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ON COMPOSITION OF FUNCTIONS AND MEASURES

RYSZARD RUDNICKI

Institute of Mathematics, Polish Academy of Sciences
Staromiejska 8/6, 40-013 Katowice, Poland

Institute of Mathematics, Silesian University
Bankowa 14, 40-007 Katowice, Poland

Abstract. Let μ be a vector measure with values in $\mathbb{R}^k$ and $f : \mathbb{R}^k \to \mathbb{R}$. A measure ν is called the composition of f and μ if for every δ-sequence (δ_n) the sequence $f(\mu * \delta_n)$ is weakly convergent to ν. We give sufficient conditions for the existence of the composition and examples of geometric applications.

Introduction. The classical formula for the arc length of a curve

$$l = \int_a^b \sqrt{1 + f'^2(x)}\, dx \tag{1}$$

can only be applied for curves given by smooth functions. We can generalize this formula to non-smooth functions if we replace the function f by the sequence of convolutions of f with a δ-sequence (δ_n). Then the length of the curve is the limit

$$l = \lim_{n\to\infty} \int_a^b \sqrt{1 + (f * \delta_n)'^2(x)}\, dx. \tag{2}$$

This formula follows from the fact that $(f * \delta_n)$ is a sequence of smooth functions converging uniformly to f.

If f is a function of bounded variation then we can introduce a measure μ given by $\mu([a, x)) = f(x) - f(a)$ and replace $(f * \delta_n)'$ in (2) by the convolution $\mu*\delta_n$. This leads to the following problem. Let μ be a vector Borel measure on the space $X = \mathbb{R}^p$ with the values in the space $Y = \mathbb{R}^k$ and let f be a real function defined on the space Y. Find conditions on f and μ so that for any δ-sequence

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and $\varphi \in C_0(X)$ the limit

$$\lim_{n\to\infty} \int\limits_X f(\mu * \delta_n(x))\varphi(x)\,dx \tag{3}$$

exists. If there exists a vector measure ν on X such that the last limit equals $\int_X \varphi(x)\,\nu(dx)$, then the measure ν can be called the *composition* of f and μ. In this paper we give sufficient conditions for the existence of this composition.

The question of the existence of the composition of a function and a measure is a part of a more general problem: Let f and g be two distributions, $f_n = f * \delta_n$ and $g_n = g * \delta_n$, where (δ_n) is a δ-sequence of smooth functions. When is the sequence of compositions $f_n \circ g_n$ distributionally convergent? The limit of this sequence is called the composition of f and g. In [2] it is proved that if g is a smooth function with non-vanishing derivative, then $f \circ g$ exists and coincides with earlier definitions of composition (see [1]). However, we cannot expect that the composition of two distributions exists for a large class of distributions. For example, in [2] it was shown that if $f(x) = |x|$, then the composition $f \circ g$ exists if and only if g is a measure. This is the reason why we restrict our investigation to the composition of a function and a measure.

In this paper we do not assume that the functions in a δ-sequence are smooth. There are two reasons for this. Firstly, smoothness does not play any role in proofs. The more important reason is that our theorems can easily be generalized to measures on spaces without any linear structure, e.g. topological groups or even topological spaces. In the second case we need to replace the convolutions $\mu * \delta_n$ by a sequence of integrable functions g_n converging weakly to μ. We resign from this general approach to avoid unnecessary technical difficulties.

The paper is divided into three sections. The main results are formulated in Section 1. Section 2 contains some remarks and examples concerning our theorems. In this section we also give two geometrical applications of our results. Section 3 contains some auxiliary lemmas and proofs.

1. Main results. Before formulating our theorems we introduce some preliminary notions.

Let $X = \mathbb{R}^p$, $Y = \mathbb{R}^k$ and let $\mathcal{B}(X)$ be the family of Borel subsets of X. Denote by m the Lebesgue measure on X. By a *δ-sequence* we understand a sequence $\delta_n : X \to \mathbb{R}$ of Borel functions integrable with respect to m such that

(a) $\delta_n \ge 0$,
(b) $\int_X \delta_n\, dm = 1$,
(c) $\operatorname{supp} \delta_n \subset K\big(0, \frac{1}{n}\big)$,

where $K(0, r)$ is the closed ball in X with radius r.

Let $\mathcal{B}(X)$ be the family of Borel subsets of X and let $\mu = (\mu_1, \ldots, \mu_k) : \mathcal{B} \to Y$ be a vector Borel measure on X. Let $\mu_i = \mu_i^+ - \mu_i^-$, where μ_i^+ and μ_i^- are the positive and negative variations of μ_i. If A is any Borel subset of X, then by

$\|\mu\|(A)$ we denote the total variation of the measure μ on A, i.e.

$$\|\mu\|(A) = \sum_{i=1}^{k} \mu_i^+(A) + \mu_i^-(A).$$

If $A = X$, then we will write $\|\mu\|$ instead of $\|\mu\|(X)$. A vector Borel measure $\mu : \mathcal{B} \to Y$ is called *finite* if $\|\mu\| < \infty$ and *locally finite* if $\|\mu\|(A) < \infty$ for every bounded Borel set A. Let μ be a vector Borel measure on X. We say that μ is *concentrated* on a Borel set A if $\|\mu\|(X \setminus A) = 0$. Two measures μ_1 and μ_2 are called *mutually singular* if there exists disjoint Borel subsets A and B of X such that μ_1 is concentrated on A and μ_2 is concentrated on B. A vector Borel measure μ on X is called *singular* if the measures μ and m are mutually singular. For any vector Borel measure μ on X, we denote by μ_r and μ_s the regular and singular parts of μ, i.e. $\mu_r = (\mu_{1r}, \ldots, \mu_{kr})$, $\mu_s = (\mu_{1s}, \ldots, \mu_{ks})$ and for each $1 \le i \le k$, $\mu_i = \mu_{ir} + \mu_{is}$ is the Lebesgue decomposition of μ_i relative to m.

Denote by $\|\cdot\|$ the norm in Y given by $\|x\| = |x_1| + \ldots + |x_k|$. Let $g : Y \to \mathbb{R}$ be a continuous function. The function g is called *homogeneous* if satisfies the condition $g(tx) = tg(x)$ for every $x \in Y$ and $t \ge 0$. Let $f, g : Y \to \mathbb{R}$ be continuous functions and let g be a homogeneous function. We will say that f is *asymptotically homogeneous* if

$$\lim_{\|x\|\to\infty} \frac{f(x) - g(x)}{\|x\|} = 0. \tag{4}$$

Let μ be a locally finite vector measure on X. If the functions δ_n satisfy conditions (a)–(c), then from Fubini's theorem it follows that

$$\mu * \delta_n(x) = \int \delta_n(x - t)\, \mu(dt)$$

is a locally integrable function with respect to m. If μ is a finite vector measure, then $\mu * \delta_n$ is an integrable function with respect to m.

Let $f : Y \to \mathbb{R}$ be a continuous function and let μ be a locally finite vector measure on X. Assume that there exists a locally finite vector measure ν on X such that

$$\lim_{n\to\infty} \int f(\mu * \delta_n(x))\varphi(x)\, m(dx) = \int \varphi(x)\, \nu(dx) \tag{5}$$

for every δ-sequence (δ_n) and $\varphi \in C_0(X)$. Then we say that the composition of f and μ exists and the measure ν is called the *composition* of f and μ.

The following theorems show when the composition of a function and a measure exists.

THEOREM 1. *Let $f, g : Y \to \mathbb{R}$ be continuous functions and let g be a homogeneous function. Assume that f and g satisfy* (4). *Let ν be a finite non-negative Borel measure on X. Assume that the measure ν is singular. Let $h_1, h_2 : X \to Y$ be Borel functions with bounded supports such that $\|h_1\|$ is integrable with respect to the Lebesgue measure m and $\|h_2\|$ is integrable with respect to the measure ν.*

Let μ be a vector measure on X such that $d\mu = h_1 dm + h_2 d\nu$. Then

$$\lim_{n\to\infty} \int |f(\mu * \delta_n) - f(h_1) - (g(h_2)\nu) * \delta_n|\, dm = 0 \tag{6}$$

for every δ-sequence (δ_n).

THEOREM 2. *Let $f : Y \to \mathbb{R}$ be an asymptotically homogeneous continuous function. If μ is a locally finite vector measure on X, then the composition of f and μ exists.*

Remark 1. It is easy to observe that if f is a Lipschitz function and for every $x \in Y$ the limit $\lim_{t\to\infty} \frac{1}{t} f(xt) = g(x)$ exists, then the functions f and g satisfy the assumptions of Theorems 1 and 2.

Remark 2. If f is a homogeneous function then the composition of f and μ was introduced in [3; Part 5.5.9] in a different way. Namely, let ν be a positive measure and $h : X \to Y$ be a function such that $d\mu = h d\nu$. Then the composition of f and ν is defined by $f(\mu) = f(h)\nu$. It is easy to check that this definition does not depend on the choice of the measure ν. According to Theorem 1, if f is a homogeneous function then our definition coincides with that of Bourbaki.

2. Remarks and examples. The following example shows that in Theorem 1 we cannot omit the assumption that μ is concentrated on a bounded set.

EXAMPLE 1. Let $\mu = \sum_{i=1}^{\infty} \frac{1}{i^2} \delta_{\{i\}}$, where $\delta_{\{i\}}$ denotes the Dirac measure at the point i. Then μ is a finite measure on $\mathbb{R}$. Let $f(x) = \sqrt{|x|}$. Then

$$f(\mu * \delta_n) = \sum_{i=1}^{\infty} \frac{1}{i} \delta_n (x-i)^{1/2}.$$

This implies that

$$\int |f(\mu * \delta_n)|\, dm = \sum_{i=1}^{\infty} \frac{1}{i} \int \delta_n (x-i)^{1/2}\, dm = \infty.$$

Remark 3. From Theorem 1 it follows that if $f(0) = 0$, $\lim_{\|x\|\to\infty} f(x)/\|x\| = 0$ and μ is a finite and singular vector mesure concentrated on a bounded set, then $f(\mu * \delta_n) \to 0$ in $L^1(X)$. The following example shows that the above conditions do not imply that $f(\mu * \delta_n) \to 0$ almost everywhere.

EXAMPLE 2. Let μ be a singular measure on $\mathbb{R}$ concentrated on a bounded set A given by

$$A = \{\, x_{i,n} = i/2^n : n \in \mathbb{N},\ 1 \le i \le 2^n - 1,\ i \text{ is an odd number} \,\}.$$

Assume that $\mu(\{x_{i,n}\}) = n^{-2} 2^{-n}$. Then

$$\mu(\mathbb{R}) = \sum_{n=1}^{\infty} \frac{1}{2n^2} < \infty.$$

Since μ is a finite singular measure concentrated on a bounded set, $(\mu*\delta_n)^{1/2} \to 0$ in $L^1(X)$ for any δ-sequence. Let $\{\delta_n\}$ be a δ-sequence defined by the formula $\delta_n = n(n+1)\mathbf{1}_{[1/(n+1),1/n]}$. If $x \in (0,1) \setminus A$, then

$$\mu * \delta_n(x) = \int \delta_n(x-t)\,\mu(dt) = n(n+1)\mu\big(\big[x - \tfrac{1}{n}, x - \tfrac{1}{n+1}\big]\big).$$

Let $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$, where $x_i \in \{0,1\}$. Then there exists an integer $i_0 > 0$ such that $x_{i_0} = 1$. Let $j > i_0$ and let $y_j = \sum_{i=1}^{j} \frac{x_i}{2^i}$. Then $\mu(\{y_j\}) \geq j^{-2}2^{-j}$. If $n_j = \big[\frac{1}{x-y_j}\big]$, then $y_j \in [x - \frac{1}{n_j}, x - \frac{1}{n_j+1}]$. Consequently,

$$\mu * \delta_{n_j}(x) \geq n_j^2 j^{-2} 2^{-j} \geq 2^j j^{-2}.$$

Thus $\mu * \delta_{n_j}(x) \to \infty$ as $j \to \infty$.

If $Y = \mathbb{R}$, we can formulate Theorems 1 and 2 in a slightly simpler way. Namely, let μ be a Borel measure on X and let $h = \frac{d\mu_r}{dm}$.

COROLLARY 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{x\to-\infty} \frac{f(x)}{x} = a, \qquad \lim_{x\to\infty} \frac{f(x)}{x} = b. \tag{7}$$

If μ is a finite measure concentrated on a bounded set, then

$$\lim_{n\to\infty} \int |f(\mu*\delta_n) - f(h) - b\mu_s^+ * \delta_n + a\mu_s^- * \delta_n|\, dm = 0 \tag{8}$$

for every δ-sequence (δ_n). If μ is a locally finite measure on X, then the composition of f and μ exists and

$$df(\mu) = f(h)dm + bd\mu_s^+ - ad\mu_s^-. \tag{9}$$

Corollary 1 is a generalization of an earlier result obtained by Antosik [2], who considered the case $X = Y = \mathbb{R}$. We can apply Corollary 1 to compute the area of a surface of revolution.

EXAMPLE 3. Let S be the area of a surface obtained by rotating a curve $y = y(x)$, $a \leq x \leq b$ $(y(x) \geq 0)$ about the x-axis. Assume that $y(x)$ is a continuous function of bounded variation. Let $\mu([a,x)) = y(x) - y(a)$, $h = dy/dx$ and let μ_s be the singular part of the measure μ and $|\mu_s| = \mu_s^+ + \mu_s^-$. From Fubini's theorem it easily follows that the function $y*\delta_n$ is a.e. differentiable and $(y*\delta_n)' = \mu*\delta_n$. Let φ be a continuous function with compact support such that $\varphi(x) = y(x)$ for $x \in [a,b]$. Since $(y * \delta_n)$ is a sequence of a.e. differentiable functions converging uniformly to y, we can assume that

$$\begin{aligned} S &= \lim_{n\to\infty} 2\pi \int_a^b y(x)\sqrt{1 + (\mu*\delta_n(x))^2}\, dx \\ &= \lim_{n\to\infty} 2\pi \int \varphi(x)\sqrt{1 + (\mu*\delta_n(x))^2}\, dx. \end{aligned}$$

In our case $f(x) = 2\pi\sqrt{1+x^2}$. This implies that

$$df(\mu) = 2\pi\sqrt{1+h^2}\, dm + 2\pi\, d|\mu_s|.$$

Consequently,

$$S = 2\pi \int \varphi(x)\sqrt{1+h^2(x)}\,dx + 2\pi \int \varphi(x)\,|\mu_s|(dx)$$
$$= 2\pi \int_a^b y(x)\sqrt{1+h^2(x)}\,dx + 2\pi \int_a^b y(x)\,|\mu_s|(dx).$$

In a similar way we can apply Theorems 1 and 2 to compute line integrals when curves are defined by functions of bounded variation. Now we give an application of our results in case $X = \mathbb{R}^2$. Namely, we show when a surface $z = g(x, y)$ has a finite area.

Example 4. Let $g : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. For each $x \in [a, b]$ and $y \in [c, d]$ denote by $V_x g$ and $V_y g$ the total variations of the function $y \mapsto g(x, y)$ in $[c, d]$ and the function $x \mapsto g(x, y)$ in $[a, b]$, respectively. Assume that

$$\int_a^b V_x g\,dx < \infty \quad \text{and} \quad \int_c^d V_y g\,dy < \infty. \tag{10}$$

Define two interval functions by

$$\mu([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) = \int_{\alpha_1}^{\beta_1} (g(x, \beta_2) - g(x, \alpha_2))\,dx,$$
$$\nu([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) = \int_{\alpha_2}^{\beta_2} (g(\beta_1, y) - g(\alpha_1, y))\,dy,$$

where $a \le \alpha_1 \le \beta_1 \le b$ and $c \le \alpha_2 \le \beta_2 \le d$. It is easy to check that μ and ν are additive interval functions of bounded variation. From [4, Theorems 7.2.1 and 7.5.5] it follows that μ and ν are finite measures. From Fubini's theorem it follows that the derivatives $\frac{\partial}{\partial x}(g * \delta_n)$ and $\frac{\partial}{\partial y}(g * \delta_n)$ exist a.e. for any δ-function δ_n and

$$\frac{\partial}{\partial x}(g * \delta_n) = \nu * \delta_n, \qquad \frac{\partial}{\partial y}(g * \delta_n) = \mu * \delta_n.$$

Since the sequence $(g * \delta_n)$ converges uniformly to g, we can assume that the area of the surface $z = g(x, y)$, $a \le x \le b$, $c \le y \le d$, is given by

$$S = \lim_{n\to\infty} \int\int \sqrt{1 + (\nu * \delta_n)^2(x, y) + (\mu * \delta_n)^2(x, y)}\,dx\,dy.$$

From Theorem 2 it follows that the last limit exists and does not depend on the sequence (δ_n). This implies that if the function g satisfies condition (10), then the surface defined by this function has a finite area. Moreover, we can estimate S by

$$S \le (d - c)(b - a) + \int_a^b V_x g\,dx + \int_c^d V_y g\,dy.$$

3. Proofs. We split the proofs of Theorems 1 and 2 into a sequence of lemmas. Throughout the proofs d denotes the distance in X. The distance between a point and a set and the distance between two sets are defined, respectively, by

$$d(x,A)=\inf_{y\in A} d(x,y), \qquad d(A,B)=\inf_{y\in B} d(y,A).$$

LEMMA 1. *Let $f:Y\to\mathbb{R}$ be a Lipschitz function, $f(0)=0$ and let μ and ν be finite vector measures on $\mathcal{B}(X)$. If the measures μ and ν are mutually singular then*

$$\lim_{n\to\infty}\int |f(\mu*\delta_n+\nu*\delta_n)-f(\mu*\delta_n)-f(\nu*\delta_n)|\,dm=0. \tag{11}$$

Proof. Let A and B be disjoint Borel sets such that $\|\mu\|(X\setminus A)=0$ and $\|\nu\|(X\setminus B)=0$. Let ε be a positive constant. Denote by F and H compact subsets of X such that $F\subset A$, $H\subset B$ and

$$\|\mu\|(X\setminus F)<\varepsilon, \qquad \|\nu\|(X\setminus H)<\varepsilon.$$

Let $\rho=d(F,H)$ and

$$F_\rho=\{x\in X: d(x,F)\le\rho/3\}, \qquad H_\rho=\{x\in X: d(x,H)\le\rho/3\}.$$

Denote by L the Lipschitz constant corresponding to f. Let n_0 be an integer such that $\operatorname{supp}\delta_n\subset K(0,\rho/3)$ for $n\ge n_0$. Then

$$\begin{aligned}
\int |f(\mu*\delta_n&+\nu*\delta_n)-f(\mu*\delta_n)-f(\nu*\delta_n)|\,dm\\
&\le \int_{F_\rho} |f(\mu*\delta_n+\nu*\delta_n)-f(\mu*\delta_n)-f(\nu*\delta_n)|\,dm\\
&\quad+\int_{H_\rho} |f(\mu*\delta_n+\nu*\delta_n)-f(\mu*\delta_n)-f(\nu*\delta_n)|\,dm\\
&\quad+2L\int_{X\setminus(F_\rho\cup H_\rho)} \|\mu*\delta_n\|+\|\nu*\delta_n\|\,dm\\
&\le \int_{F_\rho} 2L\|\nu*\delta_n\|\,dm+\int_{H_\rho} 2L\|\mu*\delta_n\|\,dm\\
&\quad+2L\int_{X\setminus(F_\rho\cup H_\rho)} \|\mu*\delta_n\|+\|\nu*\delta_n\|\,dm\\
&=2L\Big(\int_{X\setminus H_\rho}\|\nu*\delta_n\|\,dm+\int_{X\setminus F_\rho}\|\mu*\delta_n\|\,dm\Big)\le 4L\varepsilon
\end{aligned}$$

which completes the proof.

LEMMA 2. *Let $f:Y\to\mathbb{R}$ be a Lipschitz function such that $f(0)=0$ and*

$$\lim_{\|x\|\to\infty}\frac{f(x)}{\|x\|}=0. \tag{12}$$

Assume that μ is a finite singular vector measure on $\mathcal{B}(X)$. Then

$$\lim_{n\to\infty}\int|f(\mu*\delta_n)|\,dm=0. \tag{13}$$

P r o o f. Let ε be a positive constant. Since the measure μ is singular, there exists a compact set $F\subset X$ such that

$$\|\mu\|(X\setminus F)<\varepsilon \quad\text{and}\quad m(F)=0. \tag{14}$$

Let μ_1 and μ_2 be vector measures on $\mathcal{B}(X)$ given by the formulas

$$\mu_1(A)=\mu(A\cap F),\quad \mu_2(A)=\mu(A\cap(X\setminus F))\quad\text{for } A\in\mathcal{B}(X). \tag{15}$$

From (12) it follows that there exists a constant M_ε such that

$$|f(x)|\le M_\varepsilon+\varepsilon\|x\|. \tag{16}$$

Let $F_n=\{\,x\in X: d(x,F)\le\frac{1}{n}\,\}$. Then $\operatorname{supp}\mu_1*\delta_n\subset F_n$. From (16) it follows that

$$\int|f(\mu_1*\delta_n)|\,dm\le\int_{F_n}M_\varepsilon+\varepsilon\|\mu_1*\delta_n\|\,dm\le M_\varepsilon m(F_n)+\varepsilon\|\mu_1\|. \tag{17}$$

Since f satisfies the Lipschitz condition, we have

$$\int|f(\mu_2*\delta_n)|\,dm\le L\int\|\mu_2*\delta_n\|\,dm\le L\|\mu_2\|\le L\varepsilon, \tag{18}$$

where L is the Lipschitz constant. From (17), (18) and Lemma 1 it follows that

$$\int|f(\mu*\delta_n)|\,dm\le M_\varepsilon m(F_n)+\varepsilon\|\mu_1\|+L\varepsilon. \tag{19}$$

Since $m(F_n)\to 0$ as $n\to\infty$, we obtain

$$\limsup_{n\to\infty}\int|f(\mu*\delta_n)|\,dm\le\varepsilon(L+\|\mu_1\|),$$

which completes the proof.

LEMMA 3. *Let $f:Y\to\mathbb{R}$ be a continuous function such that*

$$\lim_{\|x\|\to\infty}\frac{f(x)}{\|x\|}=0.$$

Assume that μ is a finite vector measure on $\mathcal{B}(X)$ concentrated on a bounded set. Let μ_r be the regular part of μ and $h=\frac{d\mu_r}{dm}$. Then

$$\lim_{n\to\infty}\int|f(\mu*\delta_n)-f(h)|\,dm=0. \tag{20}$$

P r o o f. Without loss of generality we can assume that $f(0)=0$. Let ε be a positive constant. Then there exists $M_\varepsilon>0$ such that $|f(x)|\le\varepsilon\|x\|$ for $\|x\|\ge M_\varepsilon$. Let $\theta:[0,\infty)\to[0,1]$ be a continuous function such that $\theta(t)=1$ for $t\in[0,1]$ and $\theta(t)=0$ for $t\ge 2$. Let f_1 and f_2 be functions given by

$$f_1(x)=\theta\left(\frac{\|x\|}{M_\varepsilon}\right)f(x)\quad\text{and}\quad f_2(x)=f(x)-f_1(x).$$

Then $|f_2(x)| \le \varepsilon\|x\|$ for every $x \in Y$. This implies that

$$\begin{aligned}\int |f_2(\mu * \delta_n)|\,dm &\le \varepsilon \int \|\mu * \delta_n\|\,dm \le \varepsilon\|\mu\|,\\ \int |f_2(h)|\,dm &\le \varepsilon\|\mu_r\| \le \varepsilon\|\mu\|.\end{aligned} \tag{21}$$

Since f_1 is a continuous function with bounded support, there exists a Lipschitz function f_ε such that $f_\varepsilon(0) = 0$ and

$$|f_1(x) - f_\varepsilon(x)| < \varepsilon \quad \text{for every } x \in Y.$$

For some R the measure μ is concentrated on $K(0,R)$. Consequently, $\operatorname{supp} \mu*\delta_n \subset K(0,R+1)$. From this it follows that

$$\int |f_1(\mu * \delta_n) - f_\varepsilon(\mu * \delta_n)|\,dm \le \int_{K(0,R+1)} \varepsilon\,dm \le \varepsilon(2R+2)^p$$

and

$$\int |f_1(h) - f_\varepsilon(h)|\,dm \le \varepsilon(2R)^p.$$

From these inequalities and from (21) it follows that

$$\begin{aligned}\int |f(\mu * \delta_n) - f(h)|\,dm \\ \le \int |f_\varepsilon(\mu * \delta_n) - f_\varepsilon(h)|\,dm + 2\varepsilon(2R+2)^p + 2\varepsilon\|\mu\|.\end{aligned} \tag{22}$$

Let $\mu_s = \mu - \mu_r$. Then μ_s is a singular measure. From Lemmas 1 and 2 it follows that

$$\lim_{n\to\infty} \int |f_\varepsilon(\mu * \delta_n) - f_\varepsilon(\mu_r * \delta_n)|\,dm = 0. \tag{23}$$

From (22) and (23) it follows that

$$\begin{aligned}\limsup_{n\to\infty} \int |f(\mu * \delta_n) - f(h)|\,dm \le \limsup_{n\to\infty} \int |f_\varepsilon(\mu_r * \delta_n) - f_\varepsilon(h)|\,dm \\ + 2\varepsilon(2R+2)^p + 2\varepsilon\|\mu\|.\end{aligned}$$

Since f_ε is a Lipschitz function, we have

$$|f_\varepsilon(\mu_r * \delta_n) - f_\varepsilon(h)| \le L_\varepsilon\|h * \delta_n - h\|$$

and, consequently,

$$\begin{aligned}\limsup_{n\to\infty} \int |f(\mu * \delta_n) - f(h)|\,dm \\ \le \lim_{n\to\infty} \int L_\varepsilon\|h * \delta_n - h\|\,dm + 2\varepsilon((2R+2)^p + \|\mu\|) \\ = 2\varepsilon((2R+2)^p + \|\mu\|).\end{aligned}$$

Since ε is any positive number, the last inequality implies (20), which completes the proof.

P r o o f o f T h e o r e m 1. Let $p(x) = f(x) - g(x)$. Then from Lemma 3 it follows that

$$\lim_{n\to\infty} \int |p(\mu * \delta_n) - p(h_1)|\, dm = 0. \tag{24}$$

Let ε be a positive constant and let S be a unit sphere in Y. Since g is a continuous function, there exists a Lipschitz function $g_\varepsilon : S \to \mathbb{R}$ such that $|g_\varepsilon(x) - g(x)| \le \varepsilon$ for $x \in S$. Setting $g_\varepsilon(x) = \|x\| g_\varepsilon(x/\|x\|)$ for $x \neq 0$ and $g_\varepsilon(0) = 0$ we obtain a Lipschitz homogeneous function such that

$$|g_\varepsilon(x) - g(x)| \le \varepsilon \|x\|. \tag{25}$$

Let μ_1 and μ_2 be vector measures such that $d\mu_1 = h_1\, dm$ and $d\mu_2 = h_2\, d\nu$. Then the measures μ_1 and μ_2 are mutually singular and from Lemma 1 it follows that

$$\lim_{n\to\infty} \int |g_\varepsilon(\mu * \delta_n) - g_\varepsilon(\mu_1 * \delta_n) - g_\varepsilon(\mu_2 * \delta_n)|\, dm = 0.$$

Since g_ε is a Lipschitz function,

$$\lim_{n\to\infty} \int |g_\varepsilon(\mu_1 * \delta_n) - g_\varepsilon(h_1)|\, dm = 0.$$

Consequently,

$$\lim_{n\to\infty} \int |g_\varepsilon(\mu * \delta_n) - g_\varepsilon(h_1) - g_\varepsilon(\mu_2 * \delta_n)|\, dm = 0. \tag{26}$$

From (24), (25) and (26) it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \int & |f(\mu * \delta_n) - f(h_1) - (g(h_2)\nu) * \delta_n|\, dm \\
&\le \limsup_{n\to\infty} \int |g_\varepsilon(\mu_2 * \delta_n) - (g_\varepsilon(h_2)\nu) * \delta_n|\, dm \\
&\quad + \varepsilon \int \|\mu * \delta_n\| + \|h_1\| + (\|h_2\|\nu) * \delta_n\, dm \\
&\le \limsup_{n\to\infty} \int |g_\varepsilon(\mu_2 * \delta_n) - (g_\varepsilon(h_2)\nu) * \delta_n|\, dm + 2\varepsilon\|\mu\|.
\end{aligned}$$

Since the last inequalities hold for any $\varepsilon > 0$, it remains to prove that for every Lipschitz homogeneous function $\overline{g}$ we have

$$\limsup_{n\to\infty} \int |\overline{g}(\mu_2 * \delta_n) - (\overline{g}(h_2)\nu) * \delta_n|\, dm = 0. \tag{27}$$

In order to do this, we first assume that $h_2 = c\mathbf{1}_A$, where $c \in Y$ and A is a Borel subset of X. Then

$$\begin{aligned}
\overline{g}(\mu_2 * \delta_n) &= \overline{g}(c(\mathbf{1}_A\nu) * \delta_n) = \overline{g}(c)((\mathbf{1}_A\nu) * \delta_n) \\
&= (\overline{g}(c)\mathbf{1}_A\nu) * \delta_n = (\overline{g}(h_2)\nu) * \delta_n.
\end{aligned}$$

If h_2 is any function satisfying the assumptions of Theorem 1, then for every $\varepsilon > 0$, there exists disjoint compact sets $F_1, \ldots, F_q$ and $c_1, \ldots, c_q \in Y$ such that

$$\int \|h_2 - h^q\|\, d\nu < \varepsilon, \tag{28}$$

where $h^q = c_1\mathbf{1}_{F_1} + \ldots + c_q\mathbf{1}_{F_q}$. Let $\rho = \min_{i\neq j} d(F_i, F_j)$. Then $\rho > 0$. Since $\operatorname{supp}\delta_n \subset K(0,\rho/3)$ for sufficiently large n (say $n \ge n_0$), we have

$$\overline{g}((h^q\nu)*\delta_n) = \sum_{j=1}^{q}\overline{g}((c_j\mathbf{1}_{F_j}\nu)*\delta_n) = \sum_{j=1}^{q}(\overline{g}(c_j\mathbf{1}_{F_j})\nu)*\delta_n = (\overline{g}(h^q)\nu)*\delta_n. \tag{29}$$

Since $\overline{g}$ is a Lipschitz function, we have

$$|\overline{g}(\mu_2*\delta_n(x)) - \overline{g}((h^q\nu)*\delta_n(x))| \le L\|\mu_2*\delta_n(x) - (h^q\nu)*\delta_n(x)\| \le L(\|h_2 - h^q\|\nu)*\delta_n(x).$$

The last inequality and (28) imply

$$\int |\overline{g}(\mu_2*\delta_n) - \overline{g}((h^q\nu)*\delta_n)|\,dm \le L\varepsilon. \tag{30}$$

Analogously,

$$\int |(\overline{g}(h_2)\nu)*\delta_n - (\overline{g}(h^q)\nu)*\delta_n|\,dm \le L\varepsilon. \tag{31}$$

From (29), (30) and (31) it follows that

$$\int |\overline{g}(\mu_2*\delta_n) - (\overline{g}(h_2)\nu)*\delta_n|\,dm \le 2L\varepsilon$$

for sufficiently large n. This implies (27) and completes the proof.

Proof of Theorem 2. Let μ_r and μ_s be the regular and singular parts of the measure μ and let $\nu = \sum_{i=1}^{k}(\mu_s)_i^+ + (\mu_s)_i^-$. If $h_1 = \frac{d\mu_r}{dm}$ and $h_2 = \frac{d\mu_s}{d\nu}$, then $d\mu = h_1\,dm + h_2\,d\nu$. Since φ has bounded support, we can assume that $\operatorname{supp}\varphi \subset K(0,r)$, where r is a positive constant. Let $\widetilde{\mu}$ be the vector measure given by

$$\widetilde{\mu}(A) = \mu(A\cap K(0,r+1)) \quad \text{for } A \in \mathcal{B}(X).$$

Then $\widetilde{\mu}*\delta_n(x) = \mu*\delta_n(x)$ for $x \in K(0,r)$. Let $\psi = \mathbf{1}_{K(0,r+1)}$. Then $d\widetilde{\mu} = \psi h_1\,dm + \psi h_2\,d\nu$. From Theorem 1 it follows that

$$\lim_{n\to\infty}\int |f(\widetilde{\mu}*\delta_n) - f(\psi h_1) - (g(\psi h_2)\nu)*\delta_n|\,dm = 0. \tag{32}$$

This implies that

$$\lim_{n\to\infty}\int f(\widetilde{\mu}*\delta_n)\varphi\,dm = \int f(\psi h_1)\varphi\,dm + \lim_{n\to\infty}\int (g(\psi h_2)\nu)*\delta_n\varphi\,dm = \int f(\psi h_1)\varphi\,dm + \int g(\psi h_2)\varphi\,d\nu.$$

Since $\widetilde{\mu}*\delta_n(x) = \mu*\delta_n(x)$ and $\psi(x) = 1$ for $x \in \operatorname{supp}\varphi$, the last condition implies that

$$\lim_{n\to\infty}\int f(\mu*\delta_n)\varphi\,dm = \int f(h_1)\varphi\,dm + \int g(h_2)\varphi\,d\nu.$$

Consequently, the composition of f and μ exists and this composition is the measure $\overline{\mu}$ satisfying the condition $d\overline{\mu} = f(h_1)\,dm + g(h_2)\,d\nu$.

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