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VARIATIONAL STABILITY AND WELL-POSEDNESS IN THE OPTIMAL CONTROL OF ORDINARY DIFFERENTIAL EQUATIONS

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Introduction

This paper is a brief survey of recent results about continuous dependence problems in optimal control of (mainly linear) ordinary differential systems. In particular, continuous dependence of the optimal controls and values is characterized for some classes of convex cost problems, and connections with the convergence of most optimization methods (well-posedness in the sense of Tihonov) is investigated. Problems of this kind are fundamental in optimal control, both for theoretical and practical reasons. Changes in the parameters and coefficients of an optimal control problem result in predictable variations of the optimal objects. Recently developed perturbational methods in optimization can be successfully applied to this class of problems. A condensed version of this paper was presented at the Mathematical Theory of Optimal Control semester of the Stefan Banach International Mathematical Center, Warsaw, December 1980.

I. LINEAR CONVEX PROBLEMS

We consider the following optimal control problem: Minimize the cost functional

$$\int_0^T f(t, x, u)\,dt \tag{1}$$

subject to the state equations

$$\begin{aligned} \dot{x} &= A(t)x+B(t)u, \quad 0 \leqslant t \leqslant T. \\ x(0) &= 0. \end{aligned} \tag{2}$$

Here A, B are matrices of suitable dimensions.

We denote by

p — the adjoint variable;

v — the (optimal) value;

$\bar{u}$ — the optimal control;

$\bar{x}$ — the optimal state.

The problem is to find convergences, denoted by $\to$, of the coefficients in the plant such that for every f in a given class,

$$A_n \to A_0, \quad B_n \to B_0$$

if and only if at least one of the following holds:

$$v_n \to v_0, \quad \bar{u}_n \to \bar{u}_0, \quad \bar{x}_n \to \bar{x}_0, \quad p_n \to p_0$$

in a prescribed sense.

Among the motivations for the problems of such type we mention
continuous dependence problems;
parametric optimization;
approximate knowledge of some coefficients;
possibility of model reduction.

An abstract functional setting

We consider:

U, X — real Hilbert spaces;

$q > 0$;

L_n: $U \to X$ — linear boundeds operators such that

$$\|L_n\| \leqslant q.$$

Here $n = 0$ corresponds to the unperturbed problem.

Given $c \in (0, 1]$ we consider the set S of all

$$F\colon U \oplus X \to (-\infty, +\infty)$$

such that

(3) F is convex, continuous and continuously Fréchet differentiable with respect to $x \in X$ for every fixed $u \in U$,

(4)
$$F\left(\frac{u+v}{2}, \frac{x+y}{2}\right) \leqslant \frac{1}{2} F(u, x) + \frac{1}{2} F(v, y) - \frac{c}{4} \|u - v\|^2 \text{ for every } u, v, x, y.$$

For any $n = 0, 1, 2, \ldots$, the *primal problem* (an abstract version of the above optimal control problem (1), (2)) consists in minimizing on U

(5) $$I_n(u) = F(u, L_n u).$$

The *dual problem* consists in maximizing on X

$$y \to F^*(L_n y, -y)$$

(F^* = Fenchel's conjugate of F). The solution of this dual problem is the Lagrange multiplier

$$(6) \qquad -\bar{y}_n = \text{grad}_x F(\bar{u}_n, L_n \bar{u}_n),$$

where

(7) $\bar{u}_n$ is the solution of the primal problem.

Convergence in the sense of Mosco

Let H be a real Hilbert space and T_n a sequence of (nonempty) closed convex subsets of H.

DEFINITION. We write $T_n \xrightarrow{M} T_0$ iff

(a) for every subsequence n_k, $x_k \in T_{n_k}$ and $x_k \rightharpoonup x_0$ imply $x_0 \in T_0$;

(b) for every $x_0 \in T_0$ there exist $x_n \in T_n$ such that $x_n \to x_0$.

Given $u \in H$, let $p_n(u)$ be the point of T_n nearest to u.

THEOREM 1. *The following conditions are equivalent:*

(a) $T_n \xrightarrow{M} T_0$;

(b) $p_n(u) \to p_0(u)$ *for every* $u \in H$;

(c) $\text{dist}(u, T_n) \to \text{dist}(u, T_0)$ *for every* $u \in H$.

Let f_n be a sequence of extended real-valued proper convex lower semicontinuous functions defined on H with epigraph denoted by $\text{epi} f_n$.

DEFINITION. We write $f_n \xrightarrow{M} f_0$ iff

$$\text{epi} f_n \xrightarrow{M} \text{epi} f_0.$$

THEOREM 2. *The following conditions are equivalent:*

(a) $f_n \xrightarrow{M} f_0$;

(b) $x_n \rightharpoonup x_0$ *implies* $\liminf f_n(x_n) \geqslant f_0(x_0)$, *and for every* $x \in H$ *there exist* $x_n \to x$ *such that* $f_n(x_n) \to f_0(x)$.

The convergence in the sense of Mosco is the main tool in the proof of the next theorem. We come back to the abstract optimization problem defined above (see (5), (6), (7)).

THEOREM 3. *The following conditions are equivalent:*

(a) $\bar{u}_n \to \bar{u}_0$ *for every* $F \in S$;

(b) $I_n \xrightarrow{M} I_0$ *for every* $F \in S$;

(c) $v_n \to v_0$ *for every* $F \in S$;

(d) $\bar{y}_n \to \bar{y}_0$ *for every* $F \in S$;

(e) *any of the above statements* (a)–(d) *for every quadratic functional* $(u, x) \to \|u - u^*\|^2 + \|x - x^*\|^2$, $u^* \in U$, $x^* \in X$;

(f) $L_n x \to L_0 x$ *and* $L_n^* y \to L_0^* y$ *for every* x *and* y.

The optimal control problem

Let us consider sequences of optimal control problems (1), (2) satisfying the following assumptions:

(a) for some $q > 0$, $\int_0^T |B_n|^2 dt \leqslant q$;

(b) A_n are equiintegrable (i.e., $\int_E A_n ds \to 0$ as $\operatorname{meas} E \to 0$, uniformly in n).

Given $c \in (0, 1]$ we consider the Carathéodory integrands

$$f = f(t, x, u) \colon [0, T] \times R^p \times R^m \to (-\infty, +\infty),$$

and we say that $f \in C$ iff

(a) $f(t, \cdot, \cdot)$ is convex with a continuous gradient $f_x(t, \cdot, \cdot)$;

(b) $a(t) \leqslant f(t, x, u) \leqslant b|u|^2 + p|x|^2 + s(t)$,

$$|f_x(t, x, u)| \leqslant r|x| + w|u| + z(t)$$

for all t, x, u and some $z, a \in L^1(0, T)$, $s \in L^2(0, T)$;

(c) $f\left(t, \frac{x+y}{2}, \frac{u+v}{2}\right) \leqslant \frac{1}{2} f(t, x, u) + \frac{1}{2} f(t, y, v) - \frac{c}{4}|u - v|^2$.

Remark. Every quadratic-type integrand

$$f(t, x, u) = [u - u^*(t)]' Q(t) [u - u^*(t)] + [x - x^*(t)]' P(t) [x - x^*(t)]$$

belongs to C for any $u^*, x^* \in L^2$ if the matrices Q, P are bounded, positive semidefinite and Q is uniformly positive definite.

Given $f \in C$, we denote by p_n (for every n) the adjoint state given by

$$\dot{p}_n + A_n'(t) p_n = f_x(t, \bar{x}_n, \bar{u}_n), \quad p_n(T) = 0.$$

THEOREM 4. *The following conditions are equivalent*:

(a) $\bar{u}_n \to \bar{u}_0$ *in* $L^2(0, T)$ *for every* $f \in C$;

(b) $\bar{u}_n \to \bar{u}_0$ *in* $L^2(0, T)$ *for every quadratic-type* f;

(c) $p_n \to p_0$ *in* AC$(0, T)$ *for every* $f \in C$;

(d) $A_n \rightharpoonup A_0$ *in* $L^1(0, T)$, $B_n \rightharpoonup B_0$ *in* $L^2(0, T)$ *and* $B_n \to B_0$ *in every* $L^2(0, T - \varepsilon)$, $0 < \varepsilon < T$.

COUNTER-EXAMPLE. The convergence of optimal controls in L^2 for every quadratic-type cost does not imply in general the weak convergence of A_n to A_0 in plant (2) if weak sequential compactness is not assumed.

Consider $B_n = 0$, $A_0 = 0$, $P = Q = 1$, and for $n = 1, 2, \dots$ let

$$A_n(t) = \begin{cases} 2^n \sin(4^n \pi t), & 2^{-n} \leqslant t \leqslant 2^{1-n}, \\ 0 & \text{otherwise}. \end{cases}$$

Minimum effort control problems

Consider the characterization of continuous dependence of the optimal controls upon the matrix B in the following optimal control problem:

state equations: $\begin{cases} \dot{x} = A(t)x + B(t)u, & 0 \leqslant t \leqslant T, \\ x(0) = 0; \end{cases}$

constraint: $x(T) = y$;

cost functional: $\left(\int_0^T \sum_{j=1}^m |u_j(t) - u_j^* t)|^p \, dt \right)^{1/p}, \quad p > 1$

Let q be such that $1/p + 1/q = 1$.

THEOREM 5. *Assume complete controllability of the pair* (A, B_n), $n = 0, 1, 2, \dots$, *and uniform boundedness of* $\int_0^T |B_n|^q \, dt$. *Then the following conditions are equivalent*:

(a) *for the optimal controls* $\bar{u}_n$, $\bar{u}_n \to \bar{u}_0$ *in* $L^p(0, T)$ *for every* y *and* $u^* \in L^p(0, T)$;

(b) $B_n \to B_0$ *in* $L^q(0, T)$.

The characterization (a)$\Leftrightarrow$(b) of Theorem 5 (with (a) holding for every u^*, x^*, y^*) is true for linear plants and cost

$$\int_0^T (|u - u^*|^p + |x - x^*|^p) \, dt + |x(t) - y^*|^p.$$

Remark. By comparing Theorems 4 and 5 we see that a final state constraint (or a final state term in the cost) forces strong convergence of the coefficients on the whole time interval $[0, T]$.

A DISCONTINUOUS DEPENDENCE EXAMPLE. We consider $M > 0$, a real Hilbert space H, a point $y^* \in H$ and a given bounded linear map L: $H \to H$.

PROBLEM. Among all $u \in H$ minimizing

$$\|Lu - y^*\| \text{ subject to } \|u\| \leqslant M$$

find the element $\bar{u}$ of minimum norm. This $\bar{u}$ may depend on L in a discontinuous way. Take $H = R^2$, $y = (1, 1)$, $M = 2$,

$$L_0(x_1, x_2) = (x_1, 0), \quad L_n = (x_1, x_2/n);$$

then

$$\|L_n - L_0\| \to 0 \quad \text{but} \quad \bar{u}_n \not\to \bar{u}_0.$$

Dependence of the value on the weighting coefficients in the quadratic cost

We consider the following optimal control problem:

state equations: $\begin{cases} \dot{x} = A(t)x + B(t)u, & 0 \leqslant t \leqslant T, \\ x(0) = 0; \end{cases}$

constraint: $x(T) = y;$

cost functional: $\int_0^T [u'Q(t)u + x'P(t)x]\,dt;$

(optimal) value: $v;$

optimal control: $\bar{u}.$

We wish to characterize the admissible tolerances on the coefficients in the matrices P, Q such that the corresponding values and optimal controls differ in a prescribed way.

Abstract setting. We are given:

X — a real Hilbert space;

Y — a real reflexive Banach space, $Y \neq \{0\}$;

L: $X \to Y$ — a linear bounded surjective operator;

$a > 0$, $\omega > 0$;

A_n: $X \to X$ — a sequence of linear bounded maps such that

$$a\|u\|^2 \leqslant f_n(u) = \tfrac{1}{2}\langle A_n u, u\rangle \leqslant \omega \|u\|^2 \tag{8}$$

for every n and u.

Let $\bar{u}_n$ be the minimum point of

(9) $f_n(u)$ — subject to $Lu = y$, $y \in Y$,
v_n — the value, z_n — the Lagrange multiplier, given by $L^* z_n = A_n \bar{u}_n$.

We wish to characterize the weakest convergence (if any)

$$f_n \to f_0$$

such that for every y we have

$$v_n = \min f_n(L^{-1}y) \to \min f_0(L^{-1}y) = v_0.$$

Weak gamma convergence

We consider a normed space U and sequences f_n: $U \to [-\infty, +\infty]$.

DEFINITION. We write $f_n \xrightarrow{\Gamma} f_0$ (more precisely $f_n \to f_0$ in the $\Gamma^-(w)$ sense) iff

(a) $x_n \rightharpoonup x_0$ implies $\liminf f_n(x_n) \geqslant f_0(x_0)$;

(b) for every x there exist $x_n \rightharpoonup x$ such that

$$f_n(x_n) \to f_0(x).$$

THEOREM 6. *Let* $f_n(u) = \frac{1}{2}\langle A_n u, u\rangle$ *under the same equipositivity assumptions* (8) *on* A_n *as above. Then*

$$f_n \xrightarrow{\Gamma} f_0 \quad \textit{iff} \quad A_n^{-1} v \rightharpoonup A_0^{-1} v \quad \textit{for every } v.$$

The Γ convergence of the quadratic functionals f_n has a relevant variational interpretation in free minimization problems, but is no longer equivalent to the convergence of the constrained values.

COUNTER-EXAMPLE. Assume

$$f_n(u) = \int_0^1 a_n(x)\dot{u}(x)^2 dx$$

to be minimized with the constraint

$$\frac{d^2}{dx^2} u = u+1, \quad u(0) = 0,$$

with

$$a_n(x) = 2 + \frac{\sin nx}{|\sin nx|}$$

Then (working within $X = \{u \in H^{1,2}(0,1): u(0) = 0\}$)

$$f_n \xrightarrow{\Gamma} f_0, \quad f_0(u) = 3/2 \int_0^1 \dot{u}(x)^2 dx$$

but for the constrained values v_n, $v_n \nrightarrow v_0$.

We come back to the above abstract setting.

THEOREM 7. *Assume L to be a compact map and let condition* (8) *be satisfied. Then* $f_n \xrightarrow{\Gamma} f_0$ *implies* (*for every* y):

(a) *for the values* v_n, $v_n \to v_0$;

(b) *for the optimal solution* $\bar{u}_n$, $\bar{u}_n \rightharpoonup \bar{u}_0$;

(c) *for the Lagrange multipliers* z_n, $z_n \rightharpoonup z_0$.

A partial converse is given by the following

THEOREM 8. *Assume L_j to be linear bounded maps for every j* (*in some index set*) *such that*

$$\bigcup_j L_j^*(Y^*) \quad \textit{is dense in } X^*.$$

Then either

$$\min\{f_n(u): L_j u = y\} \to \min\{f_0(u); L_j u = y\} \quad \textit{for every } y, j,$$

or the convergence of the Lagrange multipliers

$$z_n^j \to z_0^j \quad \textit{for every } y \textit{ and } j$$

implies $f_n \xrightarrow{\Gamma} f_0$.

COROLLARY 1. *Assume* $L\colon X \to R^m$ *to be a surjective linear operator. Then the following are equivalent:*

(a) $f_n \xrightarrow{\Gamma} f_0$;

(b) *for the values* v_n *defined in* (9), $v_n \to v_0$ *for every* y.

We apply these results to the optimal control problem described at the beginning of this section. Assume $P_n, Q_n \in L^\infty(0, T)$, P_n are positive semidefinite, Q_n are equiuniformly positive definite, $A \in L^1(0, T)$ and $B \in L^2(0, T)$. For every n, the Riccati matrix E_n is given by

$$\dot{E}_n = P_n - A'E_n - E_n A - E_n B Q_n^{-1} B' E_n, \qquad E_n(T) = 0$$

as in the corresponding free final time problem.

THEOREM 9. *The following conditions are equivalent:*

(a) *for the values* v_n, $v_n \to v_0$ *for every* y;

(b) $Q_n^{-1} \rightharpoonup Q_n^{-1}$ *in* $L^\infty(0, T)$ *and for the feedback matrices* $Q_n^{-1}B'E_n$, $Q_n^{-1}B'E_n \rightharpoonup Q_0^{-1}B'E_0$ *in* $L^2(0, T)$,

II. SOME NONLINEAR RESULTS

We consider sequences of optimal control systems and ask for conditions on the convergences of the plants which are sufficient, or equivalent, to convergence of the values, the optimal controls and states. Few results seem to be known in this area as far as nonlinear systems are concerned.

A semilinear problem

The optimal control problems are as follows:

$$\text{state equations:} \begin{cases} \dot{x} = A_n(t, x(t)) + B_n(t)u(t), & 0 \leqslant t \leqslant T; \\ x(0) = y_n; \end{cases}$$

$$\text{constraints:} \quad \int_0^T |u|^2 dt \leqslant M, \qquad u(t) \in C_n;$$

$$\text{cost:} \quad \int_0^T f_n(t, x_n(t), u(t))\, dt = I_n(u).$$

THEOREM 10. *Assume the following:* $y_n \to y_0$; *for the closed convex sets*

C_n, $C_n \xrightarrow{M} C_0$; *for every* t, $f_n(t,\cdot,\cdot) \xrightarrow{M} f_0(t,\cdot,\cdot)$ *for the convex functions* f_n; $A_n(\cdot, x(\cdot)) \to A_0(\cdot, x(\cdot))$ *in* $L^1(0,T)$ *for every continuous* x;

$$|A_n(t,x) - A_n(t,y)| \leqslant q_n(t)|x-y| \quad with \quad \int_0^T q_n(t)\,dt \leqslant C < +\infty;$$

$$0 \leqslant f_n(t,x,u) \leqslant a(t) + b|x|^2 + c|u|^2, \qquad a \in L^1(0,T).$$

Then the following hold: $I_n \xrightarrow{M} I_0$, *the values* $v_n \to v_0$; *if* $\bar{u}_n$ *are optimal controls,* $\bar{u}_{n_k} \to \bar{u}$ *in* $L^2(0,T)$, *then* $\bar{u}$ *is an optimal control for the unperturbed problem.*

A nonlinear characterization

We consider the optimal control problem P_n defined as follows:

state equations: $\begin{cases} \dot{x}(t) \in D_n(t, x(t)), & 0 \leqslant t \leqslant T, \\ x(0) \text{ given}; \end{cases}$

cost functional: $f(x)$,

where f is a continuous function from $C^0(0,T)$ to R.

THEOREM 11. *Assume that* D_n: $[0,T] \times R^m \to$ *subsets of* R^m, *are compact convex valued multifunctions, equibounded and equilipschitzian with respect to the Hausdorff distance. Then the following are equivalent:*

(a) $\min P_n \to \min P_0$ *for every continuous* f;

(b) *for every* E, $\int_E D_n(t, y(t))\,dt \to \int_E D_0(t, y(t))\,dt$ *for every continuous* y.

Moreover, each of (a) *and* (b) *implies that*

$$\max\{\text{dist}(x, \text{argmin} P_0) : x \in \text{argmin} P_n\} \to 0,$$

where dist is computed in the uniform norm.

Remark. Nonlinear dependence of the cost on the control is not allowed in Theorem 11.

III. WELL-POSEDNESS IN THE SENSE OF TIHONOV AND HADAMARD

We consider:

X – a convergence space;

$V \subset X$ – the constraint set;

I: $X \to (-\infty, +\infty]$

and the problem of minimizing

$$I(x) \text{ subject to } x \in V.$$

DEFINITION. The above minimum problem is called *Tihonov well-posed* iff

(a) there exists a unique minimum point $\bar{x}$ of I in V;

(b) every minimizing sequence converges to $\bar{x}$, i.e., $u_n \in V$, $I(u_n) \to \to I(\bar{x})$ imply $u_n \to \bar{x}$.

Roughly speaking, the problem of minimizing I on given subsets V of X is called *Hadamard well-posed* iff it has solutions continuously depending on the constraint set V.

The two notions of Tihonov and Hadamard well-posedness of a minimum problem seem at first glance to be largely independent. As a matter of fact, the main results of this section will show that for convex problems there is a deep equivalence between these two basic definitions. Moreover, we shall see that well-posedness of quadratic optimal control problems is quite stringent since it is equivalent to linearity of the system to be controlled.

EXAMPLE. A Tihonov well-posed optimal control problem with respect to strong L^2 convergence:

state equations: $\begin{cases} \dot{x} = A(t)x + B(t)u + C(t), & 0 \leqslant t \leqslant T, \\ x(0) \text{ given}; \end{cases}$

constraints: $(u, x) \in K$.

K a (nonempty) closed convex subset of $L^2(0, T) \oplus L^2(0, T)$;

$$\text{(10)} \quad \text{cost functional: } \int_0^T [(x-x)'P(t)(x-x) + (u-u)'Q(t)(u-u)]dt + \\ + [x(T) - y^*]'F[x(T) - y^*],$$

where $C, A \in L^1(0, T)$, $B \in L^2(0, T)$, P and Q bounded, F and $P(t)$ positive semidefinite, Q uniformly positive definite, any x^*, $u^* \in L^2(0, T)$ and $y^* \in R^m$.

Let us remark that existence and uniqueness of the optimal control do not imply Tihonov well-posedness even with respect to weak convergence, as is shown in the following

COUNTER-EXAMPLE. Everything is scalar, as follows:

state equation: $\ddot{x} = ux - 1/u; \quad x(0) = x(1) = 0;$

constraint: $1 - 1/\sqrt{2} \leqslant u(t) \leqslant 1 + 1/\sqrt{2};$

cost functional: $\int_0^1 (x - z)^2 dt,$

where z is the solution of the following problem:

$$\ddot{z} = z - 2, \quad z(0) = z(1) = 0.$$

Let us consider the following non linear optimal control problem: Minimize the quadratic cost (10) subject to the state equations:

$$\text{(11)} \quad \begin{cases} \dot{x} = g(t, x, u), & 0 \leqslant t \leqslant T, \\ x(0) = v. \end{cases}$$

The control is now the pair (v, u), $v \in R^m$ and $u \in L^2(0, T)$. Let us assume:

(a) existence and uniqueness in the large for every Cauchy problem

$$(12)\qquad \begin{cases} \dot{x} = g(t, x, u), \\ x(t_0) = v, \end{cases}$$

given any $t_0 \in [0, T]$, $u \in L^2(0, T)$, $v \in R^m$;

(b) continuous dependence of the solution x of (11) on v and u (strong convergence in L^2) with respect to uniform convergence on $[0, T]$;

(c) P, Q, F are uniformly positive definite.

THEOREM 12. *Under the above assumptions* (a), (b), (c), *the following are equivalent*:

(i) *the optimal control problem* (10), (11) *is Tihonov well-posed with respect to the strong convergence in* $R^m \oplus L^2(0, T)$ *for every* u^*, x^*, y^*;

(ii) *there exist matrices* A, B, C *such that*

$$g(t, x, u) = A(t)x + B(t)u + C(t)$$

for every t, x, u;

(iii) *The optimal control problem* (10), (11) *is Hadamard well-posed in the sense that the optimal control is a strongly continuous functions of the desired trajectory* (u^*, x^*, y^*).

In sharp contrast with the situation described in Theorem 12, it is easy to show that "most" optimal control problems (10), (11) (with added constraints) are Tihonov well-posed. Roughly speaking, given problem (10), (11) we can always restore existence, uniqueness and strong convergence of any minimization algorithm by a slight change of the desired trajectory.

We end the paper by stating a precise equivalence theorem between Tihonov and Hadamard well-posedness in an abstract setting. We are given:

a reflexive Banach space X;

a convex continuous function f: $X \to (-\infty, +\infty)$.

The minimum problem of the function f on the subset K of X will be briefly denoted by (K, f). The optimization problem (K, f) will be now called *Hadamard well-posed with respect to Mosco's* (*Hausdorff's*) *convergence* iff for K_n closed convex subsets of X, $K_n \xrightarrow{M} K_0$ (respectively $K_n \to K_0$ in the Hausdorff sense) implies

$$\arg\min f(K_n) \to \arg\min f(K_0).$$

In the following theorem, Tihonov well-posedness is considered with respect to the strong convergence in X.

THEOREM 13. *Assume f to be uniformly continuous on bounded sets. Then if (K, f) is Tihonov well-posed for every affine half-space K then (X, f) is Hadamard well-posed with respect to Mosco's convergence.*

Conversely, assume that f has a unique minimum point on every closed convex subset of X. Then if (X, f) is Hadamard well-posed with respect to the Hausdorff convergence then (K, f) is Tihonov well-posed for every closed convex subset K of X.

Bibliographical notes

Section I. Theorems 3 and 4 are particular cases of results obtained in [2], [16] and [11] (in the last paper the assumption of equiintegrability of A_n was inadvertently omitted). Theorem 5 was obtained in [6], where further results are given and the discontinuous dependence example is discussed.

The main tool in obtaining these results is the convergence in the sense of Mosco. The same type of convergence has shown its usefulness and applicability to the study of perturbations and continuous dependence in time optimal control problems (see [17]), asymptotic optimal control problems (see [10], [14]), convergence of exterior penalty techniques (epsilon method: see [7]). For a general survey about gamma convergences and their applications see [5]. The counter-example about lack of equiintegrability of A_n is due to G. Buttazzo–G. Dal Maso.

Section II. Theorem 10 is a particular case of results obtained in [1]. Theorem 11 is given in [12] with further results.

Section III. The notion of well-posedness was given by Tihonov in [13]. The counter-example is due to M. F. Bidaut. Theorem 12 is given in [18], with some generic results about well-posedness of nonlinear regulator problems.

An attempt to prove equivalences between Tihonov and Hadamard well-posedness is contained in [3]. Theorem 13 can be found in [9]. Extensions of the notion of Tihonov well-posedness and some characterizations are given in [15]. For applications to penalty techniques see [4]. Generic properties of the Tihonov well-posedness are obtained in [8].

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