)$ if $v > c$. Therefore

$$l(M/xM) \geqslant e(M/N)+l(N/mN) = e(M/N)+\mu(N) > \mu(N).$$

This completes the proof.

3

Let (A, m) be a local ring and M be a finitely generated A-module of dimension $\leqslant 1$. We set

$$d(M) := \operatorname{Max}\{\mu(N) \mid N \text{ is a submodule of } M\},$$

$$r(M) := \operatorname{Min}\{l(M/xM) \mid x \in m\},$$

and call them the *Dilworth number* and the *Rees number* of M, respectively. Thus $d(M) \leqslant r(M)$ by the theorem, and if the equality holds we say that M is *exact*. When $M = A$, an element x of m such that $l(A/xA) = r(A)$ is called a *general element*, and an ideal J such that $\mu(J) = d(A)$ is called a *Dilworth ideal.*

Remark 1. When the residue field A/m is finite it is better to modify the definition of $r(M)$ as follows. Take a faithfully flat extension local ring B of A with infinite residue field such that $m_B = m_A B$, and define $r(M)$ to be

$r(M \otimes B)$. When m is generated by $a_1, \ldots, a_n$, let $t_1, \ldots, t_n$ be indeterminates and let $B = A[t_1, \ldots, t_n]_{m[t]}$. Then one can show that $x = \sum t_i a_i$ is a general element; the proof is not so easy, see [2]. If A contains an infinite field k then there is a nonempty open set U of k^n such that, if $(c_1, \ldots, c_n) \in U$, then $\sum c_i a_i$ is a general element (loc. cit). In the following we shall assume that A has an infinite residue field.

Remark 2. When $\dim M = 1$ and M is Cohen–Macaulay, J. Sally proved that $d(M) = e(M)$. Indeed, if $x \in m$ is M-regular, then $xM/xN \simeq M/N$ and our proof in Case 1 shows $l(M/xM) = l(N/xN)$. Taking $N = m^v M$, and choosing x such that $xm^v M = m^{v+1} M$ for sufficiently large v (such an element x exists: replacing A by $A/\mathrm{ann}(M)$ we way assume that $\dim A = 1$, and then any minimal reduction of m is a principal ideal generated by such x), we see that $l(M/xM) = l(m^v M/m^{v+1} M) = \mu(m^v M) = e(M)$. Therefore $r(M) = d(M) = e(M)$ and M is exact. If A is one-dimensional Cohen--Macaulay, then sufficiently high powers of the maximal ideal are all Dilworth ideals, and $x \in m$ such that $xm^v = m^{v+1}$ for some v is a general element.

In particular, all one-dimensional local domains are exact since they are Cohen–Macaulay. It is not easy to find nonexact one-dimensional local rings. In fact we do not know any examples yet.

In the zero-dimensional case, the simplest example of a nonexact local ring is $A = k[X, Y, Z]/I$, where k is a field and $I = (X^3, Y^3, Z^3, XYZ) + M^4$, $M = (X, Y, Z)$. In this case $d(A) = 6$ and $r(A) = 7$. Another example is $k[X, Y, Z, W]/(X^2, Y^2, XY, Z^2, W^2, ZW)$, which has $d(A) = 4$ and $r(A) = 5$.

4

When the local ring A is exact it is easy to determine $d(A)$ $(= r(A))$: we have only to find an ideal I and an element $x \in m$ such that $\mu(I) = l(A/xA)$, and Theorems 1 and 2 give criteria for the equality. But when A is notexact we need some other methods to find $d(A)$ and $r(A)$. When A is zero-dimensional ring of *monomial type*, i.e. of the form $A = k[X_1, \ldots, X_n]/I$ where I is generated by some monomials, Watanabe has given convenient criteria. In such a case $X_1 + \ldots + X_n$ is always a general elelemt (because one knows that $\sum c_i X_i$ is a general element for suitable $0 \neq c_i \in k$, and $c_i X_i \mapsto X_i$ $(1 \leqslant i \leqslant n)$ defines an automorphism of the ring of monomial type A), so that $r(A) = l(A/(X_1 + \ldots + X_n))$. On the other hand, the monomials in $X_1, \ldots, X_n$ which are not in the ideal I form a finite poset (ordered by divisibility), say $P(A)$, and its Dilworth number in the sense of combinatorics is exactly equal to $d(A)$, see [3]. A totally ordered subset of a poset is called a chain, and a subset of a poset in which no two elements are comparable is called an antichain (or an independent set); an important theorem in combinatorics, due to Dilworth, says that the minimal number of chains into which a poset P is decomposed is equal to the maximal number of elements in an antichain in P, and this number is called the Dilworth number of P.

5

Some other important results of Watanabe for Artinian local rings:

(1) if A is homogeneous (i.e. graded in such a way that $A = A_0 + A_1 + \ldots + A_s$, $A_s \neq 0$, $A_0 = k$ and $A = k[A_1]$), and if there exists $x \in A_1$ such that the multiplication by x^{s-2i} defines an isomorphism of k-linear spaces $A_i \to A_{s-i}$ for $0 \leqslant i \leqslant [i/2]$, then A is exact and $d(A) = \mathrm{Max}\{\dim_k A_i | 0 \leqslant i \leqslant s\} = \dim A_{[s/2]}$;

(2) "most" of Gorenstein homogeneous rings satisfy the condition above, and hence are exact.

6. Open questions

(A) Are the zero-dimensional complete intersection rings exact?
(B) Are there nonexact one-dimensional local rings?
(C) Is there any generalization to higher dimensional case? Is the minimum of $l(A/q)$ where q runs over the ideals generated by system of parameters an interesting invariant?

References

[1] J. D. Sally, *Numbers of Generators of Ideals in Local Rings*, Marcel Dekker, New York and Basel 1978.
[2] J. Watanabe, *m-full ideals*, Nagoya Math. J. 106 (1987), 101–111.
[3] —, *The Dilworth number of Artinian rings and finite posets with rank function*, in: Commutative Algebra and Combinatorics, Advanced Stud. in Pure Math. 11, Kinokuniya, Tokyo and North-Holland, Amsterdam 1987, 303–312.

Added in proof (April 1990). 1. The inequality $\mu(I) \leqslant l(A/xA)$ was proved by N. V. Trung in his article *Bounds for the minimum numbers of generators of generalized Cohen–Macaulay ideals*, J. Algebra 90 (1984), 1–9, with a very short proof. Watanabe found the inequality independently, as a corollary of a more general theorem.

2. The answer to the question (B) is *yes*. Watanabe has found that the homogeneous ring $A = k[X, Y, Z, W]/I$, $I = (X, Y)^3 + (Z^3, Z^2W, ZW^2)$, satisfies $d(A) = 13$ and $r(A) = 14$.

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PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

BOUNDS FOR CASTELNUOVO'S REGULARITY AND THE GENUS OF PROJECTIVE VARIETIES

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Using intrinsic geometrical properties of projective varieties we will improve Harris' bound for the geometric genus of varieties in $\mathbf{P}^n$. Furthermore, we will get new and sharp bounds for the genus of arithmetically Buchsbaum varieties and of varieties of codimension 2. In case of Buchsbaum varieties we also prove sharp bounds for Castelnuovo's regularity. Our approach in proving such bounds is to reduce the problem to the case of a collection of points in uniform position. This means the key idea here is the uniform position principle developed by J. Harris in case of space curves. Hence the present paper relies upon an analysis of the Hilbert function of the section of a subvariety V with a generic linear subspace of dimension $=\operatorname{codim}(V)$. Finally we improve some bounds in case of space curves.

0. Introduction

The study of possible genera of irreducible space curves in $\mathbf{P}^3$ has a fairly long history (see, e.g. [14–21]). A main problem is the following:

Given integers $d, k > 0$, we wish to find the maximum genus $g = G(d, k)$ of an irreducible nonsingular curve in $\mathbf{P}^3$ of degree d which is not contained in any surface of degree $< k$. This problem is still open. Our Theorem 5 of Section 5 yields contributions to solve this problem by applying new Castelnuovo bounds. Moreover, in this paper, we will study the analogous question for projective varieties of arbitrary dimension: what is the greatest possible geometric genus of an irreducible, nondegenerate variety of degree d in $\mathbf{P}^r$? This problem was solved in 1981 by J. Harris [16] (see our Corollary 6). Using

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