ON THERMODYNAMICS AND STABILITY OF CONTINUOUS MEDIA

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1. Introduction

In the nineteenth century one of the greatest achievements in mathematical physics was the development of classical thermodynamics. Including the early years of this present century the subject established the existence of absolute temperature T and entropy S in the equilibrium states of certain types of continua. Some modern research workers in continuum thermodynamics have criticised classical thermodynamics as being merely a thermostatic theory. Although classical thermodynamics relies heavily on the notion of equilibrium this criticism is unjustified. For example, the principle of increase of entropy pertains to the initial and final states of a thermodynamic system which can experience a class of non-equilibrium transitions. However, classical thermodynamics has nothing to say about the processes connecting states and applies to a limited range of materials.

Carathéodory's ([5]) view of classical thermodynamics provides a clear distinction between the mathematical and physical contents of the subject. The theory based on inaccessibility of states of a thermodynamic system undergoing adiabatic transitions only has been well described by Buchdahl ([4]). There have been minor criticisms of Carathéodory's approach by, among others, Whaples ([17]) and Cooper ([6]), but, this approach remains sadly neglected in recent times. The work of Carathéodory has not been totally ignored in the literature. For the contributions of Valanis ([16]) and Nemat-Nasser ([11]) are based on extensions of Carathéodory's statement of the second law of thermodynamics. Unfortunately their extension is to a space of states which contains not only the usual observables of deformation

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and empirical temperature but also internal or hidden variables. These internal state variables are used to model a wide range of material behaviour including history dependence, chemical reactions and microstructural changes. Unfortunately, the extended form of the second law stated in [11], [16] needs to be able to control all the state variables. It is not clear how in practice this is to be accomplished. Indeed, Nemat-Nasser ([11]) states precisely that it is impossible to control the internal state variables.

Generally the modern phase of continuum thermodynamics has favoured clarity of exposition to physical motivation. For example it is not clear which entropy production inequality is the correct restriction on material processes. Is it the Clausius-Duhem Inequality, the Müller Inequality ([10]) or the global expressions discussed by Green and Laws ([8])? Modern thermodynamic theory takes T and S as primitive concepts and makes no attempt to relate a physical statement of the second law with an entropy production inequality. Standard practice in continuum thermodynamics today is to employ the entropy inequality as a restriction on all conceivable processes. This practice is like driving a car without ever lifting the bonnet!

Some recent contributions to thermodynamics are worthy of note. Day ([7]) starts from a Work Axiom and constructs a non-equilibrium entropy function for history-dependent materials. However, he does assume from the outset the existence of a positive absolute temperature. Silhavy ([13], [14]) has constructed non-equilibrium theories of thermodynamics based on original classical ideas. Indeed he has proposed one version of the theory which combines the first and second laws of thermodynamics.

This present series of lectures aims to describe the approach to non-equilibrium thermodynamics with which I have been associated in the recent past. The approach of Bree and Beevers ([3]) can be applied not only in non-equilibrium situations but also to a wider class of materials than does the classical theory. The basic ideas in [3] are explained and application is made to constitutive theories for elastic-plastic materials, deformable dielectrics, thermoelectric media, viscoelastic solids and general dissipative media. Some stability results for dissipative media are constructed. Asymptotically stable and Lyapounov stable classical solutions of the displacement initial-boundary value problem are found.

2. Mechanical balance laws

Let V denote the volume of a region of three dimensional Euclidean space occupied by a part P of a continuum B at time t. Let x be the position of P relative to a fixed origin and take X as the coordinates of P at some reference time $t = 0$. The equation of motion is

(2.1) $x = x(X, t)$.

Define the velocity v and deformation F by

(2.2)
$$v_k = \frac{\partial x_k}{\partial t}, \qquad F_{kK} = \frac{\partial x_k}{\partial X_K}$$

where Latin upper and lower case indices range over the values 1, 2, 3. The usual index notation operates throughout.

Conservation of mass takes the equivalent forms

(2.3)
$$\dot{\varrho} + \varrho v_{k,k} = 0, \qquad \frac{\varrho_0}{\varrho} = J = \det F > 0;$$

where ϱ and ϱ_0 are the current and reference densities of B, a superposed don't denotes the material time derivative and a comma followed by the index k indicates differentiation with respect to x_k.

The conservation of momentum has

(2.4)
$$\varrho \dot{v}_i = \sigma_{ji,j} + \varrho b_i = \frac{\varrho}{\varrho_0} (\varrho_0 b_i + \Sigma_{Ki,k})$$

where b is the external body force per unit mass, σ is the Cauchy stress and Σ is the unsymmetric Piola–Kirchhoff stress with

(2.5)
$$\sigma_{ji} = \frac{1}{J} F_{jK} \Sigma_{Ki}.$$

The conservation of angular momentum requires that

(2.6)
$$\sigma_{ij} = \sigma_{ji}.$$

It is also convenient at this point to introduce the symmetric Piola–Kirchhoff stress K with

(2.7)
$$K_{AB} = J F_{iA}^{-1} \sigma_{ij} F_{jB}^{-1}.$$

3. The laws of thermodynamics

A thermodynamic system Σ is any part P of a continuous material consisting always of the same material particles. The state variables $\xi = (\xi_1, \ldots, \xi_N)$ is that finite set of independent physical quantities which are needed to specify completely the force which holds P in equilibrium. The state variables are of two types: deformative state variables such as the volume V of a gas, and non-deformative state variables such as pressure in the gas p. At least one of the state variables must be non-deformative. If only one state variable in a set ξ is non-deformative then the set is said to be standard. A system Σ is adiabatically enclosed if its equilibrium can be disturbed by mechanical means only. It is diathermic if its equilibrium can be perturbed merely by changing its surroundings.

Classical thermodynamics relies heavily on the notion of equilibrium. For example, the zeroth law of classical thermodynamics asserts that for any system Σ in equilibrium there exists a function of state $\theta(\xi)$ which is such that the states of any two standard system Σ_A and Σ_B can coexist in equilibrium if and only if

$$(3.1) \qquad\qquad \theta_A = \theta_B.$$

θ is called the empirical temperature function. For a standard set ξ it is supposed that θ can replace any one of the state variables, ξ_N say. Then take $\xi = (\bar{\xi}, \theta)$ where $\bar{\xi} = (\xi_1, \ldots, \xi_{N-1})$ are the deformative state variables and clearly θ can be regarded as the non-deformative state variable. Further, since ξ is an independent set by definition any additional state variable must satisfy and equation of the form

$$(3.2) \qquad\qquad u = u(\xi) = u(\bar{\xi}, \theta).$$

Equations like (3.2) are called equations of state.

A compound standard system is formed when two standard systems are brought into diathermic contact. A non-standard system can be envisaged as N standard systems each separated from all the others by adiabatic partitions.

To carry over the concept of state into non-equilibrium situations it is necessary to distinguish between extensive and intensive quantities. An extensive variable, such as volume, is one whose total is the sum of its separate parts whereas an intensive variable, like pressure, is defined over some region. From an extensive variable E a corresponding intensive variable e can be defined by

$$(3.3) \qquad\qquad E = \int_V \varrho e \, dV.$$

In this way the state of a system at a point can be defined in terms of intensive variables only and irrespective of whether the system is in equilibrium or not. This is called the local state. However, the measurement of empirical temperature then poses practical problems. In fact the measurement of empirical temperature relies on the thermometer being sensitive enough. In effect the local state of a material must be in equilibrium with the measuring instrument and in many non-equilibrium situations empirical temperature can be measured. However, to the extent that such measurements may not be accurate *any* non-equilibrium theory of thermodynamics should be regarded as approximate.

For each particle of a continuous medium experiencing non-equilibrium changes associate an equivalent standard system (e.s.s). The conditions in the e.s.s. are those of the particle and are uniform across the e.s.s. Thus, as P moves from position to position its e.s.s. experiences homogeneous changes

of state from one equilibrium configuration to another. In this way the e.s.s. simulates the quasi-static changes envisaged by classical thermodynamics.

In a non-equilibrium theory additional variables are needed to describe the processes which occur. These may be gradients or time rates of change of state variables or any other quantity needed to determine the non-equilibrium changes. This approach to thermodynamics suggests that there are two types of equation in constitutive theory. We have already noted that there are equations of state and processes are governed by process equations. This is in conflict with the principle of equipresence which must be regarded as physically untenable.

The first law of thermodynamics asserts the existence of an internal energy function U the increase of which represents the energy stored in a material in a transition from one equilibrium state to another and U is a state variable. The energy equation for a non-equilibrium system Σ which occupies V at time t is

$$(3.4) \qquad \frac{d}{dt}\left(\int_V \varrho \left\{u + \tfrac{1}{2} v_k v_k\right\} dV\right) = \int_V \varrho (r + b_k v_k) dV + \int_{\partial V} \left\{\sigma_{ji} v_i - q_j\right\} n_j \, dA$$

where u is the specific internal energy, r is the rate of radiative heat supplied to Σ and q is the heat flux vector across the smooth surface ∂V of V with unit normal n.

On using the momentum equation the point form of the energy equation can be written

$$(3.5) \qquad \dot{u} = r + \dot{w} - \frac{1}{\varrho} q_{k,k} = r + \dot{w} - \frac{1}{\varrho_0} Q_{K,K}$$

where

$$(3.6) \qquad \dot{w} = \frac{1}{\varrho} \sigma_{ji} v_{i,j} = \frac{1}{\varrho_0} \Sigma_{Ki} \dot{F}_{iK}; \qquad q_k = \frac{1}{J} F_{kK} Q_K.$$

It will be shown in subsequent sections that it is always possible to put

$$(3.7) \qquad \dot{w} = f(\xi) \cdot \dot{\xi} + \dot{w}_p = \dot{w}_s + \dot{w}_p$$

where $\dot{w}_s$ is the rate of working per unit mass due to changes of state and $\dot{w}_p$ is the rate of working per unit mass due to processes. Then, since u is an additional state variable there is an equation of state

$$(3.8) \qquad u = u(\xi).$$

On combining equations (3.7) and (3.8) it follows that

$$(3.9) \qquad \left(\frac{\partial u}{\partial \xi} - f\right) \cdot \dot{\xi} + \frac{\partial u}{\partial \theta} \dot{\theta} = P(\xi) \cdot \dot{\xi} = r + \dot{w}_p - \frac{1}{\varrho} \operatorname{div} q = h$$

For the system Σ occupying volume V let

(3.10) $r = r'\omega$

where r' is the rate of exchange of radiative heat per unit mass between Σ and its surroundings whereas ω is the rate of exchange of radiative heat between any part of Σ and the remainder of Σ.

Now $\omega = \omega(x, B)$ and clearly

(3.11) $\displaystyle\int_V \varrho\omega\, dV = 0$

and ω requires its own constitutive equation. As V collapses to a point so $\omega \to 0$ and $r' \to r$.

Since conditions are uniform across the e.s.s. div $q = 0$ and in general r and $\dot{w}_p$ are different for the particle and its e.s.s. However, r is chosen so that h assumes the same value for the particle and its e.s.s.

Four kinds of transitions for Σ are now considered: an adiabatic or a-transition is one which has

(3.12) $r' \equiv 0$ in V, $q \cdot n = 0$ on ∂V.

An i_1-transition has r chosen so that

(3.13) $h \equiv 0$ in V.

An i_2-transition has $\theta = \theta(t)$ only, $q \cdot n = 0$ on ∂V and

(3.14) $\displaystyle\int_V \varrho h\, dV = 0.$

Finally, and (a, i)-transition is one any part of which is an a-transition, an i_1-transition or an i_2-transition.

The second law of thermodynamics can now be stated as follows: In the neighbourhood of any given state of a thermodynamic system Σ there are states which cannot be reached from the given state by means of an (a, i)-transition.

This statement must be considered in conjunction with Carathéodory's mathematical theorem:

If in the neighbourhood of any given point in Euclidean n-space there are points which cannot be reached along solution curves of the equation

(3.15) $P(\xi) \cdot \dot{\xi} = 0$

then there exists functions $\lambda = \lambda(\xi)$ and $\eta = \eta(\xi)$ such that

(3.16) $P \cdot \dot{\xi} = \lambda\dot{\eta}.$

In other words λ^{-1} is an integrating factor for the left hand side of equation (3.15).

4. Consequences of the second law

The second law holds for all (a, i)-transitions and in particular for an i_1-transition with $h = 0$ in V. So by Carathéodory's mathematical theorem there exists functions of state λ and η with

(4.1) $P(\xi) \cdot \dot{\xi} = h = \lambda \dot{\eta}$.

By considering two homogeneous standard systems brought into diathermic contact to form a compound standard system it can be shown that for the e.s.s.

(4.2) $h = T(\theta) \dot{s}$

where T is a universal function of empirical temperature only and s is a state variable. In fact, s can be identified as the specific entropy of the particle with total entropy in V defined by

(4.3) $S = \int\limits_V \varrho s \, dV$.

This definition is consistent with a result which can be derived for a compound standard system.

Now the local form of the energy equation becomes

(4.4) $P(\xi) \cdot \dot{\xi} = T(\theta) \dot{s}(\xi)$.

It is supposed that T is a single-valued monotone increasing function of θ so that θ can be replaced by T among the state variables. In this section $\xi = (\bar{\xi}, T)$. This cannot be deduced from this version of the second law nor is it neccessary but in this case T is called the absolute temperature function. From equations (3.9) and (4.4) it follows that

(4.5) $\dot{\psi} + s\dot{T} - f(\xi) \cdot \dot{\bar{\xi}} = 0$

where $\psi = \psi(\xi) = u - Ts$ is the specific Helmholtz free energy.

Since the $\dot{\xi}$ are arbitrary then

(4.6) $s = -\dfrac{\partial \psi}{\partial T}, \qquad f = \dfrac{\partial \psi}{\partial \bar{\xi}}$.

Notice that for an i_1-transition s is constant and the transition is locally isentropic. Whereas for an i_2-transition

(4.7) $\dot{S} = \int\limits_V \varrho \dot{s} \, dV = \dfrac{1}{T} \int\limits_V \varrho h \, dV = 0$

and hence it is globally isentropic.

Let $l_0(\bar{\xi}_0, S_0)$ be a state in a standard system Σ which after experiencing an (a, i)-transition moves to a state $l'(\bar{\xi}', S')$. The range of possible values of

S' is connected and S_0 must be an end-point of the range. For if not the second law can be contradicted. By convention S_0 is the lower end point so that during (a, i)-transitions the entropy change is always non-negative. The sign of T and $\dot{s}$ is now fixed by this convention and we can prove that $T(\theta)$ is positive provided it is assumed that neighbouring isometric states l and l' (constant deformation variables) of a standard system connected by a-transitions are such that $u' > u$.

Again, using an argument of contradiction it can be shown that the entropy change in a non-standard system undergoing (a, i)-transitions only must increase. Hence it follows for a non-equilibrium system

$$(4.8) \qquad\qquad\qquad\qquad \dot{S} \geqslant 0$$

for all (a, i)-transitions. Now

$$(4.9) \qquad\qquad S = \int_V \varrho \dot{s} \, dV = \int_V \frac{\varrho \dot{h}}{T} \, dV = R + \dot{S}_1$$

where

$$(4.10) \qquad R = \int_V \frac{\varrho r'}{T} \, dV - \int_{\partial V} \frac{q \cdot n}{T} \, dA = \int_{V_0} \frac{\varrho_0 r' \, dV_0}{T} - \int_{\partial V_0} \frac{Q \cdot N}{T} \, dA_0$$

and

$$(4.11) \qquad \dot{S}_1 = \int_V \left\{ \frac{\varrho(\omega + \dot{w}_p)}{T} - \frac{q g}{T^2} \right\} dV = \int_{V_0} \left\{ \frac{\varrho_0(\omega + \dot{w}_p)}{T} - \frac{Q \cdot G}{T^2} \right\} dV_0$$

with $g_k = T_{,k}$ and $G_K = T_{,K}$ as the spatial and referential temperature gradients. By the principle of increase of entropy for all (a, i)-transitions it is clear that $\dot{S}_1 \geqslant 0$ when $r' = 0$ in V and $q \cdot \eta = 0$ on ∂V. So, the inequality

$$(4.12) \qquad\qquad \dot{S} \geqslant \int_V \frac{\varrho r'}{T} \, dV - \int_{\partial V} \frac{q \cdot \eta}{T} \, dA$$

is a sufficient condition for consistency with the second law of thermodynamics.

Suppose the terms in $\dot{S}_1$ are independent of the rates of change of the state variables $\dot{\xi}$ then $\dot{S}_1$ does not change instantaneously when the heating is switched on. Thus, inequality (4.12) is also a necessary condition. Indeed, if $\dot{S}_1$ is independent of just one of the rates $\dot{\xi}$, say $\dot{T}$, an a-transition can be found for which the values of $\dot{\xi}$ take the same value as in $\dot{S}_1$ and the energy equation is balanced by altering $\dot{T}$. Again in this case inequality (4.12) is a necessary and sufficient condition for consistency with thermodynamic theory. However, if all the rates of change $\dot{\xi}$ appear in $\dot{S}_1$ then when r' is

switched on they can all change in value instantaneously and $\dot{S}_1$ can change sign. In this case inequality (4.12) is not a necessary condition.

Inequality (4.12) is inherently global in the sense that it cannot be reduced to an equivalent local form. At a point inequality (4.12) becomes the usual form of the Clausius–Duhem inequality

$$(4.13) \qquad \varrho \dot{s} \geqslant \frac{\varrho r}{T} - \nabla \cdot \left(\frac{q}{T}\right).$$

Notice, this is a *derived* result in this theory.

To show another difference between the Clausius–Duhem inequality and the above theory consider the example: Two standard systems Σ_α and Σ_β are at temperatures T_α, T_β adiabatically enclosed from each other and the environment. Remove adiabatic partition between them and let internal radiation be the only process to occur. Let ω_α and ω_β be radiation rates to Σ_α and Σ_β then $\omega_\alpha + \omega_\beta = 0$. Then

$$(4.14) \qquad \dot{S} = \frac{\omega_\alpha}{T_\alpha} + \frac{\omega_\beta}{T_\beta} = \frac{\omega_\alpha}{T_\alpha T_\beta} (T_\beta - T_\alpha) \geqslant 0.$$

This cannot be deduced from the Clausius–Duhem inequality.

5. Applications to constitutive theory

To complete a mathematical model of a real material a constitutive theory must be constructed. This theory should have equations which are based on physical experience, are independent of the choice of axes and be properly invariant. They must also be consistent with thermodynamic theory. The modelling in this section will lay emphasis on the latter principle and the others are automatically satisfied.

Several models are considered including finitely deformed isotropic elastic-plastic materials, deformable dielectrics, one-dimensional thermoelectric solids and one dimensional visco-elastic continua. This leads naturally to dissipative media in general.

Internal radiation effects are neglected in the rest of these lectures.

5.1. Finite deformation elastic-plastic theory. The classical theory of thermodynamics is unable to provide a sound thermodynamic foundation for the theory of elastic-plastic deformations. One of the main consequences is that the plastic work rate is not necessarily positive. This is consistent with the Bauschinger effect.

For now it is necessary to assume that elastic properties are not influenced by plastic deformation. It is necessary to assume that plastic deformations do not contribute to volume changes since that would show itself as a history dependent effect on the elastic properties. This is not a bad

assumption for metal plasticity. The material is assumed to be initially in an annealed state, which is stress free and at some reference temperature. When a stress is applied to this reference state an elastic deformation occurs with

$$(5.1) \quad E_{KL} = \tfrac{1}{2}(C_{KL} - \delta_{KL}) = E_{KL}^{(e)} = \tfrac{1}{2}(F_{kK} F_{kL} - \delta_{KL}) = \tfrac{1}{2}\{F_{kK}^{(e)} F_{kL}^{(e)} - \delta_{KL}\}$$

where $E^{(e)}$ is elastic strain, $F^{(e)}$ is elastic deformation and δ_{KL} are the components of the Kronecker delta function.

Define a Piola–Kirchhoff stress by

$$(5.2) \quad K = JF^{(e)^{-1}} \sigma F^{(e)^{-1}} = K(E^{(e)}, \theta)$$

where K is the force which holds the material in equilibrium. Provided a point (K, θ) in stress-temperature space does not cross the yield surface the deformation remains elastic. On crossing the yield surface the local reference stress free configuration is changed and becomes plastically deformed. However since the elastic properties are unaffected by plastic deformations then equation (5.2) remains valid but now referred to the local plastically deformed configuration. Now the deformation becomes

$$(5.3) \quad F_{iK} = R_{iM}^{(e)} U_{MN}^{(e)} U_{NK}^{(p)} = R_{iM} U_{MK}$$

where $U^{(p)}$ is a positive definite, symmetric plastic stretch tensor and the orientation of the plastically deformed reference configuration is chosen to be that of the original reference configuration. The plastic strains $C^{(p)}$ and $E^{(p)}$ are defined by

$$(5.4) \quad C_{KL}^{(p)} = U_{KM}^{(p)} U_{ML}^{(p)} = 2E_{KL}^{(p)} + \delta_{KL}.$$

So $U^{(p)}$, $C^{(p)}$ and $E^{(p)}$ provide local strain measures of the local plastically deformed configuration with reference to the initial reference configuration. So with $F^{(e)} = R^{(e)} U^{(e)}$ and $F^{(p)} = U^{(p)}$ then

$$(5.5) \quad F = F^{(e)} F^{(p)}$$

and $F^{(e)}$, $F^{(p)}$ are respectively the elastic and plastic deformation tensors. Since plastic deformation does not contribute to volume changes

$$(5.6) \quad \det F^{(p)} = 1, \quad J = \det F^{(e)}.$$

At the onset of plastic flow

$$(5.7) \quad U^{(p)} = C^{(p)} = I, \quad E^{(p)} = O, \quad R^{(e)} = R, \quad U^{(e)} = U$$

and these provide initial conditions which together with equation (5.2) and constitutive equations for plastic deformation rate can be used to calculate their subsequent values.

In this case then $(E^{(e)}, \theta)$ are the independent state variables. There exists an equation of state for the specific internal energy

$$(5.8) \quad u = u(E^{(e)}, \theta).$$

The rate of doing work per unit mass by the stresses in addition to that which contributes to the kinetic energy is

(5.9) $$\dot{w} = \frac{1}{\varrho_0} K_{MN} \dot{E}_{MN}^{(e)} + \frac{1}{\varrho_0} K_{MN} C_{NL}^{(e)} \dot{F}_{LR}^{(p)} F_{MR}^{(p)-1}.$$

For an isotropic material the Cauchy stress σ and the stretch tensor $V_{ij}^{(e)} = R_{iK}^{(e)} R_{jL}^{(e)} U_{KL}^{(e)}$ have the same principal directions so it can be shown that

(5.10) $$s_{MN} = K_{ML} C_{LN}^{(e)} = J R_{iM}^{(e)} R_{jN}^{(e)} \sigma_{ij}.$$

Since s is symmetric the work rate $\dot{w}$ can be written

(5.11) $$\dot{w} = f(\xi) \cdot \dot{\xi} + \frac{1}{\varrho_0} s_{KL}' D_{KL}^{(p)}$$

where

(5.12) $$D_{KL}^{(p)} = \frac{1}{2}(\dot{U}_{KM}^{(p)} U_{ML}^{(p)-1} + \dot{U}_{LM}^{(p)} U_{MK}^{(p)-1});$$

and since $D_{KK}^{(p)} = 0$ we have introduced the deviatoric stress s' with

(5.13) $$s_{KL}' = s_{KL} - \frac{1}{3} s_{pp} \delta_{KL}.$$

Thus the local form of the energy equation is

(5.14) $$P(\xi) \cdot \dot{\xi} = r + \dot{w}_p - \frac{1}{\varrho_0} Q_{K,K} = \dot{h};$$

where

(5.15) $$P = \left(\frac{\partial u}{\partial E^{(e)}} - \frac{1}{\varrho_0} K, \frac{\partial u}{\partial \theta} \right).$$

So as in the general case T and s exist with

(5.16) $$\dot{h} = T(\theta) \dot{s}(E^{(e)}, \theta).$$

It is assumed that T replaces θ in the list of state variables. Then

(5.17) $$s = -\frac{\partial \psi}{\partial T}, \qquad K = \varrho_0 \frac{\partial \psi}{\partial E^{(e)}}$$

where $\psi = u - Ts$ is the specific Helmholtz free energy.

Further, if at least one of the rates of change of the state variables is absent from the process equations then the inequality

(5.18) $$\varrho_0 \dot{s} \geqslant \frac{\varrho_0 r}{T} - \left(\frac{Q_K}{T} \right)_{,K}$$

where $Q = J F^{(e)-1} q$ is both necessary and sufficient for consistency with thermodynamics. However, if all the rates of change of the state variables do

occur in the process equations then inequality (5.18) is merely sufficient. On employing the energy equation (5.18) becomes

$$\text{(5.19)} \qquad \varrho_0 \, \dot{w}_p - \frac{Q_K \, G_K}{T} \geqslant 0,$$

where $G = F^{(e)^{-1}} g$. During elastic deformations $\dot{w}_p = 0$ and heat conduction is the only process which occurs whereas during yielding plastic flow and heat conduction are two processes each of which requires a process equation. Plasticity theory does not have a sound thermodynamic base from classical theory since $\dot{w}_p$ does not vanish in the quasi-static limit.

To complete a constitutive theory for heat conducting plasticity process equations must be prescribed for plastic flow and heat flow. An appropriate stress measure appears to be s' so take the yield surface equation as

$$\text{(5.20)} \qquad f(s'_{KL}, T, \varkappa) = g(s'_{KL}) - c(T, \varkappa) = 0$$

where $\varkappa = \varkappa(\eta_A)$, $A = 1, 2, \ldots$ and

$$\text{(5.21)} \qquad \dot{\varkappa} = 0 \qquad \text{when} \qquad D^{(p)} = O.$$

Two possible choices for the parameters η are:

$$\eta_1 = \int_0^t \{D^{(p)}_{KL} D^{(p)}_{KL}\}^{1/2} \, dt, \qquad \eta_2 = \int_0^t s'_{KL} D^{(p)}_{KL} \, dt.$$

Then, it is assumed that

$$\text{(5.22)} \qquad D^{(p)}_{KL} = \begin{cases} H_{KL}(s'_{MN}, T, \varkappa)\alpha, & \alpha \geqslant 0, f = 0, \\ 0, & f < 0 \end{cases}$$

where $\alpha = \dfrac{\partial g}{\partial s'_{MN}} \dot{s}'_{MN} - \dfrac{\partial c}{\partial T} \dot{T}$. The equations (5.22) with $\alpha > 0$, $\alpha = 0$ and f < 0 are said to hold during loading, neutral loading and unloading respectively. Here it is assumed that plastic deformation is independent of time scales. Then, with $\alpha = \lambda_{KL} D^{(p)}_{KL}$, H is not arbitrary and satisfies

$$\text{(5.23)} \qquad \lambda_{KL} H_{KL} = 1, \qquad H_{KK} = 0.$$

The equation for heat conduction is

$$\text{(5.24)} \qquad Q_K = Q_K(E^{(e)}_{MN}, T, T_{,L}).$$

During loading the process equations involve the rates of change of all the state variables so inequalities

$$\text{(5.25)} \qquad s'_{KL} H_{KL} \geqslant 0, \qquad Q_K G_K \leqslant 0$$

are sufficient conditions for consistency with thermodynamic theory and they are not necessary restrictions. Thus, the plastic work rate need not be

positive and this is consistent with the experimentally observed Bauschinger effect. Other authors account for this but only when history dependent states are considered (see [9]).

During neutral loading and unloading $Q_K G_K \leqslant 0$ is both necessary and sufficient for consistency with the thermodynamic theory (see [2]).

5.2. Deformable dielectrics. The equations of electromagnetism for a deformable dielectric have been given by Toupin [15] as

(5.26)
$$\frac{\partial b}{\partial t} + \text{curl } e = O, \qquad \text{div } b = 0;$$

$$-\frac{\partial d}{\partial t} + \text{curl } h = O, \qquad \text{div } d = 0;$$

where b, h, d and e denote the magnetic induction, magnetic field, electric displacement and electric field respectively. In a moving dielectric

(5.27) $$b = \mu(h - v \wedge p), \qquad d = \varepsilon e + p$$

where p is the polarisation vector and ε, μ are material constants. Introduce the material form of the polarisation through the equation

(5.28) $$p = J^{-1} F\pi.$$

Into this model introduce a very simple form of viscous damping so that the momentum equation can be written

(5.29) $$\varrho_0(\dot{v}_i + \lambda v_i) = \Sigma_{Ki,K} + \varrho_0 f_i - \pi_{K,K} \varepsilon_i + \varepsilon_{ijk} F_{jM} \dot{\pi}_M b_k$$

where λ is another material constant and

(5.30) $$\varepsilon = e + v \wedge b.$$

Balancing energy and taking account of electro-mechanical interactions leads to

(5.31) $$\varrho_0 \dot{u} = \varrho_0 r + K_{MN} \dot{E}_{MN} + \bar{e}_K \dot{\pi}_K - Q_{K,K} + \varrho_0 \dot{w}_p$$

where

(5.32) $$\bar{e}_K = J^{-1} F_{kK} \varepsilon_k; \qquad \dot{w}_p = \lambda v_k v_k.$$

With

(5.33) $$K = K(E, \pi, \theta)$$

the state variables are determined so $u = u(E, \pi, \theta)$. Hence,

(5.34) $$K = \varrho_0 \frac{\partial \psi}{\partial E}, \qquad \bar{e} = \varrho_0 \frac{\partial \psi}{\partial \pi}, \qquad s = -\frac{\partial \psi}{\partial T}$$

and the processes of viscous damping and heat conduction are restricted by

$$\lambda \geqslant 0, \qquad QG \leqslant 0.$$

5.3. Thermoelectricity. From Maxwell's equations in a one-dimensional, non-polarisable thermoelectric material it follows that b is a constant and the electric field e and current j satisfy $e = e(t)$ and

$$(5.35) \qquad j = -\varepsilon \frac{de}{dt}.$$

The force which holds the material in equilibrium is

$$(5.36) \qquad K = \sigma F^{-1} = K(E, \theta)$$

where

$$(5.37) \qquad F = \frac{\partial x}{\partial X}, \qquad E = \tfrac{1}{2}(F^2 - 1).$$

The specific internal energy u is a state variable with $u = u(E, \theta)$. Then, the energy equation becomes

$$(5.38) \qquad \dot{u} - \frac{1}{\varrho_0} K \dot{E} = r + \dot{w}_p - \frac{1}{\varrho} \frac{\partial q}{\partial x} = h = T\dot{s}$$

where $\varrho w_p = ej$, $T = T(\theta)$ and $s = s(E, \theta)$. By a standard argument

$$(5.39) \qquad s = -\frac{\partial \psi}{\partial T}, \qquad K = \varrho_0 \frac{\partial \psi}{\partial E}$$

where $\psi = u - Ts$ and T replaces θ in the list of state variables. It should be emphasised here that the theory of thermoelectricity cannot be treated rigorously by classical thermodynamics. As for elastic-plastic materials w_p does not vanish in the quasi-static limit. However, the above thermodynamic theory is capable of rigorous application in this case.

A further consequence of the second law is that

$$(5.40) \qquad ej - \frac{qg}{T} \geqslant 0$$

where $g = \partial T/\partial x$. The two processes of heat conduction and electrical conduction are governed by process equations.

For simplicity now assume that the material is rigid and incompressible. Take

$$(5.41) \qquad e = \frac{j}{\sigma} + Pg, \qquad q = \pi j - \varkappa g$$

where σ is the resistance, P is the thermoelectric constant π is the Seebeck

coefficient and $\varkappa$ is the thermal conductivity. It is supposed that $TP(T)$ $= \pi(T)$ so the entropy inequality reduces to

$$(5.42) \qquad \frac{j^2}{\sigma} + \frac{\varkappa}{T} g^2 \geqslant 0.$$

Thus, σ and $\varkappa$ are non-negative and inequality (5.42) is both necessary and sufficient to comply with thermodynamic theory since T does not appear in the process equations. Notice that from this simple constitutive theory electric field can exist in the absence of an electric current and heat can be driven *up* a temperature gradient. For a practical description of these effects see [12].

Finally with

$$(5.43) \qquad T\frac{dP}{dT} = \frac{d\pi}{dT} - \frac{\pi}{T} = -\beta$$

the energy equation can be written

$$(5.44) \qquad \varrho u = \varrho Ts = \varrho r + \frac{\partial}{\partial x}\left(\varkappa \frac{\partial T}{\partial x}\right) + \frac{j^2}{\sigma} + \beta j \frac{\partial T}{\partial x}.$$

The quantity β is called *the Thomson coefficient*.

5.4. Viscoelastic materials. The stress K can be split up into two parts

$$(5.45) \qquad K = K^{(e)} + K^{(v)}$$

where $K^{(e)}$ is the elastic stress and $K^{(v)}$ is the viscous stress. The force which holds the e.s.s. in equilibrium is the elastic stress and

$$(5.46) \qquad K^{(e)} = K^{(e)}(E, \theta).$$

The viscous stress is such that $K^{(v)} = K^{(v)}(E, \dot{E}, \theta)$ and $K^{(v)} \to 0$ as $\dot{E} \to 0$. The state variables are (E, θ) so the equation of state for u is

$$(5.47) \qquad u = u(E, \theta).$$

From the energy equation

$$(5.48) \qquad T(\theta) = \frac{\partial u}{\partial s}, \qquad K^{(e)} = \varrho_0 \frac{\partial \psi}{\partial E}.$$

The residual entropy inequality requires

$$(5.49) \qquad \int_L \left\{ \varrho_0 \frac{\omega}{T} + K^{(v)}\dot{E} - \frac{QG}{T^2} \right\} dX \geqslant 0; \qquad K^{(v)}\dot{E} - \frac{QG}{T} \geqslant 0.$$

Since θ does not appear in the terms in these inequalities they are both necessary and sufficient for consistency with thermodynamics.

6.   A simplified model

For ease of exposition consider a one-dimensional, linear theory for a continuum L contained in $x \in [0, 1]$. In this case the mass per unit length or density ϱ is a constant. Further the displacement between the current and initial positions of a typical particle P of L is given by

$$(6.1) \qquad\qquad d = x - X.$$

Hence the speed $v = \dot{x} = \dot{d}$. The equation of linear momentum in the absence of body forces is assumed in the form

$$(6.2) \qquad\qquad \varrho(\ddot{d} + \lambda \dot{d}) = \frac{\partial \sigma}{\partial x}$$

where σ is the only non-zero component of the Cauchy stress and λ is a material constant.

The energy equation in the absence of external heat supply is

$$(6.3) \qquad\qquad u = w - \frac{1}{\varrho} \frac{\partial q}{\partial x}$$

where

$$\varrho w = \sigma \frac{\partial v}{\partial x} + \varrho w_p = \sigma D + \varrho \lambda v^2, \qquad D = \frac{\partial d}{\partial x}.$$

In this dissipative model let

$$\sigma = \sigma(D, \theta).$$

Then, from the thermodynamic theory

$$(6.4) \qquad\qquad s = -\frac{\partial \psi}{\partial T}, \qquad \sigma = \varrho \frac{\partial \psi}{\partial D}$$

where $\psi = u - Ts$. Then, the energy equation reduces to

$$(6.5) \qquad\qquad \varrho T\dot{s} = \varrho w_p - \frac{\partial q}{\partial x}.$$

The boundary and initial conditions for equations (6.2) and (6.5) are:

$$(6.6) \qquad d(0, t) = d(1, t) = 0, \qquad T(0, t) = T(1, t) = T_0, \qquad t \in (0, \infty);$$

and

$$(6.7) \qquad d(x, 0) = d_0(x), \qquad \dot{d}(x, 0) = v_0, \qquad T(x, 0) = T_0, \qquad x \in (0, 1).$$

Introduce the following measures:

(6.8)
$$z(t) = \int_0^1 \varrho d^2(x, t)\,dx, \qquad k(t) = \tfrac{1}{2}\int_0^1 \varrho v^2(x, t)\,dx,$$

$$\phi(t) = \int_0^1 \varrho \psi(x, t)\,dx.$$

Multiply the momentum equation (6.2) by the speed d and integrate over $[0, 1]$ to obtain

(6.9)
$$\dot{k} + \dot{\phi} + \int_0^1 \varrho s \dot{\tau}\,dx = -\int_0^1 \varrho \dot{w}_p\,dx$$

where $\tau = T - T_0$.

Multiply the energy equation (6.5) by τT^{-1} and integrate over $[0, 1]$ to deduce that

(6.10)
$$\int_0^1 \varrho \tau \dot{s}\,dx = \int_0^1 \left\{ \varrho \dot{w}_p \frac{\tau}{T} + \frac{qgT_0}{T^2} \right\} dx$$

where it is assumed that T_0 is independent of x.

On adding equations (6.9) and (6.10) it is found that

(6.11)
$$\frac{dl}{dt} = \frac{d}{dt}\left\{ k + \phi + \int_0^1 \varrho s \tau\,dx \right\} = -\int_0^1 \frac{T_0}{T}\left\{ \varrho \dot{w}_p - \frac{qg}{T} \right\} dx.$$

Recall that thermodynamic theory requires

(6.12)
$$\varrho \dot{w}_p - \frac{qg}{T} \geqslant 0$$

provided $\dot{D}$ and $\dot{T}$ do not both occur in the process equations for $\dot{w}_p$ and q in this case. Hence, $l(t)$ is bounded by its initial value $l_0 = k_0 + \phi_0$.

However, better results are possible. Before equation (6.11) is exploited further multiply equation (6.2) by d and integrate over $[0, 1]$ to form

(6.13)
$$\tfrac{1}{2}\ddot{z} - 2k + \tfrac{1}{2}\lambda\dot{z} = -\int_0^1 \sigma D\,dx.$$

7. A mechanical theory

For a purely isothermal theory with zero heat flux then equation (6.11) reduces to

(7.1)
$$\dot{k} + \dot{\phi} = -\int_0^1 \varrho \dot{w}_p\,dx.$$

Moreover, for a linear model ψ is quadratic in D and so equation (6.13) provides

(7.2) $\qquad \frac{1}{2}\dot{z} - 2k + 2\phi + \frac{1}{2}\lambda\dot{z} = 0.$

In this case $\dot{w}_p = \lambda v^2$ so equation (7.1) becomes

(7.3) $\qquad \dot{k} + \dot{\phi} = -2\lambda k.$

On combining equations (7.2) and (7.3) we obtain

(7.4) $\qquad \dot{k} + \dot{\phi} + \frac{1}{4}\lambda\ddot{z} + \lambda(k + \phi + \frac{1}{4}\lambda\dot{z}) = 0.$

Clearly, then

(7.5) $\qquad k + \phi + \frac{1}{4}\lambda\dot{z} = f_0 e^{-\lambda t}$

and it is supposed that

$$f_0 = k_0 + \phi_0 + \frac{1}{4}\lambda\dot{z}_0$$

is positive. Thermodynamics requires that λ be non-negative but for subsequent development here λ is assumed to be strictly positive.

Now, from

(7.6) $\qquad d(x, t) - d(x, 0) = \int\limits_0^x D(y, t)\,dy$

it follows that

(7.7) $\qquad z(t) \leqslant \frac{1}{2}\int\limits_0^1 \varrho D^2\,dx.$

This is equivalent to a Poincaré inequality in one dimension. For this linear mechanical model

(7.8) $\qquad \phi = \frac{1}{2}\int\limits_0^1 c D^2\,dx = \frac{1}{2}\bar{c}\int\limits_0^1 \varrho D^2\,dx$

where $\bar{c} > 0$. A number of possibilities now arise and we consider two of them.

THEOREM 7.1. *With the assumptions above classical solutions of the displacement initial-boundary value problem, if they exist, are such that*

$$\lim_{t \to \infty} k(t) = 0, \qquad \lim_{t \to \infty} z(t) = 0$$

provided $0 < \lambda < 2\sqrt{\bar{c}}.$

Proof. By means of a weighted arithmetic-geometric mean inequality

(7.9) $\qquad -\frac{\lambda}{4}\dot{z} \leqslant \frac{\lambda}{4}\left(\gamma z + \frac{2k}{\gamma}\right)$

where γ is a positive weight. On choosing $\gamma = \lambda$ equation (7.5) and (7.8) and inequalities (7.7) and (7.9) combine to produce

(7.10) $\frac{1}{2}k + (\bar{c} - \frac{1}{4}\lambda^2)z \leqslant f_0 e^{-\lambda t}$.

So for small enough damping the theorem is proved.

THEOREM 7.2. *With the same assumptions as Theorem 7.1, the classical solution of the displacement initial-boundary value problem, if it exists, has* $\lim\limits_{t \to \infty} z(t) = 0$ *for all values of* λ.

Proof. We take the proof in three parts. Since k is always non-negative it can be deduced from equation (7.5) using (7.7) and (7.8) that

(7.11) $\dot{z} + \frac{4\bar{c}}{\lambda} z \leqslant \frac{4f_0}{\lambda} e^{-\lambda t}$.

(i)   For $\bar{c} > \frac{1}{4}\lambda^2$ then

(7.12) $z(t) \leqslant z_0 e^{-4\bar{c}t/\lambda} + \frac{f_0}{\bar{c} - \frac{1}{4}\lambda^2} e^{-\lambda t}$.

(ii)   If $4\bar{c} = \lambda^2$ then

(7.13) $z(t) \leqslant \left(z_0 + \frac{4f_0 t}{\lambda}\right) e^{-\lambda t}$.

(iii) Finnaly, with $\lambda^2 > 4\bar{c}$ it follows that

(7.14) $z(t) \leqslant \left(z_0 + \frac{f_0}{\frac{1}{4}\lambda^2 - \bar{c}}\right) e^{-4\bar{c}t/\lambda}$.

So for a full range of λ the theorem is proved. ∎

As a corollary of this theorem, inequality (7.10) indicates that $\lim\limits_{t \to \infty} k(t) = 0$ for all values of λ.

8.   A thermo-mechanical theory

Some of the ideas of Section 7 can be extended into a thermomechanical theory. For this, consider again equation (6.11) when it is assumed that

(8.1) $\varrho\psi = \frac{1}{2}cD^2 - bD\tau - \frac{1}{2}a\tau^2, \qquad q = -\varkappa g$;

where a, b, c and $\varkappa$ are material constants. Thermodynamic theory requires that $\varkappa$ is non-negative and a positive but for stability we will suppose $\varkappa > 0$ too. Notice that

(8.2) $\phi + \int_0^1 \varrho s\tau \, dx = p + h = \frac{1}{2}\bar{c}\delta + h \geqslant \bar{c}z + h$;

where

$$\delta = \int_0^1 \varrho D^2 \, dx, \qquad h = \tfrac{1}{2} \int_0^1 a\tau^2 \, dx.$$

It can also be demonstrated that

(8.3) $$\tfrac{1}{2} \int_0^1 g^2 \, dx \geqslant \int_0^1 \tau^2 \, dx.$$

So from (8.2) and (8.3) equality (6.11) with the same assumption as Section 7 becomes

(8.4) $$\dot{k} + \dot{p} + \dot{h} + 2\bar{\lambda}k + 2\bar{\mu}h \leqslant 0$$

where

$$\bar{\lambda} = \frac{\lambda T_0}{\max_{x \in [0,1]} T} \quad \bar{\mu} = \frac{2\varkappa T_0}{a} \left\{ \max_{x \in [0,1]} (T^2) \right\}^{-1}.$$

Further, in this linear model equation (6.13) takes the form

(8.5) $$\tfrac{1}{2}\ddot{z} - 2k + 2\phi + \int_0^1 \varrho s\tau \, dx + \tfrac{1}{2}\lambda \dot{z} = 0.$$

The approach when $\tau = 0$ has been given is Section 7.

   To illustrate the method further let us consider the problem with small thermo-mechanical coupling term b. Then

(8.6) $$2\phi + \int_0^1 \varrho s\tau \, dx = 2p - \int_0^1 bD\tau \, dx.$$

So equations (8.5) and (8.6) yield

(8.7) $$\tfrac{1}{2}\ddot{z} - 2k + 2p + \tfrac{1}{2}\lambda \dot{z} \leqslant \beta_1 p + \beta_2 h$$

where

$$\beta_1 = \frac{b}{2\varrho\bar{c}}, \qquad \beta_2 = \frac{b}{a}.$$

Inequalities (8.4) and (8.7) imply that

(8.8) $$\dot{k} + \dot{p} + \dot{h} + \tfrac{1}{4}v\ddot{z} + \alpha(k + p + h) + \tfrac{1}{4}\lambda v\dot{z} \leqslant 0$$

where

(8.9) $$v = \min(\bar{\lambda}, \bar{\mu}), \qquad \alpha = \min\{v(1 - \tfrac{1}{2}\beta_1), v(2 - \tfrac{1}{2}\beta_2)\}.$$

Notice that $\alpha \leqslant v \leqslant \bar{\lambda} \leqslant \lambda$. Thus, we can write

(8.10) $$\frac{d}{dt}\{(k + p + h + \tfrac{1}{4}v\dot{z})e^{\alpha t/2}\} + \{\tfrac{1}{2}\alpha(k + p + h) + \tfrac{1}{4}v(\lambda - \tfrac{1}{2}\alpha)\dot{z}\}e^{\alpha t/2} \leqslant 0.$$

So, we can prove

THEOREM 8.1. *Without restriction on either the coupling constant or the damping term, the classical solution of the initial-boundary value problem for linear thermoelasticity, if it exists, has*

$$\lim_{t \to \infty} k(t) = 0, \qquad \lim_{t \to \infty} h(t) = 0.$$

Proof. Integration of inequality (8.4) over $[0, t]$ leads to

$$(8.11) \qquad \int_0^t \{k(y) + h(y)\}\, dy \leqslant \frac{1}{2v}(k_0 + p_0)$$

on the additional assumption that $p \geqslant 0$. This result is similar to one obtained by Batra [1] for heat conducting, linearly viscous fluids.

Finally, we prove

THEOREM 8.2. *For sufficiently small coupling and damping the classical solution of the initial-boundary value problem for linear thermoelasticity, if it exists, has*

$$\lim_{t \to \infty} z(t) = 0, \qquad \lim_{t \to \infty} k(t) = 0, \qquad \lim_{t \to \infty} h(t) = 0.$$

Proof. From inequality (8.10) it follows that

$$(8.12) \qquad k + p + h + \tfrac{1}{4}v\dot{z} + \tfrac{1}{4}v(\lambda - \tfrac{1}{2}\alpha)\dot{z} - g_0 e^{-\alpha t/2} +$$

$$+ \tfrac{1}{2}\alpha \int_0^t \{k + p + h - \tfrac{1}{4}v(\lambda - \tfrac{1}{2}\alpha)z\}\, e^{\alpha(s-t)/2}\, ds \leqslant 0$$

where

$$g_0 = k_0 + p_0 + \tfrac{1}{4}v\dot{z}_0 + \tfrac{1}{4}v(\lambda - \tfrac{1}{2}\alpha)z_0$$

is supposed positive. Thus, for $\bar{c} > \tfrac{1}{4}v(\lambda - \tfrac{1}{2}\alpha)$ it can be shown that

$$(8.13) \qquad k + p + h + \tfrac{1}{4}v\dot{z} + \tfrac{1}{4}v(\lambda - \tfrac{1}{2}\alpha)z \leqslant g_0 e^{-\alpha t/2}.$$

With $p > \bar{c}z$ another weighted arithmetic-geometric mean inequality (as in the proof of Theorem 7.1) completes the proof. Provided the thermoelastic coupling remains small it is possible, as in Section 7, to extend this result for all values of the damping term.

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