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Pre-supports of linear probability measures and linear Lusin measurable functionals

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1. Introduction, review of the results, examples

Consider a locally convex space (X, τ) provided with a Borel probability measure μ. Let (U, ϱ) denote a locally convex space, where U is a linear Borel subset in (X, τ) and the topology ϱ is finer than τ. Denote by U' the adjoint of (U, ϱ).

The purpose of this paper is to describe and classify (U, ϱ) such that every $u' \in U'$ extends uniquely to a linear measurable functional over a measure one linear Borel subset in (X, τ) in such a way that the convergence of a $\{u'_n\} \subset U'$, which is uniform over every convex compact subset in (U, ϱ), implies the stochastic convergence of the extensions.

The spaces (U, ϱ) together with the cylinder measures ν obtained by restricting μ are in Section 5 called pre-representations of the measure. Already in Section 4, the spaces U are introduced as pre-supports of μ. Though pre-supports can be of the measure zero, it follows from Theorem 4.5 that the pre-representations which they induce determine the measure uniquely up to a certain natural equivalence relation. The spaces U' corresponding to the so-called proper standard pre-supports U can be identified with linear subspaces of the space $\Lambda(\mu)$ of so-called linear Lusin measurable functionals. It follows from Theorem 4.2 that they constitute the family of these Fréchet subspaces of the space $\Lambda(\mu)$ which carry topologies finer than that of stochastic convergence and are simultaneously stochastically dense in the space $\Lambda(\mu)$.

The examples which will come now are only meant to enrich the intuition and illustrate the coming theory. Corollaries 4.1, 4.2 and 5.3 are only providing hints for applications which will be published separately.

Suppose that we are given a completely regular topological space T, the space $C(T)$ of all continuous real valued functions over T and the Borel field $\mathscr{B}(T)$ of T.

In the sequel, let μ be a Radon probability measure on $\mathscr{B}(T)$, i.e. let the measure of every $B \in \mathscr{B}(T)$ be equal to the supremum of measures of compacts contained in B.

Assume that there exists a sequence of functions $\{f_n\} \subset C(T)$ which separate points in compacts $C_m \subset T$, with $C_m \subset C_{m+1}$ and $\lim_n \mu(C_n) = 1$. Put $W = \bigcup_1^\infty C_n$.

We can inject W into the infinite product of real lines R^∞ by use of the mapping

$$(*) \qquad W \ni t \to \{f_n t\} \in R^\infty.$$

Denote by $W^\wedge$ the image of W in R^∞ and by $\mathscr{B}(R^\infty)$ the Borel field in R^∞. Clearly, we can move the measure μ over $\mathscr{B}(R^\infty)$ by means of the mapping $(*)$ and obtain a measure $\mu^\wedge$ on $\mathscr{B}(R^\infty)$ such that $\mu^\wedge(W^\wedge) = 1$.

It is intuitively clear that looking on $\mu^\wedge$ over $\mathscr{B}(R^\infty)$, we really look on the initial measure μ, only the domain of its action has been enormously extended by adding many new zero measure sets, but getting in exchange the linearity of the domain.

Given any probability measure over R^∞, we shall consider only the so-called linear Lusin measurable functionals over R^∞. A functional is said to be *linear Lusin measurable* if it is defined on a measure one linear subset of R^∞ and continuous on compact convex subsets of measure arbitrarily close to one. In the considered case, each such functional restricted to $W^\wedge$ yields a Lusin measurable function on W.

Suppose now that linear combinations of f_n approximates uniformly continuous functions on compacts of measure arbitrarily close to one. To concentrate on a concrete example, let us take the interval $[0, 1]$ as T and the Lebesgue measure as μ. Then the functions $f_n t = t^n$ fulfil all the requirements.

Any continuous linear functional $\underline{t}$ over R^∞ is given by a sequence of numbers $\underline{t} = (t_1, \ldots, t_n, \ldots)$, almost all of which are equal to zero, and then $\underline{t}$ acts on $\underline{x} = \{x_n\} \in R^\infty$ according to the formula

$$\underline{tx} = \sum_1^\infty t_n x_n.$$

Then, given any Lusin measurable function on $W^\wedge$, we can approximate it by linear functionals uniformly on compacts of the measure arbitrarily close to one and thus uniformly on their closed convex envelopes which are then compact. The limit presents a linear Lusin measurable functional which extends the given function.

This way we have not only made the domain of the measure linear but we have simultaneously reduced considering Lusin measurable functions to linear Lusin measurable functionals.

Notice, that if we shall additionally require that $\mu(\{t \in T\colon \sum_1^n s_i f_i t = 0\}) = 1$ implies $s_1 = \ldots = s_n = 0$, which is the case in our example with the Lebesgue measure, then $\underline{t}$ vanishing almost everywhere vanishes identically. We shall refer to this situation as "proper".

It remains to point out the advantages of looking over a measure in its "linear" dressing and for linear Lusin measurable functionals instead of Lusin measurable functions.

For the future convenience, we shall introduce the following notation. Given two linear topological spaces (X, τ) and (Y, ϱ), we write

$$(X, \tau) \leqslant (Y, \varrho)$$

if Y is a linear subspace of X and the identical injection of Y into X is continuous.

Consider a probability measure μ on $\mathscr{B}(R^\infty)$. Let $\Lambda(\mu)$ denote the space of classes of linear Lusin measurable functionals, where we identify functionals equal almost everywhere. We provide $\Lambda(\mu)$ with the pseudonorm

$$s[f] = \int |f|(1+|f|)^{-1} d\mu.$$

Denote by R_0^∞ the subspace of R^∞ consisting of all sequences with almost all elements equal to zero. As was mentioned earlier, R_0^∞ can be identified with the space of all continuous linear functional on R^∞.

Assume properness of μ on R_0^∞, i.e. assume that if a functional $\underline{t} \in R_0^\infty$ vanishes almost everywhere, it vanishes identically. Then every $\underline{t} \in R_0^\infty$ belongs to exactly one class in $\Lambda(\mu)$, and in this sense we can identify R_0^∞ with a linear subset of $\Lambda(\mu)$.

Let $\mathscr{F}$ denote the family of all Fréchet spaces (Θ, ϑ) such that R_0^∞ constitutes a dense linear subset of (Θ, ϑ) and that

$$(\Theta, \vartheta) \geqslant \big(\Lambda(\mu), s[\cdot]\big).$$

In the sequel, let $\mathscr{R}$ denotes the family of linear subspaces of R^∞ which are unions of countably many compact convex subsets of R^∞. Given $U \in \mathscr{R}$, we denote by U', the space of linear functionals over U which are continuous in the topology of R^∞ over every convex compact contained in U. Provide U' with the Mackey topology $\tau(U', U)$.

As already mentioned, it is shown in Section 4 of this paper that to every $(\Theta, \vartheta) \in \mathscr{F}$ there corresponds exactly one $U \in \mathscr{R}$ such that every $u' \in U'$ extends uniquely to a linear Lusin measurable functional, the class of which we shall denote by u^* in such a way that the mapping

$$U' \ni u' \to u^* \in \Theta$$

is a one-to-one bicontinuous linear transformation of $\big(U', \tau(U', U)\big)$ onto (Θ, ϑ), i.e. an isomorphism of $\big(U', \tau(U', U)\big)$ and (Θ, ϑ).

In the third section of this paper it is shown that spaces of integrable and square integrable linear Lusin measurable functionals with the standard norms both belong to $\mathscr{F}$.

Therefore, it is easy to see that representing measures in R^∞ and considering Lusin measurable functionals instead of measurable functions, we gain the possibility of evaluating functionals from certain subspaces of $\Lambda(\mu)$ in such a way that the uniform convergence on convex compact subsets induces almost everywhere convergence. In view of the example with the Lebesgue measure, the spaces on which we evaluate shall in general have the measure zero. Also, the Lebesgue measure example shows that in general we cannot find any subspace of R^∞ on which we could evaluate all functionals from $\Lambda(\mu)$ having the evaluation continuous in $(\Lambda(\mu), s[\cdot])$.

However, there is a quite wide class of measures in R^∞, where the whole $\Lambda(\mu)$ can be evaluated.

As in the previous case, we shall confine ourselves to one concrete example, i.e. to the product of Gaussian measures $\exp(-\pi t^2)\,dt$, which will also include the case of the Wiener measure.

Denote by γ the product of $\exp(-\pi t^2)\,dt$. Then the following conditions are true (cf. [12]).

Write $l_2 = \{\underline{t} = \{t_n\} \in R^\infty\colon \sum_1^\infty t_n^2 < \infty\}$. Then

a) For every $\underline{t} = \{t_n\} \in l_2$ the series

$$\underline{tx} = \sum_1^\infty t_n x_n$$

converges γ-almost everywhere on R^∞.

b) Given a linear Lusin measurable functional $f \in \Lambda(\gamma)$, there exists exactly one $\underline{t} \in l_2$ such that $f\underline{x} = \underline{tx}$ γ-almost everywhere.

c) The space $l_2^{\hat{}} = \{\underline{t} \in R^\infty\colon \{t_n/n\} \in l_2\}$ is of measure one.

d) The space l_2 is of measure zero.

From Conditions a) and b) we can see that all f from $\Lambda(\gamma)$ are square integrable, and they can be evaluated on l_2 in such a way that the usual norm convergence in l_2 is equivalent to the stochastic convergence with respect to γ.

Now, denote by $L_2[0, \pi]$ the space of Lebesgue square integrable functions over $[0, \pi]$ and let $L_2^{\hat{}}[0, \pi] = \{f \in L_2[0, \pi]\colon \int f(t)\,dt = 0\}$. Inject $L_2^{\hat{}}[0, \pi]$ onto $l_2^{\hat{}}$ by way of the mapping

$$l_2^{\hat{}} \ni \{nx_n\} \leftrightarrow f \in L_2[0, \pi],$$

where

$$f(t) = (\pi/2)^{-1/2} \sum_1^\infty x_n \cos nt, \qquad t \in [0, \pi].$$

Then the classical Wiener measure ω is obtained by transferring γ from $l_2^{\wedge}$ to $L_2^{\wedge}[0, \pi]$ by use of the above defined isomorphism. The role of l_2 will be played then by the space $L_2^d[0, \pi]$ of all functions from $L_2[0, \pi]$, derivatives of which exist in the distribution sense and are in $L_2[0, \pi]$. We provide L_2^d with the Hilbertian norm of the derivatives in $L_2[0, \pi]$. Analogically to the previous case $\omega(L_2^d[0, \pi]) = 0$, the integral

$$(*) \qquad \int a(t)\,dx(t) = \int a(t)x'(t)\,dt,$$

where $a(\cdot) \in L_2[0, \pi]$, exists ω-almost everywhere for $x(\cdot)$ from $L_2^{\wedge}[0, \pi]$, and given $f \in \Lambda(\omega)$, there exists exactly one $a(\cdot) \in L_2[0, \pi]$ such that $fx(\cdot) = \int a(t)\,dx(t)$ almost everywhere on $L_2^{\wedge}[0, \pi]$. This is the Paley–Wiener–Zygmund stochastic integral [9].

The notion of linear Lusin measurable functionals leads to a general definition of certain stochastic integrals and, through results of Section 4, provides grounds for their existence in the form of Corollary 4.2.

By a real valued random function we shall understand a triplet (R^T, R_0^T, δ), where T is an arbitrary set, R^T is the linear space of all real valued functions on T, R_0^T is the subset of R^T consisting of all functions assuming value zero save a finite number of points, elements of which act as linear functionals over R^T by way of the evaluation

$$x(\cdot)y(\cdot) = \sum_{t \in T} x(t)y(t),$$

and, finally, a tight cylinder probability measure δ, i.e. a finite additive set function defined over the field $\mathscr{C}$ generated by all sets of the form $\{x(\cdot) \in R^T\colon x(t) < r\}$, where t is an arbitrary fixed point from T and r is an arbitrary real number, fulfilling the following condition:

(+) To every $\varepsilon > 0$ there corresponds a positive valued function $f(\cdot)$ defined on T such that if $C \in \mathscr{C}$ is disjoint with $\{x(\cdot) \in R\colon |x(t)| \leqslant f(t)\}$ for all $t \in T$, then $\delta(C) < \varepsilon$.

It is well known that condition (+) guarantees the existence of the extension of δ over the weak Borel field $\mathscr{B}(R^T, R_0^T)$ to a Radon probability measure. In this paper, we shall be concerned mainly with such random functions for which the space R_0^T provided with the metric of the stochastic convergence is separable. Notice, that producing (R^T, R_0^T, δ), we actually produce a joint distribution of a stochastic process in the sense of [5], pp. 497–498.

At the end of Section 4, we distinguish certain linear probability measures which we call stochastic processes. Theorem 2.1 of this paper shows that every stochastic process admits a so-called primitive representation, which is something like the joint distribution (R^T, R_0^T, δ) mentioned before.

The notion of pre-support introduced in Section 4 leads to the so-called Λ_p-pre-supports which for $p = 2$ generalize spaces l_2 and $L_2^d[0, \pi]$ produced above for the Gaussian and Wiener measures respectively. It is left to the reader to follow the obvious connections with the results of Gross [3].

The meaning of pre-support for measures on convex compact sets is discussed in Corollary 5.3.

2. Linear probability measures and their representations

By a dual pair we shall understand here any pair (X, X') consisting of a linear space X over the reals and a linear space X' of linear functionals over X separating in X, i.e. such that $(X, \sigma(X, X'))$ is a Hausdorff space.

Consider a dual pair (X, X') and a linear subspace S of X. A linear mapping to a linear topological space (Y, τ) with a domain containing S is said to be (X, X')-*almost uniformly continuous on* S if it is $\sigma(X, X')$ to τ continuous over each convex $\sigma(X, X')$-compact subset of S. A sequence of linear functionals with domains containing S is said to be (X, X')-*almost uniformly convergent on* S if it is uniformly convergent over every convex $\sigma(X, X')$-compact contained in S.

We shall define the (X, X')-adjoint S' of S assigning to S' all linear functionals defined on and (X, X')-almost uniformly continuous on S. It is easy to see that a sequence from S' is convergent in $(S', \tau(S', S))$ iff it is (X, X')-almost uniformly convergent on S. In particular, we denote by sX' the saturation of X', i.e. the space of all (X, X')-almost uniformly continuous functionals on X. Clearly, the topologies $\sigma(X, X')$ and $\sigma(X, sX')$ have the same convex compact sets. If $sX' = X'$, then we call (X, X') *saturated*. Notice, that $(sX', \tau(sX', X))$ is the completion of $(X', \tau(X', X))$.

A subspace S of X is said to be (X, X')-*standard* if it is a union of countably many convex $\sigma(X, X')$-*compact* subsets or, equivalently, if $(S', \tau(S', S))$, where S' is the (X, X')-adjoint of S, is a Fréchet space.

We shall often drop (X, X')-, writing briefly almost uniformly continuous on S, almost uniformly convergent on S, adjoint to S and standard in X.

Denote by $\mathscr{B}(X, X')$ the Borel field of $(X, \sigma(X, X'))$ and let μ be a probability measure on $\mathscr{B}(X, X')$ which is regular, i.e. such that for every $A \in \mathscr{B}(X, X')$ we have

$$\mu(A) = \sup\{\mu(C)\colon C \subset A,\ C \text{ is } \sigma(X, X')\text{-closed}\},$$

and concentrated on a standard subspace, i.e. such that there exists an (X, X')-standard subspace of measure one.

Triplets (X, X', μ) shall here be called *representations of linear probability measures.*

Two representations (X, X', μ) and $(Y, Y', \tilde{\mu})$ shall be considered equivalent if there exists a set S which is at the same time an (X, X')-standard subspace of X and an (Y, Y')-standard subspace of Y such that the (X, X')-adjoint of S coincide with the (Y, Y')-adjoint of S, $\mu(S) = \tilde{\mu}(S) = 1$, and measures μ and $\tilde{\mu}$ coincide on $\mathscr{B}(S, S')$. This is clearly an equivalence relation, and we shall say that S carries the equivalence. Clearly, there are many different sets carrying the same equivalence.

Denote by LPM the class of representations of linear probability measures factorized by the above defined equivalence relation. An element of LPM represented by (X, X', μ) we shall denote by (μ).

Given a representation (X, X', μ) of a $(\mu) \in$ LPM we produce its completion $(X^-, X^{-\prime}, \mu^-)$ as follows. First we define X^- as the completion of $\big(X, \tau(X, X')\big)$; then for $X^{-\prime}$ we put the extensions of functionals from X' to X^-. Since $\sigma(X, X')$-compact subsets of X are $\sigma(X^-, X^{-\prime})$-compact subsets of X^- as well, we can provide μ^- so that (X, X', μ) and $(X^-, X^{-\prime}, \mu^-)$ are equivalent.

A representation is said to be *complete* if it coincides with its completion.

Notice that given a representation (X, X', μ), we can always "saturate" it to the equivalent representation (X, sX', μ). Representations identical with their saturations we shall simply call *saturated.*

A representation (X, X', μ) of $(\mu) \in$ LPM is said to be *finer* than a representation (Y, Y', μ) of the same (μ) or, equivalently, (Y, Y', μ) is said to be *coarser* than (X, X', μ), which we write

$$(Y, Y', \mu) \leqslant (X, X', \mu),$$

if X is a subspace of Y and if the identical injection of X into $\big(Y, \sigma'(Y, Y')\big)$ is (X, X')-almost uniformly continuous.

PROPOSITION 2.1. *If a representation (X, X', μ) of a $(\mu) \in$ LPM is finer than a representation (Y, Y', μ) of the same (μ), then the restrictions of functionals from Y' to X form a dense subset of $\big(sX', \tau(sX', X)\big)$.*

Proof. Clearly, every such restriction belongs to sX'. Now, take $x' \in sX'$: Given a $\sigma(X, sX')$-compact convex C, we find that it is also $\sigma(Y, Y')$-compact and that x' is $\sigma(Y, Y')$-continuous on C, so that by Corollary I.1.5 of [1] we can approximate x' on C uniformly by functionals from Y', and this concludes the proof.

Consider a representation (X, X', μ) of a $(\mu) \in$ LPM. Every measure one (X, X')-standard subspace of X shall be called here a *standard support of μ in (X, X', μ)*. A subspace Z of X shall be called a *support of μ in (X, X', μ)* if Z contains a standard support of μ in (X, X', μ). A support Z

of μ in (X, X', μ) is said to be *proper* if every z' from the (X, X')-adjoint Z' of Z, which vanishes almost everywhere, vanishes identically.

PROPOSITION 2.2. *If U is a proper support of μ in a representation (X, X', μ) of a $(\mu) \in$ LPM, then every support V of μ in (X, X', μ) contained in U is dense in $\big(U, \tau(U, U')\big)$, where U' denotes the (X, X')-adjoint of U.*

Proof. If some $u \in U$ was not in the closure of V, then there would exist a $u' \in U'$ with $u'u = 1$ vanishing on V, and thus almost everywhere as well, which contradicts the properness of U. This concludes the proof.

Given a representation (X, X', μ) of a $(\mu) \in$ LPM and a support Z of μ in (X, X', μ), we can produce an equivalent finer representation (Z, Z', μ_r), where Z' is the (X, X')-adjoint of Z. Take the measure μ_r identical with μ on $\mathscr{B}(S, S')$, where S is a standard support in Z and S' is its (X, X')-adjoint, and put $\mu_r(Z - S) = 0$. The representation (Z, Z', μ_r) shall be called the *reduction of (X, X', μ) to Z.*

PROPOSITION 2.3. *Consider a representation (X, X', μ) of a $(\mu) \in$ LPM and a standard support Z of μ in (X, X', μ). Then there exists a proper support S closed in $\big(Z, \tau(Z, Z')\big)$, where Z' denotes the (X, X')-adjoint of Z. Moreover, for any proper support V of μ in (X, X', μ), if $V \subset Z$, then $V \subset S$.*

Proof. Take the reduction (Z, Z', μ_r) of (X, X', μ) to Z. The intersection S of all supports closed in $\big(Z, \tau(Z, Z')\big)$ is again a support closed in $\big(Z, \tau(Z, Z')\big)$. If s', taken from S' vanishes almost everywhere, then the set $\{s \in S\colon s's = 0\}$ is both closed and of measure one; so it must be identical with S. Hence S is proper. If V is another proper support contained in Z, $S + V$ is also a proper support in (Z, Z', μ_r), and by Proposition 2.2 S is dense in $S + V$, and being closed is identical with $S + V$, which concludes the proof.

Representations (X, X', μ), with X being proper, are called *proper representations.*

COROLLARY 2.1. *Every representation (X, X', μ) admits a reduction to a proper representation (Z, Z', μ).*

Proof. This is a trivial consequence of Proposition 2.3.

PROPOSITION 2.4. *Given a representation (X, X', μ) of a $(\mu) \in$ LPM and a sequence $\{S_n\}$ of standard supports of μ in (X, X', μ), the intersection $\bigcap_1^\infty S_n$ is a support of μ in (X, X', μ), i.e. it contains a standard support.*

Proof. It is sufficient to notice that $\bigcap_1^\infty S_n$ contains convex $\sigma(X, X')$-compact subsets of the measure arbitrarily close to one, and this is trivial.

A representation (X, X', μ) of a $(\mu) \in$ LPM is said to be *standard* if it is proper and if $\big(X', \tau(X', X)\big)$ is a Fréchet space. From the Krein–Šmulian theorem it follows that the minimal closed standard support is always proper.

PROPOSITION 2.5. *To every sequence of representations of the same* $(\mu) \in$ LPM *there corresponds a standard representation of* (μ) *finer than all the representations from the sequence.*

Proof. Let $\{(X_n, X'_n, \mu)\}$ be a sequence of representations of (μ) and let S_n be standard supports of μ in (X_1, X'_1, μ) carrying the equivalences with (X_n, X'_n, μ) respectively. The reduction of (X_1, X'_1, μ) to a proper standard support contained in $\bigcap_1^\infty S_n$ provides the desired representation.

A representation (X, X', μ) of a $(\mu) \in$ LPM is said to be *separable* if the space $(X', \tau(X', X))$ admits a dense countable subset. A $(\mu) \in$ LPM is said to be *separable* if it admits at least one separable representation.

A representation (X, X', μ) of a $(\mu) \in$ LPM is said to be *primitive* if it is proper and if $(X, \sigma(X, X'))$ is complete, which amounts to the fact that every linear functional F on X' is an evaluation functional, i.e. $Fx' = x'x$ for some $x \in X$ and every $x' \in X'$. This simply means that the dual pair (X, X') is isomorphic to a pair (R^T, R_0^T), where T is a certain set and R^T denotes the set of all functions from T to the real line R while R_0^T consists of those functions from R^T which vanish on all save a finite number of elements of T. Then $f \in R_0^T$ acts on $g \in R^T$ producing the value $\sum_{t \in T} f(t) g(t)$.

Separable primitive representations (X, X', μ) will thus consist of X' with the countable Hamel basis and of X which by evaluation gives all possible functionals on X', so that $(X, \sigma(X, X'))$ and $(X, \tau(X, X'))$ coincide and constitute a metrizable space.

From Proposition 2.1 it follows that the primitive representations of the same $(\mu) \in$ LPM are either uncomparable with respect to the coarser-finer relation or just identical.

THEOREM 2.1. *Given a proper representation* (X, X', μ) *of a* $(\mu) \in$ LPM *and a linear subspace* U' *of* X' *which is dense in* $(X', \tau(X', X))$, *there exists a primitive representation* (Z, Z', μ) *of* (μ) *coarser than* (X, X', μ) *and such that* Z' *consists of extensions of all functionals from* U' *over the space* Z.

Proof. Denote by V the completion of $(X, \sigma(X, U'))$ and by V' the extensions of functionals from U' over V. Every (X, X')-compact set contained in X is $\sigma(V, V')$-compact, and hence we can define $\tilde{\mu}$ over $\mathscr{B}(V, V')$ in such a way that (X, X', μ) and $(V, V', \tilde{\mu})$ are equivalent. Take now a proper support Z of $\tilde{\mu}$ in $(V, V', \tilde{\mu})$ which is closed in $(V, \sigma(V, V'))$. Denote by Z' the restrictions of functionals from V' to Z. Clearly, $(Z, \sigma(Z, Z'))$ is complete. Hence the reduction of (X, X', μ) to Z provides the desired primitive representation, and the Theorem follows.

A representation (H, H', μ) of a $(\mu) \in$ LPM is said to be *Hilbertian* if it is proper and if $(H', \tau(H', H))$ is isomorphic to a Hilbert space.

A support H of μ in a representation (X, X', μ) of $(\mu) \in$ LPM is said to be *Hilbertian* if the space $(H', \tau(H', H))$, where H' is the (X, X')-adjoint of H, is isomorphic to a Hilbert space. Of course, the reduction to a proper Hilbertian support provides a Hilbertian representation.

THEOREM 2.2. *Consider a separable primitive representation (Z, Z', μ) and a finer standard representation (U, U', μ) of a $(\mu) \in$ LPM. Given a sequence $\{u_n'\} \subset U'$, there exists a separable Hilbertian representation (H, H', μ) finer than (Z, Z', μ) and coarser than (U, U', μ), such that all $\{u_n'\}$ are $\tau(H, H')$-continuous.*

Proof. Adding some functionals, if necessary, assume that $\{u_n'\}$ contains a restriction to U of functionals from Z' spanning Z'. Take a decomposition $\{C_n\}$ of U into an ascending sequence of convex $\sigma(U, U')$-compact sets. Put

$$a_{m,n} = \sup\{|u_n' u| : u \in C_m\}$$

and take $a_n > 0$ such that for every m, $\sum_{n=1}^{\infty} (a_n a_{m,n})^2 < \infty$. Define for $x \in U$

$$\|x\| = \Big(\sum_{1}^{\infty} (a_n u_n' x)^2\Big)^{1/2}$$

We have

$$C_m \subset \Big\{x \in U : \|x\| \leqslant \Big(\sum_{n=1}^{\infty} (a_n a_{m,n})^2\Big)^{1/2}\Big\}.$$

Completing $(U, \| \ \|)$ and extending the injection $U \hookrightarrow Z$ over the completion, we take the image E of this completion in Z which thus constitutes a Hilbertian support of μ in (Z, Z', μ), containing U as a $\tau(E, E')$-dense subspace and such that all $\{u_n'\}$ are $\tau(E, E')$-continuous. Taking a proper Hilbertian support H closed in $(E, \tau(E, E'))$ and reducing (Z, Z', μ) to H, we obtain the desired representation (H, H', μ). This concludes the proof of the Theorem.

PROPOSITION 2.6. *If (X, X', μ) is a complete representation of a $(\mu) \in$ LPM such that $\tau(X, X')$ admits a countable basis of Hilbertian seminorms, then there exists a proper Hilbertian support of μ in (X, X', μ).*

Proof. Let $\{\|\cdot\|_n\}$ denote the pseudonorms introducing the topology $\tau(X, X')$. Choose $a_{n,m} > 0$ so that for every m and n we have

$$\mu(\{x \in X : 2^n a_{n,m} \|x\|_n^2 < 1\}) > 1 - 2^{-n-m}.$$

Then

$$\mu(\{x \in X : \sup_n 2^n a_{n,m} \|x\|_n^2 \leqslant 1\}) > 1 - 2^{-m},$$

and thus

$$\mu(\{x \in X\colon \sum_{n=1}^{\infty} a_{n,m}\|x\|_n^2 \leqslant 1\}) > 1-2^{-m}.$$

Selecting $a_n > 0$ in such a way that $\limsup_n a_n/a_{n,m} < \infty$ for each m, and setting $H \overset{\mathrm{df}}{=} \{x \in X\colon \|x\| < \infty\}$, where

$$\|x\| \overset{\mathrm{df}}{=} \Big(\sum_{n=1}^{\infty} a_n\|x\|_n^2\Big)^{1/2},$$

we obtain

$$H \supset \Big\{x \in X\colon \sum_{n=1}^{\infty} a_{n,m}\|x\|_n^2 \leqslant 1\Big\}$$

for every m so that $\mu(H) = 1$. Since $(H, \|\cdot\|)$ is complete and finer than $\big(X, \tau(X, X')\big)$, H constitutes a standard support of μ in (X, X', μ) and the proper standard support closed in $\big(H, (H, H')\big)$ provides the desired Hilbertian support, and the Proposition holds.

We have the following

COROLLARY 2.2. *Given a representation (X, X', μ) of a $(\mu) \in$ LPM and a sequence $\{H_n\}$ of Hilbertian supports of μ in (X, X', μ), there exists a proper Hilbertian support H of μ in (X, X', μ) contained in the intersection $\bigcap_1^{\infty} H_n$.*

Proof. The reduction of (X, X', μ) to $Z = \bigcap_1^{\infty} H_n$ fulfils the requirements of Proposition 2.6 so that we can produce the desired proper Hilbertian support, and the Corollary follows.

PROPOSITION 2.7. *Given a Hilbertian representation (H, H', μ) of a $(\mu) \in$ LPM, and a Hilbertian norm $\|\cdot\|_0$ defined and almost uniformly continuous on a proper standard support S of μ in (H, H', μ), there exists a finer Hilbertian representation (E, E', μ) of (μ) such that S is contained and dense in $\big(E, \tau(E, E')\big)$ and $\|\cdot\|_0$ is $\tau(E, E')$-continuous.*

Proof. Put $\|x\| \overset{\mathrm{df}}{=} (\|x\|_0^2 + \|x\|^2)^{1/2}$ for $x \in S$, where $\|\cdot\|$ is a Hilbertian norm inducing the topology $\tau(H, H')$. The identity on S extends to a continuous mapping of the completion $(S^-, \|\cdot\|_1)$ of $(S, \|\cdot\|_1)$ into $(H, \|\cdot\|)$. Denote by $E_1 \supset S$ the image of S^- in H. Clearly, E_1 is a Hilbertian support of μ in (H, H', μ) and it contains a proper Hilbertian support $E \supset S$. It is clear that the reduction of (H, H', μ) to E provides the desired representation (E, E', μ), and the Proposition holds.

PROPOSITION 2.8. *Given a separable Hilbertian representation* (H, H', μ) *of a* $(\mu) \in$ LPM, *there exists a finer separable Hilbertian representation* (E, E', μ) *of* (μ) *such that the identical injection of* $(E, \tau(E, E'))$ *into* $(H, \tau(H, H'))$ *is compact.*

Proof. Let us notice at first that given a double sequence of non-negative numbers $\{a_{k,m}\}$, such that for every m we have $\sum_{k=1}^{\infty} a_{k,m} < \infty$, we can find $1 < t_n \to \infty$ such that $\sum_{k=1}^{\infty} t_k a_{k,m} < \infty$ for all m. By taking $\{\sum_{i=1}^{m} a_{k,i}\}$ for $\{a_{k,m}\}$, we can always reduce it to the case of $\{a_{k,m}\}$, non-decreasing with respect to m. Then choosing an increasing $\{k_n\}$ such that $\sum_{k=k_n}^{\infty} a_{k,n} < 2^{-2n}$ and putting $t_k = 2^n$ for $k_n \leqslant k < k_{n+1}$, we find that $\{t_k\}$ constitutes a sequence as desired. Take now an orthonormal basis $\{e_n'\}$ in H' with respect to some Hilbertian norm $\| \ \|'$ inducing the topology $(H', \tau(H', H))$. Then $\|x\|^2 = \sum_{k=1}^{\infty} (e_k' x)^2 < \infty$ for every $x \in H$, and thus setting

$$a_{k,m} = \int_{K_m} (e_k' x)^2 d\mu(x)$$

for $K_m = \{x \in H\colon \|x\| < m\}$, we have $\sum_{k=1}^{\infty} a_{k,m} < \infty$ for every m, and hence for some $1 < t_n \to \infty$ we have $\sum_{k=1}^{\infty} t_k a_{k,m} < \infty$ for every m. Setting $\|x\|_0 = \big(\sum_{k=1}^{\infty} t_k (e_k' x)^2\big)^{1/2}$, we obtain $\int_{K_m} \|x\|_0^2 d\mu(x) < \infty$ for every m, and since $\mu(\bigcup_1^{\infty} K_m) = 1$, we get $\mu(H_0) = 1$, where $H_0 \overset{\mathrm{df}}{=} \{x \in H\colon \|x\|_0 < \infty\}$.

It is easy to see that H_0 is a Hilbertian support of μ in (H, H', μ), and the reduction of (H, H', μ) to a proper Hilbertian support contained in H_0 fulfils the requirements of the proposition. The proof is then complete.

3. Linear Lusin measurable functionals

Consider a representation (X, X', μ) of $(\mu) \in$ LPM. A pair $(u', D_{u'})$ consisting of a support $D_{u'}$ of μ in (X, X', μ) and a linear functional u' defined and almost uniformly continuous on $D_{u'}$, i.e. $u' \in (D_{u'})'$, is said to be a *representation in* (X, X', μ) *of a linear Lusin measurable functional.*

We say that pairs $(u', D_{u'})$ and $(v', D_{v'})$ represent the same linear Lusin measurable functional in (X, X', μ) if

$$\text{(e)} \qquad \mu(\{z \in D_{u'} \cap D_{v'}\colon u'z = v'z\}) = 1.$$

The series a) and the integral (*) from Section 1 provide examples of linear Lusin measurable functionals.

PROPOSITION 3.1. *Two pairs* $(u', D_{u'})$ *and* $(v', D_{v'})$ *represent the same linear Lusin measurable functional iff there exists a third representation* $(w', D_{w'})$ *with* (X, X')*-standard* $D_{w'}$ *such that* $D_{u'} \cap D_{v'} \supset D_{w'}$ *and such that* $w'z = u'z = v'z$ *for* $z \in D_{w'}$.

Proof. Clearly $D_{u'} \cap D_{v'}$ contains a standard support which decomposes into an ascending sequence $\{C_n\}$ of convex $\sigma(X, X')$-compacts. On each C_n, and thus as well on each $K_n \overset{\mathrm{df}}{=}$, the $\sigma(X, X')$-closure of $C_n \cap \{z\colon u'z = w'z\}$, both u' and v' are equal and $\sigma(X, X')$-continuous. Setting $D_{w'} = \bigcup_1^\infty K_n$ and $w'z = u'z = v'z$ for $z \in D_{w'}$, we obtain the desired representation $(w', D_{w'})$, and the proposition follows.

Equality (e) provides an equivalence relation between the representations $(u', D_{u'})$ and classes of equivalence with respect to this relation shall be called *linear Lusin measurable functionals*. The linear Lusin measurable functional represented by $(u', D_{u'})$ we shall denote by u^*. The set of all linear Lusin measurable functionals we shall denote by $\Lambda(\mu)$. It is easy to see that linear Lusin measurable functionals defined in two equivalent representations can be naturally identified so that $\Lambda(\mu)$ does not depend on the choice of the representation of a given (μ) depending only on (μ) itself. It is clear that $\Lambda(\mu)$ is a linear space, the addition $h = f + g$ with h having a representation $(u' + v', D_{u'} \cap D_{v'})$ for representations $(u', D_{u'})$ and $(v', D_{v'})$ of f and g respectively. The pseudonorm $s[\cdot]$,

$$s[f] = \int |fx|/(1 + |fx|)\, d\mu(x),$$

where $f \in \Lambda(\mu)$, makes a linear metric topological space $\big(\Lambda(\mu), s[\cdot]\big)$ out of $\Lambda(\mu)$.

Consider the mapping

$$X' \ni x' \to x^* \in \Lambda(\mu).$$

If (X, X', μ) is proper, the mapping constitutes a one-to-one injection of X' into $\Lambda(\mu)$. The image of X' in $\Lambda(\mu)$ by this injection shall be denoted by X^*, and the topology $\tau(X', X)$ of X' moved by the injection over X^* shall be denoted by τ_X^*. We have then

$$(X^*, \tau_X^*) \geqslant \big(\Lambda(\mu), s[\cdot]\big).$$

PROPOSITION 3.2. *Consider a* $(\mu) \in$ LPM. *If a sequence* $\{f_n\} \subset \Lambda(\mu)$ *converges almost everywhere, then there exists a standard representation* (X, X', μ) *of* (μ) *such that* $\{f_n\} \subset X^*$, *and that it converges in* (X^*, τ_X^*).

Proof. Take any representation (Z, Z', μ) of (μ) and representations (z_n', D_{z_n}') of f_n in (Z, Z', μ). Substituting, if necessary, (Z, Z', μ) by its

reduction to $\bigcap_{n=1}^{\infty} D_{\varepsilon_n'}$, we can have $\{z_n'\} \subset Z'$. Given $\varepsilon > 0$, by the Jegorov Theorem we find $W \in \mathscr{B}(Z, Z')$ with $\mu(W) > 1-\varepsilon$ such that $\{z_n'\}$ converges uniformly on W. Taking a convex $\sigma(Z, Z')$-compact C with $\mu(C) > 1-\varepsilon$, we have the sequence $\{z_n'\}$ uniformly convergent on the $\sigma(X, X')$-closure K of the convex hull of $W \cap C$ and $\mu(K) > 1-\varepsilon$. Take a proper standard support X of μ in (Z, Z', μ) on which $\{z_n'\}$ is almost uniformly convergent. The reduction of (Z, Z', μ) to X provides the desired representation.

COROLLARY 3.1. *Consider a* $(\mu) \in$ LPM *and a Cauchy sequence* $\{f_n\}$ *in* $(\Lambda(\mu), s[\cdot])$. *Then there exists a standard representation* $(X, X', \mu t$ *of* (μ) *such that* $\{f_n\} \subset X^*$, *and there exists a subsequence of* $\{f_n\}$ *convergen)* *in* $\{X^*, \tau_X^*)$.

Proof. It is sufficient to apply Riesz Theorem to extract an almost everywhere convergent subsequence of $\{f_n\}$ and then apply Proposition 3.2.

COROLLARY 3.2. *The space* $(\Lambda(\mu), s[\cdot])$ *is always complete.*

Proof. This is a direct consequence of Corollary 3.1.

PROPOSITION 3.3. *For every representation* (X, X', μ) *of a* $(\mu) \in$ LPM, *the space* X^* *is dense in* $(\Lambda(\mu), s[\cdot])$.

Proof. Given $f \in \Lambda(\mu)$, we take (Z, Z', μ) finer than (X, X', μ) and such that $f \in Z^*$. By Proposition 2.1, the restrictions to Z of elements of X' form a dense subspace of $(Z', \tau(Z', Z))$, and thus X^* must be dense in $(Z^*, s[\cdot])$ as well.

PROPOSITION 3.4. *Consider a* $(\mu) \in$ LPM. *If* $(\Lambda(\mu), s[\cdot])$ *is separable, then every primitive representation of* (μ) *must be separable.*

Proof. Up to an isomorphism we can write a primitive representation of (μ) in the form (R^T, R_0^T, μ), where $f \in R_0^T$ acts as a functional on R^T assigning to $g \in R^T$ the number $fg = \sum_{t \in T} f(t)g(t)$. From the separability assumption we conclude that there exists a sequence $\{f_n\} \subset R_0^T$ of functionals such that every other functional is the almost everywhere limit of a subsequence of $\{f_n\}$. Let T_1 be a countable subset of T such that all $f_n g$ vanish for $g \in R^T$ with the support disjoint with T_1. Take any $f_0 \in R_0^T$ vanishing on $T - \{t_0\}$ and equal to 1 on $t_0 \in T - T_1$. For every b and $\varepsilon > 0$ it is

$$\{g \in R^T\colon |(f_n - f_0)g| < \varepsilon\} \subset \{g \in R^T\colon |g(t_0)| < \varepsilon\}.$$

Hence, if $\{f_{m_n}\}$ approximates f_0 almost everywhere, it must be

$$\mu(\{g \in R^T\colon |g(t_0)| < \varepsilon\}) \geqslant \mu(\{g \in R^T\colon |(f_{m_n} - f_0)g| < \varepsilon\}),$$

and since the right-hand side tends to 1 with respect to n, we have

$$\mu(\{g \in R^T\colon |g(t_0)| < \varepsilon\}) = 1$$

for every $\varepsilon > 0$ so that $\mu(\{g \in R^T\colon g(t_0) = 0\}) = 1$, and thus the functional f_0 vanishes almost everywhere. Hence, since the representation (R^T, R_0^T, μ) is proper, f_0 vanishes identically, which leads to a contradiction. Hence $T = T_1$, and the considered representation is separable. This concludes the proof of Proposition 3.4.

COROLLARY 3.3. *A $(\mu) \in$ LPM is separable if and only if $(\Lambda(\mu), s[\cdot])$ is separable.*

Proof. This is an immediate consequence of Theorem 2.1, Proposition 3.3 and Proposition 3.4.

COROLLARY 3.4. *Proper representation of a separable $(\mu) \in$ LPM is always separable.*

Proof. It follows directly from Theorem 2.1 and Propositions 2.1 and 3.4.

THEOREM 3.1. *Given a Hilbertian representation (H, H', μ) of a $(\mu) \in$ LPM and a sequence $\{f_n\} \subset H^*$ tending to zero in $(\Lambda(\mu), s[\cdot])$, there exists a finer Hilbertian representation (E, E', μ) of (μ) such that $\{f_n\} \subset E^*$, and a subsequence of $\{f_n\}$ tending to zero in (E^*, τ_E^*).*

To prove this Theorem we shall need a lemma from the general theory of measure.

LEMMA 3.1. *If a sequence $\{f_n\} \subset \Lambda(\mu)$ fulfils the Cauchy condition in $(\Lambda(\mu), s[\cdot])$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that*

$$\mu(\{x \in X\colon \sum_{n=1}^{\infty} 2^{pn/2} |g_{n+1}x - g_n x|^p < \infty\}) = 1.$$

Proof. The Cauchy condition assures that to every n and q there corresponds a k such that $\mu(\{x \in X\colon |f_i x - f_j x| < 2^{-2n}\}) > 1 - 2^{-n-q}$ for $i, j > k$. Hence, for every q we can find an increasing sequence $\{k_n^q\colon n = 1, 2, \ldots\}$ such that

$$\mu(\{x \in X\colon |f_{k_{n+1}^q} x - f_{k_n^q} x| < 2^{-n}\}) > 1 - 2^{-n-q}$$

for every p. Moreover, we can choose it so that $\{k_n^{q+1}\colon n = 1, 2, \ldots\}$ is a subsequence of $\{k_n^q\colon n = 1, 2, \ldots\}$. Put $k_n \overset{\mathrm{df}}{=} k_n^n$, $n = 1, 2, \ldots$ Then for every q, $\{k_n\colon n \geqslant q\}$ is a subsequence of $\{k_n^q\colon n = 1, 2, \ldots\}$ and there exists an m_n depending on q, $n < m_n < m_{n+1}$, such that $k_n = k_{m_n}^q$ for $n \geqslant q$. Fix an arbitrary q, take $n \geqslant q$ and put $r_{n,i} \overset{\mathrm{df}}{=} k_{m_n+i}^q$. Define also $g_n = f_{k_n}$ for $n = 1, 2, \ldots$ We have

$$|g_{n+1}x - g_n x| = |f_{r_{n+1,0}} x - f_{r_{n,0}} x - f_{r_{n,0}} x| \leqslant \sum_{i=1}^{m_{n+1}-m_n} |f_{r_{n,i}} x - f_{r_{n,i-1}} x|.$$

Since

$$\sum_{i=1}^{m_{n+1}-m_n} 2^{-m_n-i} < 2^{-m_n} < 2^{-n},$$

we have

$$B_n = \bigcap_{i=1}^{m_{n+1}-m_n} \{x \in X\colon\ |f_{r_{n,i}}x - f_{r_{n,i-1}}x| < 2^{-m_n-i}\}$$
$$\subset \{x \in X\colon\ |g_{n+1}x - g_n x| < 2^{-n}\}$$

and

$$\mu(B_n) \geqslant 1 - \sum_{i=1}^{m_{n+1}-m_n} 2^{-m_n-1-q} > 1 - 2^{-n-q}$$

so that

$$\mu(\{x \in X\colon\ |g_{n+1}x - g_n x| < 2^{-n}\}) > 1 - 2^{-n-q}$$

for $n \geqslant q$. We have

$$A_q \overset{\mathrm{df}}{=} \Big\{x \in X\colon\ \sum_{n=q}^{\infty} 2^{pn/2}|g_{n+1}x - g_n x|^p < \infty\Big\} \supset \bigcap_{n=q}^{\infty} \{x \in X\colon\ |g_{n+1}x - g_n x| < 2^{-n}\}$$

so that

$$\mu(A_q) > 1 - 2^{-q} \quad \text{for every } q.$$

However, $\mu(A_1) = \mu(A_q)$ for $q = 1, 2, \ldots$, and the lemma follows.

Proof of Theorem 3.1. Applying Lemma 3.1, we find a subsequence $\{g_n\}$ of $\{f_n\}$ with the representations $\{z_n'\} \subset H'$ fulfilling the condition

$$\mu\Big(\Big\{x \in H\colon\ \sum_{n=1}^{\infty} 2^n |z_{n+1}'x - z_n'x|^2 < \infty\Big\}\Big) = 1.$$

Define

$$\|x\|_0 = \Big(\|x\|^2 + \sum_{n=1}^{\infty} 2^n |z_{n+1}'x - z_n'x|^2\Big)^{1/2},$$

where $\|\cdot\|$ is a Hilbertian norm inducing the topology $\tau(H, H')$ and put

$$H_0 = \{x \in H\colon\ \|x\|_0 < \infty\}.$$

Clearly, $(H_0, \|\cdot\|_0)$ is complete, i.e. it is a Hilbert space, $H_0 \in \mathscr{B}(H, H')$ and $\mu(H_0) = 1$. Take a proper Hilbertian support E closed in $(H_0, \|\cdot\|_0)$ and denote by y_n' the restrictions of z_n' to E. The reduction (E, E', μ) of (H, H', μ) to E fulfils the requirements of the theorem, which concludes the proof.

Consider a $(\mu) \in$ LPM. As usual, we shall say that a sequence $\{f_n\}$ is bounded in $\big(\Lambda(\mu), s[\cdot]\big)$ if for every sequence of numbers tending to zero $\{t_n\}$, we have $\lim_n s[t_n f_n] = 0$.

THEOREM 3.2. *A sequence $\{f_n\} \subset \Lambda(\mu)$ is bounded almost everywhere if and only if there exists a standard representation (X, X', μ) of (μ) such that $\{f_n\}$ is bounded in (X^*, τ_X^*).*

Proof. Consider a standard representation (X, X', μ) of (μ). If a sequence $\{x_n'\} \subset X'$ is bounded in $\big(X', \tau(X', X)\big)$, then $\{x_n^*\}$ is certainly bounded almost everywhere which verifies the sufficiency. Suppose now that $\{f_n\}$ is a sequence bounded almost everywhere. Then it follows that $\mu\big(\bigcup_{m=1}^{\infty} \bigcap_{i=1}^{\infty} \{x\colon |f_i x| \leqslant m\}\big) = 1$. Take a standard representation (Z, Z', μ) with $\{f_n\} \subset Z^*$. Then for the representations $z_n' \in Z'$ of f_n we have

$$\mu\Big(\bigcup_{m=1}^{\infty} \bigcap_{i=1}^{\infty} \{x \in Z\colon |z_i' x| \leqslant m\}\Big) = 1.$$

Therefore, we can find $\sigma(Z, Z')$-compact convex sets C of the measure arbitrarily close to one such that on every such C all z_n' are bounded by the same number. Hence, we can also select a proper standard support X of μ in (Z, Z', μ) such that on every $\sigma(Z, Z')$-compact convex subset of X, all z_n' are bounded by the same number. Taking the reduction (X, X', μ) of (Z, Z', μ), we find that $\{f_n\}$ is bounded in (X^*, τ_X^*), and the theorem follows.

Given $(\mu) \in$ LPM and p, $1 \leqslant p < \infty$, we write for $f \in \Lambda(\mu)$

$$s_p[f] = \Big(\int |fx|^p \, d\mu(x)\Big)^{1/p}$$

and

$$\Lambda_p(\mu) \stackrel{\mathrm{df}}{=} \{f \in \Lambda(\mu)\colon s_p[f] < \infty\}.$$

Clearly, $\big(\Lambda_p(\mu), s_p[\cdot]\big)$ is a Banach space, and

$$\big(\Lambda_p(\mu), s_p[\cdot]\big) \geqslant \big(\Lambda(\mu), s[\cdot]\big).$$

Hence, from the closed graph theorem it follows that for every standard representation (U, U', μ) of (μ), if $U^* \subset \Lambda_p(\mu)$, then $(U^*, \tau_U^*) \geqslant \big(\Lambda_p(\mu), s_p[\cdot]\big)$.

LEMMA 3.2. *Consider a primitive representation (Z, Z', μ) and a finer standard representation (U, U', μ) of a separable $(\mu) \in$ LPM. Let $Z^* \subset \Lambda_p(\mu)$ for a fixed $1 \leqslant p \leqslant 2$. Given $\{u_n'\} \subset U'$ with $\{u_n^*\} \subset \Lambda_p(\mu)$, there exists a Hilbertian representation (H, H', μ) of (μ) which is finer than (Z, Z', μ) and coarser than (U, U', μ) such that $\{u_n^*\} \subset H^* \subset \Lambda_p(\mu)$.*

Proof. By Theorem 2.2 there exists a Hilbertian representation (E, E', μ) finer than (Z, Z', μ) and coarser than (U, U', μ) such that u_n' are all $\tau(E, E')$-continuous. Since Z^* is contained in $\Lambda_p(\mu)$ and the restriction to U of elements from Z' constitute a dense subset of $(U', \tau(U', U))$, we can always add some elements to $\{u_n'\}$ so that it spans a dense subspace of $(U', \tau(U', U))$. Then the extensions v_n' of u_n' span a dense subspace of $(E', \tau(E', E))$. Let $\{e_n'\}$ denote the Gram-Schmidt orthonormalization of $\{v_n'\}$ with respect to a Hilbertian norm inducing the topology $\tau(E', E)$. Put

$$\|x\|_1 = \Big(\sum_1^\infty (a_n e_n' x)^2\Big)^{1/2}$$

and

$$H_1 = \{x \in E\colon \ \|x\|_1 < \infty\}.$$

Choosing $a_n > 0$ tending sufficiently fast to zero, we can have $\mu(H_1) = 1$, and since $(H_1, \| \ \|_1)$ is complete, H_1 constitutes a Hilbertian support of μ in (E, E', μ). To every $x' \in H_1'$ there corresponds a $\{t_n\}$, $\sum_1^\infty t_n^2 < \infty$, such that for every $x \in H_1$

$$x'x = \sum_1^\infty t_n a_n e_n' x,$$

and then

$$|x'x| \leqslant M\Big(\sum_1^\infty |a_n e_n' x|^p\Big)^{1/p},$$

where

$$M = \begin{cases} \sup\limits_n |t_n| & \text{for} \quad p = 1, \\ \Big(\sum\limits_1^\infty |t_n|^q\Big)^{1/p}, \ 1/p + 1/q = 1, & \text{otherwise.} \end{cases}$$

Hence, we have

$$\int |x'x|^p d\mu(x) \leqslant M^p \sum_1^\infty a_n^p \int |e_n' x|^p d\mu(x),$$

and adjusting the convergence of $\{a_n\}$ to zero to have the right-hand side finite, we arrive to $H_1^* \subset \Lambda_p(\mu)$. Taking as H the proper Hilbertian support closed in $(H_1, \|\cdot\|_1)$, we find that the reduction of (E, E', μ) to H fulfils the requirements of the lemma.

THEOREM 3.3. *Consider a separable* $(\mu) \in$ LPM *and a proper representation* (X, X', μ) *of* (μ). *If for given* p, $1 \leqslant p \leqslant 2$, *we have* $X^* \subset \Lambda_p(\mu)$, *then* X^* *is dense in* $\big(\Lambda_p(\mu), s_p[\cdot]\big)$.

Proof. By Theorem 2.1 it is sufficient to prove it for primitive (X, X', μ). By virtue of Lemma 3.2 to given $f \in \Lambda_p(\mu)$, we can assign a Hilbertian representation (H, H', μ) of (μ) such that $f \in H^* \subset \Lambda_p(\mu)$ and such that $X^* \subset H^*$. But X^* is dense in (H^*, τ_H^*), and the theorem follows.

4. Pre-supports and a modification of the definition of the linear probability measure

Consider a representation (X, X', μ) of a $(\mu) \in$ LPM and a linear subset $U \subset X$. We say that U is a pre-support of μ in (X, X', μ) if the following condition holds:

(i) To every $\varepsilon > 0$ there corresponds a convex $\sigma(X, X')$-compact $C \subset U$ such that for every x' from the polar C^0 of C in (X, X') we have $\mu(\{x \in X\colon |x'x| \leqslant 1\}) > 1 - \varepsilon$.

It is easy to see that (i) is equivalent to the following condition:

(i') There exists an (X, X')-standard subspace $S \subset U$ such that if $\{x_n'\} \subset X'$ converges to zero almost uniformly on S, then $\{x_n^*\}$ converges to zero in $\big(\Lambda(\mu), s[\cdot]\big)$. A pre-support S of μ in (X, X', μ) is said to be *proper* if the following condition holds:

(ii) Given $\{x_n'\}$ tending (X, X')-almost uniformly on U to some u'. If $\{x_n'\}$ tends almost everywhere to zero, then u' vanishes identically on U.

It is easy to see that for standard U condition (ii) is equivalent to the following one:

(ii') Consider u' from the (X, X')-adjoint U' of U. If there exists a representation $(y', D_{y'})$ in (X, X', μ) of a linear Lusin measurable functional equivalent to zero, i.e. with y' almost everywhere zero such that $U \subset D_{y'}$ and such that the restriction of y' to U coincides with u', then u' is identically zero on U.

The equivalence easily follows from Proposition 3.2 and from Corollary I.1.5 of [1].

PROPOSITION 4.1. *Consider a representation* (X, X', μ) *of a* $(\mu) \in$ LPM *and let* $U \subset X$ *be a standard pre-support of* μ *in* (X, X', μ). *Then*

1) $\big(U', \tau(U', U)\big)$ *is a Fréchet space,*

2) *to every* u' *from the* (X, X')*-adjoint* U' *of* U *there corresponds exactly one* $u^* \in \Lambda(\mu)$ *which admits a representation* $(y', D_{y'})$ *in* $(X, X',)$ *with* $U \subset D_{y'}$ *and* y' *identical with* u' *on* U,

3) *the restriction to* U *of* $x' \in X'$ *constitutes a continuous injection of* $(X', \tau(X', X))$ *onto a dense subspace of* $(U', \tau(U', U))$,

4) *if* $\{x'_n\} \subset X'$ *approximates almost uniformly on* U *some* $u' \in U'$, *then there exists a standard support of* μ *in* (X, X', μ) *on which* $\{x'_n\}$ *converges almost uniformly to some extension* y' *of* u',

5) *if* U *is proper, and some* $V \subset U$ *is another pre-support of* μ, *then* V *is dense in* $(U, \tau(U, U'))$.

Proof. We notice first that 2) implies easily 5). Indeed, if V is not dense in $(U, \tau(U, U'))$, then there is a $u' \in U'$ which vanishes on V and does not vanish on U so that its extension to some $u^* \in \Lambda(\mu)$ is different from zero. But this is also an extension of u' considered on V, and thus if V is a pre-support of V, such an extension being unique must be identically zero which contradicts $u^* \neq 0$. This shows that 2) implies 5). Given any decomposition of U into an increasing sequence of $\sigma(X, X')$-compact symmetric convex subsets, any other convex $\sigma(X, X')$-compact subset must be absorbed by at least one of the subsets from the decomposition. It follows easily from completness of normed spaces spanned by $\sigma(X, X')$-compact convex symmetric subsets and the Baire categories. Then the Mackey theorem about dual pairs yields metrizability of $(U', \tau(U', U))$. The completness is obvious. This concludes the proof of 1) to prove 2) we use Corollary I.1.5 of [1] to approximate $u' \in U'$ by a sequence $\{x'_n\} \subset X'$ converging almost uniformly on U. Due to (i), we have $\{x'_n\}$ converging in measure to some Lusin measurable functional. Passing if necessary to a subsequence, we can have by Proposition 3.2 the almost uniform convergence of $\{x'_n\}$ on a standard support S of μ in (X, X', μ), and since the convergence shall be still almost uniform on $S + U$, we can at once assume that $S \supset U$, and thus 4) follows. The limit of $\{x'_n\}$ provides the desired extension u^* of u'. If $(y', D_{y'})$ is a representation of a linear Lusin measurable functional with $D_{y'} \supset U$ and y' vanishing on U, we can approximate y' almost uniformly on $D_{y'}$ by a sequence $\{x'_n\} \subset X'$, and then, due to (i), $\{x'_n\}$ tends stochastically to zero and $y^* = 0$, which verifies the uniqueness of the extension in $\Lambda(\mu)$ and concludes the proof of 2). The continuity of the injection $X' \to U'$ is obvious, and if for $u \in U$ we have $x'u = 0$ for all $x' \in X'$, then $u = 0$ and 3) follows. This concludes the proof of the proposition.

PROPOSITION 4.2. *Consider a proper representation* (X, X', μ) of *a* $(\mu) \in$ LPM *and a Fréchet space* (Θ, ϑ) *such that* $(X^*, \tau_X^*) \geqslant (\Theta, \vartheta) \geqslant (\Lambda(\mu), s[\cdot])$ *and such that* X^* *is dense in* (Θ, ϑ). *Then there exists a standard proper pre-support* U *of* μ *in* (X, X', μ) *such that the extensions of* $u' \in U'$ *to functionals from* $\Lambda(\mu)$ *constitutes an isomorphism of* $(U', \tau(U', U))$ *onto* (Θ, ϑ).

Proof. Let Θ' be the adjoint of (Θ, ϑ). Every $F \in \Theta'$ is also continuous in (X^*, τ_X^*) so that there exists $x_F \in X$ such that $Fx^* = x'x_F$ for every

$x^* \in X^*$. Since X^* is dense in (Θ, ϑ), $Fx^* = 0$ for all $x^* \in X^*$ implies $F = 0$, and thus the mapping

$$\Theta' \ni F \to I(F) = x_F \in X$$

constitutes a one-to-one continuous imbedding of the space $(\Theta', \sigma(\Theta', \Theta))$ into $(X, \sigma(X, X'))$. Indeed, for $x_1', \ldots, x_k' \in X'$,

$$I^{-1}(\{x \in X\colon |x_i' x| < 1,\ i = 1, \ldots, k\}) = \{F \in \Theta'\colon |Fx_i^*| < 1,\ i = 1, \ldots, k\}.$$

Hence, it is as well bicontinuous, and $\sigma(\Theta', \Theta)$-compact sets in Θ' correspond by I to $\tau(X, X')$-compact convex sets in the image of U of Θ' by I. Thus, in particular, U is (X, X')-standard. Moreover, if $\{x_n'\} \subset X'$ tends almost uniformly to zero on U, then $\{x_n^*\}$ tends to zero almost uniformly on $(\Theta', \sigma(\Theta', \Theta))$ which means exactly that $\{x_n^*\}$ tends to zero in (Θ, ϑ) and consequently in $(\Lambda(\mu), s[\cdot])$ which proves (i), i.e. that U is a pre-support of μ. Suppose now that $\{x_n'\}$ tends to some u' almost uniformly on U and that $s[x_n^*]$ tends to zero. Then as above we have $\{x_n^*\}$ converging to some f in (Θ, ϑ) and thus in $(\Lambda(\mu), s[\cdot])$ as well. But in the latter space it tends to zero so that $f = 0$ and $\{x_n^*\}$ tends to zero in (Θ, ϑ). Then for $u = X_F \in U$ we have $u'u = \lim x_n' u = \lim Fx_n^* = 0$ and u' vanishes identically, which proves (ii) and at the same time verifies that the pre-support U is proper. According to Proposition 4.1, extensions of $u' \in U'$ are limits of sequences of elements of X' converging almost uniformly on U. Since such sequences converge also in (Θ, ϑ), the extensions are in Θ. To prove the converse, take any $f \in \Theta$ and approximate it by $\{x_n^*\} \subset X^*$ in (Θ, ϑ). Then $\{x_n'\}$ converges almost uniformly on U, and by 4) of Proposition 4.1 also almost uniformly on some standard $S \supset U$ with $\mu(S) = 1$, and hence to an extension of some $u' \in U'$. Clearly, the Lusin measurable linear functional represented by this extension must be equal to f. Hence the extensions of functionals from U' to representations of Lusin measurable functionals map U' onto Θ. This mapping is certainly one-to-one and constitutes an isomorphism of $(U', \tau(U', U))$ and (Θ, ϑ), which concludes the proof.

Consider a proper representation (X, X', μ) of a $(\mu) \in$ LPM. By $\mathscr{F}(X, X', \mu)$ we shall denote the family of all Fréchet spaces (Θ, ϑ) such that

$$(\Lambda(\mu), s[\cdot]) \leqslant (\Theta, \vartheta) \leqslant (X^*, \tau_X^*)$$

and such that X^* is dense in (Θ, ϑ). Furthermore, we shall denote by $\mathscr{S}(X, X', \mu)$ the family of all standard pre-supports of μ in (X, X', μ) and by $\mathscr{P}(X, X', \mu)$ the subfamily of all proper standard pre-supports of μ in (X, X', μ). For any $U \in \mathscr{S}(X, X', \mu)$ we shall denote by U^* the space of all $u^* \in \Lambda(\mu)$ admitting a representation $(y', D_{y'})$ in (X, X', μ) such that $U \subset D_{y'}$. Then, by the definition of the representations, we have y' automatically (X, X')-almost uniformly continuous on U, and then its restriction to U belongs to the (X, X')-adjoint U' of U.

THEOREM 4.1. *Consider $U \in \mathscr{S}(X, X', \mu)$. Then every u' from the adjoint U' of U can be extended to some $u^* \in U^*$. Such an extension is unique in $\Lambda(\mu)$, and for $U \in \mathscr{P}(X, X', \mu)$ the mapping*

$$U' \ni u' \to u^* \in U^*$$

constitutes an isomorphism of U' and U^*.

Proof. It follows directly from (ii) and Proposition 4.1.

Let us provide U^* with the topology τ_U^*, the shift of $\tau(U', U)$ by the mapping $u' \to u^*$. Then from (i) it follows that

$$(U^*, \tau_U^*) \geqslant \big(\Lambda(\mu), s[\cdot]\big),$$

and Lemma 4.1 guarantees that $(U^*, \tau_U^*) \in \mathscr{F}(X, X', \mu)$. Hence we have produced a mapping

$$\mathscr{S}(X, X', \mu) \ni U \to (U^*, \tau_U^*) \in \mathscr{F}(X, X', \mu).$$

THEOREM 4.2. *The mapping * restricted to $\mathscr{P}(X, X', \mu)$ constitutes a monotone isomorphism of $\mathscr{P}(X, X', \mu)$ and $\mathscr{F}(X, X', \mu)$. The monotonicity means that for $W, V \in \mathscr{P}(X, X', \mu)$,*

$$W \subset V \text{ is equivalent to } (W^*, \tau_W^*) \leqslant (V^*, \tau_V^*).$$

Proof. From Proposition 4.2 it follows that * is onto. We shall now prove that it is a monomorphism. Since for $V, W \in \mathscr{P}(X, X', \mu)$ we trivially have $V+W \in \mathscr{P}(X, X', \mu)$, it is sufficient to consider $V \subset W$ with $(V^*, \tau_V^*) = (W^*, \tau_W^*)$. Given $v' \in V'$; by Corollary I.1.5 of [1] we can find $\{x_n'\} \subset X'$ approximating v' almost uniformly on V and thus converging in $(V^*, \tau_V^*) = (W^*, \tau_W^*)$ and hence as well in $\big(W', \tau(W', W)\big)$ to some w'. Thus every $v' \in V'$ is the restriction of some $w' \in W'$ while the restriction of every $w' \in W'$ belongs to V'. This means that V is closed in $\big(W, \tau(W, W')\big)$, and if it were $V \neq W$, some $w' \in W'$ different from zero would vanish identically on V, and this contradicts the uniqueness of extensions to Lusin measurable functionals. Hence, $*$ is an isomorphism onto. It remains to prove that it is monotone. If for $W, V \in \mathscr{P}(X, X', \mu)$ we have $V \supset W$, then of course $\big(V', \tau(V', V)\big) \geqslant \big(W', \tau(W', W)\big)$, and consequently $(V^*, \tau_V^*) \geqslant (W^*, \tau_W^*)$. Conversely, assume the latter. Since $W+V \in \mathscr{S}(X, X', \mu)$, we have $(W+V)^* \subset V^*$ Take an arbitrary $v^* \in V^*$ and let $\{x_n'\} \subset X'$ approximate almost uniformly on V, the $v' \in V'$ corresponding to v^*. Since $\{x_n'\}$ converges in (V^*, τ_V^*), it also converges in (W^*, τ_W^*) and thus in $\big(W', \tau(W', W)\big)$ which means that the limit of it belongs to $(W+V)'$ after restricting to $W+V$. But its restriction to V belongs to V' and thus $V^* = (W+V)^*$. Hence $V = W+V$, and the monotonicity follows. This way the proof of Theorem 4.2 is concluded.

Take two Fréchet spaces $(Y^*, \varrho^*), (Z^*, \delta^*) \geqslant (\Lambda(\mu), s[\cdot])$. We define the inductive limit

$$(Y^*, \varrho^*) \wedge (Z^*, \delta^*) = (Y^* + Z^*, \varrho^* \wedge \delta^*),$$

setting for $\varrho^* \wedge \delta^*$ the finest locally convex topology of $Y^* + Z^*$ which is coarser than ϱ^* and δ^* on Y^* and Z^* respectively. Since $(Y^*, \varrho^*) \wedge (Z^*, \delta^*)$ identifies with the factor space $[(Y^*, \varrho^*) \times (Z^*, \delta^*)]/\{(y^*, z^*): y^* + z^* = 0\}$, we have

$$(Y^*, \varrho^*), (Z^*, \delta^*) \in \mathscr{F}(X, X', \mu) \text{ implies } (Y^*, \varrho^*) \wedge (Z^*, \delta^*) \in \mathscr{F}(X, X', \mu).$$

Thus, we have the following

THEOREM 4.3. *The family $\mathscr{P}(X, X', \mu)$ with the inclusion ordering form a lattice.*

Proof. We know that for $W, V \in \mathscr{P}(X, X', \mu)$ the space $W + V \supset W, V$ constitutes the least upper bound. Now, since $(W^*, \tau_W^*) \wedge (V^*, \tau_V^*) \in \mathscr{F}(X, X', \mu)$, there exists a U such that $(U^*, \tau_U^*) = (W^*, \tau_W^*) \wedge (V^*, \tau_V^*)$ = the greatest lower bound of (W^*, τ_W^*) and (V^*, τ_V^*). Hence, by monotonicity of * we find that U is the greatest lower bound of W and V in $\mathscr{P}(X, X', \mu)$ which concludes the proof.

Notice, that in general, the greatest lower bound of V and W in $\mathscr{P}(X, X', \mu)$ is only contained and not equal to $W \cap V$. Clearly, $W \cap V$ always belongs to $\mathscr{S}(X, X', \mu)$.

THEOREM 4.4. *To every $Y \in \mathscr{S}(X, X', \mu)$ there corresponds a $U \in \mathscr{P}(X, X', \mu)$ contained and closed in $(Y, \tau(Y, Y'))$. Moreover, such U is maximal, i.e. if $V \subset Y$ and $V \in \mathscr{P}(X, X', \mu)$, then $V \subset U$.*

Proof. Take (Y^*, τ_Y^*) and by Theorem 4.2 produce $U \in \mathscr{P}(X, X', \mu)$ such that $(U^*, \tau_U^*) = (Y^*, \tau_Y^*)$. Take $y' \in Y'$ and let $\{x_n'\} \subset X'$ approximate it almost uniformly on Y. Then $\{x_n^*\}$ must approximate y^* in $(Y^*, \tau_Y^*) = (U^*, \tau_U^*)$, and thus $\{x_n'\}$ converges almost uniformly on U so that its limit gives an extension of y' to a functional in $(U + Y)'$. Therefore Y is closed in $(U + Y, \tau(U + Y, (U + Y)'))$. Analogically, we can show that U is closed in this space. Now, suppose that U is not contained in Y. Then there exists a functional $z' \in (U + Y)'$ which vanishes on Y and is not identically zero on U. Its extension to z^* must be almost everywhere vanishing since it vanishes on the pre-support Y. This, however, contradicts (ii). If V is another proper presupport of μ contained in Y, then $V + U$ is also proper and contained in Y so that U must be dense in $V + U$. Since U is closed, it is $V + U = U$, and the theorem follows.

Two representations (X, X', μ) and $(Y, Y', \tilde{\mu})$ of linear probability measures are said to be *pre-equivalent* if there exists a linear space S such that

a) S is at the same time a linear subspace of X and a linear subspace of Y,

b) S is a standard pre-support for both of the representations,

c) S admits the same convex compact sets in the topologies $\sigma(X, X')$ and $\sigma(Y, Y')$, and these topologies coincide on the compacts,

d) for any $s_1', \ldots, s_n' \in S'$ and any Borel set B in R^n we have

$$\mu\big(\{x\colon (u_1'x, \ldots, u_n'x) \in B\}\big) = \tilde{\mu}\big(\{y\colon (v_1'x, \ldots, x_n'x) \in B\}\big),$$

where $u_1', \ldots, u_n'$ and $v_1', \ldots, v_n'$ are extensions of $s_1', \ldots, s_n'$ to linear Lusin measurable functionals in (X, X', μ) and $(Y, Y', \tilde{\mu})$ respectively. S satisfying a), b), c) and d) shall be said to *carry the pre-equivalence*. Notice, that by virtue of Theorem 4.4, S can always be chosen proper. Clearly, the relation is symmetric, reflexive and transitive, and thus it is an equivalence relation.

THEOREM 4.5. *If two separable representations (X, X', μ) and $(Y, Y', \tilde{\mu})$ are pre-equivalent and S is a pre-support which carries the pre-equivalence, then there exist linear subspaces $U \subset X$ and $V \subset Y$, $\mu(U) = \tilde{\mu}(V) = 1$, (X, X')- and (Y, Y')-standard, and containing S as $\sigma(X, X')$- and $\sigma(Y, Y')$-dense subset respectively, such that the identity on S extends to an almost uniformly continuous isomorphism J of U onto V transforming the measure $\tilde{\mu}$ into the measure μ, i.e. such that $\mu = \tilde{\mu}J$ and $\tilde{\mu} = \mu J^{-1}$ over the Borel fields of U and V respectively.*

Therefore, if we identify U and V by means of J, the representations (X, X', μ) and $(Y, Y', \tilde{\mu})$ become equivalent in the originally introduced sense.

Proof. Let us first notice that it is sufficient to verify the case where the representations (X, X', μ) and $(Y, Y', \tilde{\mu})$ are primitive. It is clear that if (X, X', μ) and $(Y, Y', \tilde{\mu})$ are pre-equivalent, then their reductions are also pre-equivalent. Hence we can assume the considered pre-equivalent representations to be proper. Then denote by (X_1, X_1', μ) and $(Y_1, Y_1', \tilde{\mu})$ primitive representations respectively coarser than (X, X', μ) and $(Y, Y', \tilde{\mu})$. Clearly, these primitive representations are pre-equivalent provided the original ones are, while as a set carrying the pre-equivalence of the primitive representations we can take any proper S which carries the pre-equivalence of the original representations. Now, if the Theorem is valid in the primitive case, we can find $U_1 \subset X_1$ and $V_1 \subset Y_1$ both containing S and fulfilling the requirements of the theorem. Denote by J_1 the extension of the identity from S. Let U be a proper standard support of μ in (X, X', μ) contained in $J_1^{-1}(Y \cap V_1)$ and containing S. Then put $V = J_1 U$ and let J be the restriction of J_1 to U. Since J_1 is almost uniformly continuous, V is standard, and both J and J^{-1} are almost

uniformly continuous between $(U, \sigma(U, U'))$ and $(V, \sigma(V, V'))$. Since $\mu(U) = 1$, we have $\tilde{\mu}(V) = 1$, $\mu = \tilde{\mu}J$ and $\tilde{\mu} = \mu J^{-1}$. Hence also V is a standard support in $(Y, Y', \tilde{\mu})$. Since both U and V are standard support, S must be dense as required, and thus U, V and J fulfils the requirements of the theorem. To conclude the proof it remains to verify the theorem under the assumption that (X, X', μ) and $(Y, Y', \tilde{\mu})$ are primitive. Then both X' and Y' are countably generated, and thus using for instance the part 4 of Proposition 4.1, we can find proper supports Q and R in (X, X', μ) and $(Y, Y', \tilde{\mu})$ respectively, such that $S \subset Q$ and $S \subset R$ and such that every $y' \in Y'$ restricted to S extends to some $v' \in Q'$ and every $x' \in X'$ restricted to S extends to some $u' \in R'$. Now, denote by Z' the subspace of all $s' \in S'$ which extend simultaneously to some functional from Q' and to some functional from R'. Hence, Z' can be simultaneously identified with $Z_X^* \subset \Lambda(\mu)$ and with $Z_Y^* \subset \Lambda(\mu)$ belonging to $\mathscr{F}(X, X', \mu)$ and $\mathscr{F}(Y, Y', \tilde{\mu})$ respectively. Applying Theorem 4.2, we find U and V from $\mathscr{P}(X, X', \mu)$ and $\mathscr{P}(Y, Y', \tilde{\mu})$ respectively such that $U^* = Z_X^* \subset Q^*$ and $V^* = Z_Y^* \subset R^*$. Due to the monotonicity of * we have $S \subset Q \subset U$ and $S \subset R \subset V$, so that both U and V are proper supports. Hence S is dense in $(U, \tau(U, U'))$ and $(V, \tau(V, V'))$. Defining the mapping J' of V' onto U', which assigns to v' the extension of U of its restriction to S, we obtain an isomorphism of $(V', \tau(V', V))$ and $(U', \tau(U', U))$, the adjoint J of which provides us with the desired isomorphism of U onto V. This concludes the proof of Theorem 4.5.

It is easy to see that every two representations, which are equivalent, are pre-equivalent as well. Then, by virtue of Theorem 4.5, we obtain equivalence of two pre-equivalent representations if we make certain natural identifications. Hence, the difference between classes of equivalence and classes of pre-equivalence of representations of separable linear probability measures is based only on the manner in which we establish elements of linear spaces to be identical not influencing the space $(\Lambda(\mu), s[\cdot])$.

Hence, from now on we shall consider separable linear probability measures as classes of pre-equivalence, and then in the next chapter we shall place distinction between pre-equivalence and equivalence introducing so-called pre-representations of linear probability measures.

The following corollary generalizes the example of series a) from Section 1 to arbitrary linear probability measures.

COROLLARY 4.1. *Consider a primitive representation* $(R^\infty, R_0^\infty, \mu)$ *of a* $(\mu) \in$ LPM, *a pre-support* $L \subset R^\infty$ *of* μ *in* $(R^\infty, R_0^\infty, \mu)$ *and an element* $\underline{t} = \{t_n\} \in R^\infty$. *If the series*

$$\underline{tx} = \sum_1^\infty t_n x_n, \qquad \underline{x} = \{x_n\} \in R^\infty,$$

converges uniformly on every compact convex set contained in L, *then it*

converges in $(\Lambda(\mu), s[\cdot])$ *to the unique extension of* $\underline{tx}$, $\underline{x} \in L$, *to a linear Lusin measurable functional.*

Proof. It follows directly from Proposition 4.1.

Let T be a fixed set and S a fixed linear space. A representation of a linear probability measure (P) of the form $(F(T), F', P)$, where $F(T)$ is a linear space of S-valued functions defined on T (called *paths*), F' is a linear space of linear functionals over $F(T)$, and P is a Radon probability measure over the weak Borel field of $F(T)$, is called a *representation of a stochastic process with the time set T and the state space S.*

A $(P) \in$ LPM is said to be a *stochastic process with the time set T and the state space S* if (P) admits a representation $(F(T), F', P)$ as described above (cf. [6]).

Given a stochastic process $(P) \in$ LPM, the elements of $\Lambda(P)$ are called Wiener stochastic integrals. The case of Ito stochastic integral (cf. [2], [7]) shall be discussed in another paper.

COROLLARY 4.2. *Consider a stochastic process (P) with the time set T and the state space S. Let $(F(T), F', P)$ be the representation of (P) and let F_0 be a pre-support of P in $(F(T), F', P)$. Denote by F_0' the $(F(T), F')$-adjoint of F_0.*

Every linear functional $I \in F_0'$ extends uniquely to a stochastic integral $\check{I}$. Moreover, if $\{I_n\} \subset F_0'$ converges $(F(T), F')$-almost uniformly on F_0 to $I \in F_0'$, then the stochastic integrals $\check{I}_n$ converge stochastically to the stochastic integral $\check{I}$.

Proof. It follows directly from Proposition 4.1.

Formula (*) of Section 1 gives the general form of the stochastic integral in the case of the Wiener process.

5. Pre-representations of linear probability measures

Consider a dual pair (U, U'). Any set of the form

$$\{u \in U\colon\ (u_1'u, \ldots, u_n'u) \in B\},$$

where $u_1', \ldots, u_n'$ is an arbitrary finite sequence of functionals from U' and B is a Borel set in R^n, shall be called a *cylinder subset of U*; the family of all cylinder subsets of U form a field which we shall denote by $\mathscr{C}(U, U')$.

Consider a non-negative finite additive set function ν defined on $\mathscr{C}(U, U')$. For arbitrary $u_1', \ldots, u_n' \in U'$ write

$$\nu_{u_1', \ldots, u_n'}(B) = \nu\big(\{u \in U\colon\ (u_1'u, \ldots, u_n'u) \in B\}\big).$$

The function ν is said to be a *probability pre-measure over $\mathscr{C}(U, U')$* if $\nu(U) = 1$ and if the following conditions hold:

(j) To every $\varepsilon > 0$ there corresponds a convex $\sigma(U, U')$-compact $C \subset U$ such that for every u' from the polar C^0 of C in (U, U'), we have $\nu(\{u \in U\colon |u'u| \leqslant 1\}) > 1 - \varepsilon$.

(k) For every finite sequence $u_1', \ldots, u_n' \in U'$, the function $\nu_{u_1', \ldots, u_n'}$ is a measure over the Borel field in R^n.

The property expressed in (j) is often referred to as the scalar concentration of ν on convex weak compact sets. (Cf. [10], [11].)

A triplet (U, U', ν) consiting of a dual pair (U, U') and a probability pre-measure on $\mathscr{C}(U, U')$ shall be called a *pre-representation of a linear probability measure.*

Due to (k), integrals with respect to ν of functions of a finite number of u' are well defined. Put

$$s[u'] \overset{\mathrm{df}}{=} \int |u'u|(1+|u'u|)^{-1} d\nu(u).$$

A pre-representation (U, U', ν) of a linear probability measure is said to be *proper* if for every $u' \in U'$ we have $s[u'] = 0$ implies $u' = 0$; (U, U', ν) is said to be *standard* if it is proper and if $\big(U', \tau(U', U)\big)$ is a Fréchet space.

A pre-representation (U, U', ν) is said to be *separable* if $\big(U', \tau(U', U)\big)$ admits a countable dense subset.

Consider a representation (X, X', μ) of a $(\mu) \in$ LPM. With arbitrarily given pre-support U of μ in (X, X', μ), we associate a pre-representation $(U, U', {}^c\mu)$ defining U' as the (X, X')-adjoint of U and then setting

$$(*) \qquad {}^c\mu\big(\{u \in U\colon (u_1'u, \ldots, u_n'u) \in B\}\big) = \mu\big(\{x \in S\colon (\overline{u}_1'x, \ldots, \overline{u}_n'x) \in B\}\big),$$

where $\overline{u}_1', \ldots, \overline{u}_n'$ denote the extensions of $u_1', \ldots, u_n' \in U'$ to some standard support S, $U \subset S \subset X$.

LEMMA 5.1. *Definition* $(*)$ *provides a probability pre-measure over* $\mathscr{C}(U, U')$.

Proof. To verify σ-additivity of ${}^c\mu_{u_1', \ldots, u_n'}$ for arbitrarily chosen $u_1', \ldots, u_n' \in U'$, it is sufficient to show that

$$T = \{x \in S\colon \overline{u}_1'x = u_i'u \text{ for some } u \in U \text{ and } i = 1, 2, \ldots, n\}$$

has the outer measure one. We shall show that it has the inner measure one. First of all, we can always take (S, S', μ) instead of (X, X', μ), and hence we at once assume that $X = S$. We do not loose generality assuming u_i' linearly independent. Take an arbitrary $\varepsilon > 0$. Then there exists a convex $\sigma(X, X')$-compact $C \subset U$ such that

$$\mu(\{x \in X\colon |x'x| \leqslant 1\}) > 1 - \varepsilon/n \quad \text{for} \quad x' \in C^0$$

and

$$\{(u_1'u, \ldots, u_n')\colon u \in C\} \supset \{(t_1, \ldots, t_n) \in R^n\colon \max_{1 \leqslant i \leqslant n} |t_i| \leqslant 1\}.$$

Hence

$$A \stackrel{\mathrm{df}}{=} \{x \in X\colon \max_{1 \leqslant i \leqslant n} |u_i' x| \leqslant 1\} \subset T.$$

The set T remains unchanged if we substitute u_i' with tu_i' for $t \neq 0$ so that we can always make $u_1', \ldots, u_n' \in C^0$ preserving the inclusion $A \subset T$. Hence $\mu(A) > 1-\varepsilon$, and the additivity follows. Condition (j) amounts to condition (i) for the pre-support, and the lemma holds.

Notice, that producing the *reduction* $(U, U', {}^c\mu)$ of a representation (X, X', μ) to a pre-representation, we cannot anymore consider $\mathscr{C}(U, U')$ as a subfield of $\mathscr{B}(X, X')$ and ${}^c\mu$ as the restriction of μ. Indeed, in many cases, U shall be of measure zero in X and ${}^c\mu$ shall be by no means the restriction of μ to $\mathscr{C}(U, U')$.

THEOREM 5.1. *Consider two representations (X, X', μ) and $(Y, Y', \tilde{\mu})$ of some linear probability measures, and let S be a linear space which is simultaneously a standard pre-support of μ in (X, X', μ) and of $\tilde{\mu}$ in $(Y, Y', \tilde{\mu})$.*

If the reduction of (X, X', μ) to S coincide with the reduction of $(Y, Y', \tilde{\mu})$ to S, then (X, X', μ) and $(Y, Y', \tilde{\mu})$ are preequivalent.

Proof. It follows immediately from the fact that S itself carries the pre-equivalence between (X, X', μ) and $(Y, Y', \tilde{\mu})$.

THEOREM 5.2. *Every separable pre-representation is a reduction of a separable representation.*

Proof. Let (U, U', ν) be a separable pre-representation, and let $\{u_n'\} \subset U'$ generate a dense subspace V' of $\big(U', \tau(U', U)\big)$. The topologies $\tau(U, V')$ and $\sigma(U, V')$ are identical and metrizable. Denote by X the completion of $\big(U, \tau(U, V')\big)$ and by X' the extensions to X of the functionals from V'. We define a pre-measure $\tilde{\nu}$ on $\mathscr{C}(X, X')$ setting for $C \in \mathscr{C}(X, X')$

$$\tilde{\nu}(C) \stackrel{\mathrm{df}}{=} \nu(C \cap U).$$

Since $\big(X, \tau(X', X)\big)$ is isomorphic with the countable product of real lines, we can apply the Kolmogorov theorem and extend $\tilde{\nu}$ to a measure μ over $\mathscr{B}(X, X')$. Consider now the representation (X, X', μ) of the $(\mu) \in$ LPM. For $C \in \mathscr{C}(X, X')$ we have $\mu(C) = \nu(C \cap U)$ so that condition (*) is fulfilled. Condition (i) for (U, U', ν) guarantees that U is a pre-support of μ un (X, X', μ). Therefore (U, U', ν) is the reduction of (X, X', μ) to a pre-support U, that is $(U, U', \nu) = (U, U', {}^c\mu)$, and the theorem holds.

From Theorem 5.1 we know that only pre-equivalent elements of LPM admit the same reductions, and on the other hand we know from Theorem 5.2 that every pre-representation is a reduction of a representation. Thus we can now say that $(U, U', {}^c\mu)$ is a pre-representation of

a $(\mu) \in$ LPM if $(U, U', {}^c\mu)$ is the reduction of a representation of (μ). Pre-representations of the same (μ) shall be called *pre-equivalent*. We define the relation "$\leqslant$, coarser-finer" in the same way as it was done for representations.

It is easy to see that Proposition 2.1 and 2.2 are valid for pre-representations, the latter being an immediate consequence of 5 of Proposition 4.1.

A standard pre-representation $(U, U', {}^c\mu)$ of a $(\mu) \in$ LPM is said to be a Λ_p-pre-representation, $1 \leqslant p < \infty$, if $U' \ni u' \to u^* \in U^*$ provides an isomorphism of $\big(U', \tau(U', U)\big)$ and $\big(\Lambda_p(\mu) \| \cdot \|_{L_p}\big)$.

THEOREM 5.3. *A separable $(\mu) \in$ LPM admits a Λ_p-pre-representation if and only if it admits a representation (X, X', μ) with $X^* \subset \Lambda_p(\mu)$.*

Proof. This is an immediate consequence of Theorems 3.3 and 4.2.

It is easy to see that if we have two versions (U, U', ν) and $(V, V', \tilde{\nu})$ of Λ_p-pre-representations of a $(\mu) \in$ LPM, then the identity from a standard pre-support of μ contained simultaneously in U and V extends to an isomorphism of $\big(U, \tau(U, U')\big)$ and $\big(V, \tau(V, V')\big)$. In this sense, we have the uniqueness of Λ_p-pre-representation $(U, U', {}^c\mu)$ of (μ), and we call U a *Λ_p-pre-support of (μ)*.

We say that a pre-representation $(U, U', {}^c\mu)$ of a $(\mu) \in$ LPM is a representation of (μ) if ${}^c\mu$ extends to a probabiity measure μ on $\mathscr{B}(U, U')$ so that (U, U', μ) becomes a representation of (μ).

A pre-representation $(U, U', {}^c\mu)$ of a $(\mu) \in$ LPM is said to be *nuclear* if $\big(U', \tau(U', U)\big)$ is a nuclear space.

THEOREM 5.4. (Cf. [4], [8].) *Every nuclear pre-representation is a representation as well.*

This theorem follows easily from the well-known Minlos theorem. However, for the sake of selfconsistency, we shall derive it from a certain measure theoretical lemma essentially due to Ito (Lectures given in Aarhus).

LEMMA 5.2. *Consider a locally convex space $(\Theta, \vartheta) \geqslant \big(\Lambda(\mu), s[\cdot]\big)$ with the topology induced by countably many Hilbertian seminorms $\|\cdot\|_k$. Let in what follows $\{e_{k,n}\}$ and $\{f_{k,n}\}$ be double sequences of elements of Θ such that*

$$\sum_{n=1}^{\infty} \|e_{k,n}\|_k^2 < \infty, \qquad \sum_{n=1}^{\infty} \|f_{k,n}\|_k^2 = 0$$

for $k = 1, 2, \ldots$ Put

$$Z_k = \Big\{x\colon \sum_{n=1}^{\infty} (e_{k,n}x)^2 < \infty,\ \sum_{n=1}^{\infty} (f_{k,n}x)^2 = 0\Big\}.$$

Then

$$\mu\Big(\bigcup_1^{\infty} Z_k\Big) = 1.$$

Proof. Take an arbitrary $\varepsilon > 0$. For $f \in \Theta$, put

$$K_f = \{x\colon\ |fx| > 1\}.$$

We have

$$1 - \mathscr{R} \int e^{i(fx)} d\mu(x) \leqslant \int |1 - e^{i(fx)}|\, d\mu(x)$$

$$\leqslant 2\mu(K_f) + \int_{X-K_f} |fx|\, d\mu(x) \leqslant 2\mu(K_f) + s[f].$$

Since $s[f] \geqslant \int_{K_f} |fx|(1+|fx|)^{-1} d\mu(x) \geqslant \frac{1}{2}\mu(K_f)$, we have $1 - \mathscr{R}\int e^{i(fx)} d\mu(x) \leqslant 4s[f]$, and thus we can find k and $\eta > 0$ such that for $f \in \Theta$

$$\|f\|_k < \eta \text{ implies } 1 - \mathscr{R}\int e^{i(fx)} d\mu(x) \leqslant \varepsilon.$$

(We can always make $\{\|\cdot\|_k\}$ pointwise non-decreasing.)

Since $\mathscr{R}\int e^{i(fx)} d\mu(x) \geqslant -1$, we have for $f \in \Theta$,

$$\mathscr{R}\int e^{i(fx)} d\mu(x) \geqslant 1 - \varepsilon - \frac{2}{\eta}(\|f\|_k)^2,$$

and thus setting for arbitrary $t_1, \ldots, t_n$, $f = \sum_{p=1}^{n} t_p f_p$, we obtain

$$\text{(I)} \qquad \mathscr{R}\int \exp\Big(i\sum_{p=1}^{n} t_p(f_p x)\Big) d\mu(x) \geqslant 1 - \varepsilon - \frac{2}{\eta}\sum_{p,q=1}^{n} t_p t_q (f_p|f_q)_k,$$

where $(\cdot|\cdot)_k$ denotes the scalar product corresponding to $\|\cdot\|_k$. Since

$$(2\pi h)^{-1/2}\int \exp\Big(i\sum_1^n t_p s_p - \tfrac{1}{2}h\sum_1^n t_p^2\Big) dt_1 \ldots dt_n = \exp\Big(-\tfrac{1}{2}h\sum_1^n s_p^2\Big),$$

integrating both sides of (I) with respect to the Gauss measure $\exp\Big(-\frac{1}{2h}\sum_1^n t_p^2\Big) dt_1 \ldots dt_n$, we obtain

$$\int \exp\Big(-\tfrac{1}{2}h\sum_1^n (f_p x)^2\Big) d\mu(x) \geqslant 1 - \varepsilon - \frac{2}{\eta} h \sum_1^n (\|f_p\|_k)^2,$$

and consequently,

$$\text{(II)} \qquad \int \exp\Big(-\tfrac{1}{2}h\sum_1^\infty (f_p x)^2\Big) d\mu(x) \geqslant 1 - \varepsilon - \frac{2}{\eta} h \sum_1^\infty (\|f_p\|_k)^2.$$

Substituting $f_p = e_{k,p}$ in (II) and passing to zero with h, we obtain

$$\mu\Big(\Big\{x\colon \sum_{p=1}^{\infty}(e_{k,p}x)^2 < \infty\Big\}\Big) \geqslant 1-\varepsilon,$$

and substituting $f_p = f_{k,p}$ in (II) and passing to infinity with h, we obtain

$$\mu\Big(\Big\{x\colon \sum_{p=1}^{\infty}(f_{k,p}x)^2 = 0\Big\}\Big) \geqslant 1-\varepsilon$$

so that $\mu(Z_k) \geqslant 1-2\varepsilon$, and the lemma follows.

Proof of Theorem 5.4. Suppose (X, X', μ) is a primitive representation of (μ) coarser than the given nuclear pre-representation $(U, U', {}^c\mu)$. We notice at first that $\big(U', \tau(U', U)\big)$ can be made a separable (F)-space by taking a finer nuclear representation. Indeed, it is sufficient to select a sequence $\{V_n\}$ of convex symmetric neighbourhoods in $\big(U, \tau(U, U')\big)$ which induce seminorms $\|\cdot\|_n$ on U in such a way that the identical injections $(U, \|\cdot\|_{n+1})$ to $(U, \|\cdot\|_n)$ are nuclear and that (j) can be fulfilled by using only the polars of V_n. As is well-known, we can take $\|\cdot\|_k$ to be Hilbertian and the identical injections to be Hilbert-Schmidt. Then, taking a linearly dense sequence in $\big(X', \tau(X', X)\big)$, we can orthogonalize it by the Gram–Schmidt method with respect to every $\|\cdot\|_k$ separately getting sequences $\{x'_{k,n}\}$ and $\{y'_{k,n}\}$ such that setting $f_{k,n} = x^*_{k+1,n}$ and $g_{k,n} = y^*_{k+1,n}$ and regarding the pseudonorms $\|\ \|_k$ transferred to $\Theta = U^*$ we find, applying Lemma 5.2, that the space

$$U = \bigcup_{k=1}^{\infty}\Big\{x \in X\colon \sum_{n=1}^{\infty}(x'_{k,n}x)^2 < \infty,\ \sum_{n=1}^{\infty}(y'_{k,n}x)^2 = 0\Big\}$$

has measure one, and the Theorem follows.

In view of Theorem 5.4 we can always say nuclear representations instead of nuclear pre-representations assuming, however, that only (j) and (k) should be satisfied and having as a consequence that the pre-measure is actually a measure. Proving Theorem 5.4, we have additionally verified the following

PROPOSITION 5.1. *Every nuclear representation admits a finer separable standard nuclear representation.*

Separable primitive representations are of course always nuclear, and thus as a consequence of the above proposition and Theorem 2.1, we obtain the following

COROLLARY 5.1. *Given a separable* $(\mu) \in$ LPM *and* $\{f_n\} \subset \Lambda(\mu)$, *there exists a standard nuclear representation* (X, X', μ) *such that* $\{f_n\} \subset X^*$.

A pre-representation $(U, U', {}^c\mu)$ is said to be *σ-Hilbertian* if it is standard and if the topology $\tau(U', U)$ can be introduced by Hilbertian

semi-norms. We shall now provide a functor which shall assign to every standard representation a finer σ-Hilbertian pre-representation. Notice, that it is not always possible to assign a finer σ-Hilbertian representation to a standard representation.

To every standard representation (X, X', μ) of a $(\mu) \in \mathrm{LPM}$ we assign the space $\Lambda_p^{\mathrm{comp}}(X, X', \mu)$ of functionals from $\Lambda(\mu)$ which are p-integrable on every $\sigma(X, X')$-compact convex subset $C \subset X$. Writing for such C and $f \in \Lambda(\mu)$

$$\|f\|_{p,C} = \Big(\int_C |f|^p d\mu\Big)^{1/p},$$

we have

$$\Lambda_p^{\mathrm{comp}}(X, X', \mu) = \{f \in \Lambda(\mu)\colon \|f\|_{p,C} < \infty,\ C \in \mathscr{R}\},$$

where $\mathscr{R}$ denotes the family of convex $\sigma(X, X')$-compact subsets of X. Since $\mathscr{R}$ contains a cofinal ascending sequence, the space $\Lambda_p^{\mathrm{comp}}(X, X', \mu)$ provided with the seminorms $\|\cdot\|_{p,C}$ constitutes an (F)-space. If, additionally, (X, X', μ) is separable and $1 \leqslant p \leqslant 2$, then using Theorem 3.3 one can easily prove that this space belongs to $\mathscr{F}(X, X', \mu)$. Hence, we have the following

PROPOSITION 5.2. *Consider a proper separable representation (X, X', μ) of a $(\mu) \in \mathrm{LPM}$. For every p, $1 \leqslant p \leqslant 2$, there uniquely corresponds a standard pre-representation $(U, U', {}^c\mu)$ of (μ) finer than (X, X', μ) such that $U^* = \Lambda_p^{\mathrm{comp}}(X, X', \mu)$.*

Proof. It is a direct application of Theorem 4.2 and reduction of (X, X', μ) to a proper standard pre-support.

In case of $p = 2$, the pre-representation $(U, U', {}^c\mu)$ is σ-Hilbertian, and thus we have the following

COROLLARY 5.2. *To every representation of a separable $(\mu) \in \mathrm{LPM}$ one can assign a finer σ-Hilbertian pre-representation.*

Consider a triplet $(C, A(C), \mu)$ consisting of a symmetric convex set C, a linear space of affine functions over C containing all constant functions and separating in C, and a probability measure μ over the Borel field of C provided with the weak topology out of $A(C)$. Assume additionally that C provided with this weak topology is compact.

Such triplets we shall call *compact convex representations of probability measures.*

Every compact convex representation $(C, A(C), \mu)$ of a probability measure uniquely induces a representation (L_C, L'_C, μ) of a linear probability measure by setting L_C for the linear span of C and for L'_C the linear space of all linear functionals on L_C with restrictions to C belonging to $A(C)$.

A convex, compact representation $(C, A(C), \mu)$ of a probability measure is said to be *proper* if elements of $A(C)$ vanishing almost everywhere vanish idenitcally. It is easy to see that given any compact convex representation $(C, A(C), \mu)$, we can find a compact convex $C_1 \subset C$ such that $\mu(C - C_1) = 0$ and such that the reduced representation $(C_1, A(C_1), \mu)$ is already proper.

An affine function defined on a convex subset of C is said to be a *representation of an affine Lusin measurable function* if it is affine continuous on compact convex subsets of C with measure arbitrarily close to one. This means that such a function is a translation of the restriction of a representation in (L_C, L'_C, μ) of a linear Lusin measurable functional. Classes of equivalence of representations of affine Lusin measurable functions identical on measure one subsets shall be called *affine Lusin measurable functions*.

Denote by $AL_p(C, \mu)$ the linear space of all affine Lusin measurable functions f such that

$$\|f\|_{L_p} = \Big(\int_C |f|^p d\mu\Big)^{1/p}$$

A compact convex representation $(C, A(C), \mu)$ of a probability measure is said to be *separable* if $A(C)$ contains a countable subset dense with respect to the stochastic convergence.

COROLLARY 5.3. *Consider a separable convex representation $(C, A(C), \mu)$ of a probability measure. Fix p, $1 \leqslant p \leqslant 2$. There exists a compact convex $K \subset C$ such that the following conditions are fulfilled.*

Denote by $A(K)$ the set of all continuous affine functions on K. Then

a) *To every $a' \in A(K)$ there corresponds uniquely an $a^* \in AL_p(C, \mu)$ such that a' extends to a representation of a^* in $(C, A(C), \mu)$,*

b) *The mapping*

$$A(K) \ni a' \to a^* \in AL_p(C, \mu)$$

is an isomorphism of $A(K)$ and $AL_p(C, \mu)$, i.e. it is one-to-one and maps $A(K)$ onto $AL_p(C, \mu)$.

c) *There exists a positive number k such that*

$$\|a^*\|_{L_p} = k \sup\{|a'x| : x \in K\}$$

for every $a' \in A(K)$.

The set K is uniquely determined by the conditions a), b), c) *and the number k.*

Proof. Let us first identify linear functionals part of $AL_p(C, \mu)$ with $\Lambda_p(\mu)$, where (μ) is a linear probability measure with representation (L_C, L'_C, μ). Now we can apply Theorem 5.3 and find the Λ_p-pre-representation $(U, U', {}^{\circ}\mu)$ of (μ) finer than (L_C, L'_C, μ). Hence, there exists $a > t$ such that for the polar $K_1 \subset U$ of the unit ball in U', provided by the norm $\|\cdot\|_{L_p}$ shifted from $\Lambda_p(\mu)$ over U', we have $tK_1 \subset C$, and this verifies c) for $K = tK_1$. Both a) and b) follows from Theorem 4.1. The uniqueness of K follows from uniqueness of the Λ_p-representation. This concludes the proof of Corollary 5.3.

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